A duality between Schrödinger operators on graphs and certain Jacobi matrices

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The known correspondence between the Kronig–Penney model and certain Jacobi matrices is extended to a wide class of Schrödinger operators on graphs. Examples include rectangular lattices with and without a magnetic field, or comb–shaped graphs leading to a Maryland–type model.

I Introduction

Schrödinger operators on $L^2(\Gamma)$, where Γ is a graph, were introduced into quantum mechanics long time ago [1]. In recent years we have wittnessed of a renewed interest to them — see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and other, often nonrigorous, studies quoted in these papers — motivated mostly by the fact that they provide a natural, if idealized, model of semiconductor "quantum wire" structures.

Jacobi matrices, on the other hand, attracted a lot of attention in the last decade, in particular, as a laboratory for random and almost periodic systems. The most popular examples are the Harper and related almost Mathieu equation dating back to [13, 14, 15]; for more recent results and an extensive bibliography see, *e.g.*, [16, 17, 18, 19, 20, 21]. The underlying lattices are mostly periodic of dimension one or two; however, more complicated examples have also been studied [22, 23].

In case when Γ is a line with an array of point interactions, *i.e.*, the Schrödinger operator in question is a Kronig–Penney–type Hamiltonian, there is a bijective correspondence — dubbed the *French connection* by B. Simon [16] — between such systems and certain Jacobi matrices [18, 24, 25, 26, 27, 28]. The aim of this paper is to show that the same duality can be established for a wide class of Schrödinger operators on graphs, including the case of a nonempty boundary. In general, the resulting Jacobi matrices exhibit a varying "mass".

II Preliminaries

Let Γ be a connected graph consisting of at most countable families of *vertices* $\mathcal{V} = \{\mathcal{X}_j : j \in I\}$ and *links (edges)* $\mathcal{L} = \{\mathcal{L}_{jn} : (j,n) \in I_{\mathcal{L}} \subset I \times I\}$. We suppose that each pair of vertices is connected by not more than one link. The set $\mathcal{N}(\mathcal{X}_j) = \{\mathcal{X}_n : n \in \nu(j) \subset I \setminus \{j\}\}$ of *neighbors* of $\mathcal{X}_j$, *i.e.*, the vertices connected with $\mathcal{X}_j$ by a single link, is nonempty by hypothesis. The graph *boundary* $\mathcal{B}$ consists of vertices having a single neighbor; it may be empty. We denote by $I_{\mathcal{B}}$ and $I_{\mathcal{I}}$ the index subsets in I corresponding to $\mathcal{B}$ and the graph *interior* $\mathcal{I} := \mathcal{V} \setminus \mathcal{B}$, respectively.

Γ has a natural ordering by inclusions between vertex subsets. We assume that it has a *local* metric structure in the sense that each link $\mathcal{L}_{jn}$ is isometric with a line segment $[0, \ell_{jn}]$. The graph can be also equipped with a *global* metric, for instance, if it is identified with a subset of $I\!R^\nu$. Of course, the two metrics may differ at a single link. Using the local metric, we are able to introduce the Hilbert space $L^2(\Gamma) := \bigoplus_{(j,n)\in I_{\mathcal{L}}} L^2(0,\ell_{jn})$; its elements will be written as $\psi = \{\psi_{jn} : (j,n) \in I_{\mathcal{L}}\}$ or simply as $\{\psi_{jn}\}$. Given a family of functions $V := \{V_{jn}\}$ with $V_{jn} \in L^\infty(0,\ell_{jn})$, we define the operator $H_{\mathcal{C}} \equiv H(\Gamma, \mathcal{C}, V)$ by

$$H_{\mathcal{C}}\{\psi_{jn}\} := \{ -\psi''_{jn} + V_{jn}\psi_{jn} : (j,n) \in I_{\mathcal{L}} \} \tag{2.1}$$

with the domain consisting of all ψ with $\psi_{jn} \in W^{2,2}(0,\ell_{jn})$ subject to a set $\mathcal{C}$ of boundary conditions at the vertices which connect the boundary values

$$\psi_{jn}(j) := \lim_{x\to 0+} \psi_{jn}(x), \qquad \psi'_{jn}(j) := \lim_{x\to 0+} \psi'_{jn}(x)\,; \tag{2.2}$$

we identify here the point $x = 0$ with $\mathcal{X}_j$.

There are many ways how to make the operator (2.1) self–adjoint; by standard results [18, 29] each vertex $\mathcal{X}_j$ may support an N_j^2 parameter family of boundary conditions, where $N_j := \mathrm{card}\,\nu(j)$. These self–adjoint extensions were discussed in detail in [4]; here we restrict ourselves to one of the following two possibilities which represent in a sense extreme cases [10, 11]:

(a) δ *coupling:* at any $\mathcal{X}_j \in \mathcal{I}$ we have $\psi_{jn}(j) = \psi_{jm}(j) =: \psi_j$ for all $n,m \in \nu(j)$, and

$$\sum_{n\in\nu(j)} \psi'_{jn}(j) = \alpha_j \psi_j \tag{2.3}$$

for some $\alpha_j \in I\!R$.

(b) δ'_s *coupling:* $\psi'_{jn}(j) = \psi'_{jm}(j) =: \psi'_j$ for all $n,m \in \nu(j)$, and

$$\sum_{n\in\nu(j)} \psi_{jn}(j) = \beta_j \psi'_j \tag{2.4}$$

for $\beta_j \in I\!R$.

The relations (2.3) and (2.4) are independent of V, since potentials are supposed to be essentially bounded. At the graph boundary we employ the usual conditions,

$$\psi_j \cos\omega_j + \psi'_j \sin\omega_j = 0\,, \tag{2.5}$$

which can be written in either form (infinite values allowed); for the sake of brevity we denote the set $\mathcal{C}$ as $\alpha=\{\alpha_j\}$ or $\beta=\{\beta_j\}$, respectively, with the index running over $\mathcal{V}$, and use the shorthands $H_\alpha := H(\Gamma,\{\alpha_j\},V)$ and $H_\beta := H(\Gamma,\{\beta_j\},V)$. Looking for solutions to the equations

$$H_{\mathcal{C}}\psi \;=\; k^2\psi\,, \quad \mathcal{C}=\alpha,\beta, \tag{2.6}$$

we shall consider the class $D_{loc}(H_{\mathcal{C}})$ which is the subset in $\bigvee_{(j,n)\in I_{\mathcal{L}}} L^2(0,\ell_{jn})$ (the direct sum) consisting of the functions which satisfy all requirements imposed at $\psi\in D(H_{\mathcal{C}})$ except the global square integrability.

The conditions (2.3) and (2.4) define self–adjoint operators also if the coupling constants are formally put equal to infinity. We exclude this possibility, which corresponds, respectively, to the Dirichlet and Neumann decoupling of the operator at $\mathcal{X}_j$ turning the vertex effectively into N_j points of the boundary. On the other hand, we need it to state the result. Let H_α^D and H_β^N be the operators obtained from H_α, H_β by changing the conditions (2.3), (2.4) at the points of $\mathcal{I}$ to Dirichlet and Neumann, respectively, while at the boundary they are kept fixed. We define $\mathcal{K}_\alpha := \{k:\, k^2\in\sigma(H_\alpha^D),\, \mathrm{Im}\,k\ge 0\}$ and $\mathcal{K}_\beta$ in a similar way.

III Main result

Consider the operators H_α, H_β defined above. We shall adopt the following assumptions:

(i) There is $C>0$ such that $\|V_{jn}\|_\infty\le C$ for all $(j,n)\in I_{\mathcal{L}}$.

(ii) $\ell_0 := \inf\{\,\ell_{jn}:\,(j,n)\in I_{\mathcal{L}}\}>0$.

(iii) $L_0 := \sup\{\,\ell_{jn}:\,(j,n)\in I_{\mathcal{L}}\}<\infty$.

(iv) $N_0 := \max\{\,\mathrm{card}\,\nu(j):\, j\in I\,\}<\infty$.

On $\mathcal{L}_{nj}\equiv[0,\ell_{jn}]$ (the right endpoint identified with $\mathcal{X}_j$) we shall denote as $u_{jn}^{\mathcal{C}}$, $\mathcal{C}=\alpha,\beta$, the solutions to $-f''+V_{jn}f=k^2f$ which satisfy the boundary conditions

$$u_{jn}^\alpha(\ell_{jn}) = 1-(u_{jn}^\alpha)'(\ell_{jn})=0\,, \qquad 1-u_{jn}^\beta(\ell_{jn})=(u_{jn}^\beta)'(\ell_{jn})=0$$

and

$$\begin{aligned} v_{jn}^\alpha(0)=1-(v_{jn}^\alpha)'(0)=0\,,\quad 1-v_{jn}^\beta(0)=(v_{jn}^\beta)'(0)=0 &\qquad \text{if}\quad n\in I_{\mathcal{I}}\,,\\ v_{jn}^{\mathcal{C}}(0)=\sin\omega_n\,,\qquad (v_{jn}^{\mathcal{C}})'(0)=-\cos\omega_n &\qquad \text{if}\quad n\in I_{\mathcal{B}}\,; \end{aligned}$$

their Wronskians are

$$W_{jn}^\alpha = -v_{jn}^\alpha(\ell_{jn})\,, \qquad W_{jn}^\beta=(v_{jn}^\beta)'(\ell_{jn})\,,$$

respectively, or

$$W^{\alpha}_{jn} = u^{\alpha}_{jn}(0)\,, \quad W^{\beta}_{jn} = -(u^{\beta}_{jn})'(0) \qquad \text{if} \quad n \in I_{\mathcal{I}}\,,$$
$$W^{\mathcal{C}}_{jn} = -u^{\mathcal{C}}_{jn}(0)\cos\omega_n - (u^{\mathcal{C}}_{jn})'(0)\sin\omega_n \qquad \text{if} \quad n \in I_{\mathcal{B}}\,.$$

If not necessary we do not mark explicitly the dependence of these quantities on k.

Theorem III.1 *(a) Let $\psi \in D_{loc}(H_\alpha)$ solve (2.6) for some $k \notin \mathcal{K}_\alpha$ with $k^2 \in I\!R$,* $\mathrm{Im}\, k \ge 0$. *Then the corresponding boundary values (2.2) satisfy the equation*

$$\sum_{n\in\nu(j)\cap I_{\mathcal{I}}} \frac{\psi_n}{W^{\alpha}_{jn}} - \left(\sum_{n\in\nu(j)} \frac{(v^{\alpha}_{jn})'(\ell_{jn})}{W^{\alpha}_{jn}} - \alpha_j \right) \psi_j \;=\; 0\,. \tag{3.7}$$

Conversely, any solution $\{\psi_j : j \in I_{\mathcal{I}}\}$ to (3.7) determines a solution of (2.6) by

$$\psi_{jn}(x) = \frac{\psi_n}{W^{\alpha}_{jn}}\, u^{\alpha}_{jn}(x) - \frac{\psi_j}{W^{\alpha}_{jn}}\, v^{\alpha}_{jn}(x) \qquad \text{if} \quad n \in \nu(j)\cap I_{\mathcal{I}}\,, \tag{3.8}$$

$$\psi_{jn}(x) = -\frac{\psi_j}{W^{\alpha}_{jn}}\, v^{\alpha}_{jn}(x) \qquad \text{if} \quad n \in \nu(j)\cap I_{\mathcal{B}}\,. \tag{3.9}$$

(b) For a solution $\psi \in D_{loc}(H_\beta)$ of (2.6) with $k \notin \mathcal{K}_\beta$, the above formulae are replaced by

$$\sum_{n\in\nu(j)\cap I_{\mathcal{I}}} \frac{\psi'_n}{W^{\beta}_{jn}} + \left(\sum_{n\in\nu(j)} \frac{v^{\beta}_{jn}(\ell_{jn})}{W^{\beta}_{jn}} + \beta_j \right) \psi'_j \;=\; 0\,. \tag{3.10}$$

and

$$\psi_{jn}(x) = -\frac{\psi'_n}{W^{\beta}_{jn}}\, u^{\beta}_{jn}(x) + \frac{\psi'_j}{W^{\beta}_{jn}}\, v^{\beta}_{jn}(x) \qquad \text{if} \quad n \in \nu(j)\cap I_{\mathcal{I}}\,, \tag{3.11}$$

$$\psi_{jn}(x) = \frac{\psi'_j}{W^{\beta}_{jn}}\, v^{\beta}_{jn}(x) \qquad \text{if} \quad n \in \nu(j)\cap I_{\mathcal{B}}\,. \tag{3.12}$$

(c) Under (i), (ii), $\psi \in L^2(\Gamma)$ implies that the solution $\{\psi_j\}$, $\{\psi'_j\}$ of (3.7) and (3.10), respectively, belongs to $\ell^2(I_{\mathcal{I}})$.
(d) The opposite implication is valid provided (iii), (iv) also hold, and k has a positive distance from from $\mathcal{K}_{\mathcal{C}}$.

Proof. (a,b) We shall consider H_α throughout, the argument for H_β is analogous; for simplicity we drop the superscript α. If $n \in I_{\mathcal{I}}$, the transfer matrix on $\mathcal{L}_{nj}$ is

$$T_{nj}(x,0) \;=\; W^{-1}_{jn} \begin{pmatrix} u_{jn}(x) - u'_{jn}(0)v_{jn}(x) & u_{jn}(0)v_{jn}(x) \\ \\ u'_{jn}(x) - u'_{jn}(0)v'_{jn}(x) & u_{jn}(0)v'_{jn}(x) \end{pmatrix} \;;$$

the Wronskian is nonzero for $k \notin \mathcal{K}_\alpha$. This yields an expression of $\psi_{jn}(x)$ in terms of $\psi_{jn}(n) =: \psi_n$ and $\psi'_{jn}(n)$, in particular,

$$\psi_j \;:=\; \psi_{jn}(j) = u'_{jn}(0)\psi_n + v_{jn}(\ell_{jn})\psi'_{jn}(n)\,,$$

$$-\psi'_{jn}(j) \;=\; \frac{1-u'_{jn}(0)v'_{jn}(\ell_{jn})}{W_{jn}}\psi_n + v'_{jn}(\ell_{jn})\psi'_{jn}(n)\,;$$

the sign change at the *lhs* of the last condition reflects the fact that (2.2) defines the outward derivative at $\mathcal{X}_j$. We express $\psi'_{jn}(n)$ from the first relation and substitute to the second one; this yields

$$\psi'_{jn}(j) \;=\; -\,\frac{\psi_n}{W_{jn}} \;+\; \frac{v'_{jn}(\ell_{jn})}{W_{jn}}\,\psi_j\,.$$

If $n \in I_{\mathcal{B}}$ we have instead

$$\psi_j = u'_{jn}(0)\psi_n - u_{jn}(0)\psi'_{jn}(n)\,,$$

$$-\psi'_{jn}(j) \;=\; \frac{u'_{jn}(0)v'_{jn}(\ell_{jn}) - \cos\omega_n}{W_{jn}}\,\psi_n + \frac{u_{jn}(0)v'_{jn}(\ell_{jn}) - \sin\omega_n}{W_{jn}}\,\psi'_{jn}(n)\,;$$

we may write, of course, $\psi'_n := \psi'_{jn}(n)$. Using the relations $\psi_n\cos\omega_n + \psi'_n\sin\omega_n = 0$, we find

$$\psi'_{jn}(j) \;=\; \frac{v'_{jn}(\ell_{jn})}{W_{jn}}\,\psi_j\,,$$

so (3.7) follows from (2.2). The transfer–matrix expression of $\psi_{jn}(x)$ together with the mentioned formula for $\psi'_{jn}(n)$ yield (3.8); (3.9) is checked in the same way.
(c) Without loss of generality we may consider real ψ only. Any solution to the Schrödinger equation on $\mathcal{L}_{nj}$ can also be written as

$$\psi_{jn}(x) \;=\; \psi_{jn}(n)w_{jn}(x) + \psi'_{jn}(n)v_{jn}(x)\,,$$

where w_{jn} is the normalized Neumann solution at $\mathcal{X}_n$, $w'_{jn}(0) = 1 - w_{jn}(0) = 0$. The argument of [18, Sec.III.2.1] cannot be used here, even in the classically allowed region. Instead we employ a standard result of the Sturm–Liouville theory [30, Sec.1.2] by which

$$v_{jn}(x) \;=\; \frac{\sin kx}{k} \;+\; \int_0^x K_D(x,y)\,\frac{\sin ky}{k}\,dy\,, \tag{3.13}$$

and similar representations are valid for w_{jn} and their derivatives; the kernels have an explicit bound in terms of the potential V_{jn}. The latter is essentially bounded, by (i) uniformly over $\mathcal{L}$. Hence there is a positive $\ell_1 < \frac{1}{2}\,\ell_0$ such that

$$\max\Big\{\,|v_{jn}(x)|,\; |v'_{jn}(x)-1|,\; |w_{jn}(x)-1|,\; |w'_{jn}(x)|\,\Big\} \;<\; \frac{1}{10}$$

for $x \in [0,\ell_1)$ and any $(j,n) \in I_{\mathcal{L}}$; we infer that

$$\begin{aligned}\psi_{jn}(x)^2 &= \psi_{jn}(n)^2 w_{jn}(x)^2 + \psi'_{jn}(n)^2 v_{jn}(x)^2 + 2\psi_{jn}(n)\psi'_{jn}(n) w_{jn}(x) v_{jn}(x) \\ &\geq \frac{9}{20}\,\psi_{jn}(n)^2 - \frac{37}{100}\,\psi'_{jn}(n)^2 \,.\end{aligned}$$

An analogous estimate can be made for $\psi'_{jn}(n)^2$; summing both of them we get

$$\psi_{jn}(x)^2 + \psi'_{jn}(x)^2 \;\geq\; \frac{2}{25}\,\left(\psi_{jn}(n)^2 + \psi'_{jn}(n)^2\right)\,.$$

If $\psi \in L^2(\Gamma)$, it belongs to $D(H_\alpha)$, so $\psi' \in L^2(\Gamma)$ also holds. Let $\mathcal{L}_{jn}(\ell_1)$ be the link $\mathcal{L}_{jn}$ with the middle part $(\ell_1, \ell_{jn} - \ell_1)$ deleted; then

$$\begin{aligned}\|\psi\|^2_{L^2(\Gamma)} + \|\psi'\|^2_{L^2(\Gamma)} &= \sum_{(j,n)\in I_{\mathcal{L}}} \int_{\mathcal{L}_{jn}} (\psi_{jn}(x)^2 + \psi'_{jn}(x)^2)\,dx \\ &\geq \sum_{(j,n)\in I_{\mathcal{L}}} \int_{\mathcal{L}_{jn}(\ell_1)} (\psi_{jn}(x)^2 + \psi'_{jn}(x)^2)\,dx \\ &\geq \frac{4\ell_1}{25} \sum_{(j,n)\in I_{\mathcal{L}}} (\psi_{jn}(n)^2 + \psi'_{jn}(n)^2) \;\geq\; \frac{4\ell_1}{25} \sum_{j\in I_{\mathcal{I}}} \psi_j^2 \,,\end{aligned}$$

which yields the result.

(d) First we need to show that W_{jn}^{-1} has a uniform bound for $k \notin \overline{\mathcal{K}}_\alpha$ Without loss of generality we may suppose that $\inf \mathcal{K}_\alpha > 0$; otherwise we shift all potentials by a constant. In view of the Sturm comparison theorem [31, Sec.V.6], see also [29, Secs.XIII.1,15], the zeros of $v_{jn}(\ell_{jn},\cdot)$ and $v'_{jn}(\ell_{jn},\cdot)$ form a switching sequence; hence it is sufficient to prove that $\frac{dW_{jn}(k)}{dk}$ is bounded uniformly away from zero around each root $k_0 \in \mathcal{K}_\alpha$ of $W_{jn}(k) = 0$. Using the known identity [19, Sec.I.5], [29, Sec.XIII.3],

$$\frac{dv_{jn}(x,k)}{dk^2}\,v'_{jn}(x,k) \;-\; \frac{dv'_{jn}(x,k)}{dk^2}\,v_{jn}(x,k) \;=\; \int_0^x v_{jn}(x,k)^2\,dx\,,$$

at $x = \ell_{nj},\ k = k_0$, we find

$$\frac{dv_{jn}(\ell_{jn},k_0)}{dk} \;=\; \frac{2k_0}{v'_{jn}(\ell_{jn},k_0)} \int_{\mathcal{L}_{jn}} v_{jn}(x,k_0)^2\,dx\,.$$

We have supposed that $k_0 \geq \inf \mathcal{K}_\alpha > 0$; furthermore, by the argument of the preceding part we have $\int_{\mathcal{L}_{jn}} v_{jn}(x,k_0)^2 dx \geq \frac{81}{200}\,\ell_1^2$. At the same time, using the representation (3.13) we find that $|v'_{jn}(\ell_{jn},k_0)| \leq C_1$ for a positive C_1 independent of j,n . Hence to a given $k \notin \overline{\mathcal{K}}_\alpha$ there is a $C_2 > 0$, again independent of j,n , such that $W_{jn} \geq C_2$. The representation (3.13) also yields another uniform bound,

$$C_3 \;:=\; \sup\Big\{\, \|u_{jn}\|_{L^2},\ \|v_{jn}\|_{L^2},\,:\ (j,n) \in I_{\mathcal{L}} \Big\} \;<\; \infty\,.$$

The relations (3.8) and (3.9) together with (iv) now yield

$$\|\psi\|^2 \;=\; \sum_{(j,n)\in I_{\mathcal{L}}} \|\psi_{jn}\|^2 \;\le\; \sum_{j,n\in I_{\mathcal{I}}} |W_{jn}|^{-2} \left\{ |\psi_n|^2 \|u_{jn}\|^2 + |\psi_j|^2 \|v_{jn}\|^2 \right\}$$

$$+ \sum_{n\in I_{\mathcal{B}}} |\psi_j|^2 \frac{\|v_{jn}\|^2}{|W_{jn}|^2} \;\le\; 2N_0 \left(\frac{C_3}{C_2}\right)^2 \sum_{j\in I_{\mathcal{I}}} |\psi_j|^2 \,,$$

so the result follows. ∎

Remarks. 1. Since the operators $H_{\mathcal{C}}$ are below bounded, one can express for $k^2 \in \rho(H_{\mathcal{C}}) \cap \rho(H_0)$ the resolvent difference $(H_{\mathcal{C}} - k^2)^{-1} - (H_0 - k^2)^{-1}$, where $H_0 = H_\alpha^D,\, H_\beta^N$, by Krein's formula [32, 33, 34]; this expression becomes singular under the condition (3.7) or (3.10), respectively.
2. If two vertices are joined by more than a single link, the theorem can be applied if a vertex with the free δ coupling, $\alpha_j = 0$, is added at each extra link. Vertices with just two neighbors are also useful, if we want to amend the potentials V_{jn} with a family of point interactions — for an example see [35].
3. If (i)–(iv) are relaxed, the implications (b), (c) may still hold with $\ell^2(I_{\mathcal{I}})$ replaced by a weighted ℓ^2 space. If Γ is equipped with a global metric, one can establish a relation between the exponential decay of ψ and of the boundary–value sequences.

IV Examples

Example IV.1 *(rectangular lattices):* Let Γ be a planar lattice graph whose basic cell is a rectangle of sides ℓ_j, $j = 1, 2$, and suppose that apart of the junctions, the motion on Γ is free, $V_{jn} = 0$. By the proved theorem, the equation (2.6) leads in this case to

$$\begin{aligned} &(\phi_{n,m+1} + \phi_{n,m-1}) \sin k\ell_1 + (\phi_{n+1,m} + \phi_{n-1,m}) \sin k\ell_2 \\ &\qquad + \Big(V_{nm}^{\mathcal{C}}(k) \mp 2 \sin k(\ell_1 + \ell_2) \Big)\, \phi_{nm} \;=\; 0 \end{aligned} \tag{4.14}$$

in the δ and δ_s' case, respectively, where

$$V_{nm}^\alpha(k) \;:=\; -\,\frac{\alpha_{nm}}{k} \sin k\ell_1 \sin k\ell_2 \,, \qquad V_{nm}^\beta(k) \;:=\; -\,\beta_{nm} k\, \sin k\ell_1 \sin k\ell_2 \,, \tag{4.15}$$

and $\phi_{nm} = \psi_{nm},\, \psi'_{nm}$. In particular, for a square lattice we get

$$(h_0\psi)_{nm} - \left(4\cos k\ell + \frac{\alpha_{nm}}{k} \sin k\ell \right) \psi_{nm} \;=\; 0$$

and an analogous equation in the δ_s' case, where h_0 is the conventional two–dimensional Laplacian on $\ell^2(\mathbb{Z}^2)$. On the other hand, if $\ell_1 \ne \ell_2$, the free operator has a periodically modulated "mass" which leads to nontrivial spectral properties even if the underlying graph Hamiltonian is periodic, *i.e.*, $\alpha_{nm} = \alpha$ or $\beta_{nm} = \beta$, with a non–zero coupling parameter. A detailed discussion of this situation can be found in [10, 11, 12].

Example IV.2 *(magnetic field added):* Suppose that Γ is embedded into $I\!R^\nu$ in which there is a magnetic field described by a vector potential A. The boundary conditions (2.3) and (2.4) are modified replacing $\psi'_{jn}(j)$ by $\psi'_{jn}(j)+iA_{jn}(j)$, where $A_{jn}(j)$ is the tangent component of A to $\mathcal{L}_{jn}$ at $\mathcal{X}_j$; we suppose conventionally that $e=-1$. For the δ coupling this is well known [2]; in the δ'_s case one can check it easily.

There in no need to repeat the above argument, however, since the magnetic case can be handled with the help of the unitary operator $U\,:\; L^2(\Gamma)\to L^2(\Gamma)$ defined by

$$(U\psi)_{jn}(x)\;:=\;\exp\left(i\int_{x_{jn}}^{x}A_{jn}(y)\,dy\right)\,\psi_{jn}(x)\,,$$

where A_{jn} is again the tangent component of the vector potential and x_{jn} are fixed reference points. Then the functions $(U\psi)_{jn}$ satisfy (2.3) and (2.4), respectively, and the equations (3.7) and (3.10) are replaced by

$$\sum_{n\in\nu(j)\cap I_{\mathcal{I}}}\frac{e^{iA_n}}{W_{jn}^{\alpha}}\,\psi_n\;-\;\left(\sum_{n\in\nu(j)}\frac{(v_{jn}^{\alpha})'(\ell_{jn})}{W_{jn}^{\alpha}}-\alpha_j\right)e^{iA_j}\psi_j\;=\;0\,,$$

$$\sum_{n\in\nu(j)\cap I_{\mathcal{I}}}\frac{e^{iA_n}}{W_{jn}^{\beta}}\,\psi'_n\;+\;\left(\sum_{n\in\nu(j)}\frac{v_{jn}^{\beta}(\ell_{jn})}{W_{jn}^{\beta}}+\beta_j\right)e^{iA_j}\psi'_j\;=\;0\,,$$

provided the magnetic phase factors A_j obey the consistency conditions

$$A_j-A_n\;=\;\int_{\mathcal{L}_{jn}}A_{jn}(y)\,dy$$

following from the continuity requirement.

In particular, if Γ is a rectangular lattice considered above, $A_{nm}:=\Phi(n-m)/2$ corresponds to a homogeneous magnetic field in the circular gauge, with the flux $\Phi:=B\ell_1\ell_2$ through a cell, and no other potential is present, the equation (4.14) is replaced by

$$\begin{aligned}&\left(e^{i\Phi m/2}\phi_{n,m+1}+e^{-i\Phi m/2}\phi_{n,m-1}\right)\sin k\ell_1+\left(e^{-i\Phi n/2}\phi_{n+1,m}+e^{i\Phi n/2}\phi_{n-1,m}\right)\sin k\ell_2\\&\qquad+\left(V_{nm}^{\mathcal{C}}(k)\mp 2\sin k(\ell_1+\ell_2)\right)\phi_{nm}\;=\;0\,,\end{aligned}\tag{4.16}$$

where the discrete potential is again given by (4.15). If $\ell_1=\ell_2$ and the coupling constants vanish, $\alpha_{nm}=\beta_{nm}=0$, we obtain in this way the discrete magnetic Laplacian of [21] (or the Harper operator of [13, 19] for the Landau gauge), while for a general rectangle the parameter λ representing the anisotropy in [21] is replaced again by the periodically modulated "mass".

Example IV.3 *(comb–shaped graphs):* Let Γ consist of a line to which at the points $\mathcal{X}_j:=jL$ line segments of lengths ℓ_j are joined; at their ends we impose the conditions (2.5). The equation (2.6) now yields

$$(h_0\phi)_j+\left(V_j^{\mathcal{C}}(k)\mp 2\cos kL\right)\phi_j\;=\;0\tag{4.17}$$

in the δ and δ'_s case, respectively, where $\phi_j = \psi_j,\, \psi'_j$, and

$$V_j^\alpha(k) \,:=\, -\left(\frac{v'_j(\ell_j)}{v_j(\ell_j)} + \alpha_j \right) \frac{\sin kL}{k}\,, \quad V_j^\beta(k) \,:=\, -\left(\frac{v_j(\ell_j)}{v'_j(\ell_j)} + \beta_j \right) k\, \sin kL\,.$$

In particular, in the absence of an external potential on Γ the last relations become

$$V_j^\alpha(k) = -\left(\cot(k\ell_j+\eta_j) + \frac{\alpha_j}{k} \right) \sin kL\,, \quad V_j^\beta(k) = -\left(\tan(k\ell_j+\eta_j) + \beta_j k \right) \sin kL\,,$$

where $\eta_j := \arctan(k\tan\omega_j)$. If the coupling is ideal, $\alpha_j = 0$ or $\beta_j = 0$, the loose ends correspond to the Dirichlet condition, $\omega_j = 0$, and the "tooth" lengths are $\ell_j := |j|\ell$, we get thus an equation reminiscent of the Maryland model [17, 36] with a fixed coupling strength and an additional periodic modulation of the potential.

Acknowledgments. The author is grateful for the hospitality extended to him in the Institute of Mathematics, University of Ruhr, Bochum, where this work was done. The research has been partially supported by the Grant AS No.148409 and the European Union Project ERB–CiPA–3510–CT–920704/704.

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