ON TRACE FORMULAS FOR SCHRÖDINGER-TYPE OPERATORS

F. GESZTESY[1] AND H. HOLDEN[2]

Abstract. We review a variety of recently obtained trace formulas for one- and multi-dimensional Schrödinger operators. Some of the results are extended to Sturm-Liouville and matrix-valued Schrödinger operators. Furthermore, we recall a set of trace formulas in one, two, and three dimensions related to point interactions as well as a new uniqueness result for three-dimensional Schrödinger operators with spherically symmetric potentials.

1. Introduction. It is a well-established fact by now that trace formulas are of great importance in solving inverse spectral problems for Schrödinger operators. This is demonstrated in great detail in [7] in the context of short-range inverse scattering theory and in [9], [11], [26], [33], [38] in connections with the inverse periodic spectral problem. Historically these trace formulas originated in the works of Gelfand and Levitan [14] (see also [8], [12], [13]) for Schrödinger operators on a finite interval. Subsequent developments extended the range of validity of trace formulas in a variety of directions including algebro-geometric quasi-periodic finite gap potentials and certain classes of almost periodic potentials [6], [27], [30]–[32], [39]. Moreover, trace formulas proved to be a vital ingredient in descriptions of the isospectral manifold of quasi-periodic finite-gap potentials and some of their limiting cases as well as in the corresponding Cauchy problem for the Korteweg-de Vries equation. Due to the somewhat special nature of the potentials covered in the references cited thus far, it seemed natural to search for extensions of these trace formulas to a large class of potentials. This was the point of departure of our recent program which led to a trace formula for any continuous potential bound from below and subsequent generalizations to higher-order trace formulas in one dimension and certain multi-dimensional generalizations [15]–[19], [21]–[25],[37]. In the simplest case, the main new strategy is to compare the $L^2(\mathbb{R})$ Schrödinger operators $H = -\frac{d^2}{dx^2} + V$ and $H_y^D = -\frac{d^2}{dx^2} + V$, the corresponding operator with an additional Dirichlet boundary condition at the point $y \in \mathbb{R}$. The spectral characteristics of H and H_y^D, especially the Krein spectral shift function $\xi(\lambda, y)$ associated with the pair (H_y^D, H), then allows one to recover the potential $V(y)$.

In Section 2 we extend the results of [22] and [24] to Sturm-Liouville operators of the type $r^{-2}[-(p^2 f')' + qf]$ in $L^2(\mathbb{R}; r^2 dx)$ and consider general self-adjoint boundary conditions $\psi'(y) + \beta\psi(y) = 0$, $\beta \in \mathbb{R}$ in addition to the Dirichlet case $\beta = \infty$. Section 3 sketches an extension of the trace formula to matrix-valued Schrödinger operators in the Dirichlet case. Section 4 briefly reviews the multi-dimensional trace formulas in [25] and illustrates a possible abstract approach to some of these trace formulas in the special noninteracting case. In Section 5 we recall a different type of trace formula first derived in [17] in dimensions one, two, and three based on point interactions. Section 6 finally describes a new uniqueness result for three-dimensional Schrödinger operators with spherically symmetric potentials originally proven in [19].

2. Trace Formulas for Sturm-Liouville Operators.

Let $p, q, r \in C^\infty(\mathbb{R})$ be real-valued, $p, r > 0$ and q bounded from below.

We then define the self-adjoint Sturm-Liouville operator in $L^2(\mathbb{R}; r^2 dx)$ by

$$\begin{aligned} hf &= \frac{1}{r^2}[-(p^2 f')' + qf], \qquad (2.1)\\ f \in \mathcal{D}(h) &= \{g \in L^2(\mathbb{R}; r^2 dx) | g, g' \in AC_{\text{loc}}(\mathbb{R}),\ hg \in L^2(\mathbb{R}; r^2 dx)\}, \end{aligned}$$

where $AC_{\text{loc}}(\Omega)$ denotes the set of locally absolutely continuous functions in $\Omega \subseteq \mathbb{R}$. In addition, we define the Dirichlet Sturm-Liouville operator

$$\begin{aligned} &h_y^D f = \frac{1}{r^2}[-(p^2 f')' + qf], \qquad (2.2)\\ &f \in \mathcal{D}(h_y^D) = \{g \in L^2(\mathbb{R}, r^2 dx) | g, g' \in AC_{\text{loc}}(\mathbb{R}\backslash\{y\}),\ \lim_{\epsilon \downarrow 0} g(y \pm \epsilon) = 0,\ h_y^D g \in L^2(\mathbb{R}; r^2 dx)\}. \end{aligned}$$

In order to derive trace formulas we will compare the resolvents of h and h_y^D. Let $g(z, x, x')$ and $g_y^D(z, x, x')$ denote the Green's functions (i.e., the integral kernels of the resolvents) of h and h_y^D respectively,

$$g(z, x, x') = (h - z)^{-1}(x, x'), \quad g_y^D(z, x, x') = (h_y^D - z)^{-1}(x, x'). \qquad (2.3)$$

One verifies

$$g_y^D(z, x, x') = g(z, x, x') - \frac{g(z, x, y) g(z, y, x')}{g(z, y, y)}, \qquad (2.4)$$

and hence

$$\text{Tr}[(h_x^D - z)^{-1} - (h - z)^{-1}] = -\frac{d}{dz} \ln[g(z, x, x)]. \qquad (2.5)$$

To proceed further, we need a high-energy expansion, i.e., $z \to \infty$, of the diagonal Green's function $g(z, x, x)$. For that purpose we shall exploit the Liouville-Green transformation to find a Schrödinger operator H which is unitarily equivalent to h and hence use known results for Schrödinger operators derived in [21], [22],[24].

Define the change of variable

$$t = t(x) = \int_{x_0}^{x} dx' \frac{r(x')}{p(x')} \qquad (2.6)$$

for an arbitrary but fixed point $x_0 \in \mathbb{R}$. Write

$$P(t) = p(x(t)),\ Q(t) = q(x(t)),\ R(t) = r(x(t)) \qquad (2.7)$$

and introduce the unitary operator

$$\begin{aligned} &U : L^2(\mathbb{R}; r^2 dx) \longrightarrow L^2(\mathbb{R}; dt) \qquad (2.8)\\ &(Uf)(t) = [P(t)R(t)]^{1/2} F(t),\ F(t) = f(x(t)),\ f \in L^2(\mathbb{R}; r^2 dx). \end{aligned}$$

Theorem 2.1. ([10], see also [20]) The operator $H = UhU^{-1}$ in $L^2(\mathbb{R}; dt)$ explicitly reads

$$\begin{aligned} Hf &= -f'' + Vf, \qquad (2.9)\\ f \in \mathcal{D}(H) &= \{g \in L^2(\mathbb{R}; dt) \,|\, g, g' \in AC_{\text{loc}}(\mathbb{R}), Hg \in L^2(\mathbb{R}; dt)\}, \end{aligned}$$

where

$$\begin{aligned} V(t) &= \frac{Q(t)}{R(t)^2} + \frac{1}{(R(t)P(t))^2}\left[\frac{1}{2}(R(t)P(t))(R(t)P(t))_{tt} - \frac{1}{4}((R(t)P(t))_t^2\right] \\ &= \frac{q(x)}{r(x)} + \frac{p(x)}{2r(x)^3}(r(x)p(x))_{xx} + \frac{(r(x)p(x))_x}{2r(x)^2}\left(\frac{p(x)}{r(x)}\right)_x - \frac{1}{4r(x)^4}(r(x)p(x))_x^2 \\ &:= v(x), \quad x = x(t). \end{aligned} \tag{2.10}$$

Furthermore,

$$g(z,x,x') = \frac{G(z,t(x),t(x'))}{[r(x)p(x)r(x')p(x')]^{1/2}}, \ x,x' \in \mathbb{R}, \ z \in \mathbb{C}, \tag{2.11}$$

where G in the Green's function of H. Moreover,

$$H_u^D = U h_y^D U^{-1} = -\frac{d^2}{dt^2} + V \tag{2.12}$$

with V given by (2.10), is the Schrödinger operator with a Dirichlet boundary condition imposed at the point $u = \int_{x_0}^y dx[p(x)/r(x)]$. Let G_u^D denote the Green's function of H_u^D. Then

$$g_y^D(z,x,x') = \frac{G_u^D(z,t(x),t(x'))}{[p(x)r(x)p(x')r(x')]^{1/2}}. \tag{2.13}$$

Hence we find, using known results for H [22], [24] that

$$\mathrm{Tr}[e^{-\tau h_x^D} - e^{-\tau h}] \underset{\tau\downarrow 0}{\widetilde{\ \ }} \sum_{\ell=0}^{\infty} s_\ell(x)\tau^\ell, \tag{2.14}$$

$$\mathrm{Tr}[(h_x^D - z)^{-1} - (h-z)^{-1}] \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}\setminus C_\epsilon}}{\widetilde{\ \ }} \sum_{j=0}^{\infty} r_j(x) z^{-j-1}, \tag{2.15}$$

where C_ϵ is a cone with apex at $E_0 := \inf\{\sigma(H)\}$ and opening angle $\epsilon > 0$. Recursion relations for s_ℓ and r_j are given by (cf. [22],[24])

$$s_\ell(x) = (-1)^{\ell+1}\frac{r_\ell(x)}{\ell!}, \quad \ell \in \mathbb{N}_0, \tag{2.16}$$

$$r_0(x) = \frac{1}{2}, \ r_1(x) = \frac{1}{2}v(x), \tag{2.17}$$

$$\begin{aligned} &r_j(x) = j\gamma_j(x) - \sum_{\ell=1}^{j-1}\gamma_{j-\ell}(x)r_\ell(x), \ j = 2,3,\dots, \\ &\quad \gamma_0 = 1, \ \gamma_1 = \frac{1}{2}v, \end{aligned} \tag{2.18}$$

$$\gamma_{j+1} = -\frac{1}{2}\sum_{\ell=1}^{j}\gamma_\ell\gamma_{j+1-\ell} + \frac{1}{2}\sum_{\ell=0}^{j}\left[v\gamma_\ell\gamma_{j-\ell} + \frac{1}{4}\gamma_{\ell,x}\gamma_{j-\ell,x} - \frac{1}{2}\gamma_{\ell,xx}\gamma_{j-\ell}\right], \ j = 1,2,\dots.$$

Explictly, one computes

$$s_0 = -\frac{1}{2}, \ s_1(x) = \frac{1}{2}v(x), \quad \text{etc.} \tag{2.19}$$

The proof of (2.17) in [22] follows from the well-known differential equation for $\Gamma(z,t) = G(z,t,t)$, namely

$$-2\Gamma_{tt}(z,t)\Gamma(z,t) + \Gamma_t(z,t)^2 + 4[V(t)-z]\Gamma(z,t)^2 = 1 \tag{2.20}$$

and the asymptotic expansion

$$\Gamma(z,t) \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}\backslash C_\epsilon}}{\widetilde{\quad}} \frac{i}{2} z^{-1/2} \sum_{j=0}^{\infty} \Gamma_j(t) z^{-j}, \tag{2.21}$$

with $\Gamma_j(t)$ defined in (2.18) but $v(x)$ replaced by $V(t)$.

The next ingredient concerns the fact that $g(z,x,x)$ is a Herglotz function for all $x \in \mathbb{R}$, i.e., $g(\cdot,x,x)$: $\mathbb{C}_+ \to \mathbb{C}_+$ is analytic, $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Hence g allows a representation [3]

$$g(z,x,x) = \exp\Big\{c(x) + \int_{\mathbb{R}} d\lambda \Big[\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big]\xi(\lambda,x)\Big\}, \tag{2.22}$$

where $\xi(\lambda,x)$ is Krein's spectral shift function for the pair (h_x^D, h) [28], satisfying $0 \leq \xi(\lambda,x) \leq 1$, $\xi(\cdot,x) \in L^1_{\mathrm{loc}}(\mathbb{R}; d\lambda)$, and $\int_{\mathbb{R}} d\lambda (1+\lambda^2)^{-1}\xi(\lambda,x) < \infty$. Although it will not be subsequently used, for completeness we show how to obtain an expression for $c(x)$. Let $z = i$ in (2.22). By taking realparts of (2.22) one infers that

$$c(x) = \operatorname{Re}\{\ln[g(i,x,x)]\}. \tag{2.23}$$

Fatou's lemma permits the explicit representation

$$\xi(\lambda,x) = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \arg[g(\lambda + i\epsilon, x, x)] \text{ for a.e. } \lambda \in \mathbb{R} \tag{2.24}$$

and all $x \in \mathbb{R}$. We will normalize $\xi(\lambda,x)$ to be zero below the spectrum of h, i.e., $\xi(\lambda,x) = 0$ for $\lambda < E_0$. Using the spectral shift function, one can show that

$$\operatorname{Tr}[F(h_x^D) - F(h)] = \int_{E_0}^{\infty} d\lambda F'(\lambda)\xi(\lambda,x) \tag{2.25}$$

whenever $F \in C^2(\mathbb{R})$, $(1+\lambda^2)F^{(j)} \in L^2((0,\infty))$, $j = 1,2$ and $F(\lambda) = (\lambda - z)^{-1}$, $z \in \mathbb{C} \setminus [E_0, \infty)$.

In particular,

$$\operatorname{Tr}[e^{-\tau h_x^D} - e^{-\tau h}] = -\tau \int_{E_0}^{\infty} d\lambda e^{-\tau\lambda}\xi(\lambda,x), \quad \tau > 0, \tag{2.26}$$

$$\operatorname{Tr}[(h_x^D - z)^{-1} - (h-z)^{-1}] = -\int_{E_0}^{\infty} d\lambda \frac{\xi(\lambda,x)}{(\lambda - z)^2}, \quad z \in \mathbb{C}\backslash\{\sigma(h_x^D) \cup \sigma(h)\}. \tag{2.27}$$

Combining (2.14) and (2.26) we obtain the general trace formula for Sturm-Liouville operators

$$2s_1(x) = v(x) = E_0 + \lim_{\tau\downarrow 0} \int_{E_0}^{\infty} d\lambda e^{-\tau\lambda}[1 - 2\xi(\lambda,x)]. \tag{2.28}$$

The Abelian regularization cannot be removed in general, see [18].

Higher-order trace formulas are given in the next theorem.

Theorem 2.2. One infers

$$s_0(x) = -\frac{1}{2}, \quad s_\ell(x) = \frac{(-1)^{\ell-1}}{\ell!}\left\{\frac{E_0^\ell}{2} + \ell \lim_{t\downarrow 0} \int_{E_0}^{\infty} d\lambda e^{-t\lambda}\lambda^{\ell-1}\left[\tfrac{1}{2} - \xi(\lambda,x)\right]\right\}, \quad \ell \in \mathbb{N}. \tag{2.29}$$

¿From the high-energy behavior of the Green's function we find that

$$p(x)r(x) = i\{\lim_{z\downarrow -\infty} [\sqrt{z} g(z,x,x)]\}^{-1}. \tag{2.30}$$

In contrast to the Schrödinger case, the spectral shift function $\xi(\lambda, x)$ does not contain all the information necessary to construct both p and q in the Sturm-Liouville case, given the weight r. From (2.11) and (2.24) we see that in fact the spectral shift functions Ξ and ξ of H and h respectively, are identical in the sense that $\xi(\lambda, x) = \Xi(\lambda, t(x))$. For a given V we may construct $\Xi(\lambda, t)$ associated with (H_t^D, H). By choosing *any* positive $p \in C^\infty(\mathbb{R})$ we may define the Sturm-Liouville operator h using (2.10) (or (2.11) for the Green's function). By construction, the pair (h_x^D, h) will have $\xi(\lambda, x)$ as the corresponding spectral shift function.

The behavior of $\xi(\lambda, x)$ is particularly simple in spectral gaps of h. Since p, q, and r are real-valued, $g(\lambda + i0, x, x)$ is real-valued for $\lambda \in \mathbb{R}\backslash\sigma(h)$. More precisely, suppose $(\lambda_1, \lambda_2) \subset \mathbb{R}\backslash\sigma(h)$ and assume that $\mu(x) \in (\lambda_1, \lambda_2)$ is an eigenvalue of h_x^D. Then one has

$$\xi(\lambda, x) = \begin{cases} 0, & \lambda_1 < \lambda < \mu(x) \\ 1, & \mu(x) < \lambda < \lambda_2. \end{cases} \tag{2.31}$$

Next, assume that p, q, and r are periodic, i.e.,

$$p(x + a) = p(x),\ q(x + a) = q(x),\ r(x + a) = r(x), \quad x \in \mathbb{R} \tag{2.32}$$

for some $a > 0$. Then Floquet theory implies that

$$\sigma(h) = \bigcup_{n=1}^{\infty} [E_{2(n-1)}, E_{2n-1}], \quad E_0 < E_1 \leq E_2 < E_3 \leq \cdots \tag{2.33}$$

and

$$\sigma(h_x^D) = \sigma(h) \cup \{\mu_n(x)\}_{n\in\mathbb{N}}, \quad E_{2n-1} \leq \mu_n(x) \leq E_{2n},\ x \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.34}$$

In the periodic case $g(\lambda + i0, x, x)$ is purely imaginary on the spectrum, and hence

$$\xi(\lambda, x) = \begin{cases} 0, & \lambda < E_0,\ \mu_n(x) < \lambda < E_{2n},\ n \in \mathbb{N} \\ 1, & E_{2n-1} < \lambda < \mu_n(x),\ n \in \mathbb{N} \\ \frac{1}{2}, & E_{2(n-1)} < \lambda < E_{2n-1},\ n \in \mathbb{N} \end{cases}. \tag{2.35}$$

Combining (2.29) and (2.35) we obtain the following result.

Theorem 2.3. Let $p, q, r \in C^\infty(\mathbb{R})$, $p, r > 0$ be periodic, $p(x+a) = p(x)$, $q(x+a) = q(x)$, $r(x + a) = r(x)$ for some $a > 0$. Then

$$2(-1)^{\ell+1}\ell! s_\ell(x) = E_0^\ell + \sum_{n=1}^{\infty} [E_{2n-1}^\ell + E_{2n}^\ell - 2\mu_n(x)^\ell],\ \ell \in \mathbb{N},\ x \in \mathbb{R}. \tag{2.36}$$

In particular,

$$2s_1(x) = v(x) = E_0 + \sum_{n=1}^{\infty} [E_{2n-1} + E_{2n} - 2\mu_n(x)]. \tag{2.37}$$

Finally, we turn to the case where the Dirichlet boundary condition is replaced by a family of (Robin-type) self-adjoint boundary conditions. Define

$$\begin{aligned} &h_{\beta,y} f = \frac{1}{r^2}[-(p^2 f')' + qf], \\ &f \in \mathcal{D}(h_{\beta,y}) = \{g \in L^2(\mathbb{R}; r^2 dx) \,|\, g, g' \in AC([y, \pm R]), \quad R > 0, \\ &\qquad \lim_{\epsilon\downarrow 0}[g'(y \pm \epsilon) + \beta g(y \pm \epsilon)] = 0,\ h_{\beta,y} g \in L^2(\mathbb{R}; r^2 dx)\}. \end{aligned} \tag{2.38}$$

($\beta = 0$ corresponds to a Neumann boundary condition at y.)

$h_{\beta,y}$ is unitarily equivalent (using the operator U in (2.8)) to the Schrödinger operator

$$\begin{aligned} &H_{\nu(\beta,u),u} = -\frac{d^2}{dt^2} + V, \\ &\mathcal{D}(H_{\nu(\beta,u),u}) = \{g \in L^2(\mathbb{R}; dt) \mid g, g' \in AC([u, \pm R]), \quad R > 0, \\ &\qquad \lim_{\epsilon \downarrow 0} [g'(u \pm \epsilon) + \nu(\beta, u) g(u \pm \epsilon)] = 0, \ H_{\nu(\beta,u),u} g \in L^2(\mathbb{R}; dt)\}, \end{aligned} \tag{2.39}$$

where V is given by (2.10), the boundary condition is located at

$$u(y) = \int_{x_0}^{y} dx \frac{r(x)}{p(x)}, \tag{2.40}$$

and $\nu(\beta, u)$ depends on u as well as on β, viz.,

$$\nu = \nu(\beta, u) = \Big[\frac{p}{r}\beta - \frac{(pr)_x}{2r^2}\Big]\Big|_{x=y} = \Big[\frac{P}{R}\beta - \frac{(PR)_t}{2PR}\Big]\Big|_{t=u}. \tag{2.41}$$

The Green's function of $h_{\beta,y}$ is given by

$$\begin{aligned} g_{\beta,y}(z, x, x') &= (h_{\beta,y} - z)^{-1}(x, x') \\ &= g(z, x, x') - \frac{(\beta + \partial_2) g(z, x, y)(\beta + \partial_1) g(z, y, x')}{(\beta + \partial_1)(\beta + \partial_2) g(z, y, y)}, \end{aligned} \tag{2.42}$$

where we abbreviate

$$\partial_1 g(z, y, x') = \partial_x g(z, x, x')|_{x=y}, \ \partial_2 g(z, x, y) = \partial_{x'} g(z, x, x')|_{x'=y}, \text{ etc.} \tag{2.43}$$

In this case $-(\beta + \partial_1)(\beta + \partial_2) g(z, y, y)$ is a Herglotz function such that $\mathrm{Im}[(\beta + \partial_1)(\beta + \partial_2) g(\lambda + i0, y, y)] < 0$ for $-\lambda > 0$ large enough. Krein's spectral shift function for the pair $(h_{\beta,x}, h)$ then reads

$$\xi_\beta(\lambda, x) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \{\arg[(\beta + \partial_1)(\beta + \partial_2) g(\lambda + i\epsilon, x, x)]\} - 1, \ \beta \in \mathbb{R}, \ x \in \mathbb{R}, \ \lambda \in \mathbb{R}, \tag{2.44}$$

and it satisfies

$$\xi_\beta(\lambda, x) = 0 \text{ for } \ \lambda < \zeta_{\beta,0}(x) := \inf\{\sigma(h_{\beta,x})\}, \tag{2.45}$$

$$\mathrm{Tr}[F(h_{\beta,x}) - F(h)] = \int_{\zeta_{\beta,0}(x)}^{\infty} d\lambda F'(\lambda) \xi_\beta(\lambda, x) \tag{2.46}$$

for functions F as in (2.25). In particular, we find

$$\mathrm{Tr}[e^{-\tau h_{\beta,x}} - e^{-\tau h}] \underset{\tau \downarrow 0}{\widetilde{\ \ }} \sum_{\ell=0}^{\infty} s_{\beta,\ell}(x) \tau^\ell, \tag{2.47}$$

where

$$s_{\beta,\ell}(x) = (-1)^{\ell+1} \frac{r_{\beta,\ell}(x)}{\ell!}, \quad \ell \in \mathbb{N}_0, \tag{2.48}$$

with (cf. [22],[24]),

$$
\begin{aligned}
&r_{\beta,0}(x) = -\frac{1}{2},\ r_{\beta,1}(x) = \nu(\beta, u(x))^2 - \frac{1}{2}v(x),\\
&r_{\beta,j}(x) = j\gamma_{\beta,j-1}(x) - \sum_{\ell=1}^{j-1}\gamma_{\beta,j-\ell-1}(x)r_{\beta,\ell}(x),\ j=2,3,\dots,
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
&\gamma_{\beta,-1} = 1,\ \gamma_{\beta,0} = \nu^2 - \frac{1}{2}v,\ \gamma_{\beta,1} = \frac{1}{2}\nu^2 v + \frac{1}{2}\nu v_x - \frac{1}{8}v^2 + \frac{1}{8}v_{xx},\\
&\gamma_{\beta,2} = -\frac{1}{16}v^3 + \frac{3}{8}\nu^2 v^2 + \frac{3}{16}v_x(4\nu v + v_x) + \frac{1}{8}v_{xx}(v-\nu^2) - \frac{1}{8}\nu v_{xxx} - \frac{1}{64}v_{xxxx},\\
&\gamma_{\beta,j+1} = \frac{1}{8}\sum_{\ell=1}^{j}\big[2(v-\nu^2)\gamma_{\beta,\ell-1}\gamma_{\beta,j-\ell,xx} - (v-\nu^2)\gamma_{\beta,\ell-1,x}\gamma_{\beta,j-\ell,x}\\
&\quad - 4\gamma_{\beta,\ell}\gamma_{\beta,j-\ell+1} - 4v(v-\nu^2)\gamma_{\beta,\ell-1}\gamma_{\beta,j-\ell} - 2v_x\gamma_{\beta,\ell-1}\gamma_{\beta,j-\ell,x} + \gamma_{\beta,\ell-1}\gamma_{\beta,j-\ell}\big]\\
&\quad + \frac{1}{8}\sum_{\ell=0}^{j}\left[\gamma_{\beta,\ell,x}\gamma_{\beta,j-\ell,x} - 2\gamma_{\beta,\ell}\gamma_{\beta,j-\ell,xx} - 4(\nu^2-2v)\gamma_{\beta,\ell}\gamma_{\beta,j-\ell}\right],\ j=2,3,\dots.
\end{aligned} \tag{2.50}
$$

Explicitly, one computes

$$
s_{\beta,0}(x) = \frac{1}{2},\ s_{\beta,1}(x) = \nu(\beta,u(x))^2 - \frac{1}{2}v(x),\ \text{etc.} \tag{2.51}
$$

The proof of (2.49) in [22] is based on the differential equation for $\Gamma_\nu(z,t) = (\nu+\partial_1)(\nu+\partial_2)G(z,t,t)$, namely

$$
\begin{aligned}
&2[V(t)-\nu^2-z]\Gamma_{\nu,tt}(z,t)\Gamma_\nu(z,t) - [V(t)-\nu^2-z]\Gamma_{\nu,t}(z,t)^2 - 2V_t(t)\Gamma_{\nu,t}(z,t)\Gamma_\nu(z,t)\\
&-4\{[V(t)-z][V(t)-\nu^2-z] - \nu V_t(t)\}\Gamma_\nu(z,t)^2 = -[V(t)-\nu^2-z]^3
\end{aligned} \tag{2.52}
$$

and the asymptotic expansion

$$
\Gamma_\nu(z,t) \underset{\substack{|z|\to\infty\\ z\in\mathbb{C}\backslash C_\epsilon}}{\widetilde{\ \ }} \frac{i}{2}z^{-1/2}\sum_{j=-1}^{\infty}\Gamma_{\nu,j}(t)z^{-j}, \tag{2.53}
$$

with $\Gamma_{\nu,j}(t)$ defined as in (2.50) with β replaced by ν and $v(x)$ by $V(t)$.

The analog of Theorem 2.2 now reads

$$
s_{\beta,\ell}(x) = \frac{(-1)^\ell}{\ell!}\left\{\frac{\zeta_{\beta,0}(x)^\ell}{2} + \ell\lim_{\tau\downarrow 0}\int_{\zeta_{\beta,0}(x)}^{\infty} d\lambda e^{-\tau\lambda}\lambda^{\ell-1}\left[-\tfrac{1}{2} + \xi_\beta(\lambda,x)\right]\right\}, \quad \ell\in\mathbb{N}, \tag{2.54}
$$

and, in particular,

$$
\begin{aligned}
s_{\beta,1}(x) &= \nu(\beta,u(x))^2 - \frac{1}{2}v(x)\\
&= -\frac{1}{2}\zeta_{\beta,0}(x) - \lim_{\tau\downarrow 0}\int_{\zeta_{\beta,0}(x)}^{\infty} d\lambda e^{-\tau\lambda}\left[-\tfrac{1}{2} + \xi_\beta(\lambda,x)\right].
\end{aligned} \tag{2.55}
$$

Our last example in this section will be the periodic case, assuming (2.32) to hold.

In this case

$$
\begin{aligned}
&\sigma(h_{\beta,x}) = \sigma(h) \cup \{\zeta_{\beta,n}(x)\}_{n\in\mathbb{N}_0},\\
&\qquad \zeta_{\beta,0}(x) \le E_0,\ E_{2n-1} \le \zeta_{\beta,n}(x) \le E_{2n}, \quad x\in\mathbb{R},\ n\in\mathbb{N},
\end{aligned} \tag{2.56}
$$

with $\sigma(h)$ given as in (2.33). The spectral shift function now reads

$$\xi_\beta(\lambda,x)=\begin{cases}0, & \lambda<\zeta_{\beta,0}(x),\ E_{2n-1}<\lambda<\zeta_{\beta,n}(x),\ n\in\mathbb{N}\\ -1, & \zeta_{\beta,n}(x)<\lambda<E_{2n},\ n\in\mathbb{N}_0\\ -\frac{1}{2}, & E_{2(n-1)}<\lambda<E_{2n-1},\ n\in\mathbb{N}\end{cases} \tag{2.57}$$

and the trace formula (2.54) in the periodic case now equals

$$2(-1)^\ell \ell! s_{\beta,\ell}(x)=2\zeta_{\beta,0}(x)^\ell-E_0^\ell+\sum_{n=1}^\infty[2\zeta_{\beta,n}(x)^\ell-E_{2n-1}^\ell-E_{2n}^\ell],\quad \ell\in\mathbb{N},\ x\in\mathbb{R}. \tag{2.58}$$

In the case $\ell=1$ we find

$$\begin{aligned}-2s_{\beta,1}(x)=v(x)&=\frac{q(x)}{r(x)}+\frac{p(x)}{2r(x)^3}(r(x)p(x))_{xx}+\frac{(r(x)p(x))_x}{2r(x)^2}\left(\frac{p(x)}{r(x)}\right)_x-\frac{(r(x)p(x))_x^2}{4r(x)^4}\\ &=2\left(\frac{p(x)}{r(x)}\beta-\frac{(p(x)r(x))_x^2}{2r(x)^2}\right)^2+2\zeta_{\beta,0}(x)-E_0+\sum_{n=1}^\infty[2\zeta_{\beta,n}(x)-E_{2n-1}-E_{2n}].\end{aligned} \tag{2.59}$$

Subtracting this equation from (2.37) yields

$$-\left(\frac{p(x)}{r(x)}\beta-\frac{(p(x)r(x))_x^2}{2r(x)^2}\right)^2=E_0-\zeta_{\beta,0}(x)+\sum_{n=1}^\infty[E_{2n-1}+E_{2n}-\mu_n(x)-\zeta_{\beta,n}(x)]. \tag{2.60}$$

3. Matrix-Valued Schrödinger Operators.

In this section we extend the trace formula (2.28) to self-adjoint matrix-valued Schrödinger operators. General background on matrix-valued differential expressions can be found, e.g., in [1], [40]. Unlike all other sections in this contribution, the material below is in a preliminary stage with more details appearing elsewhere.

Let H in $L^2(\mathbb{R})^m\cong L^2(\mathbb{R})\otimes\mathbb{C}^m$ be a self-adjoint operator defined by

$$\begin{aligned}&Hf=-I_mf''+Qf, \\ &\qquad f\in\mathcal{D}(H)=\{g\in L^2(\mathbb{R})^m\mid g_j,g_j'\in AC_{\mathrm{loc}}(\mathbb{R}),1\le j\le m;Hg\in L^2(\mathbb{R})^m\},\end{aligned} \tag{3.1}$$

where $f=(f_1,\dots,f_m)^T$, I_m denotes the identity in $\mathbb{C}^m$, and $Q=(Q_{j,k})_{1\le j,k\le m}$ denotes a self-adjoint matrix satisfying

$$Q_{j,k}\in C(\mathbb{R})\text{ bounded from below, }1\le j,k\le m. \tag{3.2}$$

Closely associated with the equation

$$Hf=zf \tag{3.3}$$

is the first-order $2m\times 2m$ system

$$L(z)(f,f')^T=0, \tag{3.4}$$

where $(f,f')^T=(f_1,\dots,f_m,f_1',\dots,f_m')^T$ and

$$L(z)=I_{2m}\frac{d}{dx}-A(z),\quad A(z)=\begin{pmatrix}0 & I_m\\ Q-z & 0\end{pmatrix}, \tag{3.5}$$

with I_{2m} the identity in $\mathbb{C}^{2m}$. If $\Psi(z,x)$ denotes a fundamental matrix for $L(z)$, that is,

$$L(z)\Psi(z)=0, \tag{3.6}$$

or equivalently,

$$\Psi'(z,x)=A(z,x)\Psi(z,x), \tag{3.7}$$

then $\tilde{\Psi}(z,x)$ defined by

$$\tilde{\Psi}(z,x) = \Psi(z,x)^{-1} \tag{3.8}$$

satisfies the adjoint system

$$\tilde{\Psi}'(z,x) = -\tilde{\Psi}(z,x)A(z,x). \tag{3.9}$$

Moreover, the fundamental matrices $\Psi(z,x)$ and $\tilde{\Psi}(z,x)$ are of the form

$$\Psi(z,x) = \begin{pmatrix} \psi_1(z,x) & \psi_2(z,x) \\ \psi_1'(z,x) & \psi_2'(z,x) \end{pmatrix}, \quad \tilde{\Psi}(z,x) = \begin{pmatrix} \tilde{\psi}_2'(z,x) & -\tilde{\psi}_2(z,x) \\ -\tilde{\psi}_1'(z,x) & \tilde{\psi}_1(z,x) \end{pmatrix}, \tag{3.10}$$

and one verifies that

$$-\psi_j''(z,x) + Q(x)\psi_j(z,x) = z\psi_j(z,x), \quad -\tilde{\psi}_j''(z,x) + \tilde{\psi}_j(z,x)Q(x) = z\tilde{\psi}_j(z,x), \quad j=1,2. \tag{3.11}$$

In particular, assuming $\psi_j(z)$, $\tilde{\psi}_j(z)$ to be unique solutions of (3.11) (up to right resp. left multiplication of matrices constant with respect to x) satisfying

$$\begin{aligned} &\psi_{\frac{1}{2}}(z,\cdot) := \psi_\pm(z,\cdot) \in L^2([R,\pm\infty))^m, \\ &\tilde{\psi}_{\frac{1}{2}}(z,\cdot) := \tilde{\psi}_\pm(z,\cdot) \in L^2([R,\pm\infty))^m, \quad R \in \mathbb{R},\ z \in \mathbb{C}\setminus\sigma(H), \end{aligned} \tag{3.12}$$

the Green's matrix $G(z,x,x')$ of H becomes

$$G(z,x,x') = \begin{cases} \psi_+(z,x)\tilde{\psi}_-(z,x'), & x \geq x' \\ \psi_-(z,x)\tilde{\psi}_+(z,x'), & x \leq x' \end{cases} \tag{3.13}$$

and hence the resolvent of H is given by

$$((H-z)^{-1}f)(x) = \int_{\mathbb{R}} dx'\, G(z,x,x')f(x'),\ f \in L^2(\mathbb{R})^m,\ z \in \mathbb{C}\setminus\sigma(H). \tag{3.14}$$

Since

$$-\psi_j''(\bar{z},x)^* + \psi_j(\bar{z},x)^*Q(x) = z\psi_j(\bar{z},x)^*,\ j=1,2, \tag{3.15}$$

$\tilde{\psi}_j(z,x)$ are of the type

$$\tilde{\psi}_j(z,x) = A_{j,1}(z)\psi_1(\bar{z},x)^* + B_{j,2}(z)\psi_2(\bar{z},x)^*,\ j=1,2 \tag{3.16}$$

for matrices $A_{j,k}(z)$, $B_{j,k}(z)$, $1 \leq j,k \leq 2$ in $\mathbb{C}^m$ constant with respect to x. Introducing the "Wronskian" $W(\phi,\psi)(x)$ of $m \times m$ matrices ϕ and ψ by

$$W(\phi,\psi)(x) = \phi(x)\psi'(x) - \phi'(x)\psi(x), \tag{3.17}$$

one verifies that

$$\frac{d}{dx}W(\phi(\bar{z})^*,\psi(z))(x) = 0 \tag{3.18}$$

for solutions $\psi(z,x)$ and $\phi(\bar{z},x)^*$ of

$$-\psi''(z,x) + [Q(x)-z]\psi(z,x) = 0, \quad -\phi''(\bar{z},x)^* + \phi(\bar{z},x)^*[Q(x)-z] = 0. \tag{3.19}$$

Relations (3.8), (3.12), (3.15), and (3.16) then yield

$$\tilde{\psi}_\pm(z,x) = \pm W(\psi_\pm(\bar{z})^*,\psi_\mp(z))^{-1}\psi_\pm(\bar{z},x)^* \tag{3.20}$$

and hence

$$G(z,x,x) = -\psi_+(z,x)W(\psi_-(\bar z)^*,\psi_+(z))^{-1}\psi_-(\bar z,x)^*$$
$$= \psi_-(z,x)W(\psi_+(\bar z)^*,\psi_-(z))^{-1}\psi_+(\bar z,x)^*. \tag{3.21}$$

The corresponding matrix-valued Dirichlet Schrödinger operator H_y^D in $L^2(\mathbb{R})^m$ then reads

$$\begin{aligned} &H_y^D f = -I_m f'' + Qf,\\ &f \in \mathcal{D}(H_y^D) = \{g \in L^2(\mathbb{R})^m \mid g_j \in AC_{\text{loc}}(\mathbb{R}), g_j' \in AC_{\text{loc}}(\mathbb{R}\setminus\{y\}),\\ &\lim_{\epsilon\downarrow 0} g_j(y\pm\epsilon) = 0, H_y^D g \in L^2(\mathbb{R})^m\} \end{aligned} \tag{3.22}$$

and its Green's matrix $G_y^D(z,x,x')$, the analog of (2.4), is given by

$$G_y^D(z,x,x') = G(z,x,x') - G(z,x,y)G(z,y,y)^{-1}G(z,y,x'). \tag{3.23}$$

The analog of (2.5) then becomes

$$\begin{aligned} &Tr[(H_x^D - z)^{-1} - (H-z)^{-1}] = -Tr[G(z,\cdot,x)G(z,x,x)^{-1}G(z,x,\cdot)]\\ &= -Tr[G(z,x,x)^{-1}G(z,x,\cdot)G(z,\cdot,x)] = -Tr_{\mathbb{C}^m}\{G(z,x,x)^{-1}[\frac{d}{dz}G(z,x,x)]\}\\ &= -\frac{d}{dz}Tr_{\mathbb{C}^m}\{\ln[G(z,x,x)]\} = -\frac{d}{dz}\ln\{\det{}_{\mathbb{C}^m}[G(z,x,x)]\}, \end{aligned} \tag{3.24}$$

where we used cyclicity of the trace,

$$(H-z)^{-2}(x,x')_{j,k} = \frac{d}{dz}G(z,x,x')_{j,k} = \sum_{\ell=1}^m \int_{\mathbb{R}} dx''\, G(z,x,x'')_{j,\ell}G(z,x'',x')_{\ell,k}\,, \tag{3.25}$$

and $Tr_{\mathbb{C}^m}[\ln(M)] = \ln[\det_{\mathbb{C}^m}(M)]$ for matrices M in $\mathbb{C}^m$. Moreover, $Tr(\cdot)$ and $Tr_{\mathbb{C}^m}(\cdot)$ in (3.24) denote the trace in $L^2(\mathbb{R})^m$ and $\mathbb{C}^m$, respectively.

Introducing the matrix-valued Green's kernel diagonal with respect to x (cf. (3.21))

$$\Gamma(z,x) = G(z,x,x), \tag{3.26}$$

the matrix analog of (2.20) reads

$$\begin{aligned} &-\Gamma(z,x)\Gamma_{xx}(z,x) - \Gamma_{xx}(z,x)\Gamma(z,x) + \Gamma_x(z,x)^2 + \Gamma(z,x)^2Q(x)\\ &\quad + Q(x)\Gamma(z,x)^2 + 2\Gamma(z,x)Q(x)\Gamma(z,x) - 4z\Gamma(z,x)^2 = I_m \end{aligned} \tag{3.27}$$

and considerations along the lines of (2.20), (2.21) then yield

$$\Gamma(z,x) \underset{\substack{|z|\to\infty\\ z\in\mathbb{C}\setminus C_\epsilon}}{\widetilde{\ \ }} \frac{i}{2}z^{-1/2}\sum_{j=0}^\infty \Gamma_j(x)z^{-j}, \tag{3.28}$$

with

$$\Gamma_0(x) = I_m, \quad \Gamma_1(x) = \frac{1}{2}Q(x), \text{ etc.} \tag{3.29}$$

Similarly,

$$-\frac{d}{dz}\ln[G(z,x,x)] \underset{\substack{|z|\to\infty\\ z\in\mathbb{C}\setminus C_\epsilon}}{\widetilde{\ \ }} \sum_{j=0}^\infty R_j(x)z^{-j-1}, \tag{3.30}$$

where

$$R_0(x) = \frac{1}{2}I_m, \quad R_1(x) = \frac{1}{2}Q(x), \text{ etc.} \tag{3.31}$$

Next, define for all $x \in \mathbb{R}$ the analog of (2.24) by

$$\Xi(\lambda, x) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \operatorname{Im}\{\ln[G(\lambda + i\epsilon, x, x)]\} \text{ for a.e. } \lambda \in \mathbb{R},$$
$$\Xi(\lambda, x) = 0, \ \lambda < E_0 := \inf\{\sigma(H)\}, \tag{3.32}$$

where $\operatorname{Im}(M)$, $\operatorname{Re}(M)$, in obvious notation, abbreviate

$$\operatorname{Im}(M) = \frac{1}{2i}(M - M^*), \quad \operatorname{Re}(M) = \frac{1}{2}(M + M^*) \tag{3.33}$$

for matrices M in $\mathbb{C}^m$. It follows from the results in [5] that

$$0 \le \Xi(\lambda, x) \le I_m \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{3.34}$$

In the following denote by $C_{R,\epsilon}$ the counter-clockwise oriented contour

$$\begin{aligned} C_{R,\epsilon} = \{z = E_0 + \epsilon e^{i\phi} \,|\, \tfrac{3\pi}{2} \ge \phi \ge \tfrac{\pi}{2}\} \cup \{z = E_0 + \lambda + i\epsilon \,|\, 0 \le \lambda \le R\} \\ \cup \{z = E_0 + Re^{i\phi} \,|\, \arctan(\epsilon/R) \le \phi \le 2\pi - \arctan(\epsilon/R)\} \\ \cup \{z = E_0 + \lambda - i\epsilon \,|\, 0 \le \lambda \le R\}, \ R > \epsilon > 0. \end{aligned} \tag{3.35}$$

Applying the residue theorem, taking into account that $G(z, x, x)$, $x \in \mathbb{R}$, is analytic in $z \in \mathbb{C} \setminus \sigma(H)$ and $\det[G(z, x, x)] \ne 0$ for $z \in \mathbb{C} \setminus \sigma(H)$ (cf. (3.23)), then yields

$$\begin{aligned} \{\ln[G(z, x, x)]\}_{j,k} &= \frac{1}{2\pi i} \oint_{C_{R,\epsilon}} dz' \, \frac{\{\ln[G(z', x, x)]\}_{j,k}}{z' - z} \\ &= \frac{1}{2\pi i} \oint_{C_{R,\epsilon}} dz' \, \{\ln[G(z', x, x)]\}_{j,k} \frac{z'}{1 + z'^2} \\ &\quad + \frac{1}{2\pi i} \oint_{C_{R,\epsilon}} dz' \, \{\ln[G(z', x, x)]\}_{j,k} \left[\frac{1}{z' - z} - \frac{z'}{1 + z'^2}\right] \\ &= \operatorname{Re}\{\ln[G(i, x, x)]\}_{j,k} + \frac{1}{\pi} \int_{E_0}^{R} d\lambda \operatorname{Im}\{\ln[G(\lambda + i0, x, x)]\}_{j,k} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\right] \\ &\quad + o(\epsilon) + o(R^{-1}) \\ &\xrightarrow[R \to \infty, \, \epsilon \downarrow 0]{} \operatorname{Re}\{\ln[G(i, x, x)]\}_{j,k} + \int_{E_0}^{\infty} d\lambda \, \Xi(\lambda, x)_{j,k} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\right], \\ &\quad 1 \le j, k \le m. \end{aligned} \tag{3.36}$$

Thus

$$\frac{d}{dz} \ln[G(z, x, x)] = \int_{E_0}^{\infty} d\lambda \, \Xi(\lambda, x)(\lambda - z)^{-2}, \tag{3.37}$$

and the matrix analog of (2.28) then reads

$$Q(x) = E_0 I_m + \lim_{z \to i\infty} \int_{E_0}^{\infty} d\lambda \, z^2 (\lambda - z)^{-2} [I_m - 2\Xi(\lambda, x)], \tag{3.38}$$

where we used a resolvent instead of a heat kernel regularization.

Defining

$$\xi(\lambda, x) = Tr_{\mathbb{C}^m}[\Xi(\lambda, x)], \tag{3.39}$$

one infers from (3.24) that

$$Tr[(H_x^D - z)^{-1} - (H - z)^{-1}] = -\int_{E_0}^{\infty} d\lambda \xi(\lambda, x)(\lambda - z)^{-2} \tag{3.40}$$

and that

$$\xi(\lambda, x) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \arg\{\det_{\mathbb{C}^m}[G(\lambda + i\epsilon, x, x)]\}, \quad 0 \leq \xi(\lambda, x) \leq m \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{3.41}$$

Further details and applications of this formalism to inverse spectral problems will appear elsewhere.

4. Multi-Dimensional Trace Formulas.

First, reporting on recent work in [25], we attempt to extend the leading behavior in (2.14),

$$2Tr[e^{-\tau H} - e^{-\tau H_x^D}] = 1 - \tau V(x) + o(\tau) \text{ as } \tau \downarrow 0 \tag{4.1}$$

to arbitrary space dimensions $\nu \in \mathbb{N}$. The key to such an extension is an appropriate combination of Dirichlet and Neumann boundary conditions on various hyperplanes through the point $x \in \mathbb{R}^\nu$ taking into account that (4.1) is equivalent to

$$Tr[e^{-\tau H_x^N} - e^{-\tau H_x^D}] = 1 - \tau V(x) + o(\tau) \text{ as } \tau \downarrow 0, \tag{4.2}$$

where $H_x^N = H_x^0$ denotes the operator (2.39) with a Neumann boundary condition at $x \in \mathbb{R}$. We start by introducing proper notations. In the following let V be a real-valued continuous function on $\mathbb{R}^\nu$ bounded from below and define the self-adjoint operator

$$H = -\Delta \dot{+} V \tag{4.3}$$

as a form sum in $L^2(\mathbb{R}^\nu)$. Next, let $A \subseteq \{1, ..., \nu\}$ and denote by $|A|$ the number of elements of A. Moreover, let $B_\alpha^{(x)}, \alpha \subseteq \{1, ..., \nu\}$ be the 2^ν blocks obtained by removing the hyperplanes $\mathcal{P}_j^{(x)} = \{y \in \mathbb{R}^\nu \mid y_j = x_j\}$ from $\mathbb{R}^\nu$, that is, $B_\alpha^{(x)} = \{y \in \mathbb{R}^\nu \mid y_\ell > x_\ell \text{ if } \ell \in \alpha, y_\ell < x_\ell \text{ if } \ell \notin \alpha\}$ and denote by $\mathcal{P}_\nu$ the power set of $\{1, ..., \nu\}$. The operator $H_{A;x}$ is then defined to be $-\Delta + V$ on $\bigoplus_{\alpha \in \mathcal{P}_\nu} L^2(B_\alpha^{(x)})$ with Dirichlet boundary conditions on $\{P_j^{(x)}\}_{j \in A}$ and Neumann boundary conditions on $\{P_j^{(x)}\}_{j \notin A}$.

Theorem 4.1. [25] Define $C_\tau = \sum_{A \in \mathcal{B}_\nu} (-1)^{|A|} e^{-\tau H_{A;0}}, \tau > 0$. Then the integral kernel of C_τ is given by

$$C_\tau(x, x') = \begin{cases} 2^\nu e^{-\tau H}(x, -x'), & x, x' \text{ in the same orthant} \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Moreover, $C_\tau, \tau > 0$ is a trace class operator in $L^2(\mathbb{R}^\nu)$ and

$$Tr(C_\tau) = 2^\nu \int_{\mathbb{R}^\nu} d^\nu x e^{-tH}(x, -x), \quad \tau > 0. \tag{4.5}$$

The proof of (4.4) in [25] is based on the method of images while the trace class property of C_τ and (4.5) follow from the direct sum decomposition of C_τ in $\bigoplus_{\alpha \in \mathcal{P}_\nu} L^2(B_\alpha^{(x)})$.

Applying a Feynman-Kac-type analysis then yields the following ν-dimensional generalization of (4.2).

Theorem 4.2. [25]

$$Tr\Big(\sum_{A \in \mathcal{P}_\nu} (-1)^{|A|} e^{-\tau H_{A;x}}\Big) = 1 - \tau V(x) + o(\tau) \text{ as } \tau \downarrow 0. \tag{4.6}$$

While Theorem 4.2 represents a multidimensional trace formula for Schrödinger operators associated with unbounded regions in $\mathbb{R}^\nu$, one can also prove new trace formulas for Schrödinger operators defined in boxes. One obtains, e.g.,

Theorem 4.3. [25] Let V be continuous on $[0,1]^\nu$. For $A \subseteq \{1,...,\nu\}$, let H_A be $-\Delta+V$ on $L^2([0,1]^\nu)$ with Dirichlet boundary conditions on the hyperplanes with $x_j = 0$ or 1 and $j \in A$ and Neumann boundary conditions on the hyperplanes with $x_j = 0$ or 1 and $j \notin A$. Let $\langle V\rangle$ be the average of V at the 2^ν corners of $[0,1]^\nu$. Then

$$\sum_{A\in\mathcal{P}_\nu} (-1)^{|A|} Tr(e^{-\tau H_A}) = 1 - \tau\langle V\rangle + o(\tau) \text{ as } \tau \downarrow 0. \tag{4.7}$$

This result holds also for rectangular boxes $\times_{j=1}^{\nu}[a_j, b_j]$ but the rectangular symmetry is crucial in the proof of [25]. Similarly, one can prove

Theorem 4.4. [25] Let V be continuous on $[0,1]^\nu$. For $A \subseteq \{1,...,\nu\}$ let $\tilde{H}_A$ be $-\Delta + V$ on $L^2([0,1]^\nu)$ with Dirichlet boundary conditions on the hyperplanes with $x_j = 0$ for $j \in A$ and Neumann boundary conditions on the hyperplanes with $x_j = 0$ for $j \notin A$ or $x_k = 1$ for all $k \in \{1,...,\nu\}$. Then

$$\sum_{A\in\mathcal{P}_\nu} (-1)^{|A|} Tr(e^{-\tau \tilde{H}_A}) = 2^{-\nu}[1 - \tau V(0) + o(\tau)] \text{ as } \tau \downarrow 0. \tag{4.8}$$

Finally, we mention an Abelianized version of a trace formula that Lax [29] derived formally in two dimensions.

Theorem 4.5. [25] Let V be a continuous periodic function on $\mathbb{R}^2$ with $V(x_1+n_1, x_2+n_2) = V(x_1, x_2)$ for all $(x_1, x_2, n_1, n_2) \in \mathbb{R}^2 \times \mathbb{Z}^2$. Let $H_P, H_A, H_{AP}, H_{PA}, H_N,$ and H_D be the operators $-\Delta + V$ on $L^2([0,1]^2)$ with periodic, antiperiodic, AP, PA, Neumann, and Dirichlet boundary conditions respectively, where AP (resp. PA) means antiperiodic in the x_1 (resp. x_2) direction and periodic in the x_2 (resp. x_1) direction. Then

$$Tr[e^{-\tau H_P} + e^{-\tau H_A} + e^{-\tau H_{AP}} + e^{-\tau H_{PA}} - 2e^{-\tau H_N} - 2e^{-\tau H_D}] = -1 + \tau V(0) + o(\tau) \text{ as } \tau \downarrow 0. \tag{4.9}$$

For a different kind of two-dimensional trace formula for $V(x)$ comparing the heat kernels for $H = -\Delta + V$ and $H_0 = -\Delta$ with Dirichlet boundary conditions on a rectangular box, see [34]. Trace formulas for heat kernels of multi-dimensional Schrödinger operators in the short-range case have also recently been derived in [4].

Finally, we illustrate a possible new abstract approach to the trace formulas (4.7) based on certain commutation (supersymmetric) techniques in the noninteracting case where $V(x) = 0$, $x \in \mathbb{R}^\nu$. We need a bit of notation. Let $\mathcal{H}$ be a (complex separable) Hilbert space, F a closed densely defined linear operator in $\mathcal{H}$ and define the self-adjoint operators

$$H_1 = F^*F\ ,\ H_2 = FF^* \tag{4.10}$$

in $\mathcal{H}$ and

$$Q = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix},\ P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.11}$$

in $\mathcal{H} \oplus \mathcal{H}$. Moreover, we denote by $tr(.)$ the trace in $\mathcal{H}$, by $Tr(.)$ the trace in $\mathcal{H} \oplus \mathcal{H}$, and by $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{B}_1(\mathcal{H})$) the set of bounded (resp. trace class) operators in $\mathcal{H}$.

Lemma 4.6. One infers that

(i)

$$QP + PQ = 0. \tag{4.12}$$

(ii)

$$Q^2 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{4.13}$$

(iii)

$$F e^{-tH_1} \supseteq e^{-tH_2} F, \ F^* e^{-tH_2} \supseteq e^{-tH_1} F^*. \tag{4.14}$$

Proof. While (i) and (ii) are obvious, (iii) follows from

$$Q e^{-tQ^2} \supseteq e^{-tQ^2} Q.$$

Lemma 4.7. Assume $B \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is bounded and commutes with Q, i.e., $QB \supseteq BQ$. Suppose e^{-tQ^2}, $Q^2 e^{-tQ^2} \in \mathcal{B}_1(\mathcal{H} \oplus \mathcal{H})$, $t > 0$. Then

$$\frac{d}{dt} Tr[P e^{-tQ^2} B] = 0. \tag{4.15}$$

Proof.

$$\begin{aligned} &\frac{d}{dt} Tr[P e^{-tQ^2} B] = -Tr[P Q^2 e^{-tQ^2} B] \\ &= Tr[P Q e^{-tQ^2} Q B] = \cdots = Tr[P Q^2 e^{-tQ^2} B] \end{aligned} \tag{4.16}$$

using commutativity of Q and B and anticommutativity of Q and P in (4.12) and cyclicity of the trace. The fact that Q is unbounded is offset by the trace class hypotheses in Lemma 4.7. In fact, rewriting

$$-Tr[P Q^2 e^{-tQ^2} B] = -Tr[P Q (1 + |Q|)^{-1} Q (1 + |Q|) e^{-tQ^2} B]$$

enables one to prove (4.16) in a trivial manner by reshuffling $Q(1 + |Q|)^{-1} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ as opposed to Q in (4.16).

Next we introduce the closed densely defined linear operators F_n, $1 \leq n \leq \nu$ as in $\mathcal{H}$ and define $H_{1,n} = F_n^* F_n$, $H_{2,n} = F_n F_n^*$, $1 \leq n \leq \nu$ as in (4.10). Moreover, assume

$$e^{-tH_{j,n}}, \qquad H_{j,n} e^{-tH_{j,n}} \in \mathcal{B}_1(\mathcal{H}), \qquad 1 \leq n \leq \nu$$

and

$$[F_m, F_n] \subseteq 0, \qquad [F_m, F_n^*] \subseteq 0, \qquad m \neq n$$

implying

$$[H_{j,m}, H_{\ell,n}] \subseteq 0, \qquad j, \ell = 1, 2, \qquad m \neq n.$$

We also denote

$$Q_n = \begin{pmatrix} 0 & F_n^* \\ F_n & 0 \end{pmatrix}, \qquad 1 \leq n \leq \nu \tag{4.17}$$

in $\mathcal{H} \oplus \mathcal{H}$ as in (4.11) and define for any $A \in \mathcal{P}_\nu$ (the power set of $\{1, 2, \ldots, \nu\}$) the self-adjoint operator

$$H_A^0 = \sum_{n \in A} H_{1,n} + \sum_{n \notin A} H_{2,n}. \tag{4.18}$$

Then an abstract version of (4.7) in the noninteracting case reads as follows.

Theorem 4.8.

$$\sum_{A \in \mathcal{P}_\nu} (-1)^{|A|} tr(e^{-tH_A^0}) = \sum_{A \in \mathcal{P}_\nu} (-1)^{|A|} \dim \operatorname{Ran}[P_{H_A^0}(\{0\})], \tag{4.19}$$

where $P_{H_A^0}(\Omega)$, $\Omega \subseteq \mathbb{R}$, denote the spectral projections of H_A^0.

Proof. One computes

$$\frac{d}{dt}\sum_{A\in\mathcal{P}_\nu}(-1)^{|A|}tr(e^{-tH_A^0}) = -\sum_{A\in\mathcal{P}_\nu}(-1)^{|A|}tr(H_A^0 e^{-tH_A^0}) \\ = -\sum_{n=1}^{\nu} Tr[PQ_n^2 e^{-tQ_n^2}B_{\nu,n}] = 0 \tag{4.20}$$

by (4.15), where

$$B_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ B_{\nu,n} = \begin{pmatrix} b_{\nu,n} & 0 \\ 0 & b_{\nu,n} \end{pmatrix}, \quad b_{\nu,n} = -\prod_{\substack{m=1 \\ m\neq n}}^{\nu}(e^{-tH_{2,m}} - e^{-tH_{1,m}}), \qquad \nu \geq 2 \tag{4.21}$$

are bounded and commute with Q_n. Thus the left-hand-side in (4.19) is independent of t and taking $t \uparrow \infty$ then determines the right-hand-side of (4.19).

Identifying $A_n = 1\otimes\cdots\otimes 1\otimes \frac{\partial}{\partial x_n}\Big|_D \otimes 1\otimes\cdots\otimes 1$ in $L^2([0,1])^\nu)$ with

$$\frac{\partial}{\partial x_n}\Big|_D = \overline{\frac{d}{dx}\Big|_{C_0^\infty((0,1))}}, \quad 1\leq n\leq \nu \tag{4.22}$$

in $L^2([0,1])$ then yields (4.7) in the case $V(x)=0$ since only the zero-energy eigenvalue of the Neumann operator H_ϕ^0 contributes on the right-hand-side of (4.19). More generally, if A_n has the tensor product structure

$$A_n = 1\otimes\cdots\otimes 1\otimes a_n\otimes 1\otimes\cdots\otimes 1$$

in $\mathcal{H} = \mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_\nu$, then clearly $[A_m, A_n]\subseteq 0$ and one evaluates

$$\sum_{A\in\mathcal{P}_\nu}(-1)^{|A|}tr(e^{-tH_A^0}) = \prod_{n=1}^{\nu} tr(e^{-ta_n a_n^*} - e^{-ta_n^* a_n}). \tag{4.23}$$

In the special case (4.22), where $a_n = \frac{\partial}{\partial x_n}\Big|_D$, one confirms that

$$tr\left(e^{-ta_n a_n^*} - e^{-ta_n^* a_n}\right) = 1, \qquad 1\leq n\leq\nu.$$

5. Trace Formulas and Point Interactions in Dimensions One, Two, and Three. In this section we describe a different kind of multi-dimensional trace formula based on point interactions [2] and hence rank-one perturbations of resolvents first derived in [17] in a slightly different form. Since point interactions (also called contact interactions or δ-interactions) are limited to $\nu = 1,2,3$ space dimensions, so will be our approach below.

Assuming V to be real-valued, continuous and bounded from below on $\mathbb{R}^\nu, \nu = 1,2,3$, we introduce $H = -\Delta \dotplus V$ as in (4.3). The resolvent of the self-adjoint Hamiltonian $H_{\alpha,x}$, modeling H plus a point interaction centered at $x\in\mathbb{R}^\nu$ (whose strength is parameterized in terms of $\alpha\in\mathbb{R}$), is defined as follows (see, e.g., [2], [42])

$$(H_{\alpha,x} - z)^{-1} = (H-z)^{-1} + D_{\alpha,x}(z)^{-1}(\overline{G(z,x,.)},.)G(z,.,x), \quad z\in\mathbb{C}\backslash\{\sigma(H_{\alpha,x})\cup\sigma(H)\}, \tag{5.1}$$

where

$$D_{\alpha,x}(z) = \begin{cases} -\alpha^{-1} - \Gamma_\nu(z,x), & \nu = 1,\ \alpha \in \mathbb{R} \cup \{\infty\}, \alpha \neq 0 \\ \alpha - \Gamma_\nu(z,x), & \nu = 2,3,\ \alpha \in \mathbb{R}, \end{cases} \tag{5.2}$$

$$\Gamma_1(z,x) = G(z,x,x), \quad \Gamma_2(z,x) = \lim_{|\epsilon|\downarrow 0}[G(z,x,x+\epsilon) - (2\pi)^{-1}\ln(|\epsilon|)], \tag{5.3}$$
$$\Gamma_3(z,x) = \lim_{|\epsilon|\downarrow 0}[G(z,x,x+\epsilon) - (4\pi\,|\epsilon|)^{-1}],$$

and $G(z,x,x')$ denotes the Green's function of H. In analogy to (2.5) one then computes

$$Tr[(H_{\alpha,x} - z)^{-1} - (H-z)^{-1}] = -\frac{d}{dz}\ln[D_{\alpha,x}(z)]. \tag{5.4}$$

Krein's spectral shift function for the pair $(H_{\alpha,x}, H)$ is then introduced via

$$Tr[(H_{\alpha,x} - z)^{-1} - (H-z)^{-1}] = -\int_{E_{\alpha,x,0}}^{\infty} d\lambda \frac{\xi_{\alpha,x}(\lambda)}{(\lambda - z)^2}, \tag{5.5}$$

with $E_{\alpha,x,0} = \inf\{\sigma(H_{\alpha,x}) \cup \sigma(H)\}$ and the normalization

$$\xi_{\alpha,x}(\lambda) = 0, \quad \lambda < E_{\alpha,x,0}. \tag{5.6}$$

$\xi_{\alpha,x}(\lambda)$ is related to $D_{\alpha,x}(z)$ as $\xi(\lambda,x)$ is to $g(z,x,x)$ in (2.24). The high-energy expansion (see, e.g., [35], [41])

$$\lim_{|\epsilon|\downarrow 0}[G(z,x,x+\epsilon) - G^{(0)}(z,x,x+\epsilon)] = -V(x)\begin{cases} 1/4(-z)^{3/2} + o(z^{-3/2}), & \nu = 1 \\ -1/4\pi z + o(z^{-1}), & \nu = 2 \\ 1/8\pi(-z)^{1/2} + o(z^{-1/2}), & \nu = 3 \end{cases} \tag{5.7}$$

then yields

$$D_{\alpha,x}(z) = \begin{cases} -\alpha^{-1} - \frac{i}{2}z^{-1/2} - \frac{i}{4}V(x)z^{-3/2} + o(z^{-3/2}), & \nu = 1 \\ (2\pi)^{-1}\ln((-iz)^{1/2}) + \tilde{\alpha} - (4\pi)^{-1}V(x)z^{-1} + o(z^{-1}), & \nu = 2 \\ -i(4\pi)^{-1}z^{1/2} + \alpha + i(8\pi)^{-1}V(x)z^{-1/2} + o(z^{-1/2}), & \nu = 3 \end{cases}, \tag{5.8}$$

where

$$\tilde{\alpha} = \alpha + (2\pi)^{-1}\gamma - (2\pi)^{-1}\ln(2)$$

with $\gamma = .5772\ldots$ being Euler's constant. A combination of (5.4), (5.5), and (5.8) then implies the following trace formula.

Theorem 5.1. [17]

$\nu = 1$:

$$V(x) = \begin{cases} \lim_{z\downarrow -\infty}\left\{-z - 2\int_{\inf[\sigma(H)]}^{\infty} d\lambda z^2(\lambda - z)^{-2}\xi_{\infty,x}(\lambda)\right\}, & \alpha = \infty, \\ \frac{1}{6}\alpha^2 + \lim_{z\downarrow -\infty}\left\{-\frac{2}{3}z + \frac{i}{3}\alpha z^{1/2} + \frac{8i}{3}\alpha^{-1}z^{5/2}\int_{E_{\alpha,x,0}}^{\infty} d\lambda(\lambda - z)^{-2}\xi_{\alpha,x}(\lambda)\right\}, & \\ \qquad \alpha \in \mathbb{R}\backslash\{0\}. \end{cases} \tag{5.9}$$

$\nu = 2$:

$$V(x) = \lim_{z\downarrow -\infty}\left\{-z + 4\pi[(2\pi)^{-1}\ln(-iz^{1/2}) + \tilde{\alpha}]\int_{E_{\alpha,x,0}}^{\infty} d\lambda z^2(\lambda - z)^{-2}\xi_{\alpha,x}(\lambda)\right\}. \tag{5.10}$$

$\nu = 3$:

$$V(x) = 16\pi^2\alpha^2 + \lim_{z\downarrow -\infty}\left\{-z + 4\pi i\alpha z^{1/2} + 2\int_{E_{\alpha,x,0}}^{\infty} d\lambda z^2(\lambda - z)^{-2}\xi_{\alpha,x}(\lambda)\right\}. \tag{5.11}$$

Using the systematic high-energy expansion of $\lim_{|\epsilon|\downarrow 0}[G(z,x,x+\epsilon) - G^{(0)}(z,x,x+\epsilon)]$ in terms of (multi-dimensional) KdV invariants (see, e.g., [35], [41]) one can extend Theorem 5.1 to higher-order trace relations in analogy to (2.29) and (2.54).

In the special case where $V^{(0)} \equiv 0$, one obtains explicitly,

$$D_\alpha^{(0)}(z) = \begin{cases} -\alpha^{-1} - (-4z)^{-1/2}, & \nu = 1 \\ \tilde\alpha + (2\pi)^{-1}\ln((-z)^{1/2}), & \nu = 2 \\ \tilde\alpha + (4\pi)^{-1}(-z)^{1/2}, & \nu = 3, \end{cases} \tag{5.12}$$

and

$$Tr[(H_{\alpha,x}^{(0)} - z)^{-1} - (H^{(0)} - z)^{-1}] = -\int_{E_{\alpha,0}^{(0)}}^{\infty} d\lambda \frac{\xi_\alpha^{(0)}(\lambda)}{(\lambda - z)^2}. \tag{5.13}$$

Here, for $\nu = 1$,

$$\xi_\alpha^{(0)}(\lambda) = \underset{\alpha<0}{\begin{cases} 0, & \lambda < -\alpha^2/4 \\ -1, & -\alpha^2/4 < \lambda < 0 \\ a_\alpha(\lambda), & \lambda > 0 \end{cases}} \quad \underset{\alpha=0}{\begin{cases} 0, & \lambda < 0 \\ \frac{1}{2}, & \lambda > 0 \end{cases}} \quad \underset{\alpha\in(0,\infty]}{\begin{cases} 0, & \lambda < 0 \\ 1 + a_\alpha(\lambda), & \lambda > 0 \end{cases}} \tag{5.14}$$

writing $a_\alpha(\lambda) = -\pi^{-1}\arctan(|\alpha|/2\lambda^{1/2})$, and, for $\nu = 2$,

$$\xi_\alpha^{(0)}(\lambda) = \begin{cases} 0, & \lambda < -e^{-4\pi\tilde\alpha} \\ -1, & -e^{-4\pi\tilde\alpha} < \lambda \leq 0 \\ -\pi^{-1}\arctan[\pi/(4\pi\tilde\alpha + \ln(\lambda))] - 1, & 0 \leq \lambda \leq e^{-4\pi\tilde\alpha} \\ -\pi^{-1}\arctan[\pi/(4\pi\tilde\alpha + \ln(\lambda))], & \lambda \geq e^{-4\pi\tilde\alpha} \end{cases} \tag{5.15}$$

and, finally, for $\nu = 3$,

$$\xi_\alpha^{(0)}(\lambda) = \underset{\alpha<0}{\begin{cases} 0, & \lambda < -(4\pi\alpha)^2 \\ -1, & -(4\pi\alpha)^2 < \lambda < 0 \\ A_\alpha(\lambda), & \lambda > 0 \end{cases}} \quad \underset{\alpha=0}{\begin{cases} 0, & \lambda < 0 \\ -\frac{1}{2}, & \lambda > 0 \end{cases}} \quad \underset{\alpha>0}{\begin{cases} 0, & \lambda < 0 \\ A_\alpha(\lambda), & \lambda > 0 \end{cases}}, \tag{5.16}$$

writing $A_\alpha(\lambda) = -\pi^{-1}\arctan(\lambda^{1/2}/4\pi\,|\alpha|)$, and

$$E_{\alpha,0}^{(0)} = \underset{\nu=1}{\begin{cases} -\alpha^2/4, & \alpha < 0 \\ 0, & \alpha \in [0,\infty] \end{cases}} \quad \underset{\nu=2}{\left\{ -e^{-4\pi\tilde\alpha} \right.} \quad \underset{\nu=3}{\begin{cases} -(4\pi\alpha)^2, & \alpha < 0 \\ 0, & \alpha \geq 0 \end{cases}}. \tag{5.17}$$

6. A Uniqueness Result for Three-Dimensional Schrödinger Operators.

Finally, we briefly sketch a uniqueness result in the context of three-dimensional Schrödinger operators with spherically symmetric potentials originally derived in [19]. Consider the potential $V : \mathbb{R}^3 \to \mathbb{R}$,

$$V(x) = v(|x|), \qquad v \in L^1([0,R)]) \text{ for all } R > 0 \tag{6.1}$$

and define the self-adjoint Schrödinger operator H in $L^2(\mathbb{R}^3)$ associated with the differential expression $-\Delta + v(|x|)$ by decomposition with respect to angular momenta. This represents H as an infinite direct sum of half-line operators in $L^2((0,\infty); r^2dr)$ associated with differential expressions of the type

$$\hat{\tau}_\ell = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + v(r), \qquad r = |x| > 0, \qquad \ell \in \mathbb{N}_0. \tag{6.2}$$

A simple unitary transformation (see, e.g., [36], Appendix to Sect. X.1) reduces (6.2) to

$$\tau_\ell = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + v(r) \tag{6.3}$$

and associated Hilbert space $L^2((0,\infty); dr)$. Next, let $G(z,x,x')$, $x \neq x'$ denote the Green's function of H and define $H_{\alpha,0}$ in $L^2(\mathbb{R}^3)$, $\alpha \in \mathbb{R}$ as in (5.1) (with $x = 0$) and the corresponding Krein spectral shift function $\xi_{\alpha,0}(\lambda)$ as in (5.5), i.e.,

$$\xi_{\alpha,0}(\lambda) = \lim_{\epsilon\downarrow 0} \pi^{-1} \operatorname{Im}\{\ln[D_{\alpha,0}(\lambda + i\epsilon)]\} \qquad \text{a.e.} \tag{6.4}$$

Then the following uniqueness result holds.

Theorem 6.1. [19] Define $H_j, H_{j,\alpha_j,0}$, $\alpha_j \in \mathbb{R}$ associated with $-\Delta + v_j(|x|)$, $x \in \mathbb{R}^3$, $j = 1,2$ as above and introduce Krein's spectral shift function $\xi_{j,\alpha_j,0}(\lambda)$ for the pair $(H_{j,\alpha_j,0}, H_j)$, $j = 1,2$. Then the following are equivalent:

(i)

$$\xi_{1,\alpha_1,0}(\lambda) = \xi_{2,\alpha_2,0}(\lambda) \qquad \text{for a.e. } \lambda \in \mathbb{R}. \tag{6.5}$$

(ii)

$$\alpha_1 = \alpha_2 \qquad \text{and} \qquad V_1(x) = V_2(x) \qquad \text{for a.e. } x \in \mathbb{R}^3. \tag{6.6}$$

The proof of this result in [19] is based on detailed Weyl-m-function investigations associated with the angular momentum channel $\ell = 0$.

Acknowledgments. We are indebted to B. Simon and Z. Zhao for joint collaborations which led to most of the results presented in this contribution and to B. M. Levitan for discussions which inspired Section 3. We are also grateful to G. Stolz for discussions on matrix-valued differential expressions. F.G. would like to thank A. Friedman and the Institute for Mathematics and its Applications, IMA, University of Minnesota, USA and the Department of Mathematical Sciences, NTH, University of Trondheim, Norway for the great hospitality extended to him during a month long stay at each institution in the summer of 1995. F.G. also thanks B. Simon and D. Truhlar for their kind invitation to the IMA workshop "Multiparticle Quantum Scattering with Applications to Nuclear, Atomic and Molecular Physics". Moreover, we are both indebted to T. Hoffmann-Ostenhof for the kind invitation to the Erwin Schrödinger International Institute for Mathematical Physics, ESI, Vienna, Austria for a period in June and July of 1995. Financial support by the IMA, the Norwegian Research Council, and the ESI is gratefully acknowledged.

References

1. Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory*, Gordon and Breach, New York, 1963.
2. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer, New York, 1988.

3. N. Aronszajn and W. F. Donoghue, On exponential representations of analytic functions in the upper half-plane with positive imaginary part, *J. Anal. Math., 5*: 321–388 (1957).
4. R. Banuelos and A. Sa Barreto, On the heat trace of Schrödinger operators, preprint 1995.
5. R. W. Carey, A unitary invariant for pairs of self-adjoint operators, *J. reine angew. Math., 283*: 294–312 (1976).
6. W. Craig, The trace formula for Schrödinger operators on the line, *Commun. Math. Phys., 126*: 379–407 (1989).
7. P. Deift and E. Trubowitz, Inverse scattering on the line, *Commun. Pure Appl. Math., 32*: 121–251 (1979).
8. L. A. Dikii, Trace formulas for Sturm-Liouville differential operators, *Amer. Math. Soc. Transl. Ser. (2), 18*: 81–115 (1961).
9. B. A. Dubrovin, Periodic problems for the Korteweg-de Vries equation in the class of finite band potentials, *Funct. Anal. Appl., 9*: 215–223 (1975).
10. N. Dunford and J. T. Schwartz, *Linear Operators II: Spectral Theory, Self-Adjoint Operators in Hilbert Space*, Interscience, New York, 1988.
11. H. Flaschka, On the inverse problem for Hill's operator, *Arch. Rat. Mech. Anal., 59*: 293–309 (1975).
12. I. M. Gelfand, On identities for the eigenvalues of a second-order differential operator, *Uspehi Mat. Nauk, 11:1*: 191–198 (1956) (Russian). English translation in *Izrail M. Gelfand, Collected Papers Vol. I* (S. G. Gindikin, V. W. Guillemin, A. A. Kirillov, B. Kostant, S. Sternberg, eds.), Springer, Berlin, 1987, pp. 510–517.
13. I. M. Gelfand and L. A. Dikii, Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equations, *Russ. Math. Surv., 30:5*: 77–113 (1975).
14. I. M. Gelfand and B. M. Levitan, On a simple identity for the eigenvalues of a second-order differential operator, *Dokl. Akad. Nauk SSSR, 88*: 593–596 (1953) (Russian). English translation in *Izrail M. Gelfand, Collected Papers Vol. I* (S. G. Gindikin, V. W. Guillemin, A. A. Kirillov, B. Kostant, S. Sternberg, eds.), Springer, Berlin, 1987, pp. 457–461.
15. F. Gesztesy, New trace formulas for Schrödinger operators, *Evolution Equations*, G. Ferreyra, G. Ruiz Goldstein, F. Neubrander (eds.), Marcel Dekker, New York, 1995, pp. 201–221.
16. F. Gesztesy and H. Holden, Trace formulas and conservation laws for nonlinear evolution equations, *Rev. Math. Phys., 6*: 51–95 (1994).
17. F. Gesztesy and H. Holden, On new trace formulae for Schrödinger operators, *Acta Appl. Math., 39:* 315–333 (1995).
18. F. Gesztesy and B. Simon, The *xi* function, *Acta Math.*, to appear.
19. F. Gesztesy and B. Simon, Uniqueness theorems in inverse spectral theory for one-dimensional Schrödinger operators, *Trans. Amer. Math. Soc.*, to appear.
20. F. Gesztesy and K. Unterkofler, Isospectral deformations for Sturm-Liouville and Dirac-type operators and associated nonlinear evolution equations, *Rep. Math. Phys., 31*: 113–137 (1992).
21. F. Gesztesy, H. Holden, and B. Simon, Absolute summability of the trace relation for certain Schrödinger operators, *Commun. Math. Phys., 168:* 137–161 (1995).
22. F. Gesztesy, R. Ratnaseelan, and G. Teschl, The KdV hierarchy and associated trace formulas, to appear in the proceedings of the International Conference on Applications of Operator Theory (I. Gohberg, P. Lancaster, P. N. Shivakumar, eds.), *Operator*

Theory: Advances and Applications, Birkhäuser.
23. F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, Trace formulae and inverse spectral theory for Schrödinger operators, *Bull. Amer. Math. Soc., 29*: 250–255 (1993).
24. F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, Higher order trace relations for Schrödinger operators, *Rev. Math. Phys.*, to appear.
25. F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, A trace formula for multidimensional Schrödinger operators, preprint.
26. H. Hochstadt, On the determination of a Hill's equation from its spectrum, *Arch. Rat. Mech. Anal., 19*: 353–362 (1965).
27. S. Kotani and M. Krishna, Almost periodicity of some random potentials, *J. Funct. Anal., 78*: 390–405 (1988).
28. M. G. Krein, Perturbation determinants and a formula for the traces of unitary and self-adjoint operators, *Sov. Math. Dokl., 3*: 707–710 (1962).
29. P. D. Lax, Trace formulas for the Schroedinger operator, *Commun. Pure Appl. Math., 47*: 503–512 (1994).
30. B. M. Levitan, On the closure of the set of finite-zone potentials, *Math. USSR Sbornik, 51*: 67–89 (1985).
31. B. M. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press, Utrecht, 1987.
32. V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Basel, 1986.
33. H. P. McKean and P. van Moerbeke, The spectrum of Hill's equation, *Invent. Math., 30*: 217–274 (1975).
34. V. Papanicolaou, Trace formulas and the behavior of large eigenvalues, *SIAM J. Math. Anal., 26:* 218–237 (1995).
35. A. M. Perelomov, Schrödinger equation spectrum and Korteweg–de Vries type invariants, *Ann. Inst. H. Poincaré, Sect. A, 24*: 161–164 (1976).
36. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.
37. B. Simon, Spectral analysis of rank one perturbations and applications, *Proc. Mathematical Quantum Theory II: Schrödinger Operators*, J. Feldman, R. Froese, L. M. Rosen (eds.), Amer. Math. Soc., Providence, RI, to appear.
38. E. Trubowitz, The inverse problem for periodic potentials, *Commun. Pure Appl. Math., 30*: 321–337 (1977).
39. S. Venakides, The infinite period limit of the inverse formalism for periodic potentials, *Commun. Pure Appl. Math., 41*: 3–17 (1988).
40. J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, Vol. **1258**, Springer, Berlin, 1987.
41. S. F. J. Wilk, Y. Fujiwara, and T. A. Osborn, N-body Green's function and their semiclassical expansion, *Phys. Rev., 24A*: 2187–2202 (1981).
42. J. Zorbas, Perturbation of self-adjoint operators by Dirac distributions, *J. Math. Phys., 21*: 840–847 (1980).

[1] Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: `mathfg@mizzou1.missouri.edu`

[2] Department of Mathematical Sciences, Norwegian Institute of Technology, University of Trondheim, N–7034 Trondheim, Norway
E-mail address: `holden@imf.unit.no`