A semi-classical trace formula for Schrödinger operators in the case of a critical energy level

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Abstract

Let $\widehat{H} = -\frac{\hbar^2}{2}\Delta + V(x)$ be a Schrödinger operator on $\mathbb{R}^n$, with smooth potential $V(x) \to +\infty$ as $|x| \to +\infty$. The spectrum of $\widehat{H}$ is discrete, and one can study the asymptotic of the smoothed spectral density

$$\Upsilon(E,\hbar) = \sum_k \varphi\left(\frac{E_k(\hbar) - E}{\hbar}\right),$$

as $\hbar \to 0$. Here, $\{E_k(\hbar)\}_{k\in\mathbb{N}}$ is the spectrum of $\widehat{H}$ and $\hat{\varphi} \in C_0^\infty(\mathbb{R})$. We investigate the case where E is a critical value of the symbol H of $\widehat{H}$ and, extending the work of Brummelhuis, Paul and Uribe in [3], we prove the existence of a full asymptotic expansion for Υ in terms of $\sqrt{\hbar}$ and $\ln\hbar$ and compute the leading coefficient. We consider some new Weyl-type estimates for the counting function : $N_{E_c,\rho}(\hbar) = \#\{k \in \mathbb{N} \ / \ |E_k(\hbar) - E| \leq \rho\hbar\}$.

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1 Introduction and main statements.

Consider a Schrödinger operator

$$\widehat{H} = -\frac{\hbar^2}{2}\Delta + V(x) \quad \text{on } I\!R^n \qquad\qquad (n \geq 1)$$

with $V \in C^\infty(I\!R^n)$ and $\lim_{|x|\to+\infty} V(x) = +\infty$. Then, from well known results, the spectrum of $\widehat{H}$ consists in a discrete set of real eigenvalues : $E_0(\hbar) \leq E_1(\hbar) \leq \cdots$ where we count each eigenvalue according to its (finite) multiplicity.

It is still an open problem to find, in all the cases, an asymptotic expansion, as $\hbar \to 0$, of these eigenvalues in terms of the classical dynamic in phase space $I\!R^{2n}$ generated by the corresponding Hamiltonian : $H(x,\xi) = \xi^2/2 + V(x)$. This problem is only solved in the case of an integrable classical system with the WKB method which leads to the EBK (or Bohr-Sommerfeld) quantization. For more generic Hamiltonians, the semi-classical trace formula gives us one of the most important tool in this direction. There has been many mathematical publications on the subject which lead, many years after Gutzwiller has introduced it [15], to a rigorous version of this formula [19, 14, 4, 18, 21, 20, 7, 8]. Let us now describe in detail this result.

For every φ in the Schwartz space $\mathcal{S}(I\!R)$, we will note $\hat{\varphi}$ for the Fourier transform of φ

$$\hat{\varphi}(t) = \int_{-\infty}^{+\infty} e^{-i\tau t}\varphi(\tau)d\tau$$

and $\check{\varphi}$ for the inverse Fourier transform, so that $\check{\varphi}(t) = (2\pi)^{-1}\hat{\varphi}(-t)$.

Consider a strictly positif real number λ and define the smoothed spectral density

$$\Upsilon(E,\hbar) = \sum_{E_k(\hbar)\in[E-\lambda,E+\lambda]} \varphi\left(\frac{E_k(\hbar)-E}{\hbar}\right)$$

for any $\hat{\varphi} \in C_0^\infty(I\!R)$ and any $E \in I\!R$.

Consider the following two assumptions

H1 *E is a regular value of the classical Hamiltonian H,*

H2 *the Hamiltonian flow ϕ_t associated to H is a clean flow on $\Sigma_E = H^{-1}(E)$.*

We recall that ϕ_t is a clean flow on Σ_E if
- $\mathcal{P}_E = \{(t,x,\xi) \in I\!R \times \Sigma_E \;\; / \;\; \phi_t(x,\xi) = (x,\xi)\}$ is a smooth submanifold of $I\!R \times \Sigma_E$ and
- $\forall (t,x,\xi) \in \mathcal{P}_E$, $T_{t,x,\xi}\mathcal{P}_E = \{(\delta t, \delta x, \delta\xi) \;\; / \; \delta t.\Xi(x,\xi) + d\phi_t(x,\xi).(\delta x, \delta\xi) = (\delta x, \delta\xi)\}$
where $\Xi(x,\xi)$ is the infinitesimal generator of the flow at (x,ξ).

Then the semiclassical trace formula says that if (H1) and (H2) are satisfied, one can find a sequence of distribution $\{d_{l,k}\}$ such that, as $\hbar \to 0$,

$$\Upsilon(E,\hbar) = \sum_{l=1}^{L} e^{\frac{i}{\hbar}S_l} \hbar^{\frac{1-\dim Y_l}{2}} \left(\sum_{k \in I\!N} < d_{l,k}, \hat{\varphi} > \hbar^k \right) + \mathcal{O}(\hbar^\infty)$$

where
- $(Y_1, \cdots, Y_L)$ are the connected components of the periodic orbit manifold $\mathcal{P}_E$ such that $Y_l \cap \mathrm{supp}\; \hat{\varphi} \neq \emptyset$ and
- S_l is the action of any arbitrary periodic orbit of Y_l.

The leading distribution $d_{l,0}$ of each connected component can be effectively calculated and related to an intrinsic density defined on Y_l. If, for example, we assume that the periodic points of non-zero period are of zero (Liouville) measure on Σ_E, we obtain

$$\Upsilon(E,\hbar) = \hat{\varphi}(0) \frac{\mathrm{LVol}(\Sigma_E)}{(2\pi)^n} \hbar^{-n+1} + \mathcal{O}(\hbar^{-n+2}).$$

where $\mathrm{LVol}(\Sigma_E)$ stands for the Liouville measure of Σ_E.

Remark 1.1 In the case of strictly positive Liouville measure of the periodic points of Σ_E, we have :

$$\Upsilon(E,\hbar) = \nu_0^{E,\hbar}(\hat{\varphi}) \hbar^{-n+1} + \mathcal{O}(\hbar^{-n+2}) \tag{1}$$

where $\nu_0^{E,\hbar}(\hat{\varphi})$ is a density which is oscillating in $\hbar$ and supported by the periods (non necessarily discrete) of the periodic orbits (including the zero period). If we assume, for example, that Σ_E is fully periodic with a constant period equal to T, then (cf [8])

$$\nu_0^{E,\hbar}(\hat{\varphi}) = \frac{\mathrm{LVol}(\Sigma_E)}{(2\pi)^n} \sum_{k \in Z\!\!\!Z} e^{-\frac{i}{\hbar}k(S+TE) - ik\pi\mu/2} \hat{\varphi}(kT)$$

where S and μ are the common action and Maslov index of Σ_E. In [22] was find an explicit expression for $\nu_0^{E,\hbar}(\hat{\varphi})$ in the general case, and it is proven that formula (1) remains valid even if the Hamiltonian flow is not clean (with a worse error term) : cf Th. 1.4 of [22].

The proof of all these results is based on the theory of FIO (Fourier Integral Operators) where $\Upsilon(E,\hbar)$ can be seen "asymptoticaly" as the composition of two such FIO. The assumptions (H1) and (H2) are just the conditions that allow us to apply the general theorem for the clean composition of two semi-classical FIO (cf [18]). In this paper, as in [3], we wish to eliminate the hypothesis (H1) which will lead to dealing with degenerate phase functions in the oscillatory integrals that we will meet. The theory of oscillatory integrals with degenerate phase functions has an important background in the mathematical literature. A general review may be found in the book of Arnold, Gusein-Zade and Varchenko [1] which shows the complexity of the problem. In order to deal with simple degeneracies, we will make the following assumptions on Θ which is the set of critical points of H and which is simply, in our case,

$$\Theta = \{(x,0) \in I\!R^{2n} \;\; / \;\; dV(x) = 0\},$$

H3 Θ is a compact submanifold (without boundary) of $I\!R^{2n}$,

H4 H has a non degenerate normal hessian on Θ, that is $d^2H(x,0)$ restricted to $T_{x,0}I\!R^{2n}/T_{x,0}\Theta$ is non degenerate for all $(x,0) \in \Theta$,

H5 the multiplicity of the eigenvalues of the normal hessian of H is independent of $(x,0) \in \Theta$.

Remark 1.2 These are the same assumptions as in [3] and, because the connected components of Θ are contained in level sets of H and since we are interested in estimates that are local in energy, we will also assume, with no loss of generality, that Θ is connected and included in $\Sigma_{E_c} = H^{-1}(E_c)$ for a certain value, E_c, of H. We will suppose moreover that Σ_{E_c} is also connected. Because our proofs use only micro-local technics and not global properties, we only need to sum up, in the case of many connected components of Θ or Σ_{E_c}, all the contributions coming from these connected components. We remark that in this general case, the assumptions (H3), (H4) and (H5) need only to be satisfied for the part of Θ contained in a small neighborhood of Σ_{E_c}

In [3] was shown, under the assumptions (H3), (H4) and (H5), that $\Upsilon(E_c, \hbar)$ has an asymptotic expansion in terms of $\sqrt{\hbar}$ and $\ln \hbar$, but it was assumed that the support of $\hat{\varphi}$ does not contain any period of the linearised flow [1] on Θ (see appendix B).

As we will see later, we will have to include the contribution of the periods of the linearised flow in the asymptotic expansion of $\Upsilon(E_c, \hbar)$ in order to eliminate the hypothesis on the support of $\hat{\varphi}$. When (H1) is not assumed, the hypothesis (H2) can be not true (except for very special cases like the top of a double well in one dimension), and we have to weaken (H2) in the following way :

H2bis the Hamiltonian flow ϕ_t is a clean flow [2] on $\Sigma_{E_c} \setminus \Theta$.

Let

$$(\alpha_1(x)^2, \cdots, \alpha_r(x)^2, -\alpha_{r+1}(x)^2, \cdots, -\alpha_{r+\nu}(x)^2, 0, \cdots, 0) \qquad (\alpha_i(x) > 0)$$

be the eigenvalues of $d^2V(x)$ for all $(x,0) \in \Theta$ and define the codimension of Θ in $I\!R^{2n}$ by

$$N := \mathrm{codim}\Theta = 2n - \dim\Theta = n + r + \nu.$$

The assumptions (H3), (H4) and (H5) tell us that N, r and ν are independent of the point $(x,0) \in \Theta$.

We can now state our main result :

Theorem 1.3 *Let φ be a test function such that $\hat{\varphi} \in C_0^\infty(I\!R)$. Assume, for simplicity, that the periodic points on $\Sigma_{E_c} \setminus \Theta$ of non zero period are of zero (Liouville)-measure. Then, under the assumptions (H2bis), (H3), (H4) and (H5), we have, as $\hbar \to 0$,*

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[1] We say that T is a period of the linearised flow at $z \in \Theta$ if there exist a vector $u \in T_zI\!R^{2n} \setminus T_z\Theta$ so that $d\phi_T(z).u = u$.

[2] i.e.

- $\mathcal{P}_{E_c\setminus\Theta} = \{(t,x,\xi) \in I\!R \times (\Sigma_{E_c} \setminus \Theta) \ / \ \phi_t(x,\xi) = (x,\xi)\}$ is a submanifold of $I\!R^{1+2n}$
- $\forall (t,x,p) \in \mathcal{P}_{E_c\setminus\Theta}$, $T_{t,x,\xi}\mathcal{P}_{E_c\setminus\Theta} = \{(\delta t, \delta x, \delta\xi) \in T_{(t,x,\xi)}(I\!R \times \Sigma_{E_c}) \ / \ \delta t.\Xi(x,\xi) + d\phi_t(x,\xi).(\delta x, \delta\xi) = (\delta x, \delta\xi)\}$

1. *if $\nu \geq 1$ and if ν and $N-\nu$ are odd, then N is even and*

$$\Upsilon(E_c,\hbar) \sim \hbar^{-n+1}\left[\sum_{j=0}^{N/2-2} c_j\hbar^j + \hbar^{N/2-1}\sum_{j=0}^{\infty}\sum_{l=0,1} c_{j,l}\hbar^j(-\ln\hbar)^l\right], \tag{2}$$

with leading coefficients

$$a)\ if \quad N \geq 4 : c_0 = \frac{LVol(\Sigma_{E_c})}{(2\pi)^n}\hat{\varphi}(0) \tag{3}$$

$$b)\ if \quad N = 2 : c_{0,1} = -\frac{1}{\pi\alpha_1}\hat{\varphi}(0)\ \ and \tag{4}$$

$$c_{0,0} = \int \varphi(\tau)f(\tau)d\tau. \tag{5}$$

where f is analytic in the region $|Im\tau| < \frac{\alpha_1}{2}$ and $f(\tau) = o(|\tau|)$ at infinity.

2. *if $\nu \geq 1$ and if ν or $N-\nu$ is even, then $N \geq 3$ and*

$$\Upsilon(E_c,\hbar) \sim \hbar^{-n+1}\left[\frac{LVol(\Sigma_{E_c})}{(2\pi)^n}\hat{\varphi}(0) + \sum_{j=1}^{+\infty} c_j\hbar^{j/2}\right] \tag{6}$$

3. *if $\nu = 0$, then $\Theta = \Sigma_{E_c}$ and*

$$\Upsilon(E_c,\hbar) \sim \hbar^{N/2-n}\sum_{j=0}^{\infty} c_j\hbar^j \tag{7}$$

with leading coefficient

$$c_0 = (2\pi)^{N/2-n}2^{n-N}e^{-i\pi N/4}\int_\Theta < \frac{1}{(t-i0)^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i0))}, \check{\varphi}(t) > dx. \tag{8}$$

Remark 1.4 1. Formula (41) gives a more explicit expression for f.

2. The distribution introduced in formula (8) is studied in Theorem 3.1.11 of [16] and can be defined as

$$\begin{aligned} &< \frac{1}{(t-i0)^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i0))}, \check{\varphi}(t) > \\ = &\lim_{\epsilon\to 0+}\int_{I\!R} \frac{\check{\varphi}(t)}{(t-i\epsilon)^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i\epsilon))}dt. \end{aligned}$$

3. In the case of strictly positive measure of the periodic points, the theorem is the same except for the leading coefficients of (3) and (6) where they have to be replaced as in the remark (1.1).

4. The only possibility for the case 1)b) is n=1 and $\dim\Theta = 0$. This corresponds to the top(s) of a double-well (or multiple-well in the case of many connected components of Θ).

5. The proposition 2.1 is a crucial point in the demonstration of this theorem because it tells us that any periodic point of Σ_{E_c} of period smaller than a given constant stands outside a sufficiently small neighborhood of Θ, this neighborhood depending only on that constant. This is wrong for a general Hamiltonian as is shown in the remark 2.2 ; that's why we had to restrict ourselves to Schrödinger-type operators.

6. The contribution coming from the periods of the linearised flow appears as a regularization of the Duistermaat-Guillemin density which has been defined in [11], [14],[3] and is studied in appendix A. This regularization appears explicitly in (8), but it is also present in the other cases as a contribution to lower orders. They are nevertheless different because they make use of all the support of $\hat{\varphi}$ and not only the period of the periodic orbits. This is because every real can be seen as the period of a point on Θ.

In the third case of this theorem, we can express the leading coefficient as the action of a density on φ in the following way:

Theorem 1.5 *With the hypothesis of Theorem 1.3 and in the case where $\nu = 0$, we have*

$$\Upsilon(E_c, \hbar) \sim \hbar^{N/2-n} \sum_{j=0}^{\infty} c_j \hbar^j \tag{9}$$

with leading coefficient

$$\text{1) if } d > 0 \ : \ c_0 = \frac{(2\pi)^{-d/2}}{\Gamma(d/2)} \int_{\mathbb{R}} \left[\int_{\Theta} \sum_{k \in \mathbb{N}^{N-n}} (\tau - \beta_k(x))_+^{d/2-1} dx \right] \varphi(\tau) d\tau \tag{10}$$

$$\text{2) if } d = 0 \ : \ c_0 = \sum_{k \in \mathbb{N}^{N-n}} \varphi(\beta_k) \tag{11}$$

where d is the dimension of Θ,

$$\beta_k(x) = \alpha_1(x)(k_1 + \frac{1}{2}) + \cdots + \alpha_{N-n}(x)(k_{N-n} + \frac{1}{2}) \qquad \text{if } k = (k_1, \cdots, k_{N-n})$$

and $(\tau - \beta)_+^{d/2-1}$ is the function equal to $(\max(0, \tau - \beta))^{d/2-1}$.

Remark 1.6 1. The sum of (10) is necessarily a finite sum.

2. In this theorem, the hypothesis (H2bis) is trivially satisfied because $\Theta = \Sigma_{E_c}$.

3. The case $d = 0$ corresponds to the case where E_c is the energy of the bottom of a potential well. This case was studied more precisely in [24].

Using these results, one would like to obtain some Weyl-type estimates for the counting function

$$N_{E_c,\rho}(\hbar) := \#\{k \in \mathbb{N} \ / \ |E_k(\hbar) - E_c| < \rho\hbar\}$$

by letting φ equal to the characteristic function : $\chi_{[-\rho,+\rho]}$. In the case where $N = 2$, it is proved in [3] that this is possible because the linearised flow does not have any non-zero periods so that we can apply Theorem 1.3 of [3]. If this could be done in the other cases, we would obtain for example, in the case where $\nu = 0$ and with some easy calculations

$$N_{E_c,\rho}(\hbar) = \frac{(2\pi\hbar)^{-d/2}}{\Gamma(d/2+1)} \int_{\Theta} \sum_{k \geq 0} (\rho - \beta_k(x))_+^{d/2} dx + o(\hbar^{-d/2}) \tag{12}$$

Unfortunately, the Tauberian lemma of [3] does not apply in this case and we didn't achieve in generalizing it ; so that the formula (12) is only a conjecture. The fact that this result is true in the case where $d = 0$ is a simple corollary of the results of [24]. In the last part of this paper, we will study the example of an integrable Hamiltonian satisfying the hypothesis of this conjecture with $d = 1$ and we will prove, by studying directly the eigenvalues of the associated quantum problem, that (12) is verified. We would like also to remark that a density similar to the one defined on Θ in (12) has already been introduced in the context of a Schrödinger operator with a magnetic field (cf [5, 25]).

The paper is organised as follows : In section 2 of this paper, we will prove a particular result on the classical dynamic which will prove that the points $(t, x, \xi) \in \mathcal{P}_{E_c}$ where $(t, x, \xi) \in$ supp $\hat{\varphi} \times (\Sigma_{E_c} \setminus \Theta)$ can not accumulate on Θ. In section 3, we use the fundamental theorem of this paper (proved in section 4), to prove the results announced in this introduction. Section 5 is devoted to the case with an observable.

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2 A result concerning the classical dynamic.

Let H be a classical Hamiltonian of the form $H(x,p) = p^2/2 + V(x)$ where we assume that $V \in C^4(\mathbb{R}^n)$ and let E_c be a critical value of H.

We will assume moreover that :

- Θ, the set of critical points of H, is a compact submanifold of $\mathbb{R}^{2n}$ contained in $\Sigma_{E_c} = H^{-1}(E_c)$,
- $d^2H(x,0)|_{T\mathbb{R}^{2n}/T\Theta}$ is non degenerate for all $(x,0) \in \Theta$.

In the following, when we will talk about periodic orbits, we will only consider periodic orbits which are not fixed points.

In this section, we show that there exists a neighborhood U of Θ in $\mathbb{R}^{2n}$ such that every periodic orbit on Σ_{E_c} with a period smaller than a given constant will never meet U. Of course, U depends on this constant. This result will allow us, in the following parts, to isolate, on the one hand, the periodic orbit of $\Sigma_{E_c} \setminus \Theta$ which will be studied as in the regular trace formula, and, on the other hand, the singularity coming from Θ.

Proposition 2.1 *Let $T > 0$.*
There exists U_T, a neighborhood of Θ in $\mathbb{R}^{2n}$ such that for every periodic orbit $\gamma \subset \Sigma_{E_c}$ with period $T_\gamma \in]0,T]$, we have :

$$\gamma \cap U_T = \emptyset$$

Remark 2.2 This proposition is no more true in the case of a general Hamiltonian. Indeed, if we choose : $H(x,p) = p_1^2/2 - p_2^2/2 + x_1^2/2 - x_2^2/2$ in a neighborhood of $(0,0)$, then $\Theta = \{(0,0)\}$ and all the points $(x_1 = \epsilon, x_2 = \epsilon, p_1 = 0, p_2 = 0)$, where $\epsilon > 0$ belong to $H^{-1}(0)$ and are periodic with period equal to 2π.

We will now be concerned, until the end of this part, with the proof of this proposition.

Let $(x_0, 0) \in \Theta$. Because Θ is compact, it is sufficient to prove that we can find a neighbourhood of $(x_0, 0)$ in $\mathbb{R}^{2n}$ which verify the proposition. We will first prove the following lemma :

Lemma 2.3 *There exists Ω, a neighborhood of $(x_0, 0)$ in $\mathbb{R}^{2n}$ such that every periodic orbit of Σ_{E_c} will not stay forever in Ω.*

Proof of the lemma: We can assume that $E_c = 0$ and $x_0 = 0$. We can moreover assume that there exists U, a neighborhood of $(x_0 = 0)$ in $\mathbb{R}^n$ such that

$$\Theta \cap (U \times \mathbb{R}^n) = \{(x', x'', 0) \in \mathbb{R}^s \times \mathbb{R}^{n-s} \times \mathbb{R}^n / x' = f(x'')\} \cap (U \times \mathbb{R}^n)$$

where $\dim\Theta = n-s$, $f(0)=0$, $df(0)=0$ and $d^2V(0,0)=$diag$(-a_1^2,\cdots,-a_r^2,a_{r+1}^2,\cdots,a_s^2,0,\cdots,0)$ with $a_i>0$. This can be easily obtain by some translation and rotation in the x-space.
Because a canonical change of coordinate will not affect the periodic orbits, we construct now a symplectomorphism defined on a neighborhood of $(0,0)$ in order to make Θ flat. Let's consider:

$$\begin{aligned} \phi: \qquad & U \longrightarrow \phi(U)=\tilde{U} \\ & (x',x'') \mapsto (x'-f(x''),x'') \end{aligned}$$

It is easy to see that ϕ is a diffeomorphism and that

$$d\phi(x) = \begin{pmatrix} Id & -df(x'') \\ 0 & Id \end{pmatrix}$$

does not depend on x'.

Let's write now:

$$\begin{aligned} \psi \ : \ & U\times I\!R^n \longrightarrow \tilde{U}\times I\!R^n \\ & (x,p)\mapsto (\phi(x),[{}^t d\phi(x)]^{-1}.p)=(q,\pi). \end{aligned}$$

We can notice that ψ is a diffeomorphism and that

$$\begin{aligned} \psi^*(\pi dq).(\delta x,\delta p) &= <\pi dq, d\psi(x,p).(\delta x,\delta p)> \\ &= ([{}^t d\phi(x)]^{-1}.p).(d\phi(x).\delta x) \\ &= p.\delta x. \end{aligned}$$

We can thus deduce that ψ is an (exact) symplectomorphism.

If we proceed to this change of coordinate, we get the following new Hamiltonian $\tilde{H}$ (where we will also write (x,p) for the new coordinates) :

$$\tilde{H}(x,p) = H(\psi^{-1}(x,p)) = \frac{1}{2}<R(x'')p,p> + \tilde{V}(x',x'')$$

with $\tilde{V}(x',x'')=V(x'+f(x''),x'')$ and

$$R(x'') = \begin{pmatrix} Id+df(x'').{}^t df(x'') & -df(x'') \\ -{}^t df(x'') & Id \end{pmatrix}.$$

Because $df(0)=0$ and with a neighborhood $\tilde{U}$ of 0 that may have been reduced, we can assume that

$$\frac{1}{2}<R(x'')p,p> \geq \frac{p^2}{4} \qquad \forall (x,p)\in\tilde{U}\times I\!R^n.$$

In particular, $R(x'')$ is a positive definite matrix on $\tilde{U}$ and so, $\forall (x,p)\in\tilde{U}\times I\!R^n$,

$$\begin{aligned} d\tilde{H}(x,p)=0 \ &\Leftrightarrow \ \left\{ \begin{array}{l} p=0 \\ \partial_{x'}V(x'+f(x''),x'')=0 \\ \frac{1}{2}<\partial_{x''}R(x'')p,p> + \partial_{x'}V.\partial_{x''}f+\partial_{x''}V=0 \end{array}\right. \\ &\Leftrightarrow \ \left\{ \begin{array}{l} p=0 \\ dV(x'+f(x''),x'')=0 \end{array}\right. \\ &\Leftrightarrow \ \left\{ \begin{array}{l} p=0 \\ x'=0 \end{array}\right. . \end{aligned}$$

In these coordinates, we have $\Theta \cap (\tilde{U} \times I\!R^n) = \{(0, x'', 0) \in \tilde{U} \times I\!R^n\}$.
One can write:

$$\tilde{H}(x,p) = \frac{1}{2} < R(x'')p, p > + \frac{1}{2} < d^2_{x'}\tilde{V}(0,x'')x', x' > + L(x) \tag{13}$$

where

$$L(x) = \tilde{V}(x', x'') - \frac{1}{2} < d^2_{x'}\tilde{V}(0,x'')x', x' > .$$

As $\tilde{V}(0,0) = 0$ (because $E_c = 0$), $d\tilde{V}(0,x'') = 0$ for all $(0,x'') \in \tilde{U}$ and $d^2_{x'}\tilde{V}(0,0) = d^2_{x'}V(0,0)$ is non degenerate, we obtain that $|L(x)| = \mathcal{O}(||x'||^3)$ uniformly in x''. Moreover,

$$||\partial_{x'}L(x)|| = ||\partial_{x'}\tilde{V}(x',x'') - d^2_{x'}\tilde{V}(0,x'').x'|| = \mathcal{O}(||x'||^2)$$

and

$$||\partial_{x''}L(x)|| = ||\partial_{x''}\tilde{V}(x',x'') - \frac{1}{2} < \partial_{x''}d^2_{x'}\tilde{V}(0,x'')x', x' > || = \mathcal{O}(||x'||)$$

because $V \in C^4(I\!R^n)$

Let's consider

$$\begin{aligned} A &= \sup_{i=1,\cdots,r} a_i^2, \\ B &= \inf_{i=1,\cdots,s} \frac{1}{2}a_i^2 \text{ and} \\ x' &= (x_1, \cdots, x_r, x_{r+1}, \cdots, x_s) = (x'_-, x'_+). \end{aligned}$$

We thus have, for all $(x,p) \in \tilde{H}^{-1}(0) \cap (\tilde{U} \times I\!R^n)$,

$$\begin{aligned} A||x'_-||^2 &= A\sum_{i=1}^r x_i^2 \\ &\geq \sum_{i=1}^r a_i^2 x_i^2 \\ &= \tilde{H}(x,p) + \sum_{i=1}^r a_i^2 x_i^2 \\ &= \frac{1}{2} < R(x'')p, p > + \frac{1}{2} < Q(x'')x', x' > + L(x) \\ &\geq \frac{p^2}{4} + \frac{1}{2} < Q(x'')x', x' > + L(x) \end{aligned}$$

where

$$Q(x'') = d^2_{x'}\tilde{V}(0,x'') + \begin{pmatrix} 2a_1^2 & & & \\ & \ddots & & \\ & & 2a_r^2 & \\ & & & 0 \end{pmatrix}.$$

We notice that $Q(0) = \text{diag}(a_1^2, \cdots, a_s^2)$ and so, with $\tilde{U}$ that may have been reduced, we get

$$\frac{1}{2} < Q(x'')x', x' > \geq \frac{B}{2}||x'||^2.$$

As $|L(x)| = \mathcal{O}(||x'||^3)$ and with $\tilde{U}$ eventually again reduced, we also have

$$|L(x)| \leq \frac{B}{4}||x'||^2.$$

We can finally conclude that there exists a constant C, strictly positive, such that for all $(x,p) \in \tilde{H}^{-1}(0) \cap (\tilde{U} \times I\!R^n)$,

$$||x'_-||^2 \geq C(||p||^2 + ||x'||^2). \tag{14}$$

We choose $\Omega = \tilde{U} \times I\!R^n$. This is not the final Ω, because we may reduce it (implicitly) from line to line.
Let's finish the proof of the lemma by assuming that there exists $\gamma = (x(t), p(t))_{t \in [0, T_\gamma]}$, a periodic orbit, which stay forever in $\tilde{H}^{-1}(0) \cap \Omega$.
Thus, there exists t_0 such that:
- $\frac{1}{2}||x'_-(t)||^2$ is maximum at t_0 and
- $||x'_-(t_0)||^2 > 0$ (or else we would stay on Θ).
This implies that

$$\frac{d^2}{dt^2}(\frac{1}{2}||x'_-(t)||^2)(t_0) = ||\dot{x}'_-(t_0)||^2 + x'_-(t_0).\ddot{x}'_-(t_0) \leq 0. \tag{15}$$

Let's show now that $x'_-(t_0).\ddot{x}'_-(t_0) > 0$, which will contradict (15) and will end the proof of the lemma.

If we note

$$\begin{aligned} p = ((p_1, \cdots, p_r), (p_{r+1}, \cdots, p_n)) &= (p'_-, \tilde{p}) \\ R(x'').p &= \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} \begin{pmatrix} p'_- \\ \tilde{p} \end{pmatrix} \\ d^2_{x'}\tilde{V}(0, x'').x' &= \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} x'_- \\ x'_+ \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned} \dot{x}'_-(t) = \partial_{p'_-}\tilde{H}(x,p) &= R_1.p'_- + R_2.\tilde{p} \\ &= p'_- + (R_1 - Id).p'_- + R_2.\tilde{p}, \end{aligned}$$

and thus

$$\ddot{x}'_-(t_0) = \dot{p}'_-(t_0) + (R_1 - Id).\dot{p}'_- + \partial_{x''}R_1.\dot{x}''.p'_- + R_2.\dot{\tilde{p}} + \partial_{x''}R_2.\dot{x}''.\tilde{p}.$$

But

$$\dot{p}'_-(t_0) = -\partial_{x'_-}\tilde{H} = -V_1.x'_- - V_2.x'_+ + \partial_{x'_-}L, \tag{16}$$

and as $V_1(0) =$diag$(-a_1^2, \cdots, -a_r^2)$, $V_2(0) = 0$ and $||\partial_{x'_-}L(x)|| = \mathcal{O}(||x'||^2)$, we obtain

$$x'_-(t_0).\dot{p}'_-(t_0) \geq \alpha||x'_-(t_0)||^2$$

for a strictly positive α.
Let's see now the remaining terms.
We know that $||p|| \leq C||x'_-||$ from (14) and from the Hamilton equations we have $||\dot{x}''|| \leq ||d\tilde{H}||$; we thus conclude that

$$\begin{aligned} ||\partial_{x''}R_1.\dot{x}''.p'_-|| &\leq \frac{\alpha}{8}||x'_-|| \\ ||\partial_{x''}R_2.\dot{x}''.\tilde{p}|| &\leq \frac{\alpha}{8}||x'_-||. \end{aligned}$$

Moreover, because $R_1(0) = Id$ and $||\dot{p}'_-|| \leq \alpha_1 ||x'_-||$ for a strictly positive α_1 from (16), we have

$$||(R_1 - Id).\dot{p}'_-|| \leq \frac{\alpha}{8}||x'_-||.$$

It remains to bound the term $R_2.\dot{\tilde{p}}$. But

$$\dot{p}'_+ = -\partial_{x'_+}\tilde{H} = -V_3.x'_- - V_4.x'_+ - \partial_{x'_+}L$$

and $||\partial_{x'_+}L|| = \mathcal{O}(||x'||^2)$; thus

$$||\dot{p}'_+|| \leq \beta||x'_-||$$

for a constant $\beta > 0$.
Similarly,

$$\dot{p}'' = -\partial_{x''}\tilde{H} = \frac{1}{2} < \partial_{x''}R.p, p > + \frac{1}{2} < \partial_{x''}d^2_{x'}\tilde{V}(o, x'').x', x' > + \partial_{x''}L$$

and $||\partial_{x''}L|| = \mathcal{O}(||x'||)$; thus

$$||\dot{p}''|| \leq \beta_1||p||^2 + \beta_2||x'||^2 + \beta_3||x'|| \leq \beta_4||x'_-||$$

for some strictly positive constants β_1, β_2, β_3 and β_4.
Using the fact that $R_2(0) = 0$ we have:

$$||R_2.\dot{\tilde{p}}|| \leq \frac{\alpha}{8}||x'_-||,$$

which finally gives us, by gathering all the inequalities:

$$x'_-(t_0).\ddot{x}'_-(t_0) \geq \frac{\alpha}{2}||x'_-(t_0)||^2 > 0.$$

♣

End of the proof of the proposition : We can assume that the Ω of the preceding lemma is equal to the ball $B(0, \rho)$ (with $\rho > 0$) and that for all $z = (x, p)$ in $B(0, \rho)$ we have:

$$||\dot{z}(t)|| = ||dH(z)|| \leq c||z|| \tag{17}$$

for some $c > 0$.
Let's take $U_T = B(0, e^{-cT}\rho)$, and assume that there exists $\gamma = (z(t))_{t\in[0,T_\gamma]}$, a periodic orbit of Σ_{E_c} with $T_\gamma \in]0, T]$ and such that there exists a time t_1 for which $||z(t_1)|| < e^{-cT}\rho$.
¿From the preceding lemma, there exist t_0 such that $||z(t_0)|| = \rho$. One can moreover choose t_0 so that $0 \leq t_1 - t_0 \leq T$ and such that for all $t \in [t_0, t_1]$, we have $z(t) \in B(0, \rho)$.
Let's write

$$f(t) = e^{2ct}||z(t)||^2.$$

Then, for all t with $z(t) \in B(0, \rho)$, we have :

$$\begin{aligned} f'(t) &= 2e^{2ct}(c||z(t)||^2 + z(t).\dot{z}(t)) \\ &\geq 2e^{2ct}(||z(t)||||\dot{z}(t)|| + z(t).\dot{z}(t)) \qquad \text{from (17)} \\ &\geq 0, \end{aligned}$$

so f does not decrease in $[t_0, t_1]$.
We thus have

$$e^{2ct_1}||z(t_1)||^2 \geq e^{2ct_0}||z(t_0)||^2$$

which implies that,

$$||z(t_1)|| \geq e^{c(t_0 - t_1)}||z(t_0)|| \geq e^{-cT}\rho$$

and leads to a contradiction. ♣

3 Microlocal decomposition and Proof of the main statements.

In this section, we will show that $\Upsilon(E_c,\hbar)$ is equal, modulo $\mathcal{O}(\hbar^\infty)$, to a finite sum of oscillatory integral with large parameter $1/\hbar$. Some of these integrals will correspond to the ones encountered in the proof of the regular trace formula (with non-degenerate critical points). The others will have degenerate phases and will be studied in the next section. In order to study $\Upsilon(E_c,\hbar)$, we will have to use the theory of FIO (Fourier Integral Operators) depending on a large parameter. We won't recall here the results of this theory and we refer the reader to the following non exhaustive list of references : [9, 10, 17] for the standard theory and [18, 7, 8, 20] when there is a large parameter. Define

$$\Upsilon(\hbar) := \Upsilon(E_c,\hbar) = \sum_{E_k(\hbar)\in[E_c-\lambda,E_c+\lambda]} \varphi\left(\frac{E_k(\hbar)-E_c}{\hbar}\right),$$

where $\hat{\varphi}\in C_0^\infty([-T,T])$ for some $T>0$ and $\lambda>0$.
If not explicitly mentioned, T, φ and λ will remain fixed until the end of this paper.

Let $\chi\in C_0^\infty([E_c-\lambda,E_c+\lambda])$ such that $\chi\equiv 1$ in a neighborhood of E_c and $0\le\chi\le 1$. Then, with some standard semi-classical arguments (cf [8, 20]), we have

$$\begin{aligned}\Upsilon(\hbar) &= \sum_{k\geq 0}\varphi\left(\frac{E_k(\hbar)-E_c}{\hbar}\right)\chi(E_k(\hbar)) + \mathcal{O}(\hbar^\infty)\\ &= \frac{1}{2\pi}\int_{-T}^{T}\hat{\varphi}(-t)e^{\frac{i}{\hbar}tE_c}\mathrm{Tr}(e^{-\frac{i}{\hbar}t\hat{H}}\chi(\hat{H}))dt + \mathcal{O}(\hbar^\infty)\end{aligned}$$

Following the construction of [8], one can obtain $\tilde{U}(t)$, a FIO which approximates the "localized" propagator : $U(t):=e^{-\frac{i}{\hbar}t\hat{H}}\chi(\hat{H})$ in the L^2-norm of linear operator and modulo $\mathcal{O}(\hbar^\infty)$. It is now easy to verify that $\Upsilon(\hbar)$ is equal, modulo $\mathcal{O}(\hbar^\infty)$, to the composition of two FIO.

$$\Upsilon(\hbar) = \tilde{U}\circ A + \mathcal{O}(\hbar^\infty)$$

- The first one is $\tilde{U}(t)$.
  Its lagrangian manifold is:

  $$\Lambda_U = \{(t,-H(y,\eta),\phi_t(y,\eta),(y,\eta))\ \ /\ \ t\in\mathbb{R},(y,\eta)\in\mathbb{R}^{2n}\},$$

  and its principal symbol defined on Λ_U is

  $$\sigma_U(\lambda) = (2\pi\hbar)^{-n/2}e^{\frac{i}{\hbar}\int_0^t L(\phi_s(y,\eta))ds}e^{i\frac{\pi}{2}\mu(\gamma_t)}\chi(H(y,\eta))\nu_\lambda \qquad \forall\lambda\in\Lambda_U \qquad (18)$$

  where $L(y,\eta)=\eta^2/2-V(y)$ is the Lagrangian, $\mu(\gamma_t)$ is the Maslov index of the path : $\{(\tau,-H(y,\eta),\phi_\tau(y,\eta),y,\eta)\}_{\tau\in[0,t]}$ and ν_λ is the canonical density defined on Λ_U. We will denote $K_U(t,x,y)$ for the local integral kernel of $\tilde{U}$ (for more details, see [8]).

- The second one will be denoted by A.
  Its lagrangian manifold is :

  $$\Lambda_A = \{(t,-E_c,x,\theta,x,\theta)\ \ /\ \ (t,x,\theta)\in\mathbb{R}^{1+2n}\}.$$

  and its kernel is

  $$K_A(t,x,y,\hbar) = \frac{1}{2\pi}(2\pi\hbar)^{-n}\int e^{\frac{i}{\hbar}((x-y).\theta+tE_c)}\hat{\varphi}(-t)d\theta,$$

Indeed,

$$\begin{aligned}
\tilde{U}\circ A &= \frac{1}{2\pi}\int K_U(t,x,y)(2\pi\hbar)^{-n}\int e^{\frac{i}{\hbar}((x-y).\theta+tE_c)}\hat{\varphi}(-t)d\theta dt dx dy \\
&= \frac{1}{2\pi}\int e^{\frac{i}{\hbar}tE_c}\hat{\varphi}(-t)\int K_U(t,x,x)dxdt \\
&= \frac{1}{2\pi}\int e^{\frac{i}{\hbar}tE_c}\hat{\varphi}(-t)\mathrm{Tr}(\tilde{U}(t))dt \\
&= \Upsilon(\hbar)+\mathcal{O}(\hbar^\infty).
\end{aligned}$$

If (H1) and (H2) were assumed for E_c, this would imply that the manifolds $\Lambda_U\times\Lambda_A$ and $\Delta = \{(t,\tau,x,\xi,y,\eta,t,\tau,x,\xi,y,\eta)\ /\ (t,\tau,x,\xi,y,\eta)\in I\!\!R^{2+4n}\}$ intersect cleanly (in the sense of Bott) which is equivalent to saying that the composition of $\tilde{U}$ and A is clean. As a result, we would obtain an asymptotic expansion of $\Upsilon(\hbar)$ in powers of $\hbar$.

In our case, E_c is a critical value of H and we have to study more carefully the points where this intersection is not clean.

By definition, $\Lambda_U\times\Lambda_A$ and Δ intersect cleanly if and only if

- $(\Lambda_U\times\Lambda_A)\cap\Delta$ is a submanifold of $I\!\!R^{4+8n}$ and
- $\forall\lambda\in(\Lambda_U\times\Lambda_A)\cap\Delta, \qquad T_\lambda((\Lambda_U\times\Lambda_A)\cap\Delta)=T_\lambda(\Lambda_U\times\Lambda_A)\cap T_\lambda\Delta.$

But

$$\begin{aligned}
(\Lambda_U\times\Lambda_A)\cap\Delta &= \{(t,-E_c,x,\xi,x,\xi,t,-E_c,x,\xi,x,\xi)\ / \\
&\qquad H(x,\xi)=E_c, \phi_t(x,\xi)=(x,\xi) \text{ and } (t,x,\xi)\in I\!\!R^{1+2n}\},
\end{aligned}$$

and

$$\begin{aligned}
T_\lambda(\Lambda_U\times\Lambda_A) &= \{(\delta t,-dH(x,\xi).(\delta x,\delta\xi),(J\nabla H(x,\xi))\delta t+d\phi_t(x,\xi).(\delta x,\delta\xi),\delta x,\delta\xi, \\
&\qquad \delta t,0,\delta x,\delta\xi,\delta x,\delta\xi)\ /\ (\delta t,\delta x,\delta\xi)\in T_{t,x,\xi}I\!\!R^{1+2n}\}.
\end{aligned}$$

Thus, if we denote

$$P_{E_c}=\{(t,x,\xi)\in I\!\!R\times\Sigma_{E_c}\ /\ \phi_t(x,\xi)=(x,\xi)\}$$

we get that $\Lambda_U\times\Lambda_A$ and Δ intersect cleanly if and only if

C1 P_{E_c} is a submanifold of $I\!\!R^{1+2n}$ and

C2 $T_{(t,x,\xi)}P_{E_c}=\{(\delta t,\delta x,\delta\xi)\in T_{(t,x,\xi)}I\!\!R^{1+2n}\ /\ dH(x,\xi).(\delta x,\delta\xi)=0$ and $(J\nabla H(x,\xi))\delta t+d\phi_t(x,\xi).(\delta x,\delta\xi)=(\delta x,\delta\xi)\}$.

Let's write

$$(\Lambda_U\times\Lambda_A)\cap\Delta=\Delta_1\cup\Delta_2\cup\Delta_3\cup\Delta_4$$

where

$$\begin{aligned}
\Delta_1 &= \{(t,-E_c,x,0,x,0,t,-E_c,x,0,x,0)\ /\ (t,x,0)\in([-T,T]\setminus\{0\})\times\Theta\}, \\
\Delta_2 &= \{(t,-E_c,x,\xi,x,\xi,t,-E_c,x,\xi,x,\xi)\ /\ (t,x,\xi)\in P_{E_c}\cap([-T,T]\setminus\{0\})\times(\Sigma_{E_c}\setminus\Theta)\}, \\
\Delta_3 &= \{(0,-E_c,x,\xi,x,\xi,0,-E_c,x,\xi,x,\xi)\ /\ (x,\xi)\in\Sigma_{E_c}\}, \\
\Delta_4 &= \{(t,-E_c,x,\xi,x,\xi,t,-E_c,x,\xi,x,\xi)\ /\ (t,x,\xi)\in P_{E_c} \text{ and } t\notin[-T,T]\}.
\end{aligned}$$

a) We don't have to study the points of Δ_4 because for such points, $t \notin \text{supp}\hat{\varphi}$ and thus, they will give no contribution to $\tilde{U} \circ A$.

b) The contribution of the points of Δ_3 are calculated in [3] (cf Appendix B).

c) If we assume (H1bis), we easily get that on every points of Δ_2, the intersection of $\Lambda_U \times \Lambda_A$ and Δ is clean and this gives us a contribution coming from the periodic orbits of $\Sigma_{E_c} \setminus \Theta$ in the same way as in the regular trace formula.

d) We only need now to study the intersection of $\Lambda_U \times \Lambda_A$ and Δ at a point $(t, -E_c, x, 0, x, 0, t, -E_c, x, 0, x, 0) \in \Delta_1$.

We first use the result of the proposition 2.1 to say that there exist Ω, a neighborhood in $I\!R^{1+2n}$ of $(t, x, 0)$ such that

$$\Omega \cap P_{E_c} = \Omega \cap \{(t, x, 0) \in I\!R \times \Theta\}.$$

This implies that the condition (C1) is fulfilled locally at $(t, x, 0)$. Moreover, we have

$$T_{(t,x,0)} P_{E_c} = I\!R \times T_{(x,0)}\Theta.$$

Since $dH(x, 0) = 0$, we get that (C2) is fulfilled if and only if

$$\forall\ (\delta x, \delta\xi) \in T_{(x,0)} I\!R^{2n}, \quad (d\phi_t(x, 0) - Id).(\delta x, \delta\xi) = 0 \Rightarrow (\delta x, \delta\xi) \in T_{(x,0)}\Theta.$$

This is exactly the condition (5.7) of [14] where this case is studied.

Let's study more precisely the linearised flow $d\phi_t(x, 0)$ at a point $(x, 0) \in \Theta$ to detect when the condition (C2) is not fulfilled.
We have

$$d\phi_t(x, 0) = e^{tJd^2H(x,0)}$$

where

$$d^2H(x, 0) = \begin{pmatrix} Id & 0 \\ 0 & d^2V(x) \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

$d^2V(x)$ is a symmetrical matrix whose eigenvalues have multiplicities which does not depend on $(x, 0) \in \Theta$ (from (H5)). We can thus find an orthogonal matrix $R(x)$ which is C^∞ in x and such that

$${}^tR(x) d^2V(x) R(x) = \text{diag}[\alpha_1^2(x), \cdots, \alpha_r^2(x), -\alpha_{r+1}^2(x), \cdots, -\alpha_s^2(x), 0, \cdots, 0].$$

with all the $\alpha_i(x)$ strictly positive. We observe that with (H4), we necessarily have $n - s = \dim\Theta$. If we define

$$P(x) = \begin{pmatrix} R(x) & 0 \\ 0 & R(x) \end{pmatrix},$$

it is then easy to verify that $P(x)$ is a symplectic and orthogonal matrix. We thus have

$$J {}^tP = P^{-1} J.$$

Then

$$\begin{aligned} P^{-1}(x) d\phi_t(x, 0) P(x) &= e^{tP^{-1}(x)Jd^2H(x,0)P(x)} = e^{tJ {}^tP(x)d^2H(x,0)P(x)} \\ &= \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix}, \end{aligned}$$

where

$$A(x,t) = D(x,t) = \text{diag}[\cos(\alpha_1(x)t), \cdots, \cos(\alpha_r(x)t), \cosh(\alpha_{r+1}(x)t), \cdots, \cosh(\alpha_s(x)t), 1, \cdots, 1]$$

$$\begin{aligned} B(x,t) &= \text{diag}[\frac{1}{\alpha_1(x)}\sin(\alpha_1(x)t), \cdots, \frac{1}{\alpha_r(x)}\sin(\alpha_r(x)t), \\ &\quad \frac{1}{\alpha_{r+1}(x)}\sinh(\alpha_{r+1}(x)t), \cdots, \frac{1}{\alpha_s(x)}\sinh(\alpha_s(x)t), t, \cdots, t], \end{aligned} \tag{19}$$

and

$$\begin{aligned} C(x,t) &= \text{diag}[-\alpha_1(x)\sin(\alpha_1(x)t), \cdots, -\alpha_r(x)\sin(\alpha_r(x)t), \\ &\quad \alpha_{r+1}(x)\sinh(\alpha_{r+1}(x)t), \cdots, \alpha_s(x)\sinh(\alpha_s(x)t), 0, \cdots, 0]. \end{aligned}$$

We finally see that

$$(Ker(d\phi_t(x,0) - Id) \neq T_{(x,0)}\Theta) \Leftrightarrow (\exists i, \ \ 1 \leq i \leq r, \exists k \in \mathbb{Z} \ \ / \ \ t = \frac{2\pi k}{\alpha_i(x)}).$$

These t will be called the periods of the linearised flow at $(x,0)$ and we will write

$$\{t_j(x)\}_{-m(x) \leq j \leq m(x)} = \{\frac{2\pi k}{\alpha_i(x)} \ \ / \ \ k \in \mathbb{Z}, 1 \leq i \leq r\} \cap [-T,T],$$

for the finite set of periods of the linearised flow at $(x,0) \in \Theta$ which belong to $[-T,T]$.

Of course, $t_0(x) = 0$ is always a period of the linearised flow and this singularity is exactly the one studied in [3]. We thus have to find an asymptotic expansion for $\tilde{U} \circ A$ in a microlocal neighborhood of the points $(t_j(x), -E_c, x, 0, x, 0, t_j(x), -E_c, x, 0, x, 0)$ with $(x,0) \in \Theta$ and $j \neq 0$.

Let's write precisely this microlocal decomposition.

Let $\Pi(t,E,x,\xi,y,\eta) = (t,x,y)$ denote the projection on $\mathbb{R}^{1+2n}$.
Since $\tilde{U}$ is a FIO associated to the lagrangian manifold Λ_U, there exists

- $\{U_\alpha\}_{\alpha \in I}$ a covering of open sets of Λ_U,
- $$\begin{aligned} \psi_\alpha \ : \ & \mathbb{R}^{1+2n+n_\alpha} \longrightarrow \mathbb{R} \\ & (t,x,y,\theta_\alpha) \longmapsto \psi_\alpha(t,x,y,\theta_\alpha) \end{aligned}$$

  a family of non degenerate phase functions associated to the U_α and parametrizing Λ_U and
- $$\begin{aligned} a_\alpha \ : \ & \mathbb{R}^{1+2n+n_\alpha+1} \longrightarrow \mathbb{R} \\ & (t,x,y,\theta_\alpha,\hbar) \longmapsto a_\alpha(t,x,y,\theta_\alpha,\hbar) \end{aligned}$$

  a family of classical symbol of order : $ord(a_\alpha) = -(n/2 + n_\alpha/2)$

such that for all $(\theta_\alpha, \hbar) \in I\!R^{n_\alpha+1}$, we have

$$\text{supp } a_\alpha(.,\theta_\alpha,\hbar) \subset \Pi(U_\alpha)$$

and such that

$$\begin{aligned} K_U(t,x,y,\hbar) &= \sum_{\alpha\in I} \int_{I\!R^{n_\alpha}} e^{\frac{i}{\hbar}\psi_\alpha(t,x,y,\theta_\alpha)} a_\alpha(t,x,y,\theta_\alpha,\hbar) d\theta_\alpha \\ &= \sum_{\alpha\in I} K_\alpha(t,x,y). \end{aligned}$$

We thus can write

$$\begin{aligned} \Upsilon(\hbar) &= \sum_{\alpha\in I} \int_{I\!R^{1+2n}} K_\alpha(t,x,y,\hbar) K_A(t,x,y,\hbar) dtdxdy + \mathcal{O}(\hbar^\infty) \\ &= \sum_{\alpha\in I} \frac{1}{2\pi} \int_{I\!R^{1+n}} e^{\frac{i}{\hbar}tE_c} \hat{\varphi}(-t) K_\alpha(t,x,x,\hbar) dtdx + \mathcal{O}(\hbar^\infty) \\ &= \sum_{\alpha\in I} I_\alpha(\hbar) + \mathcal{O}(\hbar^\infty), \end{aligned}$$

Since $[-T,T]\times\Sigma_{E_c}$ is compact, we can assume that I is a finite set. Moreover, for the α such that U_α does not contain any point of the form $(t_j(x), -E_c, x, 0, x, 0)$, with $(x,0)\in\Theta$ and $j \in [-m(x), m(x)]\setminus\{0\}$, the asymptotic expansion of $I_\alpha(\hbar)$ corresponds to the cases already studied : cf for example [8, 3] and part 5 of [14].
We need now to calculate the asymptotic expansion of $I_{\alpha_0}(\hbar)$ for a α_0 such that there exists a $(x_0,0)\in\Theta$ and a $j_0 \in [-m(x_0), m(x_0)]\setminus\{0\}$ for which $(t_{j_0}(x_0), -E_c, x_0, 0, x_0, 0)$ belongs to U_{α_0}.

Remark 3.1 Since the eigenvalues of $d^2V(x)$ do not cross when $(x,0)$ stays on Θ, we can assume, if the covering $\{U_\alpha\}_{\alpha\in I}$ is well chosen, that there exists a unique $j = j_0$ such that there exists a $(x,0)\in\Theta$ with $(t_j(x), -E_c, x, 0, x, 0) \in U_{\alpha_0}$.

We thus have to find an asymptotic expansion of

$$I_{\alpha_0}(\hbar) = \frac{1}{2\pi} \int_{I\!R^{1+n+n_\theta}} e^{\frac{i}{\hbar}(\psi_{\alpha_0}(t,x,x,\theta)+tE_c)} a_{\alpha_0}(t,x,x,\theta,\hbar) \hat{\varphi}(-t) d\theta dx dt \tag{20}$$

where the support of $a_{\alpha_0}(.,\theta,\hbar)$ is contained in a small neighborhood of $(t_{j_0}(x_0), x_0, x_0)$, and ψ_{α_0} satisfies :

$$\{(t, \partial_t\psi(t,x,y,\theta), x, \partial_x\psi(.), y, -\partial_y\psi(.)) \;/\; \partial_\theta\psi(.) = 0, (t,x,y,\theta) \in \Omega\times I\!R^{n_\theta}\} = \Lambda_U \cap \tilde{\Omega}$$

with Ω a neighborhood of $(t_{j_0}(x_0), x_0, x_0)$ and $\tilde{\Omega}$ a neighborhood of $(t_{j_0}(x_0), -E_c, x_0, 0, x_0, 0)$.

Let's define

$$\begin{aligned} N &:= 2n - \dim\Theta = n + s \quad \text{the codimension of } \Theta \text{ in } I\!R^{2n} \text{ and} \\ f(t,x) &= \frac{1}{t^{n-N/2} \prod_{i=1}^{r} \sin(\frac{\alpha_i(x)}{2}t) \prod_{i=r+1}^{N-n} \sinh(\frac{\alpha_i(x)}{2}t)}. \end{aligned}$$

for all $(t,x,0) \in I\!R\times\Theta$ sufficiently close to $(t_{j_0}(x_0), x_0, 0)$ and such that $t \neq t_{j_0}(x)$.

We notice that $f(.,x)$ is meromorphe with a finite pole at $t_{j_0}(x)$ so that the distribution $f(t-i0,x)$ is well defined by Theorem 3.1.11 of [16].

We can state the following result :

Theorem 3.2 *As $\hbar$ goes to 0,*

$$I_{\alpha_0}(\hbar) \sim \hbar^{N/2-n} \sum_{j=0}^{\infty} d_j \hbar^j \tag{21}$$

with

$$d_0 = (2\pi)^{N/2-n} 2^{n-N} e^{i\pi\sigma/4} \int_{\Theta} < f(t-i0,x), \check{\varphi}(t)\chi_{\alpha_0}(t,x) > dx. \tag{22}$$

where σ is an integer and χ_{α_0} comes from the decomposition of unity associated to the open covering $\{U_\alpha\}$ which means that χ_{α_0} is a smooth function equal to 1 in a neighborhood of $(t_{j_0}(x_0), x_0)$.

The proof of this theorem, which is the central theorem of this paper, will be done in the next section.

End of the proof of theorem 1.3 : We can now end the proof of Theorem 1.3 by adding all the contribution coming from the microlocal decomposition. We use Theorem B.1 for the contribution coming from the singularity at $t = 0$, Theorem 5.6 of [14] for the contribution coming from Θ and far from the period of the linearised flow, Theorem 2.1 of [4] for the periodic orbits of $\Sigma_{E_c} \backslash \Theta$ and finally Theorem 3.2 for the contribution coming from the periods of the linearised flow. Since we assumed that the periodic orbits of $\Sigma_{E_c} \backslash \Theta$ were of zero Liouville measure, their contribution does not appear at the first order of the different asymptotic expansions. Moreover, these asymptotic expansions must not depend on the microlocal decomposition, so that the addition of all the contributions must be "smooth" which implies that, in the case $\nu = 0$, we necessarily have from theorem B.1 : $\sigma = -N$. ♣

Proof of theorem 1.5 : we recall that $d = 2n - N$.
The case $d = 0$ is just a corollary of the following lemma

Lemma 3.3 *$\forall \varphi \in \mathcal{S}(I\!R)$ and $\forall \alpha_i > 0$ ($1 \leq i \leq n$), we have :*

$$\sum_{k_1,\cdots,k_n \geq 0} \varphi(\alpha_1(k_1 + \frac{1}{2}) + \cdots + \alpha_n(k_n + \frac{1}{2})) = \frac{1}{(2i)^n} < \frac{1}{\prod_{i=1}^n \sin(\frac{\alpha_i}{2}(t-i0))}, \check{\varphi}(t) > .$$

Proof: Let's define the tempered distribution

$$u = \sum_{k_1,\cdots,k_n \geq 0} \delta_{\alpha_1(k_1+\frac{1}{2})+\cdots+\alpha_n(k_n+\frac{1}{2})}$$

where δ_{τ_0} is the Dirac distribution supported in $\{\tau = \tau_0\}$.
We then have to show that

$$\hat{u} = \frac{1}{(2i)^n} \frac{1}{\prod_{i=1}^n \sin(\frac{\alpha_i}{2}(t-i0))}.$$

Since

$$\begin{aligned} u &= \lim_{\epsilon \to 0^+} e^{-\epsilon\tau} u \\ &= \lim_{\epsilon \to 0^+} \sum_{k_1,\cdots,k_n \geq 0} e^{-\epsilon\tau} \delta_{\alpha_1(k_1+\frac{1}{2})+\cdots+\alpha_n(k_n+\frac{1}{2})}, \end{aligned}$$

we have

$$\begin{aligned}
\hat{u} &= \lim_{\epsilon\to 0^+} \sum_{k_1,\cdots,k_n\geq 0} \int_{I\!R} e^{-i\tau t-\epsilon\tau}\delta_{\alpha_1(k_1+\frac{1}{2})+\cdots+\alpha_n(k_n+\frac{1}{2})}d\tau \\
&= \lim_{\epsilon\to 0^+} \sum_{k_1,\cdots,k_n\geq 0} e^{-i(\alpha_1k_1+\cdots+\alpha_nk_n)(t-i\epsilon)-i(\frac{\alpha_1}{2}+\cdots+\frac{\alpha_n}{2})(t-i\epsilon)} \\
&= \lim_{\epsilon\to 0^+} \prod_{j=1}^{n} (\sum_{k\geq 0} e^{-i\frac{\alpha_j}{2}(t-i\epsilon)}e^{-i\alpha_jk(t-i\epsilon)}) \\
&= \lim_{\epsilon\to 0^+} \prod_{j=1}^{n} (\frac{e^{-i\frac{\alpha_j}{2}(t-i\epsilon)}}{1-e^{-i\alpha_j(t-i\epsilon)}}) \\
&= \lim_{\epsilon\to 0^+} \prod_{j=1}^{n} (\frac{1}{2i\sin(\frac{\alpha_j}{2}(t-i\epsilon))}).
\end{aligned}$$

♣

For the case $d > 0$, we use the fact that

$$\mathcal{F}^{-1}\left((t-i0)^{-(n-N/2)}\right)(\tau) = \frac{e^{i\pi(n-N/2)/2}}{\Gamma(n-N/2)}\tau_+^{n-N/2-1} \qquad \text{(see [16])}$$

and that

$$\mathcal{F}^{-1}\left(\frac{1}{\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i0))}\right) = (2i)^{N-n}\sum_{k\in I\!N^{N-n}}\delta_{\beta_k(x)} \qquad \text{(see the previous lemma)}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and the $\beta_k(x)$ are defined in theorem 1.5. Since these two distributions have support contained in $I\!R^+$, we can calculate the convolution which gives

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\frac{1}{(t-i0)^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i0))}\right)(\tau) \\
&= \mathcal{F}^{-1}\left((t-i0)^{-(n-N/2)}\right)\star\mathcal{F}^{-1}\left(\frac{1}{\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i0))}\right)(\tau) \\
&= \frac{e^{i\pi(n-N/2)/2}}{\Gamma(n-N/2)}(2i)^{N-n}\sum_{k\in I\!N^{N-n}}\int(\tau-\tau')_+^{n-N/2-1}\delta_{\beta_k(x)}(\tau')d\tau' \\
&= \frac{e^{i\pi N/4}}{\Gamma(n-N/2)}2^{N-n}\sum_{k\in I\!N^{N-n}}(\tau-\beta_k(x))_+^{n-N/2-1}.
\end{aligned}$$

Using the expression of c_0 in the 3) of theorem 1.3 gives the desired result. ♣

4 Proof of Theorem 3.2

In order to simplify the notations of the previous section, we won't write anymore the index α_0 and we will write t_0, $t_0(x)$, R, P and α_i instead of $t_{j_0}(x_0)$, $t_{j_0}(x)$, $R(x_0)$, $P(x_0)$ and $\alpha_i(x_0)$. We will also use the following notations :
For any set of indices $K \subset [1,n]$, any $n \times n$-matrix A and any n-vector δx, $|K|$ will be the cardinal of K, A_K will be the $|K| \times |K|$-matrix obtained from A by suppressing the lines and columns whose indices are not in K and δx_K the $|K|$-vector obtained in the same way. K^c will be equal to $[1,n] \setminus K$.

In this section, we would like to prove the theorem 3.2 by studying the asymptotic expansion of the oscillating integral defined by formula (20) and which is, with the new notations,

$$I(\hbar) = \frac{1}{2\pi} \int_{I\!R^{1+n+n_\theta}} e^{\frac{i}{\hbar}(\psi(t,x,x,\theta)+tE_c)} a(t,x,x,\theta,\hbar)\hat{\varphi}(-t)d\theta dx dt \tag{23}$$

where the support of $a(.,\theta,\hbar)$ is contained in a small neighborhood of (t_0,x_0,x_0), and ψ satisfies :

$$\{(t, \partial_t\psi(t,x,y,\theta), x, \partial_x\psi(.), y, -\partial_y\psi(.)) \ / \ \partial_\theta\psi(.) = 0, (t,x,y,\theta) \in \Omega \times I\!R^{n_\theta}\} = \Lambda_U \cap \tilde{\Omega}$$

with Ω a neighborhood of (t_0,x_0,x_0) and $\tilde{\Omega}$ a neighborhood of $(t_0,-E_c,x_0,0,x_0,0)$.

The proof will be divided in three parts : the first one will consist in finding a new phase function, "equivalent" to $\psi + tE_c$, for which we will be able to identify the critical points and to express the hessian at these critical points. In the second part, we will use a parameters depending Morse Lemma to express the phase function as the sum of a normal form and a remainder. We will then obtain an asymptotic expansion for $I(\hbar)$ with methods similar to the ones developed in Section 4 of [3]. Finally, the last part of this section will be devoted to the identification of the leading term of this asymptotic expansion.

4.1 The phase function

In order to diagonalise $d^2V(x_0)$, we make the following linear change of variables in (23):

$$q = R^{-1}.x$$

¿From $|\det R| = 1$, we get

$$I(\hbar) = \frac{1}{2\pi} \int_{I\!R^{1+n+n_\theta}} e^{\frac{i}{\hbar}(\tilde{\psi}(t,q,q,\theta)+tE_c)} \tilde{a}(t,q,q,\theta,\hbar)\hat{\varphi}(-t)d\theta dq dt$$

where

$$\tilde{\psi}(t,q,q',\theta) = \psi(t,R.q,R.q',\theta)$$

and

$$\tilde{a}(t,q,q',\theta,\hbar) = a(t,R.q,R.q',\theta,\hbar).$$

$\tilde{\psi}$ is a non degenerate phase function whose lagrangian manifold is

$$\begin{aligned}
\Lambda_{\tilde{\psi}} &= \{(t,\partial_t\tilde{\psi}(t,q,q',\theta), q, \partial_q\tilde{\psi}(.), q', -\partial_{q'}\tilde{\psi}(.)) \ / \ \partial_\theta\tilde{\psi}(.) = 0\} \\
&= \{(t,\partial_t\psi(t,R.q,R.q',\theta), q, \partial_x\psi(.).R, q', -\partial_y\psi(.).R) \ / \ \partial_\theta\psi(.) = 0\} \\
&= \{(t,\partial_t\psi(t,x,y,\theta), R^{-1}.x, {}^tR.\partial_x\psi(.), R^{-1}.y, -{}^tR.\partial_y\psi(.)) \ / \ \partial_\theta\psi(.) = 0\} \\
&= \{(t,-H(y,\eta), P^{-1}.\phi_t(y,\eta), P^{-1}.(y,\eta))\}
\end{aligned}$$

in a neighborhood of $(t_0, -E_c, R^{-1}.x_0, 0, R^{-1}.x_0, 0)$.

$\forall\ (x,0) \in \Theta$, we denote $\lambda(t,x) = (t, -E_c, R^{-1}.x, 0, R^{-1}.x, 0)$ and since $dH(x,0) = 0$, we obtain

$$\begin{aligned} T_{\lambda(t,x)}\Lambda_{\tilde{\psi}} &= \{(\delta t, 0, P^{-1}.d\phi_t(x,0).(\delta y, \delta\eta), P^{-1}.(\delta y, \delta\eta))\} \\ &= \{(\delta t, 0, P^{-1}.d\phi_t(x,0).P.(\delta y, \delta\eta), (\delta y, \delta\eta))\} \end{aligned} \tag{24}$$

and thus,

$$T_{\lambda(t,x_0)}\Lambda_{\tilde{\psi}} = \{(\delta t, 0, A(t).\delta y + B(t).\delta\eta, C(t).\delta y + D(t).\delta\eta, \delta y, \delta\eta)\}$$

where $A(t) = A(x_0, t)$ is defined by (19). $B(t)$, $C(t)$ et $D(t)$ are defined in the same way.

Let's define

$$K = \{k \in [1,r]\ \ /\ \ \cos(\alpha_k t_0) \neq 0\}.$$

We can observe that $A_{K^c}(t_0) = 0$ and $A_K(t)$ and $C_{K^c}(t)$ are inversible when t is sufficiently close to t_0.

Lemma 4.1

$$\begin{aligned} \Pi\ \ :\ \ & T_{\lambda(t_0,x_0)}\Lambda_{\tilde{\psi}} \longrightarrow I\!R^{2n+1} \\ & (\delta t, 0, \delta x, \delta\xi, \delta y, \delta\eta) \longmapsto (\delta t, \delta x_K, \delta\xi_{K^c}, \delta\eta) \end{aligned}$$

is an automorphism.

Proof : We just need to prove that Π is surjective.
Let $(\tau, v, w) \in I\!R^{1+2n}$. Define $\delta t = \tau$, $\delta\eta = w$, and

$$\delta y = \begin{pmatrix} A_K^{-1}(t_0).v_K - A_K^{-1}(t_0).B_K(t_0).w_K \\ C_{K^c}^{-1}(t_0).v_{K^c} - C_{K^c}^{-1}(t_0).D_{K^c}(t_0).w_{K^c} \end{pmatrix}.$$

We thus have (t_0 is not written)

$$\begin{aligned} & (A.\delta y + B.\delta\eta, C.\delta y + D.\delta\eta) \\ = \ & (A_K.\delta y_K + B_K.w_K, A_{K^c}.\delta y_{K^c} + B_{K^c}.w_{K^c}, C_K.\delta y_K + D_K.w_K, C_{K^c}.\delta y_{K^c} + D_{K^c}.w_{K^c}) \\ = \ & (v_K, A_{K^c}.\delta y_{K^c} + B_{K^c}.w_{K^c}, C_K.\delta y_K + D_K.w_K, v_{K^c}), \end{aligned}$$

which shows that $\Pi(\delta t, 0, A.\delta y + B.\delta\eta, C.\delta y + D.\delta\eta, \delta y, \delta\eta) = (\tau, v, w)$. ♣

$\Lambda_{\tilde{\psi}}$ is a lagrangian manifold. By using the preceding lemma and a classical result of symplectic geometry (cf Theorem 5.2 of [13]), we can find Ω a neighborhood of $\lambda(t_0, x_0)$ in $I\!R^{4n+2}$ and $S \in \mathcal{C}^\infty(I\!R^{1+2n}; I\!R)$ such that

$$\Lambda_{\tilde{\psi}} \cap \Omega = \{(t, \partial_t S(t, q_K, \xi_{K^c}, \eta), q_K, \partial_{\xi_{K^c}} S(.), \partial_{q_K} S(.), -\xi_{K^c}, \partial_\eta S(.), \eta\} \cap \Omega. \tag{25}$$

We can notice that S is defined with an arbitrary constant.
It is then easy to verify that if we define

$$\Psi(t, q, q', \xi_{K^c}, \eta) = S(t, q_K, \xi_{K^c}, \eta) - q_{K^c}.\xi_{K^c} - q'.\eta,$$

Ψ is a non degenerate phase function which parametrize $\Lambda_{\tilde{\psi}}$ in a neighborhood of $\lambda(t_0, x_0)$ (in this case, the fiber of Ψ is $\theta = (\xi_{K^c}, \eta)$).
Denote θ_0 a point such that $\partial_\theta \tilde{\psi}(t_0, q_0, q_0, \theta_0) = 0$ and

$$(t_0, \partial_t \tilde{\psi}(t_0, q_0, q_0, \theta_0), q_0, \partial_q \tilde{\psi}(.), q_0, -\partial_{q'} \tilde{\psi}(.)) = \lambda(t_0, x_0).$$

Then, we define the arbitrary constant of S in order to have

$$\Psi(t_0, q_0, q_0, 0, 0) = \tilde{\psi}(t_0, q_0, q_0, \theta_0)$$

Now, since Ψ and $\tilde{\psi}$ are both non degenerate phase functions whose associated lagrangian manifold are identical in a neighborhood of $\lambda(t_0, x_0)$, and because we manage to make them coincide at one point, we can use the proposition 1.3.1 of [10] to say that there exists a classical symbol b of order : $ord(b) = -(n/2 + (n + |K^c|)/2)$ such that

$$I(\hbar) = \int_{\mathbb{R}^{1+2n+|K^c|}} e^{\frac{i}{\hbar}(\Psi(t,q,q,\xi_{K^c},\eta)+tE_c)} b(t, q, q, \xi_{K^c}, \eta, \hbar) \hat{\varphi}(-t) d\xi_{K^c} d\eta dq dt + \mathcal{O}(\hbar^\infty) \tag{26}$$

where the support of $b(., \xi_{K^c}, \eta, \hbar)$ is contained in a small neighborhood of $(t_0, R^{-1}.x_0, R^{-1}.x_0)$.

Remark 4.2 Since $\int_0^t L(\phi_s(x, 0))ds = -tE_c$ for all $(x, 0) \in \Theta$ and from (18), we have

$$\sigma_U(\lambda(t_0, x_0)) = (2\pi\hbar)^{-n/2} e^{-\frac{i}{\hbar} t_0 E_c} |dt dx_K d\xi_{K^c} d\eta|^{1/2}.$$

But since the principal symbol of a FIO is nothing but the leading term of a stationary phase expansion, we necessarily have $e^{-\frac{i}{\hbar} t_0 E_c} = e^{\frac{i}{\hbar} \tilde{\psi}(t_0, q_0, q_0, \theta_0)}$. We finally obtain

$$\Psi(t_0, q_0, q_0, 0, 0) = \tilde{\psi}(t_0, q_0, q_0, \theta_0) = -t_0 E_c. \tag{27}$$

Let's denote the phase function of (26) by

$$\Phi(t, q, \xi_{K^c}, \eta) := \Psi(t, q, q, \xi_{K^c}, \eta) + tE_c$$

and define the set of indices of the "degenerate" directions by

$$K_D := \{k \in [1, r] \ / \ \cos(\alpha_k t_0) = 1\} \tag{28}$$

Let us look for the critical points of the phase function Φ of $I(\hbar)$.

For (t, q, η) close to $(t_0, R^{-1}.x_0, 0)$, we have

$$\begin{aligned}
(t, q, \xi_{K^c}, \eta) \text{ is a critical point of } \Phi \quad &\Leftrightarrow \left\{ \begin{array}{c} \partial_t S(t, q_K, \xi_{K^c}, \eta) = -E_c \\ \partial_{q_K} S(t, q_K, \xi_{K^c}, \eta) = \eta_K \\ \xi_{K^c} = -\eta_{K^c} \\ \partial_{\xi_{K^c}} S(t, q_K, \xi_{K^c}, \eta) = q_{K^c} \\ \partial_\eta S(t, q_K, \xi_{K^c}, \eta) = q \end{array} \right. \\
&\Leftrightarrow \left\{ \begin{array}{c} (t, -E_c, q, \eta, q, \eta) \in \Lambda_{\tilde{\psi}} \\ \xi_{K^c} = -\eta_{K^c} \end{array} \right. \\
&\Leftrightarrow \left\{ \begin{array}{c} H(P.(q, \eta)) = E_c \\ P^{-1}.\phi_t(P.(q, \eta)) = (q, \eta) \\ \xi_{K^c} = -\eta_{K^c} \end{array} \right.
\end{aligned}$$

$$\text{(from proposition (2.1))} \quad \Leftrightarrow \quad \left\{ \begin{array}{l} P.(q,\eta) \in \Theta \\ \xi_{K^c} = -\eta_{K^c} \end{array} \right.$$

$$\Leftrightarrow \quad \left\{ \begin{array}{c} (q,0) \in P^{-1}(\Theta) \\ \eta = 0 \\ \xi_{K^c} = 0 \end{array} \right. .$$

We go on by calculating the hessian of Φ at a critical point $(t,q,0,0)$ where $(q,0) \in P^{-1}(\Theta)$. We have

$$d^2_{q_K,\xi_{K^c},\eta_K,\eta_{K^c},q_{K^c}}\Phi(t,q,0,0) = \begin{pmatrix} \partial^2_{q_K}S & \partial^2_{q_K,\xi_{K^c}}S & (\partial^2_{q_K,\eta_K}S - Id) & \partial^2_{q_K,\eta_{K^c}}S & 0 \\ \partial^2_{q_K,\xi_{K^c}}S & \partial^2_{\xi_{K^c}}S & \partial^2_{\xi_{K^c},\eta_K}S & \partial^2_{\xi_{K^c},\eta_{K^c}}S & -Id \\ (\partial^2_{q_K,\eta_K}S - Id) & \partial^2_{\xi_{K^c},\eta_K}S & \partial^2_{\eta_K}S & \partial^2_{\eta_K,\eta_{K^c}}S & 0 \\ \partial^2_{q_K,\eta_{K^c}}S & \partial^2_{\xi_{K^c},\eta_{K^c}}S & \partial^2_{\eta_K,\eta_{K^c}}S & \partial^2_{\eta_{K^c}}S & -Id \\ 0 & -Id & 0 & -Id & 0 \end{pmatrix}$$

¿From now on, and to simplify the notations, we may forget the dependence in the variables (t, x, ...) in the various matricial expressions as it is the case in the following notation :

$$P^{-1}.d\phi_t(x,0).P \begin{pmatrix} \delta y_K \\ \delta y_{K^c} \\ \delta \eta_K \\ \delta \eta_{K^c} \end{pmatrix} := \begin{pmatrix} A_K & A_1 & B_K & B_1 \\ A_2 & A_{K^c} & B_2 & B_{K^c} \\ C_K & C_1 & D_K & D_1 \\ C_2 & C_{K^c} & D_2 & D_{K^c} \end{pmatrix} \begin{pmatrix} \delta y_K \\ \delta y_{K^c} \\ \delta \eta_K \\ \delta \eta_{K^c} \end{pmatrix}$$

¿From (24) and (25), we thus have ($x = R.q$):

$$T_{\lambda(t,x)}\Lambda_{\tilde{\psi}} = \{(\delta t, 0, \begin{pmatrix} A_K & A_1 & B_K & B_1 \\ A_2 & A_{K^c} & B_2 & B_{K^c} \\ C_K & C_1 & D_K & D_1 \\ D_2 & D_{K^c} & D_2 & D_{K^c} \end{pmatrix} \begin{pmatrix} \delta y_K \\ \delta y_{K^c} \\ \delta \eta_K \\ \delta \eta_{K^c} \end{pmatrix}, \delta y, \delta \eta)\}$$

and

$$\begin{aligned} & T_{\lambda(t,x)}\Lambda_{\tilde{\psi}} \\ = \quad & \{(\delta t, d(\partial_t S)(t,q_K,\xi_{K^c},\eta).(\delta t, \delta q_K, \delta\xi_{K^c}, \delta\eta), \delta q_K, d(\partial_{\xi_{K^c}}S), d(\partial_{q_K}S), -\delta\xi_{K^c}, d(\partial_\eta S), \delta\eta)\}. \end{aligned}$$

Now, if we identify these two equalities, we must have

$$\begin{pmatrix} \delta q_K \\ \delta \xi_{K^c} \\ \delta \eta \end{pmatrix} = \begin{pmatrix} A_K & A_1 & B_K & B_1 \\ -C_2 & -C_{K^c} & -D_2 & -D_{K^c} \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & Id \end{pmatrix} \begin{pmatrix} \delta y_K \\ \delta y_{K^c} \\ \delta \eta_K \\ \delta \eta_{K^c} \end{pmatrix} := L(t,q). \begin{pmatrix} \delta y_K \\ \delta y_{K^c} \\ \delta \eta_K \\ \delta \eta_{K^c} \end{pmatrix},$$

and thus

$$d^2_{q_K,\xi_{K^c},\eta}S(t,q_K,0,0).L(t,q) = \begin{pmatrix} C_K & C_1 & D_K & D_1 \\ A_2 & A_{K^c} & B_2 & B_{K^c} \\ Id & 0 & 0 & 0 \\ 0 & Id & 0 & 0 \end{pmatrix}.$$

Moreover, since

$$\begin{pmatrix} A_K & A_1 \\ -C_2 & -C_{K^c} \end{pmatrix} (t_0,x_0) = \begin{pmatrix} A_K(t_0) & 0 \\ 0 & -C_{K^c}(t_0) \end{pmatrix} \qquad \text{is inversible,}$$

for q close to $q_0(= R^{-1}.x_0)$ and t close to t_0, we get that $L(t,q)$ is inversible and thus

$$d^2_{q_K,\xi_{K^c},\eta}S(t,q_K,0,0) = \begin{pmatrix} C_K & C_1 & D_K & D_1 \\ A_2 & A_{K^c} & B_2 & B_{K^c} \\ Id & 0 & 0 & 0 \\ 0 & Id & 0 & 0 \end{pmatrix}.L^{-1}(t,q).$$

Then we have

$$d^2\Phi(t,q,0,0) \;=\; \begin{pmatrix} \begin{pmatrix} & d^2S(t,q_K,0,0) & \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & Id & 0 & 0 \\ 0 & 0 & 0 & 0 & Id \\ Id & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Id \\ 0 & Id & 0 & Id & 0 \end{pmatrix}$$

With some algebraic calculations and from (19) we finally obtain

$$d^2\Phi(t,q,0,0) \;=\; P_1(q).\begin{pmatrix} Id - A(x,t) & -B(x,t) & 0 \\ -C(x,t) & Id - D(x,t) & 0 \\ 0 & 0 & Id \end{pmatrix}.P_2(t,q),$$

where

$$P_1(q) = \begin{pmatrix} 0 & 0 & -Id & 0 & 0 \\ 0 & -Id & 0 & 0 & -Id \\ Id & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -Id \\ 0 & 0 & 0 & -Id & 0 \end{pmatrix}.\begin{pmatrix} \begin{pmatrix} & P^{-1}.P(x) & \end{pmatrix} & 0 \\ 0 & Id \end{pmatrix}$$

and

$$P_2(t,q) = \begin{pmatrix} \begin{pmatrix} & P^{-1}(x).P & \end{pmatrix} & 0 \\ 0 & Id \end{pmatrix}.\begin{pmatrix} Id & 0 & 0 & 0 & 0 \\ 0 & Id & 0 & 0 & 0 \\ 0 & 0 & Id & 0 & 0 \\ 0 & 0 & 0 & Id & 0 \\ 0 & -Id & 0 & 0 & Id \end{pmatrix}.\begin{pmatrix} \begin{pmatrix} & L^{-1}(t,q) & \end{pmatrix} & 0 \\ 0 & Id \end{pmatrix}.$$

We can easily see that $P_1(q)$ and $P_2(t,q)$ are inversible if t and q are sufficiently close to t_0 and q_0. Looking at the expressions of the matrices A, B, C and D in formulas (19), it is clear that we can find two inversible matrices P_3 and P_4 which do not depend on t and q such that

$$\begin{pmatrix} Id - A(x,t) & -B(x,t) & 0 \\ -C(x,t) & Id - D(x,t) & 0 \\ 0 & 0 & Id \end{pmatrix} \;=\; P_3.\begin{pmatrix} 0_{n-s} & 0 & 0 \\ 0 & M_1(q,t) & 0 \\ 0 & 0 & M_2(q,t) \end{pmatrix}.P_4$$

with

$$M_1(q,t) \;=\; \begin{pmatrix} L_{i_1}(q,t) & & 0 \\ & \ddots & \\ 0 & & L_{i_{|K_D|}}(q,t) \end{pmatrix}$$

$$M_2(q,t) \;=\; \begin{pmatrix} tId_{n-s} & & & & \\ & L_{j_1}(q,t) & & 0 & \\ & & \ddots & & \\ & 0 & & L_{j_{|K_I|}}(q,t) & \\ & & & & Id_{|K^c|} \end{pmatrix}$$

where we define

$$\begin{array}{rcl} K_D &=& \{i \in [1,r] \;/\; \cos(\alpha_i t_0) = 1\} \;:=\; \{i_1, \cdots, i_{|K_D|}\}, \\ K_I &:=& [1,s] \setminus K_D \;=\; \{j_1, \cdots, j_{|K_I|}\}, \\ L_i(q,t) &:=& \begin{pmatrix} 1-\cos(\alpha_i(x)t) & -\frac{1}{\alpha_i(x)}\sin(\alpha_i(x)t) \\ \alpha_i(x)\sin(\alpha_i(x)t) & 1-\cos(\alpha_i(x)t) \end{pmatrix} \qquad \text{for } i \in [1,r], \\ L_i(q,t) &:=& \begin{pmatrix} 1-\cosh(\alpha_i(x)t) & -\frac{1}{\alpha_i(x)}\sinh(\alpha_i(x)t) \\ -\alpha_i(x)\sinh(\alpha_i(x)t) & 1-\cosh(\alpha_i(x)t) \end{pmatrix} \qquad \text{for } i \in [r+1,s]. \end{array}$$

Remark 4.3 If we write $t_0(q)$ instead of $t_0(R^{-1}.q)$, we easily see that $M_1(t_0(q),q) = 0$, that $M_1(t,q)$ is inversible if $t \neq t_0(q)$ (with t sufficiently close to t_0) and that $M_2(t,q)$ is inversible for all t and q. $|K_D|$ represents the set of indices of the degenerated directions of the phase function.

4.2 The asymptotic expansion

This part is concerned with proving the following result on $I(\hbar)$.

Proposition 4.4 *As $\hbar$ goes to 0,*

$$I(\hbar) \sim \hbar^{N/2-n} \sum_{j=0}^{\infty} d_j \hbar^j$$

with

$$d_0 = \int_{\Theta} < (t - t_0(x) - i0)^{-|K_D|}, c_0(t,x,0)\check{\varphi}(t) > dx$$

where $c_0 \in C_0^\infty(I\!R \times \Theta)$ and does not depend on φ.

We first look for a change of variables in (26) in order to make $P^{-1}(\Theta)$ flat in a neighborhood of $(q_0, 0)$. Since $d^2V(q_0)$ is a diagonal matrix in the q-coordinates, there exists a neighborhood U of $(q_0, 0)$ in $I\!R^{2n}$ and a function $f \in C^\infty$ such that

$$P^{-1}(\Theta) \cap U \;=\; \{(q_N, q_T, 0) \in I\!R^s \times I\!R^{n-s} \times I\!R^n \;/\; q_N = f(q_T)\} \cap U$$

with $(q_0)_N = f((q_0)_T)$ and $df((q_0)_T) = 0$.
Then, we define

$$\begin{array}{rcl} P_5 &:& I\!R^n \longrightarrow I\!R^n \\ && (q_N, q_T) \longmapsto (y_N, y_T) = (q_N - f(q_T), q_T). \end{array}$$

It is easy to see that in a neighborhood of q_0, P_5 is a diffeomorphism and if we proceed to this change of variables in (26), we get

$$I(\hbar) \sim \int_{I\!R^{1+2n+|K^c|}} e^{\frac{i}{\hbar}\psi_1(t,y,\xi_{K^c},\eta)} a_1(t,y,\xi_{K^c},\eta,\hbar)\hat{\varphi}(-t)\, d\xi_{K^c} d\eta dy dt \tag{29}$$

where the support of $a_1(.,\xi_{K^c},\eta,\hbar)$ is contained in a small neighborhood of (t_0,q_0) and is of the same order as b. Moreover, ψ_1 satisfies

$$d\psi_1(t,y,\xi_{K^c},\eta) = 0 \;\Leftrightarrow\; \left\{ \begin{array}{l} y_N = 0 \\ \xi_{K^c} = 0 \\ \eta = 0 \end{array} \right.$$

and

$$d^2\psi_1(t,y_T,0,0,0) = P_6(y_T).\begin{pmatrix} 0_{n-s} & 0 & 0 \\ 0 & M_1(y_T,t) & 0 \\ 0 & 0 & M_2(y_T,t) \end{pmatrix}.P_7(t,y_T).$$

where $P_6(y_T)$ and $P_7(t,y_T)$ are inversible for all t et y_T and where we denote $M_1(y_T,t)$ instead of $M_1(P_5^{-1}(0,y_T),t)$.
Since $d^2\psi_1$ is a symmetrical matrix, we also have

$$d^2\psi_1(t,y_T,0,0,0) ={}^t P_7(t,y_T).\begin{pmatrix} 0_{n-s} & 0 & 0 \\ 0 & {}^tM_1(y_T,t) & 0 \\ 0 & 0 & {}^tM_2(y_T,t) \end{pmatrix}.{}^tP_6(y_T).$$

If we define

$$\begin{aligned} {}^tP_7(t,y_T) &:= \begin{pmatrix} P_7^1 & P_7^2 \\ P_7^3 & P_7^4 \end{pmatrix} \\ {}^tP_6(y_T) &:= \begin{pmatrix} P_6^1 & P_6^2 \\ P_6^3 & P_6^4 \end{pmatrix} \end{aligned}$$

so that

$$d^2\psi_1(t,y_T,0,0,0) = \begin{pmatrix} P_7^2.\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^3 & P_7^2.\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^4 \\ P_7^4.\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^3 & P_7^4.\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^4 \end{pmatrix}$$

and using the fact that $\partial_{y_T}(d\psi_1)(t,y_T,0,0,0) = 0$ we obtain :

$$\begin{aligned} P_7^2.\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^3 &= 0 \\ P_7^2.\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^4 &= 0 \\ P_7^4.\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^3 &= 0 \end{aligned}$$

Since P_6 is bijective, $(P_6^3\ P_6^4)$ is a surjective matrix, which, in addition with the fact that $\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}$ is inversible for all $t \neq t_0(y_T)$, gives that $P_7^2(t,y_T) = 0$ for all $t \neq t_0(y_T)$ and finally, using the continuity of P_7, for all t. We show in a similar way that $P_6^3(y_T) = 0$, and we finally conclude that

$$d^2\psi_1(t,y_T,0,0,0) = \begin{pmatrix} 0_{n-s} & 0 \\ 0 & P_7^4(t,y_T).\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix}.P_6^4(y_T) \end{pmatrix}$$

where $P_7^4(t,y_T)$ and $P_6^4(y_T)$ are inversible for all t and y_T.

We now proceed to the following change of variables in (29) (where t and y_T are fixed) :

$$\begin{aligned} P_8 \quad : \quad I\!R^s \times I\!R^{|K^c|} \times I\!R^n &\longrightarrow I\!R^{2|K_D|} \times I\!R^{n-s+2|K_I|+|K^c|} \\ \begin{pmatrix} y_N \\ \xi_{K^c} \\ \eta \end{pmatrix} &\longmapsto \begin{pmatrix} z' \\ z'' \end{pmatrix} = P_6^4(y_T).\begin{pmatrix} y_N \\ \xi_{K^c} \\ \eta \end{pmatrix} \end{aligned}$$

We thus obtain

$$I(\hbar) \sim \int_{\mathbb{R}^{1+2n+|K^c|}} e^{\frac{i}{\hbar}\psi_2(t,y_T,z',z'')} a_2(t,y_T,z',z'',\hbar)\hat{\varphi}(-t)dz''dz'dy_Tdt \tag{30}$$

where the support of $a_2(.,z',z'',\hbar)$ is contained in a small neighborhood of $(t_0,(y_T)_0)$ and is of the same order than b. Moreover, ψ_2 satisfies :

$$d\psi_2(t,y_T,z',z'') = 0 \Leftrightarrow \begin{cases} z' = 0 \\ z'' = 0 \end{cases}$$

and

$$d^2\psi_2(t,y_T,0,0) = \begin{pmatrix} 0_{n-s} & 0 \\ 0 & P_9(t,y_T).\begin{pmatrix} {}^tM_1 & 0 \\ 0 & {}^tM_2 \end{pmatrix} \end{pmatrix}$$

where $P_9(t,y_T)$ is inversible.

Moreover, since $M_1(t_0(y_T)),y_T) = 0$ for all y_T and with the same arguments as before, we must have

$$d^2_{y_T,z',z''}\psi_2(t_0(y_T),y_T,0,0) = \begin{pmatrix} 0_{n-s} & 0 & 0 \\ 0 & 0_{2|K_D|} & 0 \\ 0 & 0 & P_{10}(y_T).{}^tM_2(t_0(y_T),y_T) \end{pmatrix} \tag{31}$$

where $P_{10}(y_T)$ is inversible for all y_T.

Remark 4.5 Since all the changes of variables done until now were time-independent, we necessarily have

$$\begin{aligned} \partial_t\psi_2(t,y_T,z',z'') &= E_c + \partial_t S(t, q_K(y_T,z',z''), \xi_{K^c}(y_T,z',z''), \eta(y_T,z',z'')) \\ &= E_c - H(x(y_T,z',z''), \xi(y_T,z',z'')) \end{aligned}$$

The previous calculations give us the following lemma :

Lemma 4.6

$$\begin{array}{ll} 1) & \partial_{z''}\psi_2(t,y_T,0,0) = 0 \\ 2) & \partial^2_{z''}\psi_2(t_0,(y_T)_0,0,0) \quad \textit{is non degenerate.} \end{array}$$

This lemma allows us to apply the parameters depending Morse lemma which is proven in [9] for example. It says that

there exists V a neighborhood of $(t_0,(y_T)_0,0)$ in $\mathbb{R}^{1+(n-s)+2|K_D|}$ and
there exists W a neighborhood of 0 in $\mathbb{R}^{n-s+2|K_I|+|K^c|}$
such that for all $(t,y_T,z') \in V$, there exists a unique $z'' = z''(t,y_T,z')$ in W which satisfies

$$\partial_{z''}\psi_2(t,y_T,z',z''(t,y_T,z')) = 0. \tag{32}$$

Moreover, there exists a diffeomorphism P_{11} from $V \times W$ to U a neighborhood of $(t_0,(y_T)_0,0,0)$ in $\mathbb{R}^{1+2n+|K^c|}$ defined by

$$P_{11}(t,y_T,z',z'') = (t,y_T,z',\tilde{z}''(t,y_T,z',z''))$$

and satisfying

$$\begin{aligned}\tilde{z}''(t,y_T,z',z'') &= z''-z''(t,y_T,z')+\mathcal{O}(|z''-z''(t,y_T,z')|^2) \quad \text{and}\\ \psi_2\circ P_{11}^{-1}(t,y_T,z',\tilde{z}'') &= \psi_3(t,y_T,z')+\frac{1}{2}<Q(t,y_T,z')\tilde{z}'',\tilde{z}''>\end{aligned}$$

where

$$\psi_3(t,y_T,z')=\psi_2(t,y_T,z',z''(t,y_T,z'))$$

and

$$Q(t,y_T,z')=\partial^2_{z''}\psi_2(t,y_T,z',z''(t,y_T,z')).$$

If we proceed again to this change of variables in (30), and expanding the result with the stationary phase lemma, we obtain

$$I(\hbar)\sim\int_{\mathbb{R}^{1+(n-s)+2|K_D|}} e^{\frac{i}{\hbar}\psi_3(t,y_T,z')}a_3(t,y_T,z',\hbar)\hat{\varphi}(-t)dz'dy_Tdt \tag{33}$$

where the support of $a_3(.,z',\hbar)$ is contained in a small neighborhood of $(t_0,(y_T)_0)$ and

$$\begin{aligned}ord(a_3) &= ord(b)+(n-s+2|K_I|+|K^c|)/2\\ &= -(n+|K^c|/2)+(n-s+2|K_I|+|K^c|)/2\\ &= (-n-s+2|K_I|)/2.\end{aligned}$$

Lemma 4.7

$$\begin{aligned}&1) \qquad z''(t,y_T,0)=0\\ &2) \qquad \partial_{t,y_T,z'}z''(t_0(y_T),y_T,0)=0\end{aligned}$$

Proof: 1) is a direct implication of the previous lemma.
To prove 2), we take the derivative of (32) with respect to (t,y_T,z') to show that

$$\partial^2_{z''}\psi_2(t,y_T,z',z''(t,y_T,z')).\partial_{t,y_T,z'}z''(t,y_T,z')+\partial_{t,y_T,z'}\partial_{z''}\psi_2(t,y_T,z',z''(t,y_T,z'))=0.$$

But since $\partial_{t,y_T,z'}\partial_{z''}\psi_2(t_0(y_T),y_T,0,0)=0$ from (31) and since $\partial^2_{z''}\psi_2(t_0(y_T),y_T,0,0)$ is inversible from the previous lemma, we can conclude. ♣

Since ψ_3 and $t_0(y_T)$ are C^∞, there exists $F\in C^\infty$ such that

$$\psi_3(t,y_T,z')=(t-t_0(y_T))F(t,y_T,z')+\psi_3(t_0(y_T),y_T,z').$$

Then we have

Lemma 4.8

$$\begin{aligned}&1) \qquad \partial_{z'}F(t,y_T,0)=0\\ &2) \qquad \partial^2_{z'}F(t_0,(y_T)_0,0) \quad \textit{is negative definite.}\end{aligned}$$

Proof: 1) we have :

$$\partial_{z'}\psi_3(t,y_T,0)=\partial_{z'}\psi_2(t,y_T,0,0)+\partial_{z''}\psi_2(t,y_T,0,0).\partial_{z'}z''(t,y_T,0)=0$$

because $d\psi_2(t, y_T, 0, 0) = 0$, therefore $(t - t_0(y_T))\partial_{z'} F(t, y_T, 0) = 0$ for all t.
2) We easily verify that

$$\partial_t \psi_3(t_0(y_T), y_T, z') = F(t_0(y_T), y_T, z').$$

Moreover, we some calculations, we have

$$\partial^2_{z'} \psi_3(t, y_T, 0) = \partial^2_{z'} \psi_2(t, y_T, 0, 0) + 2\partial^2_{z',z''} \psi_2 . \partial_{z'} z'' + \partial^2_{z''} \psi_2 . \partial_{z'} z'' . \partial_{z'} z'' + \partial_{z''} \psi_2 . \partial^2_{z'} z''.$$

We then use the following equalities (already proven)

$$\begin{aligned} \partial_{z'} z''(t_0(y_T), y_T, 0) &= 0 \\ \partial^2_{z',z''} \psi_2(t_0(y_T), y_T, 0, 0) &= 0 \\ \partial_{z''} \psi_2(t, y_T, 0, 0) &= 0 \end{aligned}$$

to show that

$$\partial_t \partial^2_{z'} \psi_3(t_0(y_T), y_T, 0) = \partial_t \partial^2_{z'} \psi_2(t_0(y_T), y_T, 0, 0)$$

and therefore

$$\begin{aligned} \partial^2_{z'} F(t_0, (y_T)_0, 0) &= \partial_t \partial^2_{z'} \psi_2(t_0, (y_T)_0, 0, 0) \\ &= -(\partial^2_{z'} H(x(y_T, z'), \xi(y_T, z')))(x_0, 0). \end{aligned}$$

But since z' are the degenerate directions of the linearised flow, they must correspond to elliptic directions where the hessian of H is positive definite. ♣

If we apply again the parameters depending Morse lemma to the function F, we have that
there exists V_1 a neighborhood of $(t_0, (y_T)_0)$ in $I\!R^{1+n-s}$,
there exists W_1 a neighborhood of 0 in $I\!R^{2|K_D|}$,
there exists U_1 a neighborhood of $(t_0, (y_T)_0, 0)$ in $I\!R^{1+n-s+2|K_D|}$ and
there exists P_{12} a diffeomorphism from $V_1 \times W_1$ to U_1 defined by

$$P_{12}(t, y_T, z') = (t, y_T, \tilde{z}'(t, y_T, z'))$$

and which satisfies

$$\begin{aligned} \tilde{z}'(t, y_T, z') &= z' + \mathcal{O}(|z'|^2) \\ F \circ P_{12}^{-1}(t, y_T, \tilde{z}') &= F(t, y_T, 0) + \frac{1}{2} < \partial^2_{z'} F(t, y_T, 0)\tilde{z}', \tilde{z}' > . \end{aligned}$$

But, since $d\psi_2(t, y_T, 0, 0) = 0$, we have

$$\partial_t \psi_3(t, y_T, 0) = \partial_t \psi_2(t, y_T, 0, 0) + \partial_{z''} \psi_2 . \partial_t z'' = 0$$

which gives

$$\psi_3(t, y_T, 0) = \psi_3(t_0(y_T), y_T, 0),$$

$(t - t_0(y_T))F(t, y_T, 0) = 0$ and finally

$$F(t, y_T, 0) = 0.$$

We now proceed to this new change of variables in (33) and we obtain

$$I(\hbar) \sim \int_{I\!R^{1+(n-s)+2|K_D|}} e^{\frac{i}{2\hbar}(t-t_0(y_T))<\partial^2_{z'} F(t,y_T,0)\tilde{z}',\tilde{z}'>} e^{\frac{i}{\hbar}\psi_4(y_T,\tilde{z}')} a_4(t, y_T, \tilde{z}', \hbar)\hat{\varphi}(-t) d\tilde{z}' dy_T dt \tag{34}$$

where the support of $a_4(.,\tilde{z}',\hbar)$ is contained in a small neighborhood of $(t_0,(y_T)_0)$ and is of the same order as a_3. Moreover, ψ_4 satisfy

$$\begin{aligned} \partial_{\tilde{z}'}\psi_4(y_T,0) &= 0 \quad \text{and} \\ \partial^2_{\tilde{z}'}\psi_4(y_T,0) &= 0. \end{aligned} \tag{35}$$

Since $\partial^2_{z'}F(t,y_T,0)$ is negative definite, we can find a change of variables such that

$$< \partial^2_{z'}F(t,y_T,0)\tilde{z}',\tilde{z}' >= -|z|^2$$

and then choose the polar coordinate: $z=r\sigma$ with $r\in[0,+\infty[$ and $\sigma\in\Sigma_{2|K_D|-1}$ (the unit sphere of $I\!R^{2|K_D|}$) to obtain

$$I(\hbar)\sim\int_{I\!R^{1+(n-s)}\times[0,+\infty[\times\Sigma_{2|K_D|-1}} e^{-\frac{i}{2\hbar}(t-t_0(y_T))r^2}e^{\frac{i}{\hbar}\psi_5(y_T,r\sigma)}a_5(t,y_T,r\sigma,\hbar)\hat{\varphi}(-t)r^{2|K_D|-1}dtdy_Tdrd\sigma$$

¿From (27), we have that $\psi_5(y_T,0)=\Phi(t_0,q_0,0,0)=0$ and from (35) we have $\partial_r\psi_5(y_T,0)=0$ and $\partial_r^2\psi_5(y_T,0)=0$, we can thus find a function $\psi_6\in C^\infty$ such that :

$$\psi_5(y_T,r\sigma)=r^3\psi_6(y_T,r,\sigma)$$

¿From dilating r by $\sqrt{\hbar}$ and translating t by $t_0(y_T)$, we get :

$$I(\hbar)\sim\hbar^{|K_D|}g(\sqrt{\hbar})$$

where we define

$$g(\tau)=\int_{I\!R^{1+(n-s)}\times[0,+\infty[\times\Sigma_{2|K_D|-1}} e^{-\frac{i}{2}tr^2}e^{i\tau r^3\psi_6(y_T,\tau r,\sigma)}a_6(t,y_T,\tau r\sigma,\tau^2)r^{2|K_D|-1}dtdy_Tdrd\sigma$$

We now show that $g(\tau)$ has an asymptotic expansion when τ goes to 0 which is nothing but its Taylor expansion in $\tau=0$ because g is a smooth function whose derivatives are all bounded.

Since a_6 is a classical symbol of order $ord(a_3)$, we can forget the dependence in τ in the last variable of a_6 and take only its principal symbol : $a_{6,0}$. The other terms would be treated in a similar way. If we integrate with respect to t, we obtain :

$$g(\tau)=\int_{I\!R^{(n-s)}\times[0,+\infty[\times\Sigma_{2|K_D|-1}} e^{i\tau r^3\psi_6(y_T,\tau r,\sigma)}(\mathcal{F}_t a_{6,0})(r^2/2,y_T,\tau r\sigma)r^{2|K_D|-1}dy_Tdrd\sigma$$

where $\mathcal{F}_t a_{6,0}$ is the Fourier transform of $a_{6,0}$ with respect to t and is therefore a Schwartz function with respect to its first variable and C_0^∞ with respect to the others.

It is now clear that the integrand is C^∞ with respect to τ and that all its derivatives with respect to τ are uniformly bounded in τ by some functions in $L^1(I\!R)$ (we used here the rapid decrease of $\mathcal{F}_t a_{6,0}$ with respect to its first variable and we didn't talk about the integrations in y_T and σ because they are compact). Thus, we can differentiate under the integral, and by using the dominated convergence theorem of Lebesgue, we have that for all $k\in I\!N$, $g^{(k)}(0)$ is equal to

$$\int_{I\!R^{(n-s)}\times[0,+\infty[\times\Sigma_{2|K_D|-1}}\partial_\tau^k\left[e^{i\tau r^3\psi_6(y_T,\tau r,\sigma)}(\mathcal{F}_t a_{6,0})(r^2/2,y_T,\tau r\sigma)\right](\tau=0,r,\sigma,y_T)r^{2|K_D|-1}dy_Tdrd\sigma$$

and

$$|g^{(k)}(\tau)|\leq M \qquad \text{for } \tau<\tau_0.$$

Then, the Taylor theorem allows us to write that for all $\tau < \tau_0$,

$$\left| g(\tau) - \sum_{k=0}^{N} \frac{g^{(k)}(0)}{k!} \tau^k \right| \leq C_N \tau^{N+1}$$

Let's show now that g is an even function so that the asymptotic expansion will contain only even powers of τ. If we go back to the initial definition of g, we have

$$g(\tau) = \int_{\mathbb{R}^{1+(n-s)}\times[0,+\infty[\times\Sigma_{2|K_D|-1}} e^{-\frac{i}{2}tr^2} e^{\frac{i}{\tau^2}\psi_5(y_T,\tau r\sigma)} a_6(t,y_T,\tau r\sigma) r^{2|K_D|-1} dt dy_T dr d\sigma$$

and if we use the symmetry of $\Sigma_{2|K_D|-1}$, we see that g is an even function.

Moreover, since $|K_D| + ord(a_3) = |K_D| + |K_I| - (n+s)/2 = (s-n)/2 = N/2 - n$, we obtain

$$I(\hbar) \sim_{\hbar \to 0} \quad \hbar^{N/2-n} \sum_{j=0}^{\infty} d_j \hbar^j.$$

The calculation for the leading term gives

$$\begin{aligned}
d_0 &= \int_{\mathbb{R}^{1+(n-s)}\times[0,+\infty[\times\Sigma_{2|K_D|-1}} e^{-\frac{i}{2}(t-t_0(y_T))r^2} a_{5,0}(t,y_T,0)\hat{\varphi}(-t) r^{2|K_D|-1} dt dy_T dr d\sigma \\
&= Vol(\Sigma_{2|K_D|-1}) \int_{\mathbb{R}^{1+(n-s)+1}} e^{-i(t-t_0(y_T))s} a_{5,0}(t,y_T,0)\hat{\varphi}(-t) \max(0,s)^{|K_D|-1} dt dy_T ds \\
&= 2(\pi)^{|K_D|} e^{-i\pi|K_D|/2} \int_{\mathbb{R}^{n-s}} < (t-t_0(y_T)-i0)^{-|K_D|}, a_{5,0}(t,y_T,0)\hat{\varphi}(-t) > dy_T \qquad (36)
\end{aligned}$$

where we used the fact that

$$\mathcal{F}(\max(0,s)^{|K_D|-1})(t) = \Gamma(|K_D|) e^{-i\pi|K_D|/2} (t-i0)^{-|K_D|}$$

and

$$Vol(\Sigma_{2|K_D|-1}) = 2(\pi)^{|K_D|}/\Gamma(|K_D|).$$

Thus

$$d_0 = \int_{\mathbb{R}^{n-s}} < (t-t_0(y_T)-i0)^{-|K_D|}, c_0(t,y_T)\check{\varphi}(t) > dy_T$$

where $c_0 \in C_0^\infty(\mathbb{R}^{1+n-s})$. This ends the proof of proposition 4.4. ♣

4.3 Calculation of the leading term

This part is concerned with calculating explicitly the coefficient c_0 appearing in proposition 4.4. Of course, the first idea would be to follow the calculus of the previous parts to study more precisely the jacobian of the various change of variables introduced in $I(\hbar)$. But this is rather difficult and there is an easier way which consists in formally applying the stationary phase lemma to $I(\hbar)$ at the critical points where the normal hessian of the phase function is non degenerate in order to use the results of section 5 of [14]. This will give us the value of c_0 at every $(t,z) \in \mathbb{R} \times \Theta$ such that $t \neq t_0(z)$ (i.e. t is not a period of the linearised flow at z) and using the fact that c_0 is a C^∞ function, we will obtain the value of c_0 for all (t,z).

Let's define for all $(t,z) \in I\!\!R \times \Theta$ sufficiently close to $(t_0, z_0 = (x_0, 0))$ and such that $t \neq t_0(z)$, the function

$$f(t,z) = \frac{1}{t^{n-N/2} \prod_{i=1}^{r} \sin(\frac{\alpha_i(z)}{2} t) \prod_{i=r+1}^{N-n} \sinh(\frac{\alpha_i(z)}{2} t)}.$$

We can notice that $f(.,z)$ is meromorphe with a pole of order $|K_D|$ at $t_0(z)$ (see the definition of K_D in formula (28)).
We then show

Proposition 4.9 *Let $(t_0', z_0') \in I\!\!R \times \Theta$ be sufficiently close to t_0 et z_0, then*

$$c_0(t_0', z_0') = (2\pi)^{N/2-n} 2^{n-N} e^{i\pi\sigma/4} (t_0' - t_0(z_0'))^{|K_D|} f(t_0', z_0').$$

where σ is an integer which does not depend on (t_0', z_0').

Proof: It is sufficient to prove it for (t_0', z_0') such that $t_0' \neq t_0(z_0')$ because we can then use the fact that c_0 is C^∞ and that $(t - t_0(z))^{|K_D|} f(t,z)$ is also C^∞ and does not vanish at $t = t_0(z)$.

So let's take $(t_0', z_0') \in I\!\!R \times \Theta$ such that $t_0' \neq t_0(z_0')$.
We then choose Ω a neighborhood of z_0' in $I\!\!R^{2n}$ and J a neighborhood of t_0' in $I\!\!R$ such that

$$\forall t \in J,\ \forall z \in \Omega \cap \Theta,\ t \neq t_0(z).$$

Let $\rho \in C_0^\infty(\Omega)$.
With the notation of formula (29), we define

$$\tilde{I}(\hbar) = \int_{I\!\!R^{1+2n+|K^c|}} e^{\frac{i}{\hbar}\psi_1(t,y,\xi_{K^c},\eta)} a_1(t,y,\xi_{K^c},\eta,\hbar) \rho(y,\eta) \hat{\varphi}(-t) d\xi_{K^c} d\eta dy dt.$$

Then, for all φ such that $\check{\varphi} \in C_0^\infty(J)$, the critical points of ψ_1 which are contained in the support of the amplitude have a normal hessian which is non degenerate because $t \neq t_0(z)$. We are therefore in the context of section 5 of [14] where we can apply Theorem 5.6 to $\tilde{I}(\hbar)$ which gives us

$$\tilde{I}(\hbar) \sim \hbar^{N/2-n} (2\pi)^{N/2-n} e^{i\pi\sigma(t_0',z_0')/4} \int_{I\!\!R \times \Theta} \rho(z) \check{\varphi}(t) d\mu_t(z) dt.$$

On the other hand, the calculus of the previous subsection gives us immediately

$$\tilde{I}(\hbar) \sim \hbar^{N/2-n} \int_{I\!\!R \times \Theta} (t - t_0(z))^{-|K_D|} c_0(t,z) \rho(z) \check{\varphi}(t) dt dz + \mathcal{O}(\hbar^{N/2-n+1})$$

We thus have on one hand

$$\lim_{\hbar \to 0} \hbar^{n-N/2} \tilde{I}(\hbar) = \int_{I\!\!R \times \Theta} (t - t_0(z))^{-|K_D|} c_0(t,z) \rho(z) \check{\varphi}(t) dt dz$$

and on the other hand (using appendix A)

$$\lim_{\hbar \to 0} \hbar^{n-N/2} \tilde{I}(\hbar) = (2\pi)^{N/2-n} 2^{n-N} e^{i\pi\sigma(t_0',z_0')/4} \int_{I\!\!R \times \Theta} \rho(z) \check{\varphi}(t) |f(t,z)| dz dt.$$

If we identify these two equalities and pointing out that the result is valid for all $\check{\varphi} \in C_0^\infty(J)$ and all $\rho \in C_0^\infty(\Omega)$, we obtain

$$c_0(t_0', z_0') = (2\pi)^{N/2-n} 2^{n-N} e^{i\pi\sigma(t_0',z_0')/4} (t_0' - t_0(z_0'))^{|K_D|} |f(t_0', z_0')|.$$

Remark 4.10 $\sigma(t,z)$ is constant on each side of the hypersurface $t = t_0(z)$ but it may not be continuous on this hypersurface.

Instead of trying to calculate the discontinuity of $\sigma(t,z)$ on $t = t_0(z)$ by relating it to some Maslov index, we use the fact that c_0 is C^∞. This implies that $e^{i\pi\sigma(t,z)/4}(t-t_0(z))^{|K_D|}|f(t,z)|$ is also C^∞. Since $(t-t_0(z))^{|K_D|}f(t,z)$ is C^∞ and does not vanish on the hypersurface $t = t_0(z)$, we necessarily have

$$e^{i\pi\sigma(t,z)/4}(t-t_0(z))^{|K_D|}|f(t,z)| = e^{i\pi\sigma/4}(t-t_0(z))^{|K_D|}f(t,z).$$

for some integer σ which does not depend on t and z. ♣

We can now calculate the leading term of the asymptotic expansion of $I(\hbar)$ which becomes

$$d_0 = (2\pi)^{N/2-n}2^{n-N}e^{i\pi\sigma/4}\int_\Theta < (t-t_0(x)-i0)^{-|K_D|}, (t-t_0(x))^{|K_D|}f(t,x)\check{\varphi}(t)\chi(t,x) > dx$$

where χ is a smooth function equal to 1 in a neighborhood of (t_0, x_0).

Moreover,

$$\begin{aligned}
& < (t-t_0(x)-i0)^{-|K_D|}, (t-t_0(x))^{|K_D|}f(t,x)\check{\varphi}(t)\chi(t,x) > \\
= & \lim_{\epsilon\to 0^+}\int_{I\!R}(t-t_0(x)-i\epsilon)^{-|K_D|}(t-t_0(x))^{|K_D|}f(t,x)\check{\varphi}(t)\chi(t,x)dt \\
= & \lim_{\epsilon\to 0^+}\int_{I\!R}\frac{(t-t_0(x)-i\epsilon)^{|K_D|}}{(t-t_0(x)-i\epsilon)^{|K_D|}}f(t-i\epsilon,x)\check{\varphi}(t)\chi(t,x)dt \\
= & < f(t-i0,x), \check{\varphi}(t)\chi(t,x) > .
\end{aligned}$$

where the distribution is defined by theorem 3.1.11 of [16]. We finally get

$$d_0 = (2\pi)^{N/2-n}2^{n-N}e^{i\pi\sigma/4}\int_\Theta < f(t-i0,x), \check{\varphi}(t)\chi(t,x) > dx. \tag{37}$$

which complete the proof of Theorem 3.2.

5 The case with an observable

We would like to formulate, in this part, a microlocal version of Theorems 1.3 and 1.5 by introducing the diagonal elements of some observable in the sum defining $\Upsilon(E_c, \hbar)$.

More precisely, let us consider a classical observable $a(x,\xi) \in \mathcal{S}(I\!R^{2n})$ in the Schwartz class and denote by $A(\hbar)$ an $\hbar$-pseudodifferentiel operator associated to a by some quantification procedure as defined in [23] for example. Consider an orthonormal basis $\{\psi_k(\hbar)\}_{k\in I\!N}$ of eigenfunctions associated to the eigenvalues $\{E_k(\hbar)\}_{k\in I\!N}$ of $\widehat{H}$, and define

$$\Upsilon_a(E_c, \hbar) = \sum_{E_k(\hbar)\in[E_c-\lambda, E_c+\lambda]} \varphi\left(\frac{E_k(\hbar) - E_c}{\hbar}\right) < \psi_k(\hbar), A(\hbar)\psi_k(\hbar) >$$

for some φ such that $\hat{\varphi} \in C_0^\infty(I\!R)$ and $\lambda > 0$.

A straightforward extension of the proof of Theorems 1.3 and 1.5 to this case gives us the following result :

Theorem 5.1 *Assume that the hypothesis of theorem 1.3 are satisfied, then the asymptotic expansion of $\Upsilon_a(E_c, \hbar)$ is the same as for $\Upsilon(E_c, \hbar)$ provided the coefficients in the expansion are modified as follows :*

- *for the case 1) a) :*

$$c_0 = \frac{\hat{\varphi}(0)}{(2\pi)^n} \int_{\Sigma_{E_c}} a \, d\lambda,$$

- *for the case 1) b) :*

$$c_{0,1} = -\frac{1}{\pi\alpha_1}\hat{\varphi}(0)a(x_0, 0)$$

  if we let $\Theta = \{(x_0, 0)\}$,

- *for the case 2), the leading term is*

$$c_0 = \frac{\hat{\varphi}(0)}{(2\pi)^n} \int_{\Sigma_{E_c}} a \, d\lambda,$$

- *for the case 3), we have (with the notations of theorem 1.5)*

$$\textit{1) if } d > 0 \ : \ c_0 = \frac{(2\pi)^{-d/2}}{\Gamma(d/2)} \int_{I\!R} \left[\int_\Theta \sum_{k\geq 0} (\tau - \beta_k(x))_+^{d/2-1} a(x,0) dx \right] \varphi(\tau) d\tau$$

$$\textit{2) if } d = 0 \ : \ c_0 = a(x_0, 0) \sum_{k\geq 0} \varphi(\beta_k) \qquad (\textit{if } \Theta = \{(x_0, 0)\})$$

where $d\lambda$ denotes the Liouville measure which is well defined on Σ_{E_c} when $N > 2$.

6 An example for the case $\nu = 0$

Consider the Schrödinger operator

$$\widehat{H} = -\frac{\hbar^2}{2}\Delta + V(x_1, x_2) \quad \text{on } L^2(\mathbb{R}^2)$$

where we define

$$V(x_1, x_2) = (x_1^2 + x_2^2 - 1)^2.$$

The graph of this potential is nothing but a symmetrical double well which has been rotated around the vertical axe. The classical Hamiltonian is therefore integrable. $E_c = 0$ is a critical energy and the corresponding energy surface is $\Sigma_0 = \{(p_1 = 0, p_2 = 0, x_1, x_2) \in \mathbb{R}^4 \ / \ x_1^2 + x_2^2 = 1\}$. In this case, we have $\Theta = \Sigma_0$ and for every $(x, 0) \in \Theta$, we have

$$d^2V(x) = \begin{pmatrix} 8x_1^2 & 8x_1x_2 \\ 8x_1x_2 & 8x_2^2 \end{pmatrix}.$$

Hence, the eigenvalues of $d^2V(x)$ are 0 and 8. The hypothesis (H2bis), (H3), (H4) and (H5) are therefore satisfied and with the previous notations, we have : $\nu = 0$, $\alpha_1 = 2\sqrt{2}$ and $\dim \Theta = 1$. Applying theorem 1.3 and 1.5, and with some easy calculations on the leading coefficient, we get that for all φ such that $\hat{\varphi} \in C_0^\infty(\mathbb{R})$ and $\forall \lambda > 0$,

$$\sum_{E_k(\hbar) \in [-\lambda, \lambda]} \varphi\left(\frac{E_k(\hbar)}{\hbar}\right) \sim \hbar^{-\frac{1}{2}} \sum_{j=0}^{\infty} c_j \hbar^j \qquad \text{when } \hbar \text{ goes to } 0$$

where

$$c_0 = \sqrt{2} \int_{\mathbb{R}} \sum_{k \geq 0} (\tau - 2\sqrt{2}(k + \frac{1}{2}))_+^{-\frac{1}{2}} \varphi(\tau) d\tau.$$

Since the system is integrable (because of the symmetry of rotation), we can use the WKB method to estimate the number of eigenvalues at the bottom of this potential and "justify" a little bit the formula (12), which in this case would tell us that

$$N_{0,\rho}(\hbar) = 2\sqrt{2} \sum_{k \geq 0} \sqrt{(\rho - 2\sqrt{2}(k + \frac{1}{2}))_+}\, \hbar^{-\frac{1}{2}} + o(\hbar^{-\frac{1}{2}}). \tag{38}$$

Indeed, if we choose the polar coordinates and proceed to the usual reduction because of the symmetry (see [2] for example), we need to solve the reduced Schrödinger equation :

$$\widehat{H}_l \psi_l = \mu_l \psi_l \qquad \text{in } L^2([0, +\infty[)$$

where $l \in \mathbb{Z}$ and

$$\widehat{H}_l = -\frac{\hbar^2}{2}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2}{2}\frac{l^2 - 1/4}{r^2}$$

with $V(r) = (r^2 - 1)^2$. Since we are localized near $r = 1$, we only need to find the eigenvalues of the operator

$$-\frac{\hbar^2}{2}\frac{d^2}{dr^2} + 4(r-1)^2 + \frac{\hbar^2}{2}(l^2 - 1/4).$$

We finally obtain that the eigenvalues of $\widehat{H}$ near the bottom of the potential are approximated at the first order by the

$$\mu_{l,k}(\hbar) = \frac{\hbar^2}{2}(l^2 - 1/4) + 2\sqrt{2}(k + \frac{1}{2})\hbar \qquad (l, k) \in \mathbb{Z} \times \mathbb{N}$$

We therefore obtain

$$\begin{aligned} N_{0,\rho}(\hbar) &\sim_{\hbar\to 0^+} \#\left\{(l,k)\in \mathbb{Z}\times \mathbb{N}\ \ /\ \ \frac{\hbar^2}{2}(l^2-1/4)+2\sqrt{2}(k+\frac{1}{2})\hbar \le \rho\hbar\right\} \\ &\sim_{\hbar\to 0^+} \sum_{2\sqrt{2}(k+\frac{1}{2})\le\rho} \#\left\{l\in\mathbb{Z}\ \ /\ \ |l|\le \sqrt{2(\rho-2\sqrt{2}(k+\frac{1}{2}))\hbar^{-1}+1/4}\right\}. \end{aligned}$$

Now, since

$$\#\left\{l\in\mathbb{Z}\ \ /\ \ |l|\le \sqrt{2(\rho-2\sqrt{2}(k+\frac{1}{2}))\hbar^{-1}+1/4}\right\} \sim_{\hbar\to 0^+} 2\sqrt{2}\sqrt{\rho-2\sqrt{2}(k+\frac{1}{2})}\ \hbar^{-\frac{1}{2}},$$

we prove formula (38).

To conclude, we would like to point out an interesting fact about this example : if we calculate the phase space volume Vol of $\{(x,p)\in \mathbb{R}^4\ \ /\ \ p^2/2+V(x)\le\rho\hbar\}$, we find

$$\begin{aligned} Vol &= \int_{p_1^2/2+p_2^2/2+V(x_1,x_2)\le\rho\hbar} dx_1dx_2dp_1dp_2 \\ &= (2\pi)^2\int\int_{p^2/2+(r^2-1)^2\le\rho\hbar\ ;\ p\ge 0\ ;\ r\ge 0} r\,dr\ p\,dp \\ &= 2\pi^2\int\int_{P+(R-1)^2\le\rho\hbar\ ;\ P\ge 0\ ;\ R\ge 0} dPdR \\ &= 2\pi^2\int_{-\sqrt{\rho\hbar}}^{\sqrt{\rho\hbar}}(\rho\hbar-R^2)dR \\ &= \frac{8}{3}\pi^2(\rho\hbar)^{3/2} \end{aligned}$$

Therefore, the Heisenberg principle tells us that

$$N_{0,\rho}(\hbar)\sim_{\hbar\to 0^+}\frac{Vol}{(2\pi\hbar)^2}=\frac{2}{3}\rho^{3/2}\hbar^{-\frac{1}{2}} \tag{39}$$

which is in contradiction with formula (38). This implies that the Heisenberg principle seems no more true in the case of a critical value of the energy. This kind of phenomenon is studied in much more details in the very interesting paper of Feffermann [12].

Remark 6.1 Nevertheless, we can prove without too much difficulty that we have

$$2\sqrt{2}\sum_{k\ge 0}\sqrt{(\rho-2\sqrt{2}(k+\frac{1}{2}))_+}\sim_{\rho\to+\infty} 2\sqrt{2}\int_0^{\rho/(2\sqrt{2})-1/2}\sqrt{\rho-2\sqrt{2}(x+\frac{1}{2})}dx$$

and since

$$2\sqrt{2}\int_0^{\rho/(2\sqrt{2})-1/2}\sqrt{\rho-2\sqrt{2}(x+\frac{1}{2})}dx=\frac{2}{3}(\rho-\sqrt{2})^{3/2}\sim_{\rho\to+\infty}\frac{2}{3}\rho^{3/2},$$

we observe that the leading coefficient of (38) and (39) are equivalent when ρ goes to infinity.

A Determination of the density defined on Θ

A general construction of an intrinsic density defined on the fixed-point manifold of a symplectic transformation is presented in the paper of Duistermaat and Guillemin [11]. In our case, Θ is the fixed-point manifold of ϕ_t the Hamiltonian flow. We can thus obtain a density $d\mu_t$ defined on Θ. This construction is done in section 5 of [14] and studied in more details in appendix A of [3]. Of course, Θ has to be a clean fixed point manifold so that this density may not be defined for all t. The singularity of this density at $t = 0$ is studied, for example, in [3]. In fact, if we let $(x_0, 0)$ be a point on Θ, it is not hard to see that $d\mu_t(x_0, 0)$ can not be defined by this construction when t is a period of $d\phi_t(x_0, 0)$.

In this appendix, we compute explicitly the density function of $d\mu_t$ so that we will be able to see the singularities of this density.

Let $(x_0, 0) \in \Theta$, and let $(\alpha_1^2, \cdots, \alpha_r^2, -\alpha_{r+1}^2, \cdots, -\alpha_s^2, 0, \cdots, 0)$ denote the set of eigenvalues of $d^2V(x_0)$ where the α_i are strictly positive and $\dim \Theta = n - s$.

Proposition A.1 *For every orthonormal basis $e = (e_1, \cdots, e_{n-s})$ of $T_{(x_0,0)}\Theta$ and for all $t \notin \{\frac{2\pi k}{\alpha_i}; i \in [1, r]$ et $k \in \mathbb{Z}\}$, we have*

$$d\mu_t(x_0, 0)(e) = \frac{2^{-s}}{|t|^{(n-s)/2} \prod_{i=1}^{r} |\sin(\frac{\alpha_i}{2}t)| \prod_{i=r+1}^{s} |\sinh(\frac{\alpha_i}{2}t)|} \tag{40}$$

Proof : We can find $(e_{n-s+1}, \cdots, e_{2n})$ such that $(e_1, \cdots, e_{2n})$ is a symplectic basis of $\mathbb{R}^{2n}$ and such that in this basis, we have

$$d\phi_t(x_0, 0) = \begin{pmatrix} I_{n-s} & tI_{n-s} & & & \\ 0 & I_{n-s} & & & \\ & & M_1 & & \\ & & & \ddots & \\ & & & & M_s \end{pmatrix}$$

where, for $i \in [1, r]$,

$$M_i = \begin{pmatrix} \cos(\alpha_i t) & \frac{1}{\alpha_i}\sin(\alpha_i t) \\ -\alpha_i \sin(\alpha_i t) & \cos(\alpha_i t) \end{pmatrix}$$

and for $i \in [r+1, s]$,

$$M_i = \begin{pmatrix} \cosh(\alpha_i t) & \frac{1}{\alpha_i}\sinh(\alpha_i t) \\ \alpha_i \sinh(\alpha_i t) & \cosh(\alpha_i t) \end{pmatrix}.$$

If we define $W := T_{(x_0,0)}\Theta$ which is isotropic, it is clear that we can also choose the $(e_{n-s+1}, \cdots, e_{2n})$ such that

$$f = (e_{n-s+1}, \cdots, e_{2(n-s)})$$

is a basis of the complementary of $W^\perp$ in $\mathbb{R}^{2n}$ and such that

$$g = (e_{2(n-s)+1}, \cdots, e_{2n})$$

is a basis of the complementary of W in $W^\perp$.
If we define $T_t := Id - d\phi_t(x_0, 0)$, we observe that the matrix of T_t in the basis (f, g) and (e, g) is

$$T_{t,[f,g],[e,g]} = \begin{pmatrix} -tI_{n-s} & & & \\ & I_2 - M_1 & & \\ & & \ddots & \\ & & & I_2 - M_s \end{pmatrix}.$$

Using

$$\begin{aligned} (1-\cos(\alpha_i t))^2 + \sin^2(\alpha_i t) &= 2(1-\cos(\alpha_i t)) = 4\sin^2(\frac{\alpha_i}{2}t) \\ (1-\cosh(\alpha_i t))^2 - \sinh^2(\alpha_i t) &= 2(1-\cosh(\alpha_i t)) = -4\sinh^2(\frac{\alpha_i}{2}t), \end{aligned}$$

we obtain

$$|\det T_{t,[f,g],[e,g]}|^{1/2} = |t|^{(n-s)/2} 2^s \prod_{i=1}^{r} |\sin(\frac{\alpha_i}{2}t)| \prod_{i=r+1}^{s} |\sinh(\frac{\alpha_i}{2}t)|.$$

Since $t \notin \{\frac{2\pi k}{\alpha_i}; i \in [1,r] \text{ et } k \in \mathbb{Z}\}$, we have $|\det T_{t,[f,g],[e,g]}|^{1/2} \neq 0$ and we can now apply lemma A.5 of [3] to complete the proof. ♣

B A theorem for the singularity in $t = 0$.

In this appendix, we reformulate a result of [3] where the support of $\hat{\varphi}$ is supposed to be contained in a sufficiently small neighborhood of 0. We restrict ourselves to the case of a Schrödinger operator of the form

$$\hat{H} = -\frac{\hbar^2}{2}\Delta + V(x) \qquad \text{on } I\!\!R^n \ \ (n \geq 1).$$

We are only interested here in writing some of the results of [3] in a more explicit way in order to facilitate the proof of theorem 1.3.

Theorem B.1 *If the support of $\hat{\varphi}$ is contained in a sufficiently small neighborhood of 0 (in order not to contain any period of the flow or the linearised flow) and if we assume (H3), (H4) and (H5), we have, as $\hbar$ goes to 0,*

1. *if $\nu \geq 1$ and if ν and $N-\nu$ are odd, then N is even and*

$$\Upsilon(E_c,\hbar) \sim \hbar^{-n+1}\left[\sum_{j=0}^{N/2-2} c_j \hbar^j + \hbar^{N/2-1}\sum_{j=0}^{\infty}\sum_{l=0,1} c_{j,l}\hbar^j(-\ln\hbar)^l\right],$$

with leading coefficients

$$a)\ \text{if} \quad N \geq 4 : c_0 = \frac{LVol(\Sigma_{E_c})}{(2\pi)^n}\hat{\varphi}(0)$$

$$b)\ \text{if} \quad N = 2 : c_{0,1} = -\frac{1}{\pi\alpha_1}\hat{\varphi}(0) \ \text{ and}$$

$$c_{0,0} = \int \varphi(\tau) f(\tau) d\tau.$$

where f is analytic in the region $|Im\tau| < \frac{\alpha_1}{2}$ and $f(\tau) = o(|\tau|)$ at infinity.

2. *if $\nu \geq 1$ and if ν or $N-\nu$ is even, then $N \geq 3$ and*

$$\Upsilon(E_c,\hbar) \sim \hbar^{-n+1}\left[\frac{LVol(\Sigma_{E_c})}{(2\pi)^n}\hat{\varphi}(0) + \sum_{j=1}^{+\infty} c_j \hbar^{j/2}\right]$$

3. *if $\nu = 0$, then*

$$\Upsilon(E_c,\hbar) \sim \hbar^{N/2-n}\sum_{j=0}^{\infty} c_j \hbar^j$$

with leading coefficient

$$c_0 = (2\pi)^{N/2-n} 2^{n-N} e^{-i\pi N/4} \int_\Theta < \frac{1}{(t-i0)^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i0))}, \check{\varphi}(t) > dx.$$

Proof: 1)a) and 2) are just results of theorem 1.1 and 1.2 of [3]. To prove 1)b), we use the theorem 3.7 of [3] and the fact that in this particular case ($\nu = N - \nu = 1$), we have (with the notation of [3])

$$\begin{aligned}\alpha(t,0,0) &= 8\check{\varphi}(t)\beta(t,0) && \text{(cf formula (82) of [3])}\\ &= \frac{4}{\pi}\check{\varphi}(t)|t|d\mu_t(0) && \text{(cf formula (149) of [3])}\\ &= \frac{2}{\pi}\check{\varphi}(t)\frac{t}{\sinh(\frac{\alpha_1}{2}t)} && \text{(cf appendix A).}\end{aligned}$$

We thus have from theorem 3.7 of [3]

$$c_{0,1} = \frac{\pi}{2}\alpha(0,0,0) = \check{\varphi}(0)\lim_{t\to 0}\frac{t}{\sinh(\frac{\alpha_1}{2}t)} = \frac{1}{\pi\alpha_1}\hat{\varphi}(0)$$

and

$$c_{0,0} = -\frac{1}{2}\int \partial_t(\check{\varphi}(t)\frac{t}{\sinh(\frac{\alpha_1}{2}t)})\mathrm{sgn}t\log|t|dt + C\hat{\varphi}(0)$$

for some constant C. Hence, and because of the absolute convergence of the integrals, we have

$$\begin{aligned}c_{0,0} &= -\frac{1}{4\pi}\int_{\mathbb{R}^2}\partial_t(e^{it\tau}\varphi(x)\frac{t}{\sinh(\frac{\alpha_1}{2}t)})\mathrm{sgn}t\log|t|dtd\tau + C\hat{\varphi}(0)\\ &= \int_{\mathbb{R}} f(\tau)\varphi(\tau)d\tau\end{aligned}$$

where

$$\begin{aligned}f(\tau) &= -\frac{1}{4\pi}\int_{\mathbb{R}}\partial_t(e^{it\tau}\frac{t}{\sinh(\frac{\alpha_1}{2}t)})\mathrm{sgn}t\log|t|dt + C\\ &= \frac{i\tau}{4\pi}\mathcal{F}\left(\frac{t\ \mathrm{sgn}t\log|t|}{\sinh(\frac{\alpha_1}{2}t)}\right)(\tau) - \frac{1}{4\pi}\mathcal{F}\left(\partial_t(\frac{t}{\sinh(\frac{\alpha_1}{2}t)})\mathrm{sgn}t\log|t|\right)(\tau) + C \qquad (41)\end{aligned}$$

Now, using the fact that $\frac{t\ sgnt\log|t|}{\sinh(\frac{\alpha_1}{2}t)}$ and $\partial_t(\frac{t}{\sinh(\frac{\alpha_1}{2}t)})\mathrm{sgn}t\log|t|$ are both in L^1 and decrease much faster than $e^{-|t|(\frac{\alpha_1}{2}-\epsilon)}$ for all $\epsilon > 0$, we get that both Fourier transform appearing above are continuous functions which vanish at infinity and that f is analytic in the region $|\mathrm{Im}\tau| < \frac{\alpha_1}{2}$. To prove 3), we have to look more carefully at the proof of theorem 3.11 of [3] to notice that the α introduced in formula (105) of [3] can be extended to a function which is even in the r variable because of the symmetry of the sphere. Therefore, we have that for all j, $\alpha^{(2j+1)}(0) = 0$. This implies that the asymptotic expansion of theorem 3.11 of [3] does not contain any half integer power of $\hbar$. We thus obtain

$$\Upsilon(\hbar) = \sum_{j\geq 0} c_j\hbar^{N/2-n+j}$$

where

$$c_0 = \frac{1}{2}\Gamma(\frac{N}{2})e^{-i\pi N/4} < (t-i0)^{-N/2}, 2^{N/2}\Sigma_{N-1}\check{\varphi}(t)\beta_0(t,0) > .$$

But with the formula (149) of [3], the results of appendix A and the fact that $\Sigma_{N-1} = 2\frac{\pi^{N/2}}{\Gamma(N/2)}$, we obtain

$$c_0 = (2\pi)^{N/2-n}2^{n-N}e^{-i\pi N/4}\int_{\Theta} < (t-i0)^{-N/2}, \frac{t^{N/2}\check{\varphi}(t)}{t^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}t)} > dx$$

We can then conclude by observing that $\frac{t^{N/2}}{t^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}t)}$ is a holomorphic function with t in a neighborhood of 0, and thus

$$
\begin{aligned}
& < (t-i0)^{-N/2}, \frac{t^{N/2}\check{\varphi}(t)}{t^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}t)} > \\
= & \lim_{\epsilon\to 0^+}\int_{I\!R}(t-i\epsilon)^{-N/2}\frac{t^{N/2}\check{\varphi}(t)}{t^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}t)}dt \\
= & \lim_{\epsilon\to 0^+}\int_{I\!R}(t-i\epsilon)^{-N/2}\frac{(t-i\epsilon)^{N/2}\check{\varphi}(t)}{(t-i\epsilon)^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i\epsilon))}dt \\
= & < \frac{1}{(t-i0)^{n-N/2}\prod_{i=1}^{N-n}\sin(\frac{\alpha_i(x)}{2}(t-i0))}, \check{\varphi}(t) > .
\end{aligned}
$$

♣

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