Determinants of random Schrödinger operators arrizing from lattice gauge fields

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Abstract

For a class of bounded random selfadjoint operators $(Lu)(n) = \sum_{i=1}^{d} A_i(n)u(n + e_i) + A_i^*(n - e_i)u(n - e_i)$, determined by a discrete abelian or nonabelian $U(N)$ lattice gauge field $n \mapsto (A_1(n), \dots, A_d(n))$ on $\mathbf{Z}^d$, the potential theoretic logarithmic energy $I(L) = -\int\int \log|E - E'| \ dk(E) \ dk(E')$ of the density of states dk of L is finite and satisfies $I(L) = -\log|\det(L^{(2)}|$, where $L^{(2)}$ is the two particle Hamiltonian of L. For the n-particle Hamiltonians $L^{(n)}$ defined on the n particle subspace of the Fock space, we show the existence of ergodic or gauge invariant minimizers of the height functionals $I_n(L) = -\log|\det(L^{(n)})|$. We prove $I_n(L) \in [-\log(\sqrt{2nd}), 0]$ and $I_n(L) \sim \text{EulerGamma}/2 - \log(\sqrt{dn}) + o(1)$ for $n \to \infty$. A random walk expansion for $I_{n,\beta}(L) = -\log|\det(L^{(n)} - \beta)|$ identifies $\det(1 - zL^{(n)})$ as a dynamical zeta function.

Keywords: Random lattice gauge fields, potential theory of the spectrum, Discrete random Schrödinger operators.

1 Introduction

A class of random Schrödinger operators can be obtained from abelian or nonabelian discrete random fields on $\mathbf{Z}^d$. These operators have finite determinants. A natural problem is to minimize $-\log|\det L|$ among this class of operators, to determine the set of extrema as well to study their properties. This variational problem is mathematically similar to the entropy problem in statistical mechanics and has relations with lattice gauge fields, random matrix theory, random Schrödinger operators and potential theory in the complex plane.

Our guide is the variational problem in statistical mechanics to maximize the entropy on the set of invariant measures of an expansive topological dynamical system (X, T) with time $\mathbf{Z}^d$. Because this functional is upper semi-continuous, it takes its maximal value, the topological entropy. (See for example [22]).

Every discrete gauge field x with the structure group $U(N)$ in d dimensions defines an operator L on $\mathcal{H} = l^2(\mathbf{Z}^d, \mathbf{C}^N)$ by $(L_x u)(n) = \sum_{i=1}^{d} x_i(n)u(n + e_i) + x_i^*(n - e_i)u(n - e_i)$. Given a measure μ on X and a random variable $x \mapsto L_x$ taking values in the space of these operators, there exists a density of states dk_μ. In lattice gauge field theory, one usually considers the Wilson Hamiltonian $n \mapsto H_x(n) = (L_x^4)_{nn}$. The expectation value $< H >= \text{tr}_\mu(L^4)$

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has as a continuous function both minima and maxima. We consider here the Hamiltonian $n \mapsto -\log |L_x|_{nn}$ which is not bounded, but which has, as we will see, a finite expectation value $-\mathrm{tr}_\mu \log |L| = -\log |\det_\mu(L)|$. A "van der Monde determinant" of an operator L is obtained by applying the functional calculus for L to the "characteristic function" $\beta \mapsto \det_\mu(L-\beta)$ and taking again the determinant $\Delta_\mu(L) = \det_\mu(\det_\mu(L-L))$. While $\Delta_\mu(L)$ is not defined for many operators and not interesting for matrices, where trivially $\Delta_\mu(L) = 0$, the value $\Delta_\mu(L)$ is nonvanishing and finite for random operators obtained from lattice gauge fields and $I_2(\mu) = -\log \Delta_\mu(L)$ coincides with the potential theoretical energy of the density of states. I_2 is lower semicontinuous as a functional on the set $M(X)$ of shift-invariant measures on the set X of all gauge fields. The minimizers of this energy exist therefore by a similar reason as equilibrium measures exist for the entropy functional.

A variational problem for Schrödinger operators L arrizing in discrete Hamiltonian systems $T : \mathbf{T}^{2n} \to \mathbf{T}^{2n}$ is closely related to the entropy maximizing problem in ergodic theory because the maximal value of $\log |\det_\mu(L)|$ is the topological entropy of the map, if the maximizer μ is absolutely continuous with respect to the volume measure on $\mathbf{T}^{2n}$ and the random operator L is the Hessian of a discrete action functional at a critical point.

A finite dimensional version of the problem to minimize $I(\mu)$ is the following: consider a finite discrete torus which is the Cayley graph of the group $\mathbf{Z}_p^d$ with generators $e_1, \dots, e_p$. If we attach to each bond $(n, e+e_i)$ an element $x_i(n) \in U(N)$, this defines a finite gauge field $x(n) = (x_1(n), \dots, x_d(n))$, $n \in \mathbf{Z}_p^d$. Consider the finite dimensional operator $(L_x u)(n) = \sum_{i=1}^d x_i(n)u(n+e_i) + x_i^*(n-e_i)u(n-e_i)$ on the finite dimensional space $l^2(\mathbf{Z}_p^d, \mathbf{C}^N)$. The flux-phase problem is to minimize $-\mathrm{tr}(|L|) = -\sum_i |\lambda_i|$ and a related question is to minimize $-|\det(L)| = -\prod_i |\lambda_i|$, where λ_i are the eigenvalues of L (see [12, 14, 13]). The finite dimensional version of the problem to minimize the electrostatic energy of the density of states is to minimize $I(L) = -\log \prod_{i<j}(\lambda_i - \lambda_j)^2 = -\sum_{i \neq j} \log |\lambda_i - \lambda_j|$, where λ_i are the eigenvalues of L. In finite dimensions, the quantity $I(L)$ is the energy of a Coulomb gas in $\mathbf{C}$ consisting of finitely many equally charged particles rsp. point vortices located at λ_i. The factor $\Delta(L) = e^{-I(L)}$ appears often in random matrix theory [17], a branch of probability which shows the relation of potential theory in the plane with quantum systems and deals with the statistics of eigenvalues for a given probability measure on a group of matrices.

Determinants are natural functionals in physics or mathematics, and extremals of these functionals are expected to have special properties. In functional integration approaches to quantum gravity [5], one integrates over a probability space of manifolds with the functional $\det(L)$, a Ray-Singer regularized determinant of the Laplace-Beltrami operator L (see [24]). For the more classical variational problem of minimizing the height $-\log|\det(L)|$, it is known for example that among a class of conformal metrics giving fixed area to a compact two dimensional manifold, the minimizers are given by the metric of constant curvature (see [18] or [2] for recent results in higher dimensions). The height is a measure for the complexity of the Riemannian metric and a useful spectral invariant beside the coefficients of the heat kernel. Not only determinants of Laplace Beltrami operators can be related to Selberg zeta functions [24], Dynamical zeta functions in dynamical systems can often be written as determinants or Fredholm determinants (see [23]). In the situation here, which is more elementary than the corresponding situations in differential geometry or dynamical systems, we note that the determinants of the discrete Laplacians which occur in this article are suitable dynamical zeta functions of the random walk.

2 A first variational problem

Given the finite dimensional unitary group $U = U(N)^d$. A shift invariant measure μ on the compact metric space $X = U^{\mathbf{Z}^d}$ defines a dynamical system (X, T_i, μ) with time $\mathbf{Z}^d$. The generators T_i are the shifts $T_i(x)(n) = x(n + e_i)$, where $\{e_i, i = 1, \dots, d\}$ is the natural orthonormal basis in $\mathbf{Z}^d$. Denote by $M(X)$ the set of shift invariant probability measures on X. Every $x \in X$ can be written as $x(n) = (x_1(n), \dots, x_d(n))$, with $x_i(n) \in U(N)$, $n \in \mathbf{Z}^d$ and x defines the discrete selfadjoint Laplacian

$$(L_x u)(n) = \sum_{i=1}^{d} x_i(n) u(n + e_i) + x_i(n - e_i) u(n - e_i)$$

on the Hilbert space $\mathcal{H} = l^2(\mathbf{Z}^d, \mathbf{C}^N)$ of l^2-functions on the lattice taking values in the the spin space $\mathbf{C}^N$. The operator-valued random variable $x \mapsto L_x$ has a density of states dk_μ, which is a measure on $\mathbf{R}$ defined by the functional $f \mapsto \int \mathrm{Trace}(f(L_x)_{00})\, d\mu(x)$ on $C(\mathbf{R})$, where Trace is the normalized trace in the finite dimensional matrix algebra $M(N, \mathbf{C})$. The density of states dk_μ exists also for measures μ which are not necessarily shift invariant.

Remark. If $x \mapsto L_x$ is considered to be an operator-valued random variable, then the measure μ plays the role of the *law* of this random variable. We could look more generally at a random operator L over a $\mathbf{Z}^d$ action (Ω, T_i, μ). We will see in Lemma (8.1) that similarily as in probability theory, where every random variable Y can be realized on $(\mathbf{R}, \mu)$, given the law μ of Y, every operator-valued random variable L of our type can be defined on the probability space (X, μ).

If μ is an ergodic shift invariant measure, then L is called ergodic. By general principles, the spectrum of L_x is then the same almost everywhere and coincides with the support of the density of states [3]. However, we do consider here the set $M(X)$ of all shift invariant probability measures and not only the ergodic probability measures.

Lemma 2.1 *The logarithmic potential theoretical energy*

$$I(\mu) = -\int\int \log|E - E'|\, dk_\mu(E)\, dk_\mu(E')$$

of the density of states of the operator L takes values in $[-\log(d), 0]$.

Proof. Define $\lambda(E) = \int \log|E - E'|\, dk_\mu(E')$.

(i) $\lambda(E) \geq 0$, if μ is a shift invariant periodic measure.
Proof. Assume the measure μ has the support on configurations in X, which are k-periodic in all directions $x(n + ke_i) = x(n)$ for $i = 1, \dots, d-1$. The Thouless formula on the strip ([10, 1] generalized in Appendix A) assures that $N \cdot \lambda(E)$ is the sum of the N Lyapunov exponents of a plus the sum of the N largest Lyapunov exponents of the transfer cocycle $A_E(x)$ of a Jacobi matrix $(L_x u)(n) = a_n(x)u(n+1) + a_{n-1}(x)u(n-1) + b_n(x)u(n)$ on $l^2(\mathbf{Z}, \mathbf{C}^{(k^{d-1})})$, where $a_n(x) = a_n(T_i^k x), b_n(x) = b_n(T_i^k(x))$, $i = 1, \dots, d-1$ are matrices in $M(k^{d-1}, \mathbf{C})$, and $a_n(x) = a(T_d^n x), b_n(x) = b(T_d^n x)$.
The Lyapunov exponents of the cocycle a taking values in $M(k^{d-1}, \mathbf{C})$ are all vanishing. The reason is that $a^k(x) = a_k(x)a_{k-1}(x) \cdots a_2(x)a_1(x)$ is unitary for all $x \in X$. This can be seen in the case $d = 2$ directly because $a(x)$ is in a suitable basis a periodic $k \times k$ Jacobi matrix with only unitary off diagonal terms. In general, the claim follows by induction in

d. In each inductive step, the unitary off diagonal elements of a Jacobi matrix is replaced by a periodic $k \times k$ Jacobi matrix with unitary off diagonal elements.

(ii) Periodic shift invariant measures are dense in $M(X)$.
Proof. The proof given by Parthasarathy [19] in one dimension generalizes to higher dimensional shifts (see Appendix B).

(iii) $\lambda(E) \geq 0$ for $E \in \mathbf{C}$.
For $E \notin \mathbf{R}$, the map $\mu \mapsto \lambda_\mu(E)$ is continuous. For fixed $E \in \mathbf{R}$, the map $\mu \mapsto \lambda(E,\mu)$ is upper semi-continuous on $M(X)$ as a pointwise infimum of continuous functions

$$\mu \mapsto (\chi_\epsilon \star \lambda_\mu)(E) ,$$

where $(\chi_\epsilon \star \lambda)(E)$ is a smoothed subharmonic function in E (see for example [21] p. 49) which converge for $\epsilon \to 0$ pointwise to $\lambda(E)$. By (i),(ii), $\lambda_\mu(E) \geq 0$ on a dense set of measures in $M(X)$. By upper semicontinuity, $\lambda_\mu(E) \geq 0$ for all μ, E. Therefore, $I(\mu) = -\int \lambda(E)\, dk_\mu(E) \leq 0$.

(iv) $I(\mu) \geq -\log(d)$.
Proof. The operator L has the spectrum contained in an interval $[-2d, 2d]$ because it is the sum of $2d$ unitary operators. The capacity

$$\sup\{e^{-I(\mu)} \mid \operatorname{supp}(\mu) \subset [-2d, 2d]\}$$

of the interval $[-2d, 2d]$ is equal to d. □

Remarks.
1) In a similar set-up, it is remarked in the book of Carmona-Lacroix (see [1] p.333) that the function $E \mapsto \lambda(E) = \int_{\mathbf{C}} \log|E - E'|\, dk_\mu(E')$ is nonnegative on $\mathbf{C}$ if dk is the density of states of the discrete Schrödinger operators of the form $\Delta + V$ with the free Laplacian $L_{free} = \Delta$. As the proof of the previous lemma shows, this stays true if L_{free} is replaced by an operator arrizing from gauge fields.

Lemma 2.2 *The density of states dk is symmetric with respect to 0 in the sense that $dk(U) = dk(-U)$ for all measurable sets U on $\mathbf{R}$.*

Proof. Since L has only off-diagonal "hopping terms", $\operatorname{tr}_\mu(L^n) = \int E^n\, dk_\mu(E)$ is vanishing for odd n. (We will see that also explicitly, when giving a random walk expansion of $\operatorname{tr}(L^n)$). Therefore, for $f \in C(\mathbf{R})$

$$\int f(E)\, dk_\mu(E) = \operatorname{tr}_\mu(f(L)) = \operatorname{tr}_\mu(f(-L)) = \int f(-E)\, dk_\mu(E) = \int f(E)\, dk_\mu(-E) .$$

□

3 Multiparticle operators and a trace on the Fock space

Given a random operator L on $\mathcal{H} = l^2(\mathbf{Z}^d, \mathbf{C}^N)$, the n-particle Hamiltonian $L^{(n)} = \sum_{j=1}^n L_j$ is the sum of the n commuting operators $L_j = 1 \otimes \ldots \otimes L \otimes \ldots \otimes 1$ on the n-particle space $\mathcal{H}^{(n)} = l^2(\mathbf{Z}^{dn}, \mathbf{C}^N)$, the n-fold tensor product of $\mathcal{H}$. The operators L_j and so $L^{(n)}$ are representations of L on $\mathcal{H}^{(n)}$ and the restriction of the *bilinear Hamiltonian* of $\overline{L}$ to the

n-particle subspace. The bilinear Hamiltonian is sometimes also denoted by $d\Gamma(L)$.

Remarks.
1) The n-particle operators leave the Fermionic and Bosonic subspaces in $\mathcal{H}^{(n)}$ invariant and $e^{itL^{(n)}} = e^{itL} \otimes e^{itL} \otimes \cdots \otimes e^{itL}$.
2) The operators $L^{(n)}$ are again random gauge operators in dimension $d \cdot n$.

Given a probability distribution p_n on $\mathbf{N} = \{0, 1, 2, \dots\}$. A trace for $\overline{L}$ is defined by $\mathrm{tr}_p(\overline{L}) = \sum_n p_n \mathrm{tr}(L^n)$, where $\mathrm{tr}(L^{(n)})$ is the trace of the n-particle opeator $L^{(n)}$. ¿From the probabilistic point of view, the Poisson distribution is the most natural choice. We denote this trace by

$$\mathrm{tr}_\lambda(\overline{L}) = \sum_n e^{-\lambda} \frac{\lambda^n}{n!} \mathrm{tr}(L^n) \; .$$

The bilinear Hamiltonian $\overline{L}$ has a density of states which we denote by $\overline{dk}_\lambda$. A parameter is the expected number of particles $\mathrm{tr}_\lambda(\overline{N}) = \lambda$, where $\overline{N}$ is the *number operator* on the Fock space which takes the constant value n on $\mathcal{H}^{(n)}$.

The tensor product of ergodic operators is ergodic:

Proposition 3.1 *If K_i, $i = 1, \dots n$ are ergodic operators defined over $\mathbf{Z}^d$-dynamical system (Ω_i, T_{ij}, m_i), where $T_{ij} : \Omega_i \to \Omega_i$, $j = 1, \dots, d$. Then $K_1 \otimes \dots \otimes K_n$ is a random operator over the ergodic $\mathbf{Z}^{nd}$ dynamical system $(\prod_{i=1}^d \Omega_i, T_{ij}, m_1 \otimes \cdots \otimes m_n)$.*

Proof. Given a bounded measurable function $f : \prod_{i=1}^d \Omega_i \to \mathbf{R}$, which is invariant under all transformations T_{ij}. Since by the ergodicity of $\{T_{ij}\}_{j=1 \dots d}$ for every $j = 1, \dots, n$, the function $x_j \mapsto f(x_1, \dots, x_j, \dots, x_n)$ is constant for m_j- almost all points in Ω_j, also f is constant almost everywhere. □

Remarks.
1) It follows that if L is an ergodic operator, then $L^{(n)}$ is an ergodic operator and the spectrum of $x \mapsto L_x^{(n)}$ is almost everywhere constant.
2) For $d = 1$, multi particle operators $L^{(n)}$ allow isospectral Toda deformations. This can be seen directly or using the criterium in [7].

4 A sequence of variational problems

A random Schrödinger operator has the determinant $|\mathrm{det}_\mu L| = e^{\mathrm{tr}_\mu(\log|L|)} = e^{\int \log|E| \; dk_\mu(E)}$. Borrowing terminology from differential geometry, we call $-\log|\mathrm{det} L| = -\mathrm{tr}(\log|L|)$ a *height*. Also the operator $\overline{L}$ has a determinant with corresponding height $\overline{I} = -\log|\mathrm{det}_\lambda \overline{L}| = -\mathrm{tr}_\lambda(\log|\overline{L}|)$. We consider the problem to minimize $I_n(\mu) = -\log|\mathrm{det}_\mu(L^{(n)}|)$ or $\overline{I}$ on the set $M(X)$ of all shift invariant probability measures μ on X.

Any random operator is an operator-valued random variable $x \mapsto L_x \in \mathcal{B}(l^2(\mathbf{Z}^d, \mathbf{C}^N))$. As for real-valued random variables, two random operators are called *independent*, if the σ-algebras, they generate, are independent. For example, if L, K are random operators, then $L \otimes 1$ and $1 \otimes K$ are independent.

Lemma 4.1 *If $L_1, L_2, \dots, L_n$ are independent random operators, then the density of states of the sum $L = \sum_i L_i$ is the convolution $dk_1 \otimes \dots \otimes dk_n$ of the density of states dk_i of the individual operators L_i.*

Proof. The operators L_i are independent operator-valued random variables. The real-valued random variables $l_i : \omega \mapsto \mathrm{tr}(L_i)$ are independent too and their law is just the density of states dk_i. Since $l = \sum_{i=1}^d l_i$ is the trace of L, the claim follows from the fact that the law of a sum of independent random variables is the convolution of the laws of the summands. $\square$

Remark. It follows that the set of measures occurring as the density of states of random operators attached to gauge fields is closed under convolutions.

Corollary 4.2 *The height of $L^{(n)}$ satisfies*

$$I_n(\mu) = -\log|\mathrm{det}_\mu(L^{(n)})| = -\int_{\mathbf{R}} \log|E| dk_\mu^{(n)}(E) = -\int_{\mathbf{R}^d} \log|E_1 + \ldots + E_n| \; dk_\mu(E_1) \ldots dk_\mu(E_n) \; .$$

Proof. By definition of the convolution of measures on $\mathbf{R}$, one has $\int f(E) \; dk \otimes dk(E) = \int \int f(E+E') \; dk(E) \; dk(E')$ and more generally

$$\int_{\mathbf{R}} f(E) \; dk^{(n)}(E) = \int_{\mathbf{R}^d} f(E_1 + E_2 + \cdots + E_d) \; dk(E_1) \; \ldots \; dk(E_d) \; .$$

$\square$

Especially:

Corollary 4.3 *The potential theoretical energy of the density of states dk of L is the height of $L^{(2)}$ attached to L:*

$$I(\mu) = -\log|\det(L_\mu^{(2)})| \; .$$

Proof. Because by Lemma (2.2), dk is symmetric,

$$\begin{aligned} \log|\mathrm{det}_\mu L^{(2)}| &= \int_{\mathbf{R}} \log|E| \; dk_\mu^{(2)} = \int_{\mathbf{R}^2} \log|E+E'| \; dk_\mu(E) \; dk_\mu(E') \\ &= \int_{\mathbf{R}^2} \log|E-E'| \; dk_\mu(E) \; dk_\mu(E') = -I(\mu) \; . \end{aligned}$$

$\square$

Remarks.
1) Because the mean of dk is $\mathrm{tr}(L^{(n)}) = 0$ and the variance is $\mathrm{tr}((L^{(n)})^2) = 2dn$, the central limit theorem implies that the density of states of the operator $L^{(n)}/\sqrt{2dn}$ converges to the normalized Gauss measure on $\mathbf{R}$. If $\sum_n p_n = 1$, then $p_n \to 0$ faster than $1/\sqrt{2dn}$ so that the density of states of $p_n L^{(n)}$ converges to the Dirac measure on 0 which implies that $p_n L^{(n)}$ converges to zero in norm.
2) The Fourier transform of dk, the characteristic function $s \mapsto \hat{dk}(s) = \int e^{ist} \; dk(t) = \mathrm{tr}(e^{isL})$ contains all the information of the density of states. As in probability theory, the knowledge of the Fourier transfor $\hat{dk}$ is useful for computations of $dk^{(n)}$ which has the Fourier transform $(\hat{dk})^n(s)$.

Proposition 4.4 *Denote by γ the Euler constant. Given any operator L, we have*

$$I_n(L) - \gamma/2 + \log(\sqrt{dn}) \to 0, \; n \to \infty \; .$$

Proof. The density of states of $L^{(n)}$ is the law of the sum of n independent random variables with law dk, zero expectation and variance $\mathrm{tr}(L^2) = 2d$. For any real number, the density of states of rL is $dk(rE)$ if $dk(E)$ is the density of states of L.
The central limit theorem implies that the density of states of $L^{(n)}/\sqrt{2dn}$ converges weakly to the Gauss measure $(2\pi)^{-1/2}e^{-x^2/2}\,dx$. Therefore, for $\mathrm{Im}(E) \neq 0$, and $n \to \infty$

$$G_n(E) = -\log|\det(L^{(n)}/\sqrt{2dn} + E)| \to -(2\pi)^{-1/2}\int_{-\infty}^{\infty} \log|x+E| \cdot e^{-x^2/2}\,dx = F(E)\ .$$

By the dominated convergence theorem, the smoothed functions $(\chi_\epsilon \star F)(0)$ converge for $\epsilon \to 0$ to

$$F(0) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \log|x| \cdot e^{-x^2/2}\,dx$$

which has the value $(\gamma + \log(2))/2$. Also $(\chi_\epsilon \star G_n)(E)$ converges with fixed n for $\epsilon \to 0$ to $I_n + \log(\sqrt{2dn})$. □

Remarks.
1) The asymptotic formula in Proposition (4.4) holds for any random operator L which satisfies $\mathrm{tr}(L) = 0$ and $\mathrm{tr}(L^2) = 2d$.
2) A reformulation of the law of iterated logarithm in the present context is that the accumulation points of $\mathrm{tr}(L^{(n)})/\sqrt{2nd\log\log(n)}$ is the set $[-1, 1]$.
3) The operator $L^{(n)}$ has its spectrum in $[-2nd, 2nd]$ and for the equilibrium measures $dk_{[-2nd,2nd]}$ on this interval we get the minimal energy $-\log(nd)$ which is smaller than $-\log(\sqrt{dn})$. It follows that for large enough n, the minimizers of I_n do not achieve the potential theoretical minimum. We see below that this is true unless $dn = 1$.
4) The integral

$$\lim_{n\to\infty} \log|\det(L^{(n)}/\sqrt{2dn} - \beta)| = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \log|x-\beta| \cdot e^{-x^2/2}\,dx$$

seems to be difficult to evaluate in closed form, but it can be computed numerically.
5) Let $\overline{N}$ is the number operator on the Fock space. The determinant $\det_\lambda(\overline{L}\ (2d\overline{N})^{-1/2})$ can be computed in the limit of an infinite expectation value of particles as

$$\lim_{\lambda\to\infty} \det{}_\lambda(\overline{L}\ (d\overline{N})^{-1/2}) = e^{\gamma/2} = 1.33457\ldots\ .$$

5 Existence of minimizers

Proposition 5.1 *The functionals I_n and $\overline{I}$ take their minima on the compact metric space $M(X)$ of shift invariant measures μ on X.*

Proof. The map $\mu \mapsto dk$ is continuous because for all $f \in C(\mathbf{R})$, the map $\mu \mapsto \mathrm{tr}_\mu(f(L)) = \int f(L)_{00}\,d\mu$ is continuous. Also $\mu \mapsto dk_\mu^{(n)} = \int f(L^{(n)})_{00}\,d\mu$ is continuous.

The map $E \mapsto \lambda_{n,\mu}(E) = \int \log|E - E'|\,dk_\mu^{(n)}(E')$ is subharmonic and bounded and therefore the pointwise infimum of in E smooth functions $\lambda_{n,\mu,\epsilon}(E) = (\chi_\epsilon \star I_{n,\mu}(\mu))(E)$. These functions are continuous in μ for $\epsilon \neq 0$ because they are integrals of bounded functions which have this property almost everywhere.

Therefore, $\mu \mapsto I_n(\mu)$ is lower semicontinuous as a pointwise limit of continuous functions on the compact metric space $M(X)$ and such a function has a minimum.

The same argument holds for the density of states $dk^{(n)}$ or $\overline{dk} = \sum_{n=0}^{\infty} p_n dk^{(n)}$. □

Corollary 5.2 *There exist ergodic minimizers of $I_n, I, \overline{I}$.*

Proof. The maps $\mu \mapsto -\log|\det_\mu L^{(n)}| \in [-\log(nd), 0]$ and $\mu \mapsto -\log|\det_\mu \overline{L}| \in (-\infty, 0]$ are linear in μ if we extend it to all signed measures.
Fix n. Given a probability measure μ, on which I_n is minimal. We can write it as an integral over ergodic probability measures and the minimality assures that I_n takes the minimal value on almost all of the ergodic measures in the ergodic Choquet decomposition of μ. □

Remarks.
1) If a nonergodic minimum exists, it is not unique because it can then be written as an integral over ergodic measures, on which the functionals take the same minimal value.
2) One could try to find the minimum of I_n by determining minimizers μ_k on the set of k-periodic measures which have the support on k-periodic configurations. But there is no reason why an accumulation point of μ_k should be a minimizer for I_n. (For the entropy functional $\mu \mapsto h_\mu(T)$, there exists a dense set of periodic measures, which have zero entropy but the maximum is positive.)

6 Lower bounds on I_n

Potential theory gave the lower bound $I = I_2 \geq -\log(d)$ which would be achieved if the density of states of $L^{(n)}$ was the equilibrium measure. We saw also from the central limit theorem that $I_n \geq -\log(\sqrt{dn})$ for large enough n. Using the Jensen inequality for random operators, one can say more:

Lemma 6.1 *For all $k \in \mathbf{N} \setminus \{0\}$,*

$$I_n = -\log|\det(L^{(n)})| \geq -(1/k)\log \mathrm{tr}(|L^{(n)}|^k) .$$

Proof. (i) Jensen's inequality for the probability space $(\mathbf{R}, dk)$ gives for any random operator K and any convex bounded function f

$$\mathrm{tr}(f(K)) = \int f(E)\, dk(E) \geq f(\int E\, dk(E)) = f(\mathrm{tr}(K)) .$$

(ii) In order to apply (i) with $K = |L^{(n)}|^k$ and the convex unbounded function $f(x) = -\log|x|$, we approximate f from above by bounded continuous convex functions f_l. The dominated convergence theorem and the finiteness of $-\log|\det_\mu(L)| = -\int f(L)_{00}\, d\mu$ assures that

$$\int f_l(|L|^k)_{00}\, d\mu \to \int f(|L|^k)\, d\mu = -k \cdot \log|\det_\mu(L)| .$$

¿From (i), we know

$$\int f_l(|L|^k)_{00}\, d\mu \geq f_l(\int (|L|^k)_{00}\, d\mu) \to f(\int (|L|^k)_{00}\, d\mu) = -\log \mathrm{tr}(|L|^k) .$$

□

Remarks.
1) Applying this lemma in the case $k=1$ shows that minimization of $-\log|\det(L^{(n)})|$ tells something about the minimum of $-\log \mathrm{tr}(|L^{(n)}|)$ and so about the problem to minimize $-\mathrm{tr}|L^{(n)}|$. It follows also that $\mathrm{tr}|L^{(n)}| \geq 1$.
2) Jensen's inequality has a generalisation in form of the Peierl's-Bogoliubov inequality

$$\mathrm{tr}(f(K)e^L)/\mathrm{tr}(e^L) \geq f(\mathrm{tr}(Ke^L)/\mathrm{tr}(e^L))$$

in finite dimensions (see [26]). Using Avron-Simon's lemma [3] this can easily be generalized to the random case.

Corollary 6.2 *$I_n(\mu) \geq -\log\sqrt{2nd}$ for all n and all $\mu \in M(X)$.*

Proof. Apply b) in the case $k=2$ and use that $\mathrm{tr}((L^{(n)})^2) = 2nd$, which is the number of paths of length 2 in $\mathbf{Z}^{nd}$ starting at the origin. □

Remarks.
1) It follows that $I = I_2 \geq -\log\sqrt{4d}$ which is better than $I \geq -\log(d)$ given by the bound from the equilibrium measure. Because, an operator with the equilibrium measure on $[-2d, 2d]$ satisfies $\mathrm{tr}(L^2) = \int_{-2d}^{2d} \pi^{-1} x^2/\sqrt{4d^2 - x^2}\ dx = 2d^2$ and in our case always $\mathrm{tr}(L^2) = 2d$, the density of states $dk(L)$ is for $d > 1$ never the equilibrium measure on $[-2d, 2d]$. On the other hand, for $d = 1$, the density of states is always the equilibrium measure on $[-2, 2]$.
2) The Green function $g(E)$ of the spectrum $\sigma(L)$ is related to the electrostatic conductor potential $\lambda(E)$ by $g(z) = -u(z) + \log(\gamma)$, where γ is the capacity of the spectrum (see [27]). This shows that a random operator has the equilibrium measure on the spectrum as the density of states if and only if the function $\lambda(E)$ is constant on the spectrum (it takes then the constant value $\log(\gamma) = -I(L)$ on the spectrum).
3) The tails of the density of states can be estimated using knowledge of the moments $\mathrm{tr}(L^n)$. Let Y be a random variable with law $P = dk$. Chebychev-Markov's inequality gives for $c > 0$

$$2\int_{-\infty}^{-c} dk = P[|Y| \geq c] \leq E[|Y|^n]/c^n = \mathrm{tr}(|L|^n)/c^n \ .$$

7 Gauge invariant minimizers

The compact topological group $G = X = U(N)^{\mathbf{Z}^d}$ acts on X by gauge transformations defined by

$$x \mapsto (g(T_1)x_1g^{-1}, g(T_2)x_2g^{-1}, \dots, g(T_d)x_dg^{-1}) \ ,$$

where the transformations T_i are the shifts on X. If a measure μ is shift invariant, then, for $g \in G$, the measure $g^*\mu$ needs no more to be shift invariant in general. We still assign to $g^*\mu$ the same value $I(\mu)$ because gauge transformation induces a unitary conjugation $L_x \mapsto gL_xg^{-1}$ for each operator L_x on $\mathcal{H}$.

Proposition 7.1 *There exists a minimum of I, which is both gauge and shift invariant. The same holds for the functionals $I_n, \overline{I}$.*

Proof. The existence is shown in two steps. We first construct a gauge invariant measure μ_0 and construct from it later a measure which is also shift invariant.
The Haar measure ν on G defines a Borel measure $\rho = \phi^*\nu$ on the compact set $G^*\mu \subset M(X)$

by push-forward of the measurable map $\phi : g \mapsto g^*\mu$. The functional I takes the same constant value on the compact convex hull of $G^*\mu$. By Choquet theory (see [20]), there exists a unique measure $\mu_0 = \int_{G^*\mu} \mu' \; d\rho(\mu')$ on the convex hull of $G^*\mu$. This measure is gauge invariant. We find therefore for each measure μ a natural gauge invariant measure μ_0 such that $I(\mu) = I(\mu_0)$. If μ is a minimizer, then also the gauge invariant measure μ_0 is a minimizer.
Given a gauge invariant measure μ_0, then also $T_i^*\mu_0$ is gauge invariant and $I_n(\mu_0) = I_n(T_i^*\mu_0)$. With the van Hove sequence $\Lambda_0^n = [-n,n]^d \subset \mathbf{Z}^d$, a shift and gauge invariant measure can be constructed by taking an accumulation point of

$$|\Lambda_0^n|^{-1} \sum_{k \in \Lambda_0^n} (T^k)^* \mu_0 \; .$$

Since I_n is lower semicontinuous, the value of I_n on any of these accumulation points is equal to the minimal value $I(\mu_0)$. □

Proposition 7.2 *There exists a convex compact metric space Y of measures on X which represents the gauge equivalence classes of measures. Different points in Y represent different equivalence classes.*

Proof. Since the map $\psi : \mu \mapsto \mu_0$ described in the last proof is continuous, the image Y of ψ is compact and represents all gauge and shift invariant measures. Any convex combination of gauge invariant measures is again gauge invariant. If μ is gauge invariant, then $\mu_0 = \mu$ so that Y consists of all gauge invariant measures. □

Remarks.
1) It is not excluded that different gauge equivalence classes are represented by the same point in Y, but then, I, I_n and $\overline{I}$ take the same value on all of these equivalence classes.
2) The gauge invariant measure $\psi(\mu) \in Y$ belonging to μ might not be ergodic. We do not know whether a gauge invariant ergodic minimum exists.
3) The Haar measure μ_{Haar} on X is an ergodic shift and gauge invariant measure on X. It is a candidate for the minimum.
4) For $d = 2$, the energy of the equilibrium measure on $[2d, 2d]$ is $-\log(2) = -0.69315$. We got a numerical value $I_2(L_{free,d=2}) = -0.40533$.
5) The computation of I_n for the free operator $L_{free,d}$ can be done as follows. The density of states of $L_{free,d}^{(n)}$ is $dk_{free}^{(dn)}$, where $dk(free)$ has the arcsin distribution on $[-2, 2]$. Therefore, $dk_{free}^{(dn)}$ is the law of $\sum_{i=1}^{nd} Y_i$ of independent random variables Y_i with distribution function $F(t) = 1/2 + 1/\pi \arcsin(t/2)$. In other words, we can write the functional I_n for the free operator as the expectation

$$I_n(L_{free,d}^{(n)}) = -\log|\mathrm{det} L_{free,d}^{(n)}| = -\mathrm{E}[\log|\sum_{i=1}^{nd} Y_i|] \; .$$

For numerical purposes, it is convenient to take independent random variables Z_i with uniform distribution on $[0, 1]$ and to compute

$$J_{nd} := I_n(L_{free,d}) = -\mathrm{E}[\log(|\sum_{i=1}^{nd} 2\sin(\pi(Z_i - \frac{1}{2}))|)] \; .$$

This can be computed by a Monte Carlo method. We know that $J_{nd} = I_n(L_{free,d}) = 0$ for $nd = 1, 2$. Our numerical experiments gave $J_3 \sim -0.32, J_4 \sim -0.42, J_5 \sim -0.54, J_6 \sim -0.65$

but these data are not reliable. It would be good to have some negative upper bounds.
6) The functionals $\mathrm{tr}(L^k)$ take the maximal value for $L = L_{free,d}$ for every k. It is not sure, whether the functionals I_n have a maximum. If yes, then $L_{free,d}$ is a candidate.

8 Operators with a given curvature

Given a random operator L defined by a measure $\mu \in M(X)$. The *field* of L is defined as

$$dx_{ij}(n) = F_{ij}(n) = x_j(n)^* x_i(n+e_j)^* x_j(n+e_i) x_i(n),\ i < j$$

which is an element in $X_2 := U_2^{\mathbf{Z}^d}$, where $U_2 = \oplus_{i<j} U_{ij}$, with $U_{ij} = U(N)$. The continuous map $x \in X \mapsto dx \in X_2$ commutes with the shifts and induces so a measure $\mu_2 := d\mu_1$ on X_2 by push-forward. A cohomological question is, for which μ_2 does there exist a measure μ_1 so that $d\mu_1 = \mu_2$. It is a result of Feldman-Moore and Lind and a generalisation of it [4, 15, 8] that in the abelian case, as for $N = 1$, any field configuration (2-form) $F_{ij}(n)$ satisfying $dF_{ijk}(n) = F_{ij}(n+e_k)F_{ij}^* F_{jk}(n+e_i)F_{jk}^* F_{ki}(n+e_j)F_{ki}^* = 1$ is obtained from a gauge potential (1-form) A, if the underlying process is free. This stays true also in the nonabelian case as long as $d = 2$.

A $\mathbf{Z}^d$-dynamical system (Ω, S, m) is called *free*, if all the sets $V_n = \{\omega \in \Omega \mid S^n\omega = \omega\}$, for $n \in \mathbf{Z}^d \setminus \{0\}$ have measure zero. We say, a random operator $\omega \mapsto (L_\omega u)(n)$ is *realized* over the $\mathbf{Z}^d$-dynamical system (Ω, S, m), if there exist measurable functions $A_i : \Omega \to U(N)$ such that $(L_\omega u)(n) = \sum_{i=1}^d A_i(T^n\omega)u(n+e_i) + A_i^*(T^{n-e_i}\omega)u(n-e_i)$ for all $\omega \in \Omega$ and $u \in \mathcal{H}$.

Lemma 8.1 *Given a random operator L realized over a $\mathbf{Z}^d$ dynamical system (Ω, S, m).*
a) L can be realized as a random operator L over $(X = U^{\mathbf{Z}^d}, T, \mu)$, with $\mu \in M(X)$.
b) L can be realized over a free dynamical system.

Proof. a) By definition, L is defined by the measureable functions $A_i : \Omega \to U(N)$. For $\omega \in \Omega$, define $\phi(\omega) = x = (A_1(S^n\omega), \dots, A_d(S^n\omega)) \in X$. Let T_i, $i = 1, \dots, d$ be the shifts on X and $\mu = \phi^* m$ the push-forward measure under the measurable map $\phi : \Omega \to X$. The dynamical system (X, T, μ) is a factor of (X, S, m). The functions $x_i(x) = x_i(0)$ give the same configurations over (X, T, μ) as the functions A_i over (Ω, S, m), because $x_i(T^n x) = A_i(S^n\omega)$.
b) By changing the dynamical system (Ω, S, m) on each of the sets $V_n = \{\omega \mid S^n(\omega) = \omega\}$ by replacing S_i on V_n by $\tilde{S}_i = S_i \circ S_i'$, where $S_i' : V_n \to V_n$ is a mixing $\mathbf{Z}^d$ system, we can make it mixing on each of the sets V_n. The dynamical system $(\Omega, \tilde{S}, m)$ is then free. With the same functions A_i, the field configurations $n \mapsto A_i(n) = A_i(\tilde{S}^n) = A_i(S^n)$ are then the same. $\square$

Let U_k be the $\left(\begin{array}{c} d \\ k \end{array}\right)$-fold direct product of $U(N)$. Let X_k be the compact space $X_k = U_k^{\mathbf{Z}^d}$ consisting of configurations $x_{(i_1,\dots,i_k)}(n)$ with $i_1 < \dots < i_k$ and $n \in \mathbf{Z}^d$. Every X_k is naturally a compact topological group. Any $U(N)$-valued k-form over a $\mathbf{Z}^d$ dynamical system is determined by a shift invariant measure on X_k. Therefore, instead of speaking of k-forms as in [8], we can deal with measures μ on X_k and in the abelian case define a coboundary operator $d_k : M(X_k) \to M(X_{k+1})$ which satisfies $d_{k+1} \circ d_k = 0$ for $k \geq 1$.

Remark. The change of view by replacing measurable functions over a $\mathbf{Z}^d$ dynamical system by measures has in probability theory the analogue that random variables are determined by their laws. Also in ergodic theory, the argument has a prototype: every measure theoretical dynamical system (Ω, S, m), where Ω is a compact topological space, is isomorphic to $(\Omega^{\mathbf{Z}}, T, \mu)$, where T is the shift and μ is the push-forward measure of the measurable map $\phi : \Omega \to X, \phi(\omega)_n = T^n(\omega)$.

Corollary 8.2 *a) Assume ν is a shift-invariant measure on $X_p, p \geq 2$ which has its support on an abelian subgroup of X_p. If $d\nu = 0$, there exists a shift invariant measure μ on X_{p-1}, so that $d\mu = \nu$.*
b) In general, but with the restriction $d = 2$, any measure ν on X_2 defines a measure μ on X_1 for which $d\mu = \nu$.

Proof. This is a reformulation of the Feldmann-Moore-Lind result [4, 15] and its generalisation to higher dimensions [8]. In these results, there is the assumption that the dynamical system (X, T, μ) is free. By Lemma (8.1b), the random operator L can always be realized over a dynamical system (X, S, μ) which is free. There, we know the existence of a p-form B satisfying $dB = A$. The form A defines then by Lemma (8.1a) a measure on X_p leading to an operator with field F. $\square$

Remarks.
1) As an example, we consider in any dimension the constant field $F_{ij}(n) = c \in U(N)$. Since, $dF = 0$, there exists then a measure μ on X, such that L_x has the constant curvature F. The reason is that in a basis, in which c is diagonal, c is in an abelian group, which allows to apply case a) in the Corollary (8.2).
2) $L^{(n)}$ is again a gauge operator over a $\mathbf{Z}^{nd}$ system. It has the curvature $F^{(n)}$. Denote by $T_{i,j}$ the translation in direction i of particle j. For a fixed particle j, the field $F^{(n)}_{(i,j),(k,j)}$ is just the field F_{ik}. Let $A_{i,j} = x_i(0)$ the gauge potential of the one particle operators. These functions are invariant under space translations $T_{i,l}$ of a different particle $l \neq j$. For different particles $j \neq l$, we have therefore $F^{(n)}_{(i,j),(k,l)} = A^{-1}_{k,l} A^{-1}_{i,j} A_{k,l} A_{i,j}$ which is vanishing in the abelian case (on the level of pure gauge fields, only particles with a nonabelian field do interact).
3) Whenever ν determines a measure μ with $d\mu = \nu$, the density of states dk depends only on ν (see [8]). It follows that for $d = 2$ or in the abelian case, the simplex $M(X_2)$ represents the equivalence classes of gauge equivalent measures. Different points in $M(X_2)$ define different equivalence classes. In the nonabelian case with $d > 2$, we have to refer to Proposition (7.2).

We consider now the expectation value $\mathrm{tr}(L^4)$ of the Wilson Hamiltonian $H(x) = (L_x^4)_{00}$ and show that it is minimized by a gauge field which has constant curvature $F_{ij} = -1$. This minimization holds in the abelian case among all fields F with $dF = 0$ and in dimension 2 among all fields F. Unless in the flux-phase problem which deals with $\mathrm{tr}|L|$ (and which is settled in finite dimensional, two dimensional situations), the calculations for $\mathrm{tr}(L^4)$ is easy:

Proposition 8.3 *The expectation value $\mathrm{tr}(L^4)$ of the Wilson Hamiltonian takes values in the interval $[4d^2 + 2d, 12d^2 - 6d)]$. The minimum is achieved for $F = -1$ and the maximum for $F = 1$.*

Proof. $\mathrm{tr}(L^4)$ takes values in the interval $[A - 2B, A]$, where $A = 4d^2 + 2d(2d-1) + 4d(d-1)$ is the number of closed paths of length 4 starting at $0 \in \mathbf{Z}^d$ and $B = 4d(d-1)$ is the number of plaquettes at 0, where plaquettes with different orientation are considered as different. $F = -1$ gives $A - 2B$ and $F = 1$ gives A and no smaller or bigger values can occur. $\square$

The problem to find the extrema of $\mathrm{tr}(L^6)$ is more difficult because loops over two plaquettes and so correlations occur. In two dimensions, where it is possible to minimize over all fields F instead to minimize over the gauge potential, we have to find a minimum on $M(X_2) = M(U^{\mathbf{Z}^2})$ of the functional

$$\mu \mapsto A + B \int \mathrm{Trace}(x(0))\ d\mu(x) + C/2 \int \mathrm{Trace}(x(0)x(e_1) + x(0)x(e_2))\ d\mu(x)\ ,$$

where A is the number of paths of length 6 starting at $n = 0$, giving zero winding number to all plaquettes, $B = 128$ is the number of such paths winding around one plaquette and $C = 16$ is the number of such paths winding around two plaquettes. In the abelian case $U = U(1)$ and if we consider the much simpler one dimensional problem of a constant field $F = e^{i\alpha}$, we have to minimize

$$A + B\cos(\alpha) + C\cos(2\alpha)\ .$$

Since $B > 4C$, the minimum is $\alpha = \pi$ which corresponds to a constant field $F = -1$.

9 A random walk expansion

The functionals $I, I_n, \overline{I}$ can not be expected to be continuous in μ. (The question seems already to be unsettled in the case $d = 2$, $N = 1$ and constant curvature $e^{2\pi i\alpha}$, with irrational α, where dk is the density of states of the almost Mathieu operator and where the unknown values I_1, I_2 are believed to satisfy $I_1 = I_2 = 0$.) It is therefore convenient to consider for every $\beta \in \mathbf{C}$ the functional

$$\mu \mapsto I_{n,\beta} = -\int \log|E+\beta|\ dk_\mu^{(n)}(E) = -\log|\mathrm{det}_\mu(L^{(n)}+\beta)|$$

which has the property that $I_{n,\beta} + \log(\beta)$ is real analytic in β and continuous in μ for $|\beta| > 2dn$. It is convenient to consider also $I_{n,z}(\mu) = -\log|\mathrm{det}_\mu(1 - zL)|$ which is real analytic in a neighborhood of 0. Note that by the same arguments given below, the functionals $I_{n,\beta}$ rsp. $I_{n,z}$ have all minimizers. By an analytic continuation, the knowledge of the Taylor coefficients of the superharmonic function $\beta \mapsto I_{n,\beta} + \log(\beta)$ in a neighborhood of ∞ determines $I_n = I_{n,\beta=0}$. The Taylor coefficients can be computed in terms of the gauge fields. For sake of simplicity, we consider first the functional I.

Proposition 9.1 *For $|\beta| > 4d$, the Taylor expansion*

$$I_\beta(\mu) + \log(\beta) = \sum_{m=1}^{\infty} \left(\sum_{l=0}^{m} \binom{m}{l} \mathrm{tr}_\mu(L^l)\mathrm{tr}_\mu(L^{m-l}) \right) \frac{\beta^{-m}}{m}$$

holds. The functions $\mu \mapsto \mathrm{tr}_\mu(L^n) = \int E^n\ dk_\mu(E)$ are nonvanishing only for even n and satisfy

$$\mathrm{tr}_\mu(L^n) = \int_X \sum_{\gamma\in\Gamma_n} \mathrm{Trace}(\int_\gamma x)\ d\mu(x)\ ,$$

where Γ_n is the set of all closed paths in $\mathbf{Z}^d$ of length n, Trace is the normalized trace in $U(N)$ and $\int_\gamma x$ is the product $x_{\gamma(n)-\gamma(n-1)}(\gamma(n-1))x_{\gamma(n-1)-\gamma(n-2)}(\gamma(n-2))\dots x_{\gamma(2)-\gamma(1)}(\gamma(1))$ of the gauge field along the path γ (and where $x_{e_i} = x_i$).

Proof.

$$I_\beta(\mu) = -\int\int \log|1 - \frac{(E'+E)}{\beta}|\ dk_\mu(E)\ dk_\mu(E') - \log(\beta)\ .$$

An expansion of the logarithm gives

$$\begin{aligned} I_\beta(\mu) + \log(\beta) &= \sum_{m=1}^\infty \int\int (E+E')^m\ dk_\mu(E)\ dk_\mu(E') \frac{\beta^{-m}}{m} \\ &= \sum_{m=1}^\infty \int \mathrm{tr}_\mu((L+E')^m)\ dk_\mu(E') \frac{\beta^{-m}}{m} \\ &= \sum_{m=1}^\infty \sum_{k=0}^m \begin{pmatrix} m \\ l \end{pmatrix} \int \mathrm{tr}_\mu(L^k)(E')^{m-k}\ dk_\mu(E') \frac{\beta^{-m}}{m} \end{aligned}$$

which gives the expression in the proposition. The random walk computation of $\mathrm{tr}(L^n)$ follows from the computation of the trace of each summand in $L^n = (x_1 + x_1^* + x_2 + x_2^* \ldots + x_d + x_d^*)^n$. □

We see from the development that $\mu \mapsto I_\beta(\mu)$ behaves "quadratically" in μ. For the determinant of $L^{(n)}$, a similar result holds and $\mu \mapsto I_{\beta,n}(\mu)$ behaves as a "polynomial" of degree n:

Corollary 9.2 *For $|\beta| > 2nd$, the Taylor expansion*

$$-\log\det(L^{(n)} + \beta) + \log(\beta) = \sum_{m=1}^\infty \left(\sum_{l_1+\ldots+l_n=m} \mathrm{tr}_\mu(L^{(l_1)}) \cdots \mathrm{tr}_\mu(L^{(l_n)}) \right) \frac{\beta^{-m}}{m}$$

holds, where $\mathrm{tr}_\mu(L^{(0)}) = 1$.

Proof. The proof is easily adapted from the above proof, using

$$\int_{\mathbf{R}^n} (E_1 + E_2 \ldots + E_n)^m\ dk_\mu(E_1) \ldots dk_\mu(E_n) = \sum_{l_1+\ldots+l_n=m} \mathrm{tr}_\mu(L^{(l_1)}) \cdots \mathrm{tr}_\mu(L^{(l_n)})\ .$$

□

Remarks.
1) Since $\mathrm{tr}(L^2) = 2d$, the first interesting term is $\mathrm{tr}(L^4)$. For large $|\beta|$, an approximation of $I_\beta(\mu) = I_{2,\beta}(\mu)$ is given by

$$\log(\beta) + 2(2d)\beta^{-2} + (6(2d)^2 + 4\mathrm{tr}(L^4))\beta^{-4} + O(\beta^{-6})\ .$$

2) The knowledge of the function $\beta \mapsto I_\beta$ for fixed μ does clearly not determine μ in general, because of gauge freedom. However, the function $\beta \mapsto I_{2,\beta}$ determines the moments $\mathrm{tr}(L^n)$ of the density of states of $L = L_\mu$ and so the density of states dk_μ itself. The reason is that a sequence $a_k, k = 0, \ldots$ of real numbers is determined from the sequence $b_m = \sum_{k=0}^m \begin{pmatrix} m \\ k \end{pmatrix} a_k a_{m-k}, m = 0, 1 \ldots$ and the sign of a_0.
3) The function $\log\det(L - \beta)$ has not an an analytic continuation to the resolvent set of L and this is why the real part is usually considered. However, $\beta \mapsto \log\det(L - \beta)$ is a Herglotz function on $\mathrm{Im}(E) > 0$ and the imaginary part on the real axes is proportional to

the integrated density of states. The determinant $\beta \mapsto \det(L-\beta)$ is defined everywhere on $\mathbf{C}$ and analytic on the resolvent set of L.

In dimension $d=2$, and $U(1)$, there are some cases, where more can be said (see [8]):

1) If $L=L_{\mathrm{free}}$, then

$$\det(1-zL)=\sum_{k=1}^{\infty}\frac{z^k}{k}|\Gamma_k| \; ,$$

where Γ_k is the set of closed paths in $\mathbf{Z}^d$ of length k starting at $0\in\mathbf{Z}^d$ and where $|\Gamma_k|$ denotes its cardinality.
2) If L has curvature constant to -1, then

$$\det(1-zL)=\sum_{k=1}^{\infty}\frac{z^k}{k}\sum_{\gamma\in\Gamma_k}(-1)^{n(\gamma)} \; ,$$

where $n(\gamma)$ is the sum of the winding numbers $n(\gamma,P)$ over all plaquettes P in $\mathbf{Z}^2$, which is well defined because $n(\gamma,P)\neq 0$ only for finitely many n.
3) A generalisation is when the curvature is constant $e^{2\pi i\alpha}$. Then

$$\det(1-zL)=\sum_{k=1}^{\infty}\frac{z^k}{k}\sum_{\gamma\in\Gamma_k}\prod_P e^{2\pi i n(\gamma,P)} \; ,$$

where $n(\gamma,P)$ is the winding number of the path γ with respect to a point in the plaquette P.
If α is irrational, then the density of states of L is the same as the density of states of the one dimensional almost Mathieu operator $(K_\theta u)_n=u_{n+1}+u_{n-1}+2\cos(\theta+2\pi\alpha n)u_n$ on $l^2(\mathbf{Z},\mathbf{C})$ (see [25, 11, 6] for reviews). This is not true for rational α, where the spectrum of K_θ depends on $\theta\in\mathbf{T}^1$.
4) If μ is the Haar measure on $X_1=(U(1)\times U(1))^{\mathbf{Z}^2}$, then

$$\det(1-zL)=\sum_{k=1}^{\infty}\frac{z^k}{k}|\Gamma_k^0| \; ,$$

where Γ_k^0 is the subset of those $\gamma\in\Gamma_k$, which give zero winding numbers to all plaquettes.
5) If μ is the Haar measure on $(\mathbf{Z}_p\times\mathbf{Z}_p)^{\mathbf{Z}^2}$, then

$$\det(1-zL)=\sum_{k=1}^{\infty}\frac{z^k}{k}|\Gamma_k^p| \; ,$$

where Γ_k^p is the subset of $\gamma\in\Gamma_k$, which give modulo p a zero winding number to all plaquettes.

10 Relation with a dynamical zeta function

Consider the following general definition of a dynamical zeta function (see [23] p.1). Given a dynamical system (M,f), where M is a homeomorphism of the topological space M.

Assume the set of fixed points $\mathrm{Fix}(f^m)$ is finite for all $m \in \mathbf{N}$. If $A : M \to M(N, \mathbf{C})$ is a continuous matrix-valued function, define $\zeta(z)$ by

$$\log \zeta(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{m \in \mathrm{Fix}(f^k)} \mathrm{Trace}(\prod_{j=0}^{k-1} A(f^k m) \ .$$

Consider now the set B consisting of the $2d$ bonds connecting 0 with its neighbors in $\mathbf{Z}^d$. Define $M = B^{\mathbf{Z}} \times \mathbf{Z}^d$ and the map $f = (d, b) = (\sigma(d), b + d_0)$, where σ is the shift on $B^{\mathbf{Z}}$. Every $x \in X$ defines a map A_x on M by $A_x(d, b) = x_b(d)$, where x_{-l} is interpreted as x_l^*. The map f^k has finitely many fixed points on the set $\{(d, b) \mid b = 0\}$ and these k- periodic points of f correspond bijectively to closed paths in $\mathbf{Z}^d$ of length k starting at $n = 0$. Averaging

$$\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{m \in \mathrm{Fix}(f^k) \cap \{b=0\}} \mathrm{Trace}(\prod_{j=0}^{k-1} A_x(f^k m))$$

over X with respect to the measure μ gives $\log \det(1 - zL)$.

11 A relation with the entropy functional

Given a compact metric space X, d commuting homeomorphisms T_i on X and d continuous maps $A_i : X \to U$. For every shift invariant measure, we have the random operator $L_x = A_x + A_x^* + V_x$ with $A_x u(n) = \sum_i U_i(T^n x) u(n + e_i)$ and $V_x u(n) = V(T^n x) u(n)$. Again, the potential theoretical energy of the density of states is a lower semicontinuous functional on the compact set of T_i-invariant measures and the minimum is attained.

Consider now the case $d = 1$ and a smooth symplectic twist map $T : \mathbf{T}^{2n} \to \mathbf{T}^{2n}$ defined by

$$T : (q, p) \mapsto (2q - p - f(q), q)$$

with smooth $f : \mathbf{T}^n \to \mathbf{R}^n$. In the case $n = 1$, $f(x) = \gamma \sin(x)$, this is the Standard map. For each T-invariant measure μ, we have an ergodic Schrödinger operator $L = L(\mu)$ for which we can form the finite determinant $\det L$. The calculation of this determinant is in general difficult and it is in general not known, whether $|\det_\mu L| > 1$ or $|\det_\mu L| = 1$ for the Lebesgue measure μ on $\mathbf{T}^{2n}$ because of the following consequence of the Thouless formula.

Lemma 11.1 *$n \cdot \log |\det_\mu L|$ is the metric entropy of the invariant measure μ of twist map T if μ is absolutely continuous with respect to Lebesgue measure.*

Proof. The Thouless formula is $n \cdot \mathrm{tr}(\log |L|) = \sum_{i=1}^n \lambda_i(0)$, where $\lambda_i(0)$ are the n largest Lyapunov exponents of the transfer cocycle A_E of L for $E = 0$. But A is in the same time also the Jacobean DT of the twist map T. Pesin's formula (see [16]) states that the Lyapunov exponent is the metric entropy of the invariant measure μ, if μ is absolutely continuous with respect to the Lebesgue measure. □

Maximizing the functional $|\det L|$ for Schrödinger operators arrizing in a monotone twist (see [9] for spectral results) is related to the metric entropy problem.

Corollary 11.2 *If the maximum μ of $\mu \mapsto |\det_\mu(L)|$ is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}^2$, then $\log |\det_\mu(L)|$ is the topological entropy of the twist map.*

Proof. If μ is absolutely continuous, then $\log|\det_\mu L|$ is a metric entropy $h_\mu(T)$ and so by the variational principle $h_{top}(T) \geq \log|\det_\mu L|$. Ruelle's inequality shows that $h_\mu(T) \leq \log|\det_\mu L|$ for all invariant measures. The maximal value of $\mu \mapsto \log|\det_\mu L|$ is therefore also bigger or equal than the topological entropy $h_{top}(T)$. □

12 Questions

We would like to know explicit minima of the potential theoretical energy of the density of states. Is there a unique gauge invariant minimum? If yes, it would be ergodic. A natural guess for a unique gauge invariant ergodic minimum is the Haar measure μ on X. If U is abelian and $d = 2$, we can in this case compute the moments of the density of states but we could not determine $|\det_\mu L|$. We also do not know if we can get the maximal value 0 for the energy in higher dimensions nor do we know the minima of the functionals $\mathrm{tr}_\mu(L^{2n})$ for $n > 2$.

Is always $I_{n+1}(L) \leq I_n(L)$? We know only that it is true for large n, since we can decide, how the sequence $I_n(L)$ converges to $-\infty$. The monotonicity would have consequences like that the invertibility of L, which implies $I_1(L) < 0$, would give $I_2(L) < 0$ and so imply that the capacity of the spectrum is bigger than one.

In all the examples of operators attached to gauge fields we know, L is not invertible: examples are the Hofstadter case $d = 2$ with constant rational field $e^{2\pi i p/q}$, the Haar measure case, the free case. Does there exist $\mu \in M(X)$ such that L_μ is invertible?

In the finite dimensional case, there exists a natural Haar measure ρ on all operators L and so a natural partition function $\int e^{-\beta I(L)} \, d\rho(L) = \int |\det_\mu L^{(2)}| \, d\rho(\mu)$. The largest contribution to this partition function is expected to come from the minimizers μ, especially if the inverse temperature β is large. The determination of a natural measure ρ on Y or X_2 and the computation of the partition function is the quantum version of the variational problem to minimize the energy. It can be shown with the Ryll-Nardzewski fixed point theorem that on any compact metric space Y, there exist measures ρ which are invariant under all isometries. Such a measure ρ can be chosen in such a way that it gives positive measure to every open set. Moreover, with such a measure ρ on $M(X_2)$, the knowledge of the function $\beta \mapsto \int e^{-\beta I(\mu)} \, d\rho(\mu)$ determines the minimal value of I. Uniqueness of ρ can only be expected if the group of isometries is transitive on Y. Whatever measure ρ is taken, we expect from the asymptotic behavior of I_n that the partition function satisfies asymptotically $\int e^{-I_n(\mu)} \, d\rho(\mu) \sim e^{-\gamma/2}/\sqrt{n/2}$.

Appendix A: The Thouless formula for Schrödinger operators on the strip with off diagonal elements

Given a measure preserving invertible transformation T on a probability space (X, μ). Given $a, b \in L^\infty(X, M(N, \mathbf{R}))$ so that $a^{-1} \in L^\infty(X, M(N, \mathbf{R}))$. Consider the random operator $x \mapsto L_x$ on $\mathcal{H} = l^2(\mathbf{Z}, \mathbf{C}^N)$ given by

$$(L_x u)_n = a_n u_{n+1} + a^*_{n-1} u_{n-1} + b_n u_n \ ,$$

where $a_n = a(T^n x), b_n = b(T^n x)$. Denote by dk the density of states of L defined by the functional $f \mapsto \mathrm{tr}(f(L)) = \int \mathrm{Trace}(f(L_x))_{00} \, d\mu(x)$, where Trace is the normalized trace on $M(N, \mathbf{R})$ satisfying Trace(1) = 1. Define $\lambda(E) = \int \log |E - E'| \, dk(E')$. Denote by $\lambda_i(E), i = 1, \dots, N$ the nonnegative Lyapunov exponents of the $2N \times 2N$ transfer cocycle

$$A_E(x) = \begin{pmatrix} E - b(x) & -a(T^{-1}x)^2 \\ 1 & 0 \end{pmatrix} a(T^{-1}x) ,$$

which has the property that

$$A_{n,E} = \begin{pmatrix} E - b_n & -a_{n-1}^2 \\ 1 & 0 \end{pmatrix} a_{n-1}^{-1}$$

satisfies $A_{n,E} v_n = v_{n+1}$ if $Lu = Eu$ and $v_n = (a_n u_{n+1}, u_n)$. Denote by $\lambda_i(a)$ the N Lyapunov exponents of the cocycle $x \mapsto a(x)$.

Proposition 12.1 *The Thouless formula for random Jacobi matrices on the strip is*

$$\lambda(E) = N^{-1} \sum_{i=1}^{N} (\lambda_i(A_E) + \lambda_i(a)) ,$$

where $\lambda_i(A_E)$ are the N largest Lyapunov exponents of the $SL(2N, \mathbf{C})$ transfer cocycle A_E and $\lambda_i(a)$ are the N Lyapunov exponents of the $GL(N, \mathbf{C})$ cocycle a.

Proof. The proof is a straightforward generalisation from the proofs [3, 1, 10] in the known situations (see examples 1),2) below):
It suffices to prove the formula for $\mathrm{Im}(E) \neq 0$, where $\lambda(E)$ is analytic. The usual subharmonicity argument assures then that the formula is true everywhere on $\mathbf{C}$. By ergodic decomposition it suffices also to assume that T is ergodic. Define $a^n(x) = a_{n-1}(x) \cdots a_2(x) a_1(x)$. We have

$$A_E^n(x) = A_{n,E}(x) \cdots A_{1,E}(x) = \begin{pmatrix} P_E^n(x) & Q_E^{n-1}(x) \\ a(T^{n-1}) P_E^{n-1}(x) & a(T^{n-1}) Q_E^{n-2}(x) \end{pmatrix} ,$$

where $E \mapsto P_E^n(x), Q_E^n(x)$ are matrix coefficient polynomials of degree n of the form

$$P_E^n(x) = a^n(x) \prod_{k=1}^{n} (E - E_j^{(n)}(x)), \; Q_E^n(x) = a^n(x) \prod_{k=1}^{n} (E - F_j^{(n)}(x))$$

Then, both $(Nn)^{-1} \log |\mathrm{Det} P_E^n(x)|$ and $(Nn)^{-1} \log |\mathrm{Det} Q_E^n(x)|$, (here Det is the finite dimensional determinant), converge almost everywhere to the average of the largest N Lyapunov exponents of A_E. Also $(Nn)^{-1} \log |\det(a^n)(x)|$ converges to the average of the N Lyapunov exponents of a. (Oseledec's theorem assures that $(a^n(x)(a^n)^*(x))^{1/2n}$ converges for almost all x to a constant matrix with the N eigenvalues $\exp(\lambda_i(a))$.) Because $E_j^{(n)}(x)$ (rsp. $F_j^{(n)}(x)$) are eigenvalues of L_x with Dirichlet boundary conditions $u_0 = u_n = 0$, (rsp. $u_1, u_{n+1} = 0$), we know by the Avron-Simon lemma, that both $(Nn)^{-1} \log \prod_{k=1}^n (E - E_j^{(n)})$ and $(Nn)^{-1} \log \prod_{k=1}^n (E - E_j^{(n)})$ converges to $\int \log |E - E'| \, dk(E')$. $\square$

Examples.
1) If $N = 1$, Birkhoff's ergodic theorem gives $\lambda_1(a) = \int \log |a| \, d\mu$ and gives the known special case $\lambda(E) = \lambda(A_E) + \int \log |a| \, d\mu$.
2) If $a = 1$, then $\lambda_i(a) = 0$ and we have the known special case $\lambda(E) = N^{-1} \sum_{i=1}^N \lambda_i(A_E)$.

(The formulas in [10, 1] differ by a factor N which is due to a different Trace on finite dimensional matrices.
3) If a, b take values in diagonal matrices and a_i are the diagonal functions, then L is a direct sum of N ordinary Jacobi operators L_i with density of states dk_i and dk is the average of the dk_i and $\lambda(E)$ is therefore the average of the functions $\lambda_i(E) = \int \log|E-E'|\; dk_i(E')$. The N nonnegative Lyapunov exponents of A_E are the Lyapunov exponents of the transfer matrices $A_{E,i}$ of the operators L_i and the N Lyapunov exponents of a are given by $\lambda_i(a) = \int \log|a_i|\; d\mu$ so that $N^{-1}\lambda_i(E) = N^{-1}\sum_{i=1}^N(\lambda_i(A_E) + \lambda_i(a))$ follows from the one dimensional case.

Appendix B: A result of Parthasarthy

Parthasarthy [19] proved that if T is the shift on the product space $(X = M^{\mathbf{Z}}, \mathcal{B}^{\mathbf{Z}})$, where M is a compact metric space equiped with the Borel σ-algebra $\mathcal{B}$, then the set P of periodic shift invariant measures is dense in the set E of all ergodic measures which is itself dense in the set M of all shift invariant probability measures. Parthasarthy's original proof in the special case $d = 1$ generalizes directly to higher dimensions:

Lemma 12.2 (Parthasarthy) *P is dense in E which is dense in M.*

Proof. (i) We first show that E is dense in M. Partition $\mathbf{Z}^d$ into cubes $\Lambda_r^n = \prod_{i=1}^d [r_i - n, r_i + n]$ with $r \in \mathbf{Z}^d$. Let $\mathcal{A}_r^n$ by the sub σ-algebra of $\mathcal{A}$ generated by functions depending only on coordinates in Λ_r^n. Given $\mu \in M$, let μ_r^n be the restriction of μ to $\mathcal{A}_r^n$. Define $\nu_n = \prod_{r \in \mathbf{Z}^d} \mu_r^n$ which is defined on $\mathcal{A}$ and invariant and ergodic under the shifts T_i^{2n+1}. Define $\mu_n = (|\Lambda_0^n|)^{-1}\sum_{k\in\Lambda_0^n} \nu_n(T^k)$ which is invariant and ergodic for T. Since μ_n agrees with ν_n on $\mathcal{A}_0^n$ and if $A \in \mathcal{A}_0^k$ then $\nu_n(T^r A) = \mu(A)$ for $A \in \mathcal{A}_0^n$, we get for fixed k

$$|\mu_n(A) - \mu(A)| \le \frac{1}{|\Lambda_0^n|}\sum_{j\in\Lambda_0^n} |\nu_n(T^j A) - \mu(A)| \le \frac{2d|\Lambda_0^k|}{|\Lambda_0^n|} \to 0$$

so that $\mu_n(A) \to \mu(A)$ for $A \in \mathcal{A}_0^k$. Since this is true for every k, one has $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{A}$.
(ii) Given $\mu \in E$, there exists $x \in X$ such that $\mu_n = (1/|\Lambda_0^n|)\sum_{k\in\Lambda_0^n}\delta(T^k x)$ converges to μ. Let y be a point in X which is $2n+1$ periodic for each T_i and which coincides with x on Λ_0^n. Define $\nu_n = (1/|\Lambda_0^n|)\sum_{k\in\Lambda_0^n}\delta(T^k y)$. which is in P. The same argument as in (i) shows that $\mu_n(A) \to \nu_n(A)$ for $A \in \mathcal{A}_0^k$ and so $\nu_n \to \mu$. □

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