A POSSIBLE MECHANISM FOR THE KAM TORI BREAKDOWN

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Abstract. *Inspired by the quantum field theory, a resummation is performed in the Lindstedt series defining the KAM tori, and a possible mechanism for universality of the tori breakdown is discussed. This work is mostly based on a joint paper with G. Gallavotti.*

1. Introduction. The parametric equations of KAM tori for a ℓ degrees of freedom quasi integrable system can be shown to be one point–Schwinger functions of a suitable euclidean quantum field theory (QFT), [4], [5]. This shows that it is very natural to use the powerful techniques developed in QFT (see for instance [1]) to study the convergence of the perturbative series for the KAM tori (*Lindstedt series*), and in fact this has been done in recent proofs of the KAM theorem, (indeed in the first of such proofs, due to Eliasson, [2], this is only implicit but it appears more and more clear in following simplified proofs, [4], [6], [7], [8]). In QFT it is often necessary to perform a resummation in the perturbative expansion for the Schwinger functions, so that the original series in the perturbative parameter ε is replaced by a different power series in suitable functions of ε, called *form factors*: inspired by the analogy with QFT, we perform a resummation of the Lindstedt series, (see §3), and obtain a new expression for the sum of the series, which we call the *resummed series*. By the non resummed series, one estimates the tori analyticity domain D around $\varepsilon = 0$, with boundary $\varepsilon = \varepsilon_D(\theta)$, $\theta \in [0, 2\pi)$, with a disk of radius R not larger than the largest disk contained in D, *i.e.* $R \leq \min_\theta |\varepsilon_D(\theta)|$. Hence, if for some system $\varepsilon(0), \varepsilon(\pi) \gg \min_\theta |\varepsilon_D(\theta)|$, then, although the convergence radius of the non resummed series can be well estimated by R, the invariant tori exist for real ε bigger than R, *i.e.* R is a bad estimate of the value ε_c of ε at which there is the tori breakdown. We will see that the resummed series, at least in principle, could allow us to estimate the domain D with a domain less simple than a disk and this might allow to obtain better estimates of the value of ε_c. Finally the resummation leads to a conjecture about the mechanism of the phenomenon of the tori breakdown, (see §4).

2. Formalism. We consider for simplicity ℓ rotators with inertia momenta J, angular momenta $\vec{A} = (A_1, \ldots, A_\ell) \in \mathbf{R}^\ell$, and angular positions $\vec{a} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbf{T}^\ell$, described by the Hamiltonian $\mathcal{H} = \frac{1}{2J}\vec{A}\cdot\vec{A} + \varepsilon f(\vec{\alpha})$, where $\vec{A} \in \mathbf{R}^\ell$, $\vec{\alpha} \in \mathbf{T}^\ell$ and $f(\vec{\alpha})$ is analytic and even in its argument. Similar considerations holds for much more general hamiltonians. Let $\vec{\omega}_0 = J^{-1}\vec{A}_0$ be a rotation vector verifying for $C_0, \tau > 0$ the *diophantine property* $C_0|\vec{\omega}_0 \cdot \vec{\nu}| > |\vec{\nu}|^{-\tau}$, $\vec{0} \neq \vec{\nu} \in \mathbf{Z}^\ell$. The KAM theorem states the existence of a one parameter family $\varepsilon \to T_\varepsilon$ of tori with parametric equations $\vec{A} = \vec{A}_0 + \vec{H}(\vec{\psi})$, $\vec{a} = \vec{\psi} + \vec{h}(\vec{\psi})$, $\vec{\psi} \in \mathbf{T}^\ell$, where $\vec{H}(\vec{\psi})$ and $\vec{h}(\vec{\psi})$ are analytic functions of ε, ψ_j, $j = 1, \ldots, \ell$, divisible by ε, defined for $|\varepsilon|$, $|\mathrm{Im}\psi_j|$ small enough. Writing $\vec{H}(\vec{\psi}) = \sum_{k=1}^\infty \sum_{\vec{\nu}} e^{i\vec{\nu}\cdot\vec{\psi}} \varepsilon^k \vec{H}^{(k)}_{\vec{\nu}}$, $\vec{h}(\vec{\psi}) = \sum_{k=1}^\infty \sum_{\vec{\nu}} e^{i\vec{\nu}\cdot\vec{\psi}} \varepsilon^k \vec{h}^{(k)}_{\vec{\nu}}$, it is easy to define a graphical representation for $\vec{h}^{(k)}_{\vec{\nu}}$ and $\vec{H}^{(k)}_{\vec{\nu}}$ in the following way. Consider k oriented lines, labeled from 1 to k: the final extreme v' of a line is called the root and the other extreme v the vertex, and the line is denoted $v' \leftarrow v$. The lines are arranged on a plane by attaching in all possible ways the vertices of some lines to the roots of others, to form a connected tree: each tree, following a therminology mutuated from QFT, can be called a *Feynman graph.* In this way only one root r remains unmatched and it will be called the *root* of the graph, whose lines will be called *branches* and whose vertices other than the root will be called *nodes*. Between the nodes there is an ordering relation $\leq$; we write $w < v$ is v is closer to the root. Each node v is given a *mode* label $\vec{\nu}_v$ which is one of the Fourier modes $\vec{\nu}$'s such that $f_{\vec{\nu}} \neq 0$. We define the *momentum* flowing through the branch going from v to v' as $\vec{\nu}(v) = \sum_{w \leq v} \vec{\nu}_w$: the parity properties of the model impliy $\vec{\nu}(v) \neq \vec{0}$, for any v. To each tree we associate a *value* obtained by assigning to a branch $v' \leftarrow v$, the *line factor* $\frac{-i\vec{\nu}_{v'}\cdot iJ^{-1}\vec{\nu}_v}{(i\vec{\omega}_0\cdot\vec{\nu}(v))^2}$ for all the branches distinct from the one containing the root; to the root branch we associate, instead, the quantity $\frac{-iJ^{-1}\vec{\nu}_v}{(i\vec{\omega}_0\cdot\vec{\nu}(v))^2}$. We multiply all the above operators and multiply the resulting factor times the function $\prod_v f_{\vec{\nu}_v}$. This defines the *Feynman rules*: the $\vec{h}^{(k)}_{\vec{\nu}}$ is given by $k!^{-1}$ times the sum of all the values of all the k branches trees with momentum $\vec{\nu}$ flowing through the branch leading to the root, and $\vec{H}^{(k)}_{\vec{\nu}} = (i\vec{\omega}_0 \cdot \vec{\nu}(v))\vec{h}^{(k)}_{\vec{\nu}}$. The scaling properties of the factor $\frac{1}{(i\vec{\omega}\cdot\vec{\nu}(v))^2}$ (*propagator*) suggests decomposing it into components relative to various *scales*. We can write

$$\frac{1}{(i\vec\omega_0\cdot\vec\nu(v))^2}=\frac{\chi_1(\vec\omega_0\cdot\vec\nu(v))}{(i\vec\omega_0\cdot\vec\nu(v))^2}+\sum_{n=-\infty}^{0}\frac{\chi(2^{-n}\vec\omega_0\cdot\vec\nu(v))}{(i\vec\omega_0\cdot\vec\nu(v))^2}\ ,$$

where $\chi_1(x)$ and $\chi(x)$ are the characteristic functions of the sets $|x|\ge 1$ and, respectively, $1/2\le|x|<1$; correspondingly we can break each Feynamn graph into a sum of many terms by developing the sums. This can be simply represented by assigning to each branch λ an extra label n_λ and multiplying the factor associated to such a line times $\chi(2^{-n_\lambda}\vec\omega_0\cdot\vec\nu)$: the value of $\vec H^{(k)}_{\vec\nu},\vec h^{(k)}_{\vec\nu}$ will be the sum over all possible new graphs which once deprived of the new scale labels would become "old" graphs contributing to $\vec H^{(k)}_{\vec\nu},\vec h^{(k)}_{\vec\nu}$ respectively. The branches of the new graphs are naturally collected into connected *clusters* "of fixed scale": a cluster of scale n consists in a maximal connected set of branches with scale label $\ge n$, containing at least one line of scale n. The lines which are not contained in a cluster, but have an extreme inside the clusters will be called the external lines of the cluster: if the extreme inside the cluster is the root, they will be *incoming*, while if the extreme is the vertex they will be *outgoing*. Examining the convergence of the perturbation series it becomes clear that if one considers the sum of the contributions to $\vec H^{(k)}_{\vec\nu}$, $\vec h^{(k)}_{\vec\nu}$ by all the graphs that *do not contain clusters with just one incoming and one outgoing branch which, furthermore, have the same momentum*, then the series so generated converge for ε small, [2], [4]. Therefore the clusters with one incoming and one outgoing equal momentum branches are called *resonances* and the KAM theory can be interpreted as an analysis of the reason why the resonances do not destroy the analyticity in ε at ε small, *i.e.* of the cancellations that make the resonances give a contribution much smaller than one could fear ($\varepsilon^k C^k$ instead of $\varepsilon^k k!$).

3. Lindstedt series resummation. If we multiply each graph value by the appropriate power of ε (equal to the number of branches of the graph) we see that the values of $\vec H$ and $\vec h$ can be computed by considering all the graphs *without resonances* but modifing the line factor of scale n as:

$$\frac{\chi(2^{-n}\vec\omega_0\cdot\vec\nu(v))}{(i\vec\omega_0\cdot\vec\nu(v))^2}\,(-i\vec\nu_{v'}\cdot iJ^{-1}\vec\nu_v)\to\frac{\chi(2^{-n}\vec\omega_0\cdot\vec\nu(v))}{(i\vec\omega_0\cdot\vec\nu(v))^2}(-i\vec\nu_{v'}\cdot\frac{1}{1-\sigma_{n,\varepsilon}(\vec\omega_0\cdot\vec\nu(v))}iJ^{-1}\vec\nu_v)\ ,$$

where $\sigma_{n,\varepsilon}(\vec\omega_0\cdot\vec\nu(v))$ is a suitable matrix representing the sum of all the possible insertions of resonances on the line connecting v' with v. The function $\sigma_{n,\varepsilon}(\vec\omega_0\cdot\vec\nu(v))\equiv\sigma_{n,\varepsilon}(2^n x)$ does not vanish only for $x\in(1/2,1]$. It is possible to prove that (1) the matrix $\sigma_{n,\varepsilon}(2^n x)$ is analytic in ε for ε small, independently on n, and (2) there is a constant $\mathcal{C}$ such that $||\sigma_{n,\varepsilon}(2^n x)||<\mathcal{C}|\varepsilon|$,

[5]. Furthermore the limit $\lim_{n\to-\infty}\sigma_{n,\varepsilon}(2^n x) = \sigma_\varepsilon$ exists and is a x– independent function of ε, analytic for ε small enough and divisible by ε, [5]. The constants σ_ε will be called the *form factors.* It is reasonable to think that the singularities of $\vec{H}$, $\vec{h}$, depending on the form factors $\sigma_{n,\varepsilon}$, are the same as those of the series $\vec{H}^*$, $\vec{h}^*$ obtained replacing $\sigma_{n,\varepsilon}$ by σ_ε. Thus the series for $\vec{H}^*$, $\vec{h}^*$ depend only on the variables $\eta = \varepsilon(1-\sigma_\varepsilon)^{-1}$, and the possibility arises that a singularity is reached at a value ε^* at which σ_{ε^*} is still finite, while η crosses the boundary of the analyticity domain for $\vec{H}^*$, $\vec{h}^*$.

Suppose that the analycity domain in $\varepsilon \equiv |\varepsilon|e^{i\theta}$ of $\vec{h}$, $\vec{H}$ is given by the complex domain D whose boundary is $\varepsilon_D(\theta)$. By the study of the non resummed Lindstedt series one obtains a circular estimate for the analicity domain which will be *not larger that the biggest circle contained in* D, so that in general it is a bad estimate for the value of ε_c. From the resummed series, on the other hand, one finds, if $\|\cdot\|$ is the matrix norm, $\|\frac{\varepsilon}{1-\sigma_\varepsilon}\| \le R^*$, if R^* is the convergence radius in η of the series for $\vec{H}^*$, $\vec{h}^*$, and the ε's which verify the above inequality define a domain D' different from a disk and possibly with a very complicated form. Then the problem is reduced to the study of the form factors σ_ε appearing in the resummed series: one can expect that they produce an estimate for $\varepsilon_D(0)$, $\varepsilon_D(\pi)$ better than R, (see §1 for the notation). For instance, if one could truncate σ_ε at the first non trivial order (the second one), one would find $\frac{|\varepsilon|^2}{A+B|\varepsilon|\cos(2\theta)} \le R^*$, for some constants A, B, so that the estimated analyticity domain for the tori would be an ellipse. In general such a truncation is not justified at all, because, for ε near ε^*, σ_ε is not well approximated by the second order expansion: on the contrary the full functional dependence on ε would be required. Nevertheless one can envisage the following scenario: although the convergence radii for the tori $\vec{h}^*$, $\vec{H}^*$, and for the form factors σ_ε are probably the same, there is the possibility that σ_ε remains small *at real* ε^* *on the boundary of* D', so that a second order truncation for σ_ε represents a good approximation for σ_ε. We have not been able to prove if such a phenomenon really occurs: perhaps one could proceed as in QFT, where one defines a recursive relation (*beta function*, see [1]) for the "form factors on scale n" $\sigma_{n,\varepsilon}(2^n x)$, so obtaining useful informations about σ_ε, and in this way one to could try to really perform the above analysis. One difficulty in this direction could be that the *relevant part* of the beta function (*i.e.* the part of the beta function which one expects to describe the important features of the model, by dimensional arguments) seems to be identically vanishing: anyway a numerical study would be highly desirable.

4. Tori breakdown and universality. The possibility that the singularities of the resummed series, as functions both of ε and $\vec{\psi}$, have a *universal nature* is related to the behaviour of the large order coefficients, which are likely to be very mildly dependent on the actual values of the Fourier components $f_{\vec{\nu}}$. This can indeed be seen to happen when only the contributions to the coefficients arising from simple classes of trees are taken into account. The simplest class of graphs not giving a trivial contribution, *i.e.* contribution which is an entire function of ε, to the invariant tori is given by the set of trees of the form of a linear chain $k \leftarrow \ldots \leftarrow 2 \leftarrow 1$, see [5]. For concreteness we fix $\ell = 2$, $\vec{\omega} = (r, 1)$ with $r = \frac{\sqrt{5}-1}{2} =$ *golden mean*, and the perturbation to be an even function of $\vec{\alpha}$ only as $f(\vec{\alpha}) = a\cos\alpha_1 + b\cos(\alpha_1 - \alpha_2)$, ("Escande–Doveil pendulum", [3]). Let us call "resonant line" the line ortogonal to $\vec{\omega}$, and let (p_n, q_n) be the convergents for continued fraction for r. Any integer $s \geq 1$ can be written $s = q_n + \sigma_{n-2}q_{n-2} + \ldots + \sigma_1 q_1$, if $q_n \leq s < q_{n+1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n-2} = 0, 1$, with the constraint $\sigma_j\sigma_{j+1} = 0$, $j = 1, \ldots, n-3$. Let Λ_{q_n} be the family of self–avoiding walks on the integer lattice $\mathbf{Z}^2$ starting at $(0,0)$, ending at $(q_n, -p_n)$ and contained in the strip $0 < x \leq q_n$, except for the left extreme points. Then a self–avoiding walk joining $(0,0)$ to (s, s') with $= q_n + \sigma_{n-2}q_{n-2} + \ldots + \sigma_1 q_1$ and $s' = p_n + \sigma_{n-2}p_{n-2} + \ldots + \sigma_1 p_1$ can be obtained by simply joining a walk in Λ_{q_n}, one in $\Lambda_{q_{n-2}}$ if $\sigma_{n-2} = 1, \ldots,$ one in Λ_1 if $\sigma_1 = 1$: the latter self–avoiding walks will define the class Λ_s of walks. It is clear by the construction that the class Λ_s of self–avoiding walks contains many of the ones which have the largest products $\prod_j \frac{1}{(i\vec{\omega}\cdot\vec{\nu}(j))^2}$ of small divisors. Therefore we define:

$$\vec{Z}(\Lambda_{q_n}) = \sum_{\text{walks in }\Lambda_{q_n}} (-i\eta J^{-1}\vec{\nu}_1)\frac{f_{\vec{\nu}_1}}{(i\vec{\omega}\cdot\vec{\nu}(1))^2}\prod_{j=2}^{k}\frac{f_{\vec{\nu}_{v_j}}(\vec{\nu}_{j-1}\cdot\eta J^{-1}\vec{\nu}_j)}{(\vec{\omega}\cdot\vec{\nu}(j))^2}e^{i(q_n\psi_1 - p_n\psi_2)} ,$$

(with $\vec{\nu}(1) = (q_n, -p_n)$ which can be realized as a sum of copies of the vectors $\vec{\nu}_1 = (1,0)$ and $\vec{\nu}_2 = (1,-1)$). We expect:

$$\vec{Z}(\Lambda_{q_n}) = \vec{\zeta}\,\frac{C(\eta,f)^{q_n}}{q_n^{\delta}}\,e^{i(q_n\psi_1 - p_n\psi_2)}(1+O(q_n^{-1})) \equiv \vec{\zeta}Z(\Lambda_{q_n}) = Z_n\,e^{i(q_n\psi_1 - p_n\psi_2)} ,$$

where $\vec{\zeta}$ is a suitable unit vector, $C(\eta, f)$ is a suitable function of η and f, and δ is a "critical exponent" characteristic of the golden mean. Let $\varepsilon_n = (rp_n - q_n)$; then $Z(\Lambda_s)$ is given by $Z(\Lambda_s) \simeq Z(\Lambda_{q_n})Z(\Lambda_{q_{n-2}})^{\sigma_{n-2}} \ldots Z(\Lambda_{q_1})^{\sigma_1}$, if $s < q_{n+1}$, while $Z(\Lambda_{q_{n+1}}) \simeq Z(\Lambda_{q_n})Z(\Lambda_{q_{n-1}})\left(\frac{\varepsilon_{n-1}}{\varepsilon_{n+1}}\right)^2$, if $s = q_{n+1}$, (we can define, for consistency, $Z(\Lambda_{q_0}) = \varepsilon_0^{-2} = r^{-2}$ and $Z(\Lambda_{q_{-1}}) = (\varepsilon_0/\varepsilon_{-1})^2 = r^2$). Then the contribution to $\vec{h}^*$ due to the above classes of trees and walks can be written, approximately, if $r_n = \frac{p_n}{q_n}$, as

$$\vec{\zeta}\sum_{n=1}^{1}\frac{[C(\eta,f)e^{i(\psi_1-r_n\psi_2)}]^{q_n}}{q_n^\delta}=\vec{\zeta}\sum_{n=1}^{1}\frac{[C(\eta,f)e^{i(\psi_1-r\psi_2)}]^{q_n}}{q_n^\delta}e^{-i\psi_2 O(\frac{1}{q_n})}\;.$$

We expect that *the singularities of the resummed series, as ε grows, are the same as those of $\vec{H},\vec{h}$*, (see §3), and furthermore we expect the above considered contributions to the parametric equations of the resummed tori to be the *most singular*. Hence we interpret the above equation as saying that we should expect $\vec{h},\vec{H}$ to be, at the breakdown of the invariant torus which corresponds to $|C(\eta,f)|=1$, singular as functions of ψ_1,ψ_2 and of η (hence of ε). Furthermore (1) the set $|C(\eta,f)|=1$ is in the η–plane a natural boundary for the above approximations to the functions $\vec{h}^*,\vec{H}^*$ as functions of η, and (2) if $\eta=\frac{\varepsilon}{1-\sigma_\varepsilon}$ is supposed smooth in ε or at least Lipshitz continuous, (a possibility mentioned above), when $\varepsilon\to\varepsilon_c^-$, one has $C(\eta,f)\simeq(1-\gamma(\varepsilon_c-\varepsilon))$, so that the singularity of $\vec{h}(\psi_1,0)$ or $\vec{H}(\psi_1,0)$ in ε,ψ_1 is described by the singularity of a single function $\xi(z)$, or, respectively, $\xi'(z)$, of the single variable $z=e^{-\gamma(\varepsilon_c-\varepsilon)}e^{i\psi}$, $\xi(z)=\sum_{k=1}^\infty\frac{z^{q_k}}{q_k^\delta}$, or $\xi'(z)=\sum_{k=1}^\infty\frac{z^{q_k}}{q_k^{\delta+1}}$. This would mean that the *critical torus* has a Lipshitz continuous regularity with *any* exponent $\delta'<\delta$ in the ψ_1–variable and δ in the $\varepsilon-\varepsilon_c$ variable, [9]. For instance if we fix ψ_2, *e.g.* as $\psi_2=0$ (which can be regarded as a special Poincaré section of the invariant torus), then $\vec{h}^*$ is a $C^{\delta'}$ function and $\vec{H}^*$, which is obtained from $\vec{h}$ by applying the operator $\vec{\omega}\cdot\partial_{\vec{\psi}}$, is a $C^{1+\delta}$ function, [9]. Therefore, based on the hypothesis that the singularity of $\vec{H},\vec{h}$ and of $\vec{H}^*,\vec{h}^*$ are the same and on the above heuristic discussion, the following conjecture emerges.

5. Conjecture. *Consider the conjugacy $\vec{\alpha}\leftrightarrow\vec{\psi}$ to a pure rotation of the motion generated by the Poincaré map on a circle on the critical torus. There is $\delta>0$ such that the coniugacy is described by two functions $\vec{h},\vec{H}$ and written as $\vec{\alpha}=(\psi,0)+(h_1(\psi),h_2(\psi))$ and $\vec{A}=(H_1(\psi),H_2(\psi))$ with $\vec{h}$ Hölder continuous with exponent $\delta'<\delta$ and $\vec{H}$ of class $C^{1+\delta'}$. Furthermore the above conjugacy has a Hölder continuous regularity $\delta'<\delta$ in the $\varepsilon-\varepsilon_c$ variable.*

In particular $\vec{H}$ is "once more differentiable" than $\vec{h}$, for $\varepsilon=\varepsilon_c$; this is quite surprising and, perhaps, quite unexpected as $\vec{H}$ is obtained from $\vec{h}$, by applying to it $\vec{\omega}\cdot\partial_{\vec{\psi}}$. The idea of studying the universality of the tori breakdown from a tree expansion is a refined version of an important idea in [11]: except that we have *not* made here the simplifying assumption of absence of resonances (*i.e.* we *allow* for non zero Fourier components of opposite wave label $\pm\vec{\nu}$, and find resummations that in some sense eliminate them). If one accepts that the above pendulum system has the same critical exponents for the golden mean torus in the standard map then it follows that $\delta = 0.7120834$ by the scaling argument on p. 207 of [10].

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