The absence of the absolutely continuous spectrum for δ' Wannier–Stark ladders

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A modification of the Kronig–Penney model consisting of equidistantly spaced δ'–interactions is considered. We prove that absolutely continuous spectrum of such a system disappears under influence of an external electric field. The result extends to periodic lattices of non–identical δ' interactions and potentials which are lower unbounded and, up to a bounded term, asymptotically decreasing with bounded first two derivatives.

I Introduction

One–dimensional Schrödinger operators with potentials composed of a periodic and an aperiodic, most often linear, part have been studied by many authors — see, *e.g.*, [1–4] and references therein — most attention being paid to the resonance structure of such systems. A basic ingredient of the typical scattering picture is that the spectrum of the corresponding Hamiltonian is absolutely continuous and covers the whole real line provided the aperiodic part is below unbounded; this property can be proven under rather weak differentiability requirements on the potential.

The aim of the present paper is to show that the spectral properties may change substantially if a smooth periodic potential is replaced by an array of singular interactions. The best known system of this type is the Kronig–Penney model, *i.e.*, a sequence of equally spaced δ–interactions; its spectral properties in presence of an external field remain an open problem.

However, in one–dimensional systems there are singular interactions different from δ. Another important class is represented by the δ'–interactions specified by the boundary conditions (1) below; a detailed discussion of their properties can be found in [5]. In distinction to δ, they cannot be approximated by families of Schrödinger operators with scaled short–range potentials, instead one can use families of rank–one operators [6] or velocity–dependent potentials [7, 8]. Moreover, this does not exhaust all possible aspects of the δ'–interaction; elsewhere we have presented a heuristic argument showing that it can be regarded as a paradigm for geometric scatterers [9].

An important distinction between the two types of contact interactions is manifested in Kronig–Penney–type models (without an external force): if we replace δ by δ' we obtain the spectrum in which the band widths are asymptotically constant while the gaps are widening. To understand this difference, recall that the transmission probability through a single δ' barrier vanishes in the high–energy limit, and the boundary conditions defining the interaction represent there an effective Neumann decoupling [9]. A propagation through an array of δ' barriers is made possible by their conspiracy, but only within narrow energy windows close to the eigenvalues of the Neumann problem.

If an external potential is added, a useful insight into the problem is obtained from the heuristic tilted–band picture according to which the particle moves as quasifree in the regions where its energy finds within a tilted spectral band of the periodic problem, while the complement corresponding to the tilted gaps represents a classically forbidden region which must be tunneled through. This suggests that in the δ' case, where the gaps dominate the "periodic" spectrum at high energies, an unrestricted propagation under influence of the external force may not be possible.

Our main result confirms this conjecture: using the stability of the absolutely continuous spectrum with respect to trace–class perturbations in a way somewhat analogous to [10], we demonstrate that the spectrum of δ' Wannier–Stark–ladder Hamiltonian $H(\beta, E, a)$ is purely singular for any nonzero "coupling constant" β and external field strength E. Moreover, the validity of this result can be extended to lattices of non–identical δ' interactions and a wide family of external potentials, as we shall discuss in Section 4.

Let me add the following remark. The problem treated here was formulated in collaboration with J.E. Avron and Y. Last. Together we devised the mentioned strategy and wrote a proof for the δ' Kronig–Penney model with a linear potential. The result was announced in Ref.[9]; recently the same conclusion has been reached by a completely different method [11]. Our original proof had a flaw, however; in attempt to rectify it we formulated separately two different arguments. The one published here appeared to be applicable to a considerably wider class of operators, and my coauthors insisted that I publish it in my name. While I respect their decision and appreciate their scrupulous attitude, I want to state that the credit for the main result of the paper is shared equally by all three of us.

II The main result

Consider the free Stark Hamiltonian $H_E := -\frac{d^2}{dx^2} - Ex$ on $L^2(I\!R)$ with $D(H_E) := \{\, f \in H^{2,2}(I\!R) :\; H_E f \in L^2 \,\}$, and an equidistant lattice, $\mathcal{L} := \{na\}_{n=-\infty}^{\infty}$ with a spacing $a > 0$. Suppose that at each point of $\mathcal{L}$ we introduce the δ'–interaction of a strength β_n, *i.e.*, we define the operator $H(\{\beta_n\}, E, a)$ in the following way: it acts as H_E on the intervals $J_n := (na, (n+1)a)$ and its domain differs from $D(H_E)$ by replacing the smoothness requirement at the points of $\mathcal{L}$ by the boundary conditions

$$f'(na+) = f'(na-) =: f'(na)\,, \quad f(na+) - f(na-) = \beta_n f'(na)\,; \tag{1}$$

the β_n are real numbers or $+\infty$ in which case (1) is replaced by the Neumann condition, $f'(na) = 0$. If the interaction strength at each point is the same, $\beta_n = \beta$, we write $H(\{\beta_n\}, E, a) =: H(\beta, E, a)$; in particular, $H(0, E, a) = H_E$ and $H(\infty, E, a)$ is the orthogonal sum of the single–interval operators obtained by imposing the Neumann condition at each point of $\mathcal{L}$.

Our main result is the following.

Theorem II.1 *Let $E, \beta \neq 0$; then the absolutely continuous spectrum of the δ' Wannier–Stark Hamiltonian $H(\beta, E, a)$ is empty.*

As mentioned above, we shall demonstrate this property by using known results about the stability of the absolutely continuous spectrum. More specifically, the proof will be based on comparing the resolvent of $H(\beta, E, a)$ to that of a suitable operator with a pure point spectrum. The latter is obtained by changing the δ' conditions to the Neumann one at points of a sublattice $\tilde{\mathcal{L}} \subset \mathcal{L}$; following the heuristic tilted–band picture sketched above we switch the boundary conditions deep in the tilted–gap regions, because the wavefunctions are expected to be suppressed there due to tunneling and the effect of such "cuts" is likely to be minimal.

III The proof

To simplify the problem, one can cut the line into two halflines and to consider each of them separately. Recall that changing the boundary conditions at a point of $\mathcal{L}$ to the Neumann one means decoupling the Hamiltonian into an orthogonal sum; at the same time it represents a rank–one perturbation in the resolvent which does not change the essential spectrum. Furthermore, without loss of generality we may put $a = 1$, since the character of the spectrum does not change at scaling transformations; for the sake of simplicity we drop then a from the symbols of the operators.

III.1 The resolvent of $H(\{\beta_n\}, E)$

We shall start with the more general operator $H(\{\beta_n\}, E)$ on $L^2(\mathbb{R}^+)$ defined above and compare it to a suitable "decoupled" operator; we assume the Neumann boundary at the origin, $f'(0+) = 0$. First we have to introduce some notation.

Given $z = k^2$, we have in each interval J_n just one solution $u_n \equiv u_n(\cdot, k)$ of the equation $-u''(x) - [Ex + z]u(x) = 0$ which satisfies the boundary conditions $u_n(n+) = 1$, $u_n'(n+) = 0$. Similarly, there is exactly one solution v_n of this equation with $v_n((n+1)-) = 1$, $v_n'((n+1)-) = 0$; their Wronskian equals

$W_n := W(u_n, v_n) = v_n'(n+) = -u_n'((n+1)-)$. Each of these solutions is of the form $f(-E^{1/3}(\cdot + \frac{z}{E}))$, where f is a combination of Airy functions; we assume for definiteness that $E > 0$.

More generally, let $\tilde{\mathcal{L}} := \{n_\ell\}_{\ell=1}^\infty$ be an increasing sequence of integers with $n_1 = 0$, which specifies a sublattice of $\mathcal{L}$. We denote

$$\tilde\beta_n := \begin{cases} \infty & \dots \quad n = n_\ell \\ \beta_n & \dots \quad \text{otherwise} \end{cases}$$

and moreover, $\tilde J_\ell := (n_\ell, n_{\ell+1})$. Then to any $z = k^2$, there is just one function $\tilde u_\ell$ which satisfies in $\tilde J_\ell$ the equation $-u''(x) - [Ex+z]u(x) = 0$ together with the b.c. (1) at the points $n_\ell + 1, \dots, n_{\ell+1} - 1$ and $\tilde u_\ell(n_\ell+) = 1$, $\tilde u_\ell'(n_\ell+) = 0$. Similarly, there is a unique $\tilde v_\ell$ with $\tilde v_\ell(n_{\ell+1}-) = 1$, $\tilde v_\ell'(n_{\ell+1}-) = 0$, and the corresponding Wronskian equals

$$\tilde W_\ell := W(\tilde u_\ell, \tilde v_\ell) = \tilde v_\ell'(n_\ell+) = -\tilde u_\ell'(n_{\ell+1}-).$$

The functions $\tilde u_\ell$, $\tilde v_\ell$ can be again written explicitly in terms of Airy functions but we shall not need it. Next, we define the operator Γ on the weighted Hilbert space $\ell^2(\beta_{n_\ell} n_\ell^{-1})$ by

$$\begin{aligned} \Gamma_{\ell\ell} &:= \frac{1}{\beta_{n_\ell}} \left(\beta_{n_{\ell+1}} - \frac{\tilde v_{\ell+1}(n_{\ell+1}+)}{\tilde W_{\ell+1}} - \frac{\tilde u_\ell(n_{\ell+1}-)}{\tilde W_\ell} \right), \\ \Gamma_{\ell,\ell+1} &= \Gamma_{\ell+1,\ell} := \frac{1}{\sqrt{\beta_{n_\ell}\beta_{n_{\ell+1}}}\,\tilde W_{\ell+1}} \end{aligned} \tag{2}$$

for $\ell = 1, 2, \dots$. This allows us to express the resolvent difference.

Theorem III.1 *For any k for which the operator Γ is invertible, the difference $C := (H(\{\beta_n\}, E) - z)^{-1} - (H(\{\tilde\beta_n\}, E) - z)^{-1}$ is an integral operator with the kernel*

$$\begin{aligned} C(x,y) &= \sum_{\ell,m} \left\{ \frac{\tilde u_\ell(x)}{\sqrt{\beta_{n_\ell}}\,\tilde W_\ell \tilde W_m} \left(\frac{M_{\ell,m-1}\tilde v_m(y)}{\sqrt{\beta_{n_{m-1}}}} - \frac{M_{\ell m}\tilde u_m(y)}{\sqrt{\beta_{n_m}}} \right) \right. \\ &\left. + \frac{\tilde v_\ell(x)}{\sqrt{\beta_{n_{\ell-1}}}\,\tilde W_\ell \tilde W_m} \left(\frac{M_{\ell-1,m}\tilde u_m(y)}{\sqrt{\beta_{n_m}}} - \frac{M_{\ell-1,m-1}\tilde v_m(y)}{\sqrt{\beta_{n_{m-1}}}} \right) \right\}, \end{aligned} \tag{3}$$

where $M := \Gamma^{-1}$.

Proof: The resolvent kernel of the "chopped" operator $H_{\tilde\beta} := H(\{\tilde\beta_n\}, E)$ can be expressed as

$$G_{\tilde\beta}(x,y) = \begin{cases} -\tilde W_\ell^{-1} \tilde u_\ell(x_<)\tilde v_\ell(x_>) & \dots \quad x, y \in \tilde J_\ell \\ 0 & \dots \quad x, y \text{ belong to different } \tilde J_\ell \end{cases} \tag{4}$$

where $x_< := \min(x,y)$, $x_> := \max(x,y)$. We look for the resolvent kernel of $H_\beta := H(\{\beta_n\},E)$ in the form

$$\begin{aligned} G_\beta(x,y) \;=\; G_{\tilde\beta}(x,y) \;&+\; \sum_{\ell,m} \Big[\lambda^{11}_{\ell m}\tilde u_\ell(x)\tilde u_m(y) + \lambda^{12}_{\ell m}\tilde u_\ell(x)\tilde v_m(y) \\ &+\; \lambda^{21}_{\ell m}\tilde v_\ell(x)\tilde u_m(y) + \lambda^{22}_{\ell m}\tilde v_\ell(x)\tilde v_m(y)\Big]\,, \end{aligned} \tag{5}$$

where $\lambda^{jk}_{\ell m}$ are coefficients to be found. Since $H_{\tilde\beta}$ and $H_\beta := H(\{\beta_n\},E)$ are self–adjoint extensions of the same symmetric operator and $\tilde u_\ell$, $\tilde v_\ell$, $\ell = 1,2,\ldots$, span its deficiency subspaces, the last relation may be regarded as an application of Krein's formula. Of course, the latter is conventionally formulated for the case of finite deficiency indices [5, Appendix A] and the existing infinite–dimensional versions require one of the two extensions to be below bounded [12] or at least to have a gap [13] which might not be the case here (*cf.* Conjecture V.1). Fortunately, we do not need it: since the functions $\tilde u_\ell$, $\tilde v_\ell$, with different indices have disjoint supports, the *rhs* of (5) is a finite sum; at the same time it is the most general Ansatz for the sought kernel due to the fact that $G_\beta(x,\cdot)$ and $G_\beta(\cdot,y)$ have to solve the Schrödinger equation [14].

The coefficients satisfy the obvious symmetry requirements

$$\lambda^{jj}_{\ell m} = \lambda^{jj}_{m\ell}\,, \quad \lambda^{12}_{\ell m} = \lambda^{21}_{m\ell}\,.$$

To find them, notice that by definition $(H_\beta - z)^{-1}$ maps $L^2(I\!R^+)$ into $D(H_\beta)$; in particular, $f := G_\beta g$ must satisfy the boundary conditions (1) for any $g \in L^2(I\!R^+)$ at each point of $\tilde{\mathcal{L}}$. The relations (4) and (5) give

$$\begin{aligned} f(x) \;&=\; -\,\tilde W_\ell^{-1}\tilde u_\ell(x)\int_x^{n_{\ell+1}}\tilde v_\ell(y)g(y)\,dy \;-\; \tilde W_\ell^{-1}\tilde v_\ell(x)\int_{n_\ell}^x \tilde u_\ell(y)g(y)\,dy \\ &\quad+\sum_m \Big(\lambda^{11}_{\ell m}\tilde u_\ell(x)h^u_m + \lambda^{12}_{\ell m}\tilde u_\ell(x)h^v_m + \lambda^{21}_{\ell m}\tilde v_\ell(x)h^u_m + \lambda^{22}_{\ell m}\tilde v_\ell(x)h^v_m\Big) \end{aligned}$$

for $x \in \tilde J_\ell$, where

$$h^u_m := \int_{\tilde J_m}\tilde u_m(y)g(y)\,dy\,, \quad h^v_m := \int_{\tilde J_m}\tilde v_m(y)g(y)\,dy\,.$$

We express from here the boundary values, substitute into (1) and use the fact that h^u_m, h^v_m are independent quantities; this yields the following infinite system of linear equations

$$\lambda^{2,j}_{\ell m}\tilde v'_\ell(n_\ell+) - \lambda^{1,j}_{\ell-1,m}\tilde u'_{\ell-1}(n_\ell-) \;=\; 0\,,$$

$$\lambda^{11}_{\ell m} + \lambda^{21}_{\ell m}\tilde v_\ell(n_\ell+) - \lambda^{11}_{\ell-1,m}\tilde u_{\ell-1}(n_\ell-) - \lambda^{21}_{\ell-1,m} + \delta_{\ell-1,m}\tilde W^{-1}_{\ell-1} \;=\; \beta_{n_\ell}\lambda^{21}_{\ell m}\tilde v'_\ell(n_\ell+)\,,$$

$$\lambda^{12}_{\ell m} + \lambda^{22}_{\ell m}\tilde v_\ell(n_\ell+) - \lambda^{12}_{\ell-1,m}\tilde u_{\ell-1}(n_\ell-) - \lambda^{22}_{\ell-1,m} - \delta_{\ell m}\tilde W^{-1}_\ell \;=\; \beta_{n_\ell}\lambda^{22}_{\ell m}\tilde v'_\ell(n_\ell+)\,,$$

where $j = 1,2$, $\ell = 2,3,\dots$ and $m = 1,2,\dots$, together with the boundary requirement $\lambda^{2,j}_{1,m} = 0$. This can be simplified further by choosing

$$\lambda^{11}_{\ell m} \;=\; -\,\frac{M_{\ell m}}{\sqrt{\beta_{n_\ell}\beta_{n_m}}\tilde W_\ell \tilde W_m}\,, \qquad \lambda^{21}_{\ell m} \;=\; \frac{M_{\ell-1,m}}{\sqrt{\beta_{n_{\ell-1}}\beta_{n_m}}\tilde W_\ell \tilde W_m}\,,$$

$$\lambda^{12}_{\ell m} \;=\; \frac{M_{\ell,m-1}}{\sqrt{\beta_{n_\ell}\beta_{n_{m-1}}}\tilde W_\ell \tilde W_m}\,, \qquad \lambda^{22}_{\ell m} \;=\; -\,\frac{M_{\ell-1,m-1}}{\sqrt{\beta_{n_{\ell-1}}\beta_{n_{m-1}}}\tilde W_\ell \tilde W_m}$$

for $\ell,m = 1,2,\dots$. The first two sets of equations are then satisfied identically, while the third one can be written with the help of the identitites $\tilde W^{-1}_{\ell-1}\delta_{\ell-1,m} = \tilde W^{-1}_m \delta_{\ell-1,m}$ and $\sqrt{\beta_{n_{\ell-1}}}\,\delta_{\ell-1,m} = \sqrt{\beta_{n_m}}\,\delta_{\ell-1,m}$ as

$$\frac{M_{\ell m}}{\sqrt{\beta_{n_\ell}\beta_{n_{\ell-1}}}\tilde W_\ell} \;-\; \frac{M_{\ell-1,m}}{\beta_{n_{\ell-1}}}\left(\beta_\ell - \frac{\tilde v_\ell(n_\ell+)}{\tilde W_\ell} - \frac{\tilde u_{\ell-1}(n_\ell-)}{\tilde W_{\ell-1}}\right)$$
$$+\; \frac{M_{\ell-2,m}}{\sqrt{\beta_{n_{\ell-1}}\beta_{n_{\ell-2}}}\tilde W_{\ell-1}} \;=\; \delta_{\ell-1,m}\,,$$

or $(\Gamma M)_{\ell-1,m} = \delta_{\ell-1,m}$. The fourth set of equations leads to the same conclusion; since Γ is invertible by assumption, we have $M = \Gamma^{-1}$. $\blacksquare$

Before proceeding further, let us comment on the hypothesis of the theorem. The invertibility of Γ is related naturally to spectral properties of H_β. To see this, we write the solution to the Schrödinger equation in question for a given k as

$$f(x) \;=\; -\,\tilde W_\ell^{-1} f'(n_{\ell+1})\tilde u_\ell(x) \;+\; \tilde W_\ell^{-1} f'(n_\ell)\tilde v_\ell(x)$$

if $x \in \tilde J_\ell$, $\ell = 1,2,\dots$; it is unique up to a multiplicative constant. A straightforward modification of the argument from [5, Sec.III.2.1] shows that f satisfies the conditions

$$\sum_{m=1}^\infty \Gamma_{\ell m}\sqrt{\beta_{n_m}}\, f'(n_{m+1}) \;=\; 0 \tag{6}$$

for any $\ell = 1,2,\dots$. The operator H_β has an eigenvalue *iff* $f \in L^2(I\!R^+)$. In the situation considered below, *i.e.*, for $\mathrm{Im}\, z$ large enough and the cutting points chosen in the tilted gaps, this is further equivalent to $\{f'(n_m)\} \in \ell^2(\beta_{n_m}k^{-2}_{n_m})$ — *cf.* (11) and (13). Hence Γ is invertible in this case; below we shall demonstrate directly that it is bounded invertible.

III.2 A trace–class estimate

As we have said, the proof of Theorem II.1 can be split into two halfline problems. The growing–potential case is easy. The Kronig–Penney operator without an electric field is below bounded, $H(\beta,0) \geq -c$ for some $c \geq 0$. Changing the boundary condition to Neumann at a point $-n$ does not change the essential spectrum and a finite interval does not contribute to it, so

$$\inf \sigma_{ess}(H(\beta, E)) \geq -c + En\,.$$

Since n can be chosen arbitrarily, the spectrum is pure point.

Let us turn to the more difficult case when the potential decreases along the halfline. If the b.c. (1) are changed to the Neumann one at infinitely many points, the corresponding operator $H(\{\tilde\beta_n\}, E)$ has certainly a pure point spectrum. Our aim is to show that for a suitably chosen sequence $\{n_\ell\}$ and some $z \in \mathbb{C}$, the resolvent difference of Theorem III.1 is a trace–class operator, in which case the sought result follows from the Birman–Kuroda theorem [15, Sec.XI.3].

We denote $z = \rho + i\nu$, where both $\rho,\, \nu$ will be chosen large positive. If ρ is large, the asymptotic properties of the Airy functions [16, Chap.10] show that the elementary solutions introduced above are in the intervals J_n of the form

$$u_n(x) = \cos k_n(x-n)\,(1+\mathcal{O}(k_n^{-1}))\,, \quad v_n(x) = \cos k_n(x-n-1)\,(1+\mathcal{O}(k_n^{-1}))\,, \tag{7}$$

where $k_n := \sqrt{z + E\overline{x}_n}$ for some $\overline{x}_n \in J_n$. The momentum values behave asymptotically as

$$k_n \;=\; \left(\sqrt{E}\, n^{1/2} + \frac{z}{2\sqrt{E}}\, n^{-1/2}\right)\,(1+\mathcal{O}(n^{-1})) \tag{8}$$

for $n\to\infty$. The Wronskian is then $W_n \;=\; k_n \sin k_n\,(1+\mathcal{O}(n^{-1/2}))$; since z is complex, it is never zero and one can estimate it by

$$|W_n| \;\geq\; \frac{\nu}{2}\,\left(1+\mathcal{O}(n^{-1/2})\right)\,. \tag{9}$$

We also need a bound for

$$\frac{W_{n-1}}{W_n} \;=\; \left(1 + \frac{\sin k_{n-1} - \sin k_n}{\sin k_n}\right)\,\left(1+\mathcal{O}(n^{-1/2})\right)\;;$$

it is clear that one should pay attention only to the vicinity of the points where $\mathrm{Re}\, k_n \approx \pi\ell$. There we have

$$\left|\frac{\sin k_{n-1} - \sin k_n}{\sin k_n}\right| \;\leq\; 2\left(\frac{E}{\nu} + \frac{1}{n}\right)\,\left(1+\mathcal{O}(n^{-1})\right)\,,$$

which yields the bound

$$\left|\frac{W_{n-1}}{W_n} - 1\right| \;\leq\; \frac{2E}{\nu} \;+\; \mathcal{O}(n^{-1/2})\,. \tag{10}$$

The functions $\tilde u_\ell,\, \tilde v_\ell$ can be estimated using "transfer matrices" relating the elementary solutions in neighboring intervals. A general solution of the equation

$-f''(x)-[Ex+z]f(x)=0$ in J_n has the form $f_n(x)=\xi_n u_n(x)+\eta_n v_n(x)$; we look for T_n such that

$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = T_n \begin{pmatrix} \xi_{n-1} \\ \eta_{n-1} \end{pmatrix} .$$

Using the b.c. (1), one proves easily

$$T_n = \begin{pmatrix} -\beta_n W_{n-1} + \frac{W_{n-1}}{W_n} v_n(n+) + u_{n-1}(n-) & 1 \\ -\frac{W_{n-1}}{W_n} & 0 \end{pmatrix},$$

$$T_n^{-1} = \begin{pmatrix} 0 & -\frac{W_n}{W_{n-1}} \\ 1 & -\beta_n W_n + v_n(n+) + \frac{W_n}{W_{n-1}} u_{n-1}(n-) \end{pmatrix};$$

in our case $\beta_n=\beta$. Using now (7)–(10), we get

$$\begin{aligned} |(T_n)_{11}| &\geq |\beta|\,|W_{n-1}| - |u_{n-1}(n-)+v_n(n+)| - |v_n(n+)| \left(\frac{2E}{\nu} + \mathcal{O}(n^{-1/2})\right) \\ &\geq \left(\frac{\beta\nu}{2} - 2 - \frac{2E}{\nu}\right) \left(1+\mathcal{O}(n^{-1/2})\right), \end{aligned}$$

so choosing ν large enough, the *rhs* can be made positive and sufficiently large as $n\to\infty$. At the same time,

$$|(T_n)_{21}| \leq 1 + \frac{2E}{\nu} + \mathcal{O}(n^{-1/2}) .$$

Since the coefficients of $\binom{\xi_n}{\eta_n}$ corresponding to $\tilde u_\ell(x)$ start from $\binom{1}{0}$ at $x=n_\ell$, the above bounds tell us that for all sufficiently large n there is a number $d>1$ independent of n such that

$$|\xi_n| \geq d^j |\xi_{n-j}| , \qquad |\xi_n| \geq d|\eta_n| . \tag{11}$$

The norm of $\tilde u_\ell$ can be then estimated as follows,

$$\|\tilde u_\ell\|^2 \leq 2 \sum_{n=n_\ell}^{n_{\ell+1}-1} (|\xi_n|^2+|\eta_n|^2) \leq 2\,|\xi_N|^2\,(1+d^{-2}) \sum_{j=0}^{N'} d^{-2j} ,$$

where $N:=n_{\ell+1}-1$ and $N':=n_{\ell+1}-n_\ell-1$. Hence there is a positive d' independent of ℓ such that $\|\tilde u_\ell\|\leq d'|\xi_{n_{\ell+1}-1}|$. The coefficients of $\tilde v_\ell$ behave similarly if we change the direction and switch the roles of ξ_n, η_n ; this yields the inequality $\|\tilde v_\ell\|\leq d'|\eta_{n_\ell}|$.

For simplicity, we introduce $U_\ell := -\frac{\tilde u_\ell(\cdot)}{\tilde u'_\ell(n_{\ell+1}-)}$ and $V_\ell := \frac{\tilde v_\ell(\cdot)}{\tilde v'_\ell(n_\ell+)}$. The denominators depend clearly on the choice of the sequence $\{n_\ell\}$. We pick its points roughly in "the middle of the gaps" assuming, *e.g.*,

$$\left| k_{n_\ell} - \pi\left(\ell+\frac{1}{2}\right)\right| \leq \frac{\pi}{4} , \tag{12}$$

so that $n_\ell = \frac{\pi^2\ell^2}{E}\,(1+\mathcal{O}(\ell^{-1}))$. Since $\tilde v_{n_{\ell+1}-1}(n_{\ell+1}-) = 0$ by assumption, we have in view of (7) and (12)

$$|\tilde u'_\ell(n_{\ell+1}-)| \,\ge\, \frac{1}{\sqrt2}\,k_{n_{\ell+1}-1}|\xi_{n_{\ell+1}-1}|\,\left(1+\mathcal{O}(\ell^{-1})\right)\,.$$

In the same way, $|\tilde v'_\ell(n_\ell+)|$ has a bound proportional to $|\eta_{n_\ell}|$; using the above inequalities for the norms of these functions, we get

$$\|U_\ell\| \,\le\, \frac{\sqrt2\,d'}{k_{n_{\ell+1}-1}}\,\left(1+\mathcal{O}(\ell^{-1})\right)\,, \qquad \|V_\ell\| \,\le\, \frac{\sqrt2\,d'}{k_{n_\ell}}\,\left(1+\mathcal{O}(\ell^{-1})\right)\,. \tag{13}$$

To estimate the difference of the resolvents, we need also a bound for the coefficients $M_{\ell m}$ in (3). Since all the β_n are the same by assumption, the operator (2) can be written as $\Gamma = I + N$ with

$$N_{\ell\ell} \,:=\, -\beta^{-1}(U_\ell(n_{\ell+1}-)+V_{\ell+1}(n_{\ell+1}+))\,, \qquad N_{\ell,\ell+1} \,=\, N_{\ell+1,\ell} \,:=\, \beta^{-1}\tilde W_\ell^{-1}\,.$$

The above estimates then yield

$$\begin{aligned}|N_{\ell\ell}| \,&\le\, \frac{2}{\pi\ell\beta}\left(1+\frac{\sqrt2}{d}\right)\left(1+\mathcal{O}(\ell^{-1})\right)\,,\\ |N_{\ell,\ell+1}| \,&\le\, \frac{\sqrt2}{\pi\ell\beta}\,d^{-N'}\left(1+\mathcal{O}(\ell^{-1})\right) \,\le\, \frac{2}{\pi\ell\beta}\,e^{-2\ell\pi^2E^{-1}\ln d}\,.\end{aligned}$$

Recall again that a contribution from a finite interval cannot affect the essential spectrum. Hence we may consider $\{n_\ell\}$ with ℓ starting at a sufficiently large value, so that $\|N\|<1$. The inverse is then easily computed as a geometric series,

$$M_{\ell m} \,=\, \delta_{\ell m} \,-\, N_{\ell m} \,+\, \sum_{s=2}^{\infty}(-1)^s \sum_{r_2,\dots,r_s} N_{\ell r_2}N_{r_2r_3}\dots N_{r_sm}\,.$$

Since the elements of the r–th side diagonal contain at least r off–diagonal elements of N (recall that the latter is tridiagonal!), there are positive $C,\,d''$ such that

$$|M_{\ell,\ell+r}| \,\le\, C\,e^{-d''|r|}\,. \tag{14}$$

Putting the estimates (13) and (14) together in combination with (8) and (12), we get

$$\begin{aligned}\mathrm{Tr}\,|C| \,&\le\, \frac{1}{\beta}\sum_{\ell,m}\Big\{|M_{\ell m}|\,\|U_\ell\|\,\|U_m\| + |M_{\ell,m-1}|\,\|U_\ell\|\,\|V_m\|\\ &\quad +\, |M_{\ell-1,m}|\,\|V_\ell\|\,\|U_m\| + |M_{\ell-1,m-1}|\,\|V_\ell\|\,\|V_m\|\Big\}\\ &\le\, \frac{C(2d')^2}{\pi^2\beta}\sum_\ell\left\{\frac{1}{\ell}\sum_{r\ge0}\left(\frac{1}{\ell+r}+\frac{1}{\ell+1+r}\right)e^{-d''r}\right.\\ &\quad \left. +\, \frac{1}{\ell-1}\sum_{r\ge0}\left(\frac{1}{\ell+r}+\frac{1}{\ell-1+r}\right)e^{-d''r}\right\} \,<\, \infty\,,\end{aligned}$$

what we wanted to prove. ∎

IV Generalizations

The presented proof of Theorem II.1 extends easily at least in two directions:

(i) *the background potential need not be linear:* let $H_V := -\frac{d^2}{dx^2} + V(x)$ be the "free" Schrödinger operator with a potential V to be specified below; then we can define $H(\{\beta_n\}, V, a)$ as in Section 2, *i.e.*, through the b.c.(1).

(ii) one may consider a lattice of *non–identical* δ' *interactions*, *i.e.*, a general sequence $\{\beta_n\}$.

For simplicity, let us consider again only the halfline problem with a decreasing potential and the lattice spacing $a = 1$. The key observation is that the asymptotics (7) of the elementary solutions comes in fact from the WKB expansion, and therefore it is valid for any potential V which is decreasing and regular enough provided we define $k_n := \sqrt{z - V(\overline{x}_n)}$. Assume, *e.g.*, that

(a) V is locally bounded and there is x_0 such that for all $x > x_0$ one has $V(x) = -U(x) + W(x)$, where

  (a1) U is nondecreasing with $\lim_{x\to\infty} U(x) = \infty$,

  (a2) U is C^2 smooth with $|U'(x)| \le c$ and $|U''(x)| \le \tilde{c}U(x)$ for some $c, \tilde{c} > 0$,

  (a3) W is piecewise continuous and bounded, $\|W\|_\infty < \infty$.

Using the standard WKB error estimates [17, Sec.6.2] we can replace (7) by

$$u_n(x) = \cos k_n(x-n)\,(1 + \mathcal{O}(U(x)^{-1/2}))$$

and an analogous expression for $v_n(x)$; the momentum behaves asymptotically as

$$k_n = \left(\sqrt{U(\overline{x}_n)} + \frac{z - W(\overline{x}_n)}{2\sqrt{U(\overline{x}_n)}} \right) (1 + \mathcal{O}(U(\overline{x}_n)^{-1}))\,.$$

Since U' is bounded by (a2), the key estimate (10) is replaced by

$$\left| \frac{W_{n-1}}{W_n} - 1 \right| \le \left\{ \frac{4(c + \|W\|_\infty)}{\nu} + \frac{4c}{|U(n-1)|} \right\} (1 + \mathcal{O}(U(n)^{-1}))\,,$$

so the "transfer–matrix" analysis may be repeated. As for the "coupling constants", we adopt the following assumptions:

(b) $|\beta_n| \ge \beta > 0$ for all n,

(c) there is a monotonic sequence $\{n_\ell\} \subset \mathbb{Z}_+$ such that $\mathrm{Re}\, k_{n_\ell} = \pi(n_\ell + \epsilon_\ell)$ with $\epsilon_\ell \in (\frac{1}{4}, \frac{3}{4})$, and $\beta_{n_\ell}\beta_{n_{\ell+1}}^{-1}$ remains bounded as $n_\ell \to \infty$, The matrix elements (2) of the operator Γ can be now written as

$$\Gamma_{\ell m} = \frac{1}{\sqrt{\beta_{n_\ell}}} (B+S)_{\ell m} \frac{1}{\sqrt{\beta_{n_m}}} \,,$$

where $B := \mathrm{diag}\,\{\ldots, \beta_{n_{\ell+1}}, \ldots\}$, and $S_{\ell\ell} := -U_\ell(n_{\ell+1}-) - V_{\ell+1}(n_{\ell+1}+)$, $S_{\ell,\ell+1} = S_{\ell+1,\ell} := \tilde{W}_\ell^{-1}$. If the sequence $\{n_\ell\}$ starts with ℓ large enough, $\|B^{-1}N\| < 1$ and the inverse is given by

$$\begin{aligned} M_{\ell m} &= \beta_{n_\ell}\beta_{n_{\ell+1}}^{-1}\delta_{\ell m} - \sqrt{\beta_{n_\ell}}\beta_{n_{\ell+1}}^{-1} S_{\ell m}\beta_{n_{m+1}}^{-1}\sqrt{\beta_{n_m}} \\ &+ \sum_{s=2}^{\infty}(-1)^s \sum_{r_2,\ldots,r_s} \sqrt{\beta_{n_\ell}}\beta_{n_{\ell+1}}^{-1} S_{\ell r_2}\beta_{n_{r_2+1}}^{-1} \ldots S_{r_s m}\beta_{n_{m+1}}^{-1}\sqrt{\beta_{n_m}} \,. \end{aligned}$$

Since the numbers $\beta_{n_\ell}\beta_{n_{\ell+1}}^{-1}$ remain bounded at the cutting points in view of (c), the estimate (14) holds again.

In this way, we arrive at the following result.

Theorem IV.1 *Let $H(\{\beta_n\}, V, a)$ be the operator on $L^2(I\!R^+)$ defined above, with V and $\{\beta_n\} \subset I\!R$ satisfying the assumptions (a)–(c). Then $\sigma_{ac}(H(\{\beta_n\}, V, a) = \emptyset$.*

The result extends easily to operators on the whole real line with potentials growing in the other direction, because for the other halfline we have used just the fact that the spectrum is purely discrete when the potential tends to infinity. A more involved argument is required, however, if the sequence $\{\beta_n\}$ may approach zero, since then the point–interaction Hamiltonian without the presence of an external field need not be below bounded.

V Concluding remarks

The assumptions used here are clearly not optimal, but we are not going to push the argument further. Let us remark instead that the mentioned splitting of the problem on the line into two halflines together with Proposition IV.1 shows that the absolutely continuous spectrum is void not only for δ' Wannier–ladder "slopes" but for "hills" with the potential decreasing in both directions as well.

In addition to finding weaker restrictions on the potential decay, other generalizations are possible. For instance, one can treat similarly arrays of δ'–interactions, where the spacing assumes a finite number of different values. Furthermore, δ and δ' are extreme cases in the general four–parameter class of one–dimensional point interactions [7, 8, 18]. The behavior of bands and gaps in the absence of the external force suggests that the result will remain valid at least as long as there is a nonzero δ' component in such an interaction, *i.e.*, a discontinuity of the wavefunction at the lattice points which depends on the one–sided derivatives.

Having excluded the absolutely continuous spectrum, one asks naturally how the other parts of the spectrum look like. Consider again our basic model, *i.e.*, the operator $H(\beta, E, a)$ for nonzero E, β. The above proof shows, in particular, that the essential spectrum of $H(\beta, E, a)$ does not change if the system is "chopped" by imposing the Neumann condition at a properly chosen sequence $\{n_\ell a\}$. The discrete spectra of the corresponding "finite sections" of $H(\beta, E, a)$ can be found numerically [19]; in this way we arrive at the following

Conjecture V.1 *For nonzero* E, β,

$$\sigma_{ess}(H(\beta, E, a)) = \left\{ \frac{4}{\beta a} + \left(\frac{m\pi}{a}\right)^2 - E\left(n + \frac{1}{2}\right) a \;:\; m, n \in \mathbb{Z} \right\}.$$

In other words, the essential–spectrum points are sums of three terms: the energy step of the ladder, the eigenvalues of $H(\infty, E, a)$, and the asymptotic *halfwidth* of a band. Recall that $\sigma_{ess}((H(\beta, E, a))$ consists of all accumulation points of the spectrum, since a second–order differential operator cannot have eigenvalues of infinite multiplicity. If the conjecture is true, the spectrum exhibits an intriguing dependence on the number–theoretic properties of the external field, namely

(a) for $\gamma := \left(\frac{a}{\pi}\right)^2 Ea$ is rational, the spectrum is nowhere dense, and therefore automatically pure point.

(b) on the other hand, if γ is irrational, $\sigma((H(\beta, E, a)) = \sigma_{ess}((H(\beta, E, a)) = \mathbb{R}$.

In this way, δ' Wannier–Stark systems still represent a challenge.

Acknowledgement

As mentioned above the problem as well as the strategy of its solution resulted from collaboration with J.E. Avron and Y. Last to whom I am deeply indebted. I am grateful for the hospitality extended to me at the Israel Institute of Technology where the work started, and to the E. Schrödinger Institute in Vienna, where later parts were done. I am obliged to M. Tater for assistance in the numerical analysis supporting Conjecture V.1 and to the referee for helpful comments. The research has been supported in part by the AS CR Grant No. 148409.

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[19] There is also a formal argument supporting the conjecture which comes from rephrasing the problem in terms of discrete Schrödinger operators à la mode (6) (J.E. Avron, private communication).