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LATTICES AND THEIR CONTINUUM LIMITS

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Abstract

We address the problem of the continuum limit for a system of Hausdorff lattices (namely lattices of isolated points) approximating a topological space M. The correct framework is that of projective systems. The projective limit is a universal space from which M can be recovered as a quotient. We dualize the construction to approximate the algebra $\mathcal{C}(M)$ of continuous functions on M. In a companion paper we shall extend this analysis to systems of noncommutative lattices (non Hausdorff lattices).

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1 Introduction

Lattice discretizations have become very popular methods to approximate physical models which are too complicated to be solved analytically [1]. However, in spite of their success, there are certain features of continuum dynamics which are generally not addressed in a transparent and satisfactory way. For example, it is not obvious how to describe any topological aspect of quantum physics within a lattice approach.

A typical continuum theory is usually given by a suitable carrier space (configuration or phase space) together with a dynamics on it. Interesting properties of the physical system could come from either of them. For instance, nontrivial topological properties of the configuration space may have deep consequences even for simple dynamics. On the contrary, in the usual lattice models, these two aspects are not clearly separated.

It is interesting to formulate lattice theories in a way that dynamical and kinematical aspects remain as separated as possible. The first question one can ask is how the topology of the underlying space (-time) M arises from a lattice of points, regardless of the specific dynamics. The second, and more difficult question, refers to the topology of the (∞-dimensional) space Γ of all configurations. In typical lattice models, the only topological information refers to M and is that of nearest neighbors as encoded in the Hamiltonian. Even though this captures some of the global topological features of M it does not provide *per se* a notion of limit in which M is recovered. Moreover, this incomplete topological information has no bearings on the configuration space Γ which is topologically trivial. For instance, this is the reason why on the lattice solitons are not truly topological.

In [2, 3] we have initiated a systematic investigation of these issues. This work has been inspired by a paper of Sorkin [4], where it is shown how a Hausdorff topological space can be approximated with finite, non Hausdorff topological spaces (posets). This method gives satisfactory results under two aspects. On one side, already with a finite number of points it reproduces relevant topological properties of the space being approximated. On the other side, it gives, via the notion of projective system, a well defined concept of continuum limit from which the initial space can be reconstructed. In [2, 3] we developed the essential tools for doing quantum physics on finite topological spaces and considered the dualization of these spaces. In [3] it was observed that posets are genuine noncommutative spaces in the sense that one can associate with them a noncommutative algebra $\mathcal{A}$ playing a role analogous to that of the algebra of continuous functions for Hausdorff spaces[1]. This algebraic framework provides new and well developed tools to construct quantum mechanical and field theoretical models. Connes' noncommutative geometry [5] for example, can be immediately applied giving access to structures that retain their richness even when the geometry, though not trivial, is anyway poorer than the one of the continuum[2]. In this way, topological informations enter non trivially at all stages of

[1]These algebras contain enough information to reconstruct the lattice completely, thus providing a full dualization.

[2]For example in [6] the noncommutative geometry of the distances is applied to lattices.

the construction. In that paper we have also explicitly shown how non trivial topological effects are captured by these topological lattices and their algebras, by constructing algebraically the θ-quantizations of a particle on a circle poset.

In this and a companion paper [7], we address the question of how to recover a topological space M through a suitable notion of limit of a system of lattices Q^n. The dualization of this framework, in the spirit of noncommutative geometry, is also analyzed. We show how the algebra $\mathcal{C}(M)$ of continuous functions on M can analogously be recovered from the system of algebras $\mathcal{C}(Q^n)$.

In the present paper, we apply these methods to Hausdorff lattices, namely to lattices made of isolated points. This kind of lattices arises, for example, when discretizing a scalar field defined on a manifold M. We shall show that, even in this simple case, a structure of projective system [8] produces a topologically non trivial limit space from which a topological space M being approximated can be recovered. As we shall see, the limit space is a universal one in the sense that any two such limit spaces for different spaces M are naturally homeomorphic and can be identified with the Cantor set. The extra information which is needed to recover M is provided by a projection π from the Cantor set to M. This projection, which is not naturally built in the limit space, can be constructed starting from the projective system.

We then show that there is a structure of direct (or inductive) limit [8] on the algebra of continuous functions defined on the lattices. Although at a finite level these algebras are trivial, their inductive limit is the algebra of continuous functions on the Cantor set. The algebra $\mathcal{C}(M)$ of continuous functions on M is then the subalgebra of projectable functions with respect to π.

Since Hilbert spaces play a key role in quantum theories and also in noncommutative geometry, a similar analysis will be repeated for $L^2(M)$ through an inductive system of finite dimensional Hilbert spaces.

As mentioned before, in [7] we shall see that with the use of topological lattices the projective and the inductive limits will loose they universal character and will be naturally related to M and $\mathcal{C}(M)$ respectively.

2 Continuum limit of Hausdorff Lattices

Consider a piecewise linear space M of dimension d, or in other words a space that admits a locally finite cellular decomposition $\Sigma = \{S_\alpha, \alpha \in I, I \subset \mathbf{N}\}$. For convenience we shall use cubic cells so that the S_α's will be closed cubes. Once such a decomposition is chosen, one can associate with it a lattice Q of points. The way this is usually done is by looking at the set of vertices in the decomposition. In this paper, however, we will instead take the vertices of the dual lattice, which means that the points of Q correspond to highest dimensional cubes. With this choice it becomes possible to introduce a nontrivial notion of limit for a sequence of finer and finer decompositions.

The lattice Q is then given a Hausdorff topology. On a space with a finite (or countable) number of points there is a unique Hausdorff topology, and it is the one for which each point is open and closed at the same time. Our aim is to understand if, and to what extent, M can be recovered as a limit. We then consider a sequence Σ^n of finer and finer cubic decompositions, for example the one obtained by splitting every d-dimensional cube in two at each step, together with their associated lattices Q^n's.

Now, the simple fact of having a sequence of lattices is not sufficient yet to obtain M as its limit. What one has to do is to give this sequence a further structure which converts it into what is known in mathematics as an inverse or projective system of topological spaces, which we now pass to describe.

A *projective* (or *inverse*) *system* of topological spaces is a family of topological spaces $Y^n, n \in \mathbf{N}$ [3] together with a family of continuous projections $\pi^{(m,n)} : Y^m \to Y^n,\ n \leq m$, with the requirements that $\pi^{(n,n)} = \mathbb{I},\ \pi^{(n,m)} = \pi^{(n,p)}\pi^{(p,m)}$. The projective limit Y^∞ is defined as the set of coherent sequences, that is the set of sequences $\{x^n \in Y^n\}$ with $x^n = \pi^{(m,n)}(x^m)$.

There is a natural projection $\pi^n : Y^\infty \to Y^n$ defined as:

$$\pi^n(\{x^m \in Y^m\}) = x^n \ . \tag{2.1}$$

The space Y^∞ is given a topology, by declaring that a set $\mathcal{O}^\infty \subset Y^\infty$ is open iff it is the inverse image of an open set belonging to some Y^n or a union (finite or infinite) of such sets.

Let us then consider again the sequence of cubic decompositions $\Sigma^n = \{S^n_\alpha, \alpha \in I^n\}$, with $\Sigma^{(n+1)}$ obtained from Σ^n by subdivision of its cubes. In order to be able to correctly reproduce the space M in the limit, the sequence of cubic decompositions must be such that all cubes in it become smaller and smaller. The precise meaning of this requirement is that for any point $x \in M$ and any open set $\mathcal{O}_x$ containing x, there must exist a level of approximation such that all cubes containing x will be contained in $\mathcal{O}_x$ from that level on[4]:

$$\forall\ x \text{ and } \forall\ \mathcal{O}_x \ni x,\ \exists\ m \text{ such that } \forall\ n \geq m\ , {S^n}_\alpha \ni x\ \Rightarrow\ {S^n}_\alpha \subset \mathcal{O}_x\ . \tag{2.2}$$

This subdivision procedure naturally induces a structure of projective system on the corresponding sequence Q^n of lattices.

The projection

$$\pi^{(m,n)} : Q^m \to Q^n,\ m > n \tag{2.3}$$

associates to a d-dimensional cube of the finer subdivision the unique d-dimensional cube from which it comes.

[3] More generally, the index n could be taken in any directed set.

[4] We are really using decompositions which are 'fat' in the sense of [9].

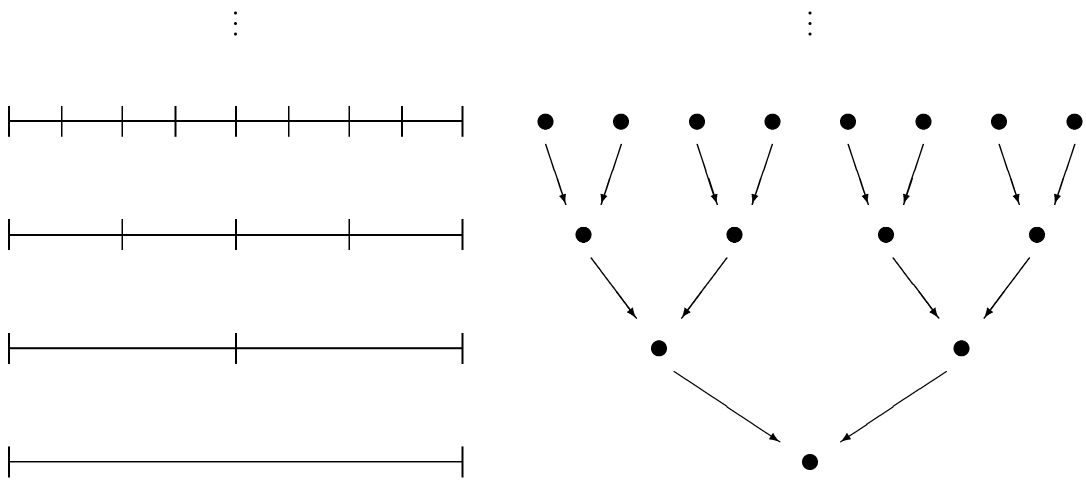

Fig. 1. (a) shows a subdivision of the interval. (b) shows the corresponding projective system.

We call $Q^\infty(M)$ the projective limit of the projective system. A point in $Q^\infty(M)$ is nothing but a decreasing sequence $\{q^n\}$ of cubes, namely a sequence such that $q^{n+1} \subset q^n$. We shall see that this space is not the original space M, but it is bigger in the sense that M can then be recovered from it as a quotient. For simplicity we will use q^n to denote both the element of Q^n and the corresponding cube ${S^n}_\alpha$. There exists a natural projection $\pi : Q^\infty(M) \to M$. It is defined as follows:

$$\pi(\{q^n\}) = \bigcap_n q^n. \tag{2.4}$$

In this manner, we get a unique point of M. That this point is unique is a consequence of condition (2.2).

In order to illustrate the topology of $Q^\infty(M)$ we will consider the case when M is the interval $I = [0, 1]$. Its decomposition and the associated projective system are shown in Fig. 1.

An element of $Q^\infty(I)$, namely a coherent sequence $q^\infty \equiv \{q^n \in Q^n\}$, can be identified with a string $\epsilon_1\epsilon_2\epsilon_3$... of 0's and 1's and the correspondence is one to one. The string can be constructed in the following way: The starting approximation Q^0 consists of a single element corresponding to the whole interval. After the first subdivision the interval I is split into two equal halves. Then we take ϵ_1 equal to 0 or 1 depending on whether q^1 is the left or right half; ϵ_2 will similarly be 0 or 1 depending on whether q^2 is the left or right half in which q^1 splits in going from Q^1 to Q^2 and so on. With the help of a decimal point on the extreme left, we can see this sequence as the binary representation of a point on the interval. Notice that this point coincides with $\pi(q^\infty)$. This labeling of $Q^\infty(I)$ makes it clear that there might be more than one point in $Q^\infty(I)$ that project to the same point of I. For example the points $.0111\cdots$ and $.1000\cdots$ both correspond to the point $1/2$. The same thing will happen for all points of I of the form $m/2^n$, with m and n integers. On the other hand, the remaining points of I are the image of a unique point of $Q^\infty(I)$, because they have a unique binary representation.

Thus $Q^\infty(I)$ is a "quasi fiber bundle" over I whose fibers contain either one or two points. Even though $Q^\infty(I)$ is in this sense bigger than I, the interval can be recovered as the quotient $Q^\infty(I)/\sim$, where $\sim$ indicates the equivalence relation defined by the projection π. From the topological point of view, this statement is nontrivial, as it is not guaranteed that $Q^\infty(I)/\sim$, endowed with the quotient topology, is homeomorphic to I. The proof that this is the case will be given below in the general case.

The space $Q^\infty(I)$ is nothing but a Cantor set which is, up to homeomorphisms, the only totally disconnected [5], perfect[6], metric topological space. A familiar realization of a Cantor set is the "middle-third Cantor set". It is obtained by starting from the interval $[0, 1]$ dividing it in three parts, removing the middle third and iterating the procedure ad libitum on each of the remaining parts. That the previous $Q^\infty(I)$ is a Cantor set can be then proven either by showing directly that it enjoys the mentioned properties or by explicitly showing that it is homeomorphic to the middle third Cantor set.

The space $Q^\infty(I)$ coming from the interval is actually "universal", in the sense that it is homeomorphic to the space $Q^\infty(M)$ associated with a generic M. A simple argument to show that this is the case goes as follows: a projective system for M can always be obtained by taking some initial decomposition of M in cubes and by refining it by suitably splitting each cube in two halves at each step. It is then clear that the corresponding projective system coincides with the one we have constructed for the interval from a certain interval on, and thus has the same projective limit. The difference lies in the projection π, whose definition, as given in equation (2.4), uses the explicit interpretation of the q^n as subset of M and thus depends on the specific M. Again $Q^\infty(M)$ is a quasi fiber bundle over M. The number of points in the fibers also depends on the projection π. In other words: *$Q^\infty(M)$ is universal, but the fibration is not.* In fact, the existence of such a projection π is not surprising due to a known result [10] that there exists a continuous projection from the Cantor set onto any compact metric topological space. For this reason, from now on, we simply write Q^∞ instead of $Q^\infty(M)$.

We now turn to the proof that M is actually homeomorphic to the quotient space $Q^\infty/\sim$.

Let us first prove that the projection π in eq.(2.4) is continuous. We have to show that the inverse image of an open set B in M is open in $Q^\infty(M)$. Let $q = \{S^n_\alpha(q)\}$ be a point belonging to $\pi^{-1}(B)$ and let $x = \pi(q)$. Because of the condition (2.2) on the sequence of cubical decompositions there exists a $j \in N$ such that $n > j$ implies that all cubes S^n_α

[5] A totally disconnected space is one for which each connected component is just a single point.

[6] A perfect space is one for which each of its points is an accumulation point for the space.

containing x are all contained in B. Consider then $\mathcal{O}^\infty = \pi^{-1}(S^n_\alpha(q))$ with $n > j$ which is an open set of $P^\infty(M))$ containing q. $\mathcal{O}^\infty$ is also entirely contained in $\pi^{-1}(B)$, in fact all its points are coherent sequences whose representatives at level n are cells fully contained in $S^n_\alpha(q)$ and since the cube $S^n_\alpha(q)$ is fully contained in B they project in B.

To prove that the topology of M is equivalent to the quotient topology on $Q^\infty/\sim$, it is then sufficient to show that the inverse image of a subset of M, which is *not* open, is not open in Q^∞ as well.

Consider then the set $\pi^{-1}(B) \subset Q^\infty$, with $B \subset M$ *not* open. We will show that the assumption that $\pi^{-1}(B)$ is open leads to a contradiction.

The statement that B is not open in the topology of M is equivalent to saying that there exists a sequence of points $\{x_i\}$ of M, not belonging to B, which converges to a point $x \in B$. From this sequence we will extract a particular subsequence $\{y_j\}$, still converging to x. We first introduce a countable basis of open neighborhoods for x, namely a countable family $\{\mathcal{O}_i\}$ of decreasing open sets containing x.

Let us start with $\mathcal{O}_1$. Due to condition (2.2), there are one or more d-cubes $S^{n(1)}_\alpha \subset \mathcal{O}_1$, with $S^{n(1)}_\alpha \ni x$. At least one of these d cubes, call it $S^{n(1)}_{\alpha(1)}$, will contain an infinite number of elements the sequence $\{x_i\}$. Then, choose y_1 to be any one of these elements.

At the next level 2 there will again be at least one d-cube $S^{n(2)}_{\alpha(2)} \subset S^{n(1)}_{\alpha(1)}$, with $S^{n(2)}_{\alpha(2)}$ still containing an infinite number of elements of the sequence x_i. Again choose y_2 as any one of these elements.

By iterating this procedure, we obtain the sequence $\{y_j\}$, which, being extracted from the original sequence, still converges to x. Moreover, $y_j \in S^{n(j)}_{\alpha(j)}$ and $\{S^{n(j)}_{\alpha(j)}\}$ is a coherent sequence [7] which thus defines a point $q \in Q^\infty$. By construction $\bigcap_j S^{n(j)}_{\alpha(j)} = x$, and consequently $\pi(q) = x$.

Since $\pi^{-1}(B)$ is assumed to be open, and recalling how the topology of Q^∞ is defined, there will be a $\overline{j}$ such that $\pi^{-1}(S^{n(j)}_{\alpha(j)}) \subset \pi^{-1}(B)$ for $j \geq \overline{j}$. But then also $\pi^{-1}(y_j)$, with $j \geq \overline{j}$, must belong to $\pi^{-1}(B)$ and this implies, contrary to the hypothesis on the sequence $\{x_i\}$, that $y_j \in B$.

3 Algebras for Hausdorff Lattices

In the previous Section, we have shown how to approximate a topological space M by a sequence of lattices and how to recover M by a limiting procedure. Here we shall dualize this construction. The projective system of lattices will be replaced by an inductive system of commutative C^*-algebras, and $Q^\infty(M)$ by the inductive limit $\mathcal{A}_\infty$. While before M could be recovered as a quotient of $Q^\infty(M)$, now $\mathcal{C}(M)$ will turn out to be a subalgebra of

[7] A little care must be taken as $n(j)$ may not coincide with the subdivision level of the lattice.

$\mathcal{A}_\infty$. Duality here is understood in the sense of the Gel'fand-Naimark theorem [11]. The idea behind this theorem is that the full topological information on a Hausdorff topological space M is encoded in the abelian C^*-algebra $\mathcal{C}(M)$ of its continuous functions. The space itself is identified with the set of all complex homomorphisms of $\mathcal{C}(M)$ and the topology is given in terms of pointwise convergence:

$$p_n \to p \qquad \Leftrightarrow \qquad f(p_n) \to f(p) \quad \forall f \in \mathcal{C}(M) \ , \quad p_n, \ p \in Hom(\mathcal{C}(M), \mathbb{C}) \ . \tag{3.1}$$

To each Q^n we now associate its algebra $\mathcal{A}_n$ of continuous functions. Since the Q^n's are discrete Hausdorff spaces, a continuous function a_n is just an assignment of a complex number to each point q^n in Q^n, and then [8] $\mathcal{A}_n \equiv \mathbb{C}^{2^n}$. We shall write any such a function as the vector

$$a_n = \{\lambda_1, \cdots, \lambda_{2^n}\} \ . \tag{3.2}$$

The norm $|| \cdot ||_n$ of a function a_n is just the sup norm.

While in the previous Section the framework for defining a limiting procedure was that of a projective system of topological spaces, here it will be that of a direct or inductive system of C^*-algebras.

An *inductive system* of C^*-algebras is a sequence of C^*-algebras $\mathcal{A}_n$, together with norm non-increasing immersions $\Phi_{(n,m)} \ : \ \mathcal{A}_n \to \mathcal{A}_m, \ n < m$, such that the composition law $\Phi_{(n,m)}\Phi_{(m,p)} = \Phi_{(m,p)}, \ n < m < p$, holds.

The inductive limit $\mathcal{A}_\infty$ is the C*-algebra consisting of equivalence classes of "Cauchy sequences" $\{a_n\}, \ \ a_n \in \mathcal{A}_n$. Here by Cauchy sequence we mean that $||\Phi_{(n,m)}(a_n) - a_m||_m$ goes to zero as n and m go to infinity. Two sequences $\{a_n\}$ and $\{b_n\}$ are equivalent if $||a_n - b_n||_n$ goes to zero. The norm in $\mathcal{A}_\infty$ is defined by

$$||a||_\infty = \lim_{n\to\infty} ||a_n||_n \tag{3.3}$$

where $\{a_n\}$ is any of the representatives of a [9].

In our case the direct system is naturally defined by the pull-backs $\Phi_{(n,m)} = \pi^{(n,m)*}$ associated with the projections in (2.3)

$$(\Phi_{(n,m)}(a_n))(q_m) = a_n(\pi^{(m,n)}(q_m)) \ . \tag{3.4}$$

The Φ's are isometric $*$-homomorphisms. Where previously there was a projection $\pi^{(n)}$ from Q^∞ to Q^n, there is now an immersion Φ_n of $\mathcal{A}_n$ in $\mathcal{A}_\infty$ defined as:

$$\Phi_n(a_n) = \{\Phi_{(n,m)}(a_n), \ n < m\}, \ a_n \in \mathcal{A}_n \ . \tag{3.5}$$

The algebra $\mathcal{A}_\infty$ is isomorphic to the C*-algebra of continuous functions on Q^∞. In order to prove this, it is useful to realize Q^∞ as the middle third Cantor set introduced

[8] Again we assume that the lattices split in half going from one level to the next, so that the total number of points at level n is 2^n.

[9] For a more detailed account of the definition see for example [11] or [12].

earlier. Now an element $a_n \in \mathcal{A}_n$ identifies an element $\Phi_n(a_n) \in \mathcal{A}_\infty$ that can be thought of as a continuous and piecewise constant function on Q^∞. The collection of all such functions is dense in $\mathcal{A}_\infty$ by the very definition of $\mathcal{A}_\infty$. Therefore every element of $\mathcal{A}_\infty$ can be thought of as a uniformly convergent sequence of functions of this type, and naturally determines a continuous function on Q^∞. Conversely, we can now prove that for any continuous function f on Q^∞, one can find a sequence $\{f_n\}$ of functions in $\mathcal{A}_\infty$ uniformly converging to it. Indeed, since the Cantor set is a compact metric space, every continuous function is also uniformly continuous, thus $\forall \epsilon > 0 \; \exists \; \delta_n$ s.t. for $|x - x'| < \delta_n \; , \; |f(x) - f(x')| < \frac{1}{2^n}$. The sequence $\{f_n\}$ is defined as follows: f_n is a continuous piecewise constant function on Q^∞ defined by an element $\overline{f}_n \in \mathcal{A}_{m(n)}$, with $m(n)$ such that $\frac{1}{3^{m(n)}} \leq \delta_n$. The value of f_n in any of the sets in which it is constant is simply any of the values of f in that set.

We now have to recognize in $\mathcal{A}_\infty$ the algebra $\mathcal{C}(M)$ of continuous functions on the space M. We will show that $\mathcal{C}(M)$ is a subalgebra of $\mathcal{A}_\infty$, which is the dual statement of the fact that M is a quotient of Q^∞.

In this respect we remind that Q^∞ is a quasi fiber bundle on M. We will now show that the algebra $\mathcal{C}(M)$ of continuous functions on M is isomorphic to the subalgebra of $\mathcal{A}_\infty$ made of projectable functions. To start with, it is obvious that the pull-back of continuous functions on M are continuous functions on $\mathcal{A}_\infty$, which take constant value on the fibers. It is sufficient to show then that they exhaust all such functions of $\mathcal{A}_\infty$. Consider then a continuous function, $f \in \mathcal{A}_\infty$ which is constant on the fibers. The function f thus defines naturally a function $\tilde{f}$ on M. The inverse image $\mathcal{O}^\infty = f^{-1}(\mathcal{O})$ of an open set, $\mathcal{O}$ of $\mathbb{C}$ is an open set of Q^∞ containing all the fibers through its points. Since we have already shown that M is homeomorphic to $Q/\sim$, then $\mathcal{O}^\infty$ projects onto an open set, $\mathcal{O}_M$, of M. Since $\mathcal{O}_M$ is the inverse image of $\mathcal{O}$ through $\tilde{f}$, $\tilde{f}$ is itself continuous.

Seen from the base, a generic element of $\mathcal{A}_\infty$ can be regarded as a multi-valued function on M while $\mathcal{C}(M)$ can be identified with the set of projectable functions on Q^∞.

We thus have proven that the algebra of continuous functions over the Cantor set Q^∞ contains as subalgebra the algebras of all continuous functions over compact topological spaces.

4 Hilbert Spaces

Since Hilbert spaces and representation of algebras of observables as operator play a prominent role in quantum mechanics, we now show how these structures fit in our scheme. In particular we will see how the space $L^2(M, \mu)$ of square integrable functions of M can be approximated by the analogous spaces $L^2(Q^n, \mu^n)$ for the lattices Q^n and recovered as an inductive limit.

Now, the algebras $\mathcal{A}_n$ have an obvious representation as diagonal matrices on the

Hilbert space $\mathcal{H}^n = \mathbb{C}^{2^n}$, given by

$$a_n = (\lambda_1, ..., \lambda_{2^n}) \in \mathcal{A}_n \ \rightarrow \hat{a}_n = \mathrm{diag}(\lambda_1, ..., \lambda_{2^n}) \in M_{2^n}(\mathbb{C}). \tag{4.1}$$

The inner product between two vectors $\phi = (\phi_1, ..., \phi_{2^n})$ and $\psi = (\psi_1, ..., \psi_{2^n})$ of $\mathcal{H}^n$ is defined as

$$< \psi_n, \phi_n >= \sum_{i=1}^{2^n} \psi_i^* \phi_i \mu_i^n \tag{4.2}$$

where μ_i^n are the normalized measures of the cubes associated with the points of $Q^{(n)}$.

What we have is a sequence $\mathcal{H}^n$ of Hilbert spaces, one for each level. The structure of inductive system on the algebras $\mathcal{A}_n$ induces an analogous structure on the Hilbert spaces. In this way, the inductive limit $\mathcal{H}^\infty$ will naturally carry a representation of $\mathcal{A}_\infty$. Moreover, $\mathcal{H}^\infty$ will have a natural realization as the space of square integrable functions $L^2(Q^\infty, \mu^\infty)$. All that is needed to carry this construction out is to introduce a suitable system of cyclic vectors $\{\chi_n\}$ one for each level. A possible choice for χ_n is the vector whose components are all equal to 1. Once the cyclic vectors have been chosen, embeddings $i_{(n,m)} : \mathcal{H}^n \rightarrow \mathcal{H}^m$, $n < m$, are defined by

$$i_{(n,m)}(\hat{a}_n)\chi_n \equiv \widehat{\Phi_{(n,m)}(a_n)}\chi_m, \quad a_n \in \mathcal{A}_n \ , \ \chi_n \in \mathcal{H}^n \ , \ \chi_m \in \mathcal{H}^m \ , \tag{4.3}$$

where $\Phi_{(n,m)}$ are the embedding between the algebras $\mathcal{A}_n$ given in (3.4). The importance of having a cyclic vector defined at each level is clear from this equation as this makes the embeddings $i_{(n,m)}$ defined on all of $\mathcal{H}^n$. With our choice of cyclic vectors these embeddings are isometries [10]. The inductive limit of the $\mathcal{H}^n$'s, namely $\mathcal{H}^\infty$, can be defined by means of Cauchy sequences of vectors $\{\psi_n\}$ as was done for the algebra $\mathcal{A}_\infty$. There exists natural isometric embeddings $i_n : \mathcal{H}^n \rightarrow \mathcal{H}^\infty$ defined as

$$i_n(\psi_n) \equiv [\{\phi_m\}] = \{i_{(n,m)}(\psi_n), \quad m > n\} \ , \qquad \phi_n \in \mathcal{H}^n \ , \tag{4.4}$$

where $[\cdot]$ denotes equivalence class.

Nevertheless, we find it convenient to present this inductive system in terms of L^2 spaces of square-integrable functions. Think of the $\mathcal{H}^n$ as $L^2(Q^n, \mu^n)$. The sequence of measures μ^n induces in turn a measure on the open sets of Q^∞, which can be extended, using standard methods [13], to a unique σ-additive absolutely continuous measure μ^∞ on Q^∞. The embeddings i_n in eq. (4.4), become the pull-backs of $L^2(Q^n, \mu^n)$ in $L^2(Q^\infty, \mu^\infty)$ of the projections π_n from Q^∞ to Q^n. Since the union of the $L^2(Q^n, \mu^n)$ is dense in $L^2(Q^\infty, \mu^\infty)$, and since $\mathcal{H}^\infty$ can be seen as the set of equivalence classes of Cauchy sequences in $L^2(Q^{(\infty)}, \mu^\infty)$, it follows that these two spaces can be identified.

Finally, recall again that $Q^{(\infty)}$ is a quasi fiber bundle on M. The set containing all the fibers consisting of more that one point is countable and thus has zero measure. It follows that $L^2(Q^\infty, \mu^\infty)$ is naturally isometric to $L^2(M, \mu)$.

[10]The general choice of cyclic vectors which makes the embeddings (4.3) isometric should satisfy the condition $|\chi_j^n|^2 = \sum_{l, \pi^{(n,m)}(q_j^n) = q_j^m} |\chi_l^m|^2 \mu_l^m$.

5 Conclusions

In this paper we have discussed some aspects of the topology of a space M when one considers lattice discretizations of it. We have seen that the space being approximated can be recovered from a projective limit of a projective system associated with the lattice discretizations.

The projective limit is basically an universal object (the Cantor set), and the information of the original topology can be encoded in a continuous projection from the Cantor set.

We have also discussed (in the spirit of Noncommutative Geometry) the dual algebra of continuous functions for the lattices at the finite level, and for the continuum limit. It is possible to recover the algebra $\mathcal{C}(M)$ of continuous functions of the space M as a subalgebra of the continuous functions on the Cantor set. Therefore even from the algebraic point of view the continuum limit of lattices is an universal object, and the information over the original starting point all lies in the choice of this subalgebra.

All of these aspects will be dealt with again in the context of noncommutative lattice in [7], where the universality of the limits will be lost and both the space M and the algebra $\mathcal{C}(M)$ will arise naturally from the projective and direct system, respectively.

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