GRAVITATIONAL IONIZATION: PERIODIC ORBITS OF BINARY SYSTEMS PERTURBED BY GRAVITATIONAL RADIATION

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Abstract. The long term perturbation of a Newtonian binary system by an incident gravitational wave is discussed in connection with the issue of gravitational ionization. The periodic orbits of the planar tidal equation are investigated and the conditions for their existence are presented. The possibility of ionization of a Keplerian orbit via gravitational radiation is discussed.

1. Introduction

A Newtonian two-body system cannot be completely isolated from all other masses in the universe as a consequence of the universality of the gravitational interaction. In fact, the attraction of the other masses would cause the binary system to move through approximately inertial spacetime. This center-of-mass motion should be distinguished from the relative motion, which is affected by the gradient of the disturbing forces. Consider, for instance, the equations of motion for an "isolated" two-body system in Newtonian mechanics

$$\begin{aligned} m_1\frac{d^2\mathbf{X}_1}{dt^2}+\frac{G_0m_1m_2}{|\mathbf{X}_1-\mathbf{X}_2|^3}(\mathbf{X}_1-\mathbf{X}_2) &= -\,m_1\nabla\Phi(\mathbf{X}_1),\\ m_2\frac{d^2\mathbf{X}_2}{dt^2}+\frac{G_0m_1m_2}{|\mathbf{X}_1-\mathbf{X}_2|^3}(\mathbf{X}_2-\mathbf{X}_1) &= -\,m_2\nabla\Phi(\mathbf{X}_2), \end{aligned} \tag{1}$$

where G_0 is Newton's constant of gravitation and

$$\Phi(\mathbf{X}):=-\sum_p\frac{G_0m_p}{|\mathbf{X}-\mathbf{X}_p|}$$

represents the combined gravitational potential of all other masses m_p at $\mathbf{X}_p$ in the universe. Here and throughout this work, the finite size of

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astronomical bodies is neglected. If, in the inertial space coordinates (X^1, X^2, X^3), the binary system is so far away from the other masses that the relative distance between the masses comprising the binary is very small compared to the distance of the center of mass of the binary to the external masses, then, to first order in this small ratio, the equation of relative motion has the form

$$\frac{d^2 r^i}{dt^2} + \frac{k r^i}{r^3} = -K_{ij}(t) r^j, \tag{2}$$

where $\mathbf{r} = (r^1, r^2, r^3) := \mathbf{X}_1 - \mathbf{X}_2$, r is the length of $\mathbf{r}$, and $k = G_0(m_1 + m_2)$. Here, K_{ij}, the tidal matrix, is given by

$$K_{ij} = \frac{\partial^2 \Phi}{\partial X^i \partial X^j}$$

evaluated at the center of mass of the binary system. In Newtonian mechanics the gravitational potential Φ is a harmonic function; therefore, the symmetric tidal matrix is trace-free.

It turns out that (2) holds approximately in general relativity as well, except that K_{ij} would be represented by the "electric" components of the Riemannian curvature of the underlying spacetime projected onto a Fermi frame along the center-of-mass worldline [9, 10]. That is, the equation of relative motion can be considered to be the Newton-Jacobi equation in the sense that once the internal Newtonian attraction is neglected, equation (2) reduces to the Jacobi equation in Fermi normal coordinates for the relative motion of two neighboring geodesics in the underlying spacetime manifold. Thus the spacetime coordinates in (2) refer to a local Fermi system established along the path of the center of mass of the system. In our approximate treatment, we neglect relativistic effects in the binary system. On the other hand, the external influences may now include gravitational radiation. It should be noted in this connection that classical celestial mechanics has been mainly concerned with the n-body problem; however, the "vacuum" between these bodies is expected to abound with gravitational radiation as well as with other radiation fields. It is therefore interesting to consider the interaction of gravitational waves with n-body systems, since it is estimated that half of all stars are members of binary or multiple systems.

In this paper, attention is focused on a Newtonian binary system that undergoes perturbation due to an incident gravitational wave. Let the spacetime metric due to the gravitational wave be given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon \chi_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric, ϵ is the strength of the perturbation, $0 < \epsilon \ll 1$, and $\chi_{\mu\nu}$ represents the gravitational radiation field. In the transverse traceless gauge, $\chi_{0\mu} = 0$ and χ_{ij} is a symmetric traceless matrix that satisfies the wave equation $\Box^2 \chi_{ij} = 0$ and the transversality condition $\partial_j \chi_{ij} = 0$. It turns out that in this gauge,

$$K_{ij}(t) = -\frac{1}{2}\epsilon \frac{\partial^2 \chi_{ij}}{\partial t^2}(t, \mathbf{X}_{\text{cm}}), \tag{3}$$

where $(m_1 + m_2)\mathbf{X}_{\text{cm}} = m_1 \mathbf{X}_1 + m_2 \mathbf{X}_2$. It is possible to fix the position of the center of mass (e.g., $\mathbf{X}_{\text{cm}} = \mathbf{0}$) in the approximation under consideration here, since K_{ij} is considered only to first order in ϵ. The perturbing field χ_{ij} may be expressed as a Fourier sum of plane monochromatic waves with wavelengths much larger than the semimajor axis of the binary system. Such waves could be generated by the motion of masses during the Hubble expansion, or could be primordial waves left over from the big bang era. It is therefore important to note that in our analysis there is no need to specify the *initial conditions* for the interaction of the waves with the binary; instead, we concentrate on the "steady-state" situation involving a dominant plane wave of frequency Ω incident on the binary. Hence, the symmetric and traceless tidal matrix in equation (2) is given in Cartesian coordinates by

$$\begin{aligned}
K_{11} =& \epsilon\alpha\Omega^2 \cos^2\Theta \cos(\Omega t), \\
K_{12} =& \epsilon\beta\Omega^2 \cos\Theta \cos(\Omega t + \rho), \\
K_{13} =& -\epsilon\alpha\Omega^2 \cos\Theta \sin\Theta \cos(\Omega t), \\
K_{22} =& -\epsilon\alpha\Omega^2 \cos(\Omega t), \\
K_{23} =& -\epsilon\beta\Omega^2 \sin\Theta \cos(\Omega t + \rho),
\end{aligned}$$

where α, β, and ρ are constants, and Θ is the polar angle from the normal to the plane of the unperturbed orbit to the propagation vector of the incident radiation. However, for the sake of simplicity we will explicitly consider only the case of normal incidence, i.e. $\Theta = 0$. In this case, the orbital plane will remain fixed; that is, when the waves are normally incident, the problem reduces to planar motion under the approximations considered here.

Let the unperturbed Keplerian motion be confined to the (x, y)-plane. The transverse nature of the radiation field implies that the orbital plane will be unchanged under the perturbation of normally incident waves. Thus in (2), we have $r^1 = x$, $r^2 = y$, and $r^3 = 0$. The nonzero elements of the tidal matrix for our single-frequency radiation are

$$\begin{aligned} K_{11} =& -K_{22} = \epsilon\alpha\Omega^2 \cos(\Omega t), \\ K_{12} =& \quad K_{21} = \epsilon\beta\Omega^2 \cos(\Omega t + \rho), \end{aligned}$$

where $\epsilon\alpha$ and $\epsilon\beta$ represent the amplitudes of the two independent linear polarization states of the low-frequency gravitational wave, and ρ represents the constant phase difference between them.

The justification for replacing the actual problem with this rather simplified nonlinear model is that it becomes amenable to mathematical analysis. It should also be remarked that—within the limitations discussed in this section—equation (2) for the relative motion holds generally in the Fermi coordinate system established along the center-of-mass worldline. Thus, for this system, $K_{ij} = R_{0i0j}$, where $R_{\mu\nu\rho\sigma}$ denotes the Riemannian curvature due to external sources projected onto the frame of the center of mass. In the Newtonian limit of general relativity, each K_{ij} reduces to a second partial derivative of the external Newtonian potential $\Phi(\mathbf{X})$ evaluated along the path of the center of mass. On the other hand, for a weak gravitational wave, equation (2) holds to first order in the amplitude of the gravitational potential. Thus, in general, the matrix (K_{ij}) is a function of the proper time along the path of the center of mass. It is always possible to diagonalize this symmetric matrix; however, its dependence upon time implies that (2) must then be written in a rotating system of coordinates.

In electrodynamics, the interaction of electromagnetic radiation with a two-body system constitutes a basic problem (e.g., the scattering and absorption of light by the Rutherford-Bohr atom). The gravitational analog of this problem in the classical regime would involve the scattering and absorption of gravitational radiation by a Keplerian two-body system. The wavelength of light is much larger than the Bohr radius of the atom; therefore, the dominant interaction takes place via the electric dipole moment of the atom since electromagnetism is a spin - 1 field. We expect by analogy that for gravitational radiation with a (reduced) wavelength that is much larger than the semimajor axis of the Keplerian orbit, the dominant interaction would involve the mass quadrupole moment of the binary since gravitation is a spin - 2 field. This approximation corresponds precisely to dropping higher-order terms in the tidal equation (2).

The reciprocity between emission and absorption of radiation should be noted. In the quadrupole approximation for the emission of gravitational radiation, the waves carry away energy and angular momentum but not linear momentum. The same holds in the inverse process as well, except that in general the system can gain or lose energy. Moreover, a Newtonian binary system emits gravitational radiation of frequency $\Omega = m\omega$, $m = 1, 2, 3 \ldots$, where ω is the Keplerian frequency of the elliptical orbit. Similarly, resonant absorption of gravitational waves by an elliptical binary occurs at $\Omega = m\omega$, $m = 1, 2, 3 \ldots$, according to the linear perturbation analysis [10].

A two-body system continuously emits gravitational radiation according to general relativity. Gravitational energy in the form of radiation is thus carried away from the system. Hence, the relative orbit evolves in such a way that the semimajor axis of the osculating ellipse monotonically shrinks. This phenomenon of inward spiraling of the members of the binary is consistent with the timing observations of the Hulse-Taylor binary pulsar [7] [14]. Direct observational evidence for gravitational radiation does not exist at present; however, efforts are under way to detect in the laboratory gravitational waves emitted by astrophysical sources with $\epsilon \approx 10^{-20}$.

In this work, we will ignore the emission of gravitational radiation by the binary system, and concentrate our attention instead on the absorption process. The flow of energy between the incident radiation and the binary is not unidirectional, however. The self-gravitating system can absorb energy from the radiation field or deposit energy into the wave so as to induce an amplification of the radiation. These issues were first discussed in connection with the problem of ionization [10] in the context of linear perturbation analysis that in general breaks down over time. Here we employ the concepts introduced by Poincaré [12] for the treatment of nonlinear problems. These enable us to prove that periodic orbits exist in the perturbed system for which energy must steadily flow back and forth between the wave and the binary. It is important to emphasize that such periodic orbits occur near resonance conditions when certain definite phase relationships are satisfied. If the binary system monotonically absorbs energy from the wave, then the semimajor axis of the osculating ellipse will grow with time and the system eventually ionizes. We provide a qualitative picture for such a process in § 6.

The gravitational quadrupole interaction may be illustrated by considering the Hamiltonian for the relative motion

$$\mathcal{H} = \frac{1}{2}p^2 - \frac{k}{r} + \frac{1}{6}K_{ij}(t)Q_{ij}, \tag{4}$$

where the quadrupole moment per unit mass is defined by $Q_{ij} = 3r^i r^j - r^2\delta_{ij}$ and in the most general case of a normally incident gravitational wave packet considered in this paper the matrix K_{ij} is a traceless symmetric matrix of periodic functions. Thus $K_{11} = -K_{22} = h_1(t)$, $K_{12} = K_{21} = h_2(t)$, and there exist τ_1 and τ_2 such that $h_1(t) = h_1(t+\tau_1)$ and $h_2(t) = h_2(t+\tau_2)$. Here h_1 and h_2 represent the amplitudes of the two linearly independent polarization states of the perpendicularly incident gravitational wave. Now let

$$\mathcal{E} = \frac{1}{2}p^2 - \frac{k}{r}, \quad \mathcal{L}^i = \epsilon_{ijk}r^j p^k, \quad \eta^i = \epsilon_{ijk}p^j\mathcal{L}^k - \frac{k}{r}r^i$$

denote the Newtonian energy, orbital angular momentum and the Runge-Lenz vector associated with the relative motion of the system (per unit reduced mass); then, these otherwise conserved quantities vary as a consequence of the coupling of the quadrupole moment of the system to the curvature of the background spacetime. Thus, in this quadrupole approximation, the Keplerian orbit exchanges energy and angular momentum with the radiation field. At each instant, the relative motion can be described by the orbital elements of the osculating ellipse. This osculating ellipse continuously makes transitions to other osculating Keplerian orbits with different energies and angular momenta as a consequence of interaction with the external wave. It is interesting to describe the path of the system in the six-dimensional manifold of orbital elements; in fact, this paper is devoted to the description of periodic paths in this manifold.

The interaction of gravitational radiation with matter may have played a significant role in the evolution of the universe. The treatment presented here is confined, however, to the interaction of an incident wave with a *Newtonian* binary system. In particular, we neglect all relativistic effects in the relative motion of the binary system. Let the system have a Keplerian frequency ω and semimajor axis a; then, by Kepler's third law, $\omega^2 = k/a^3$. Relativistic two-body effects may be neglected provided $k \ll c^2 a$, where c is the speed of light in vacuum. Moreover, the quadrupole approximation for the interaction of the system with the gravitational wave is valid if $\Omega a \ll c$. More generally, our approach is sound provided

$$\Big(\frac{k}{c^2 a}\Big)^{1/2} \ll \frac{\Omega}{\omega} \ll \Big(\frac{k}{c^2 a}\Big)^{-1/2}.$$

Furthermore, the requirement that the external wave be a small perturbation of the system implies that $\Omega/\omega \ll 1/\sqrt{\epsilon}$, since the strength of the interaction is given by $\epsilon\Omega^2/\omega^2$, a quantity that must be much smaller than unity. Our objective is to determine the conditions under which periodic Keplerian motions of the binary are continued to periodic motions under the interaction. It is an important consequence of our methods, which are originally due to Poincaré [12], that the existence of higher-order perturbing influences on the orbit, i.e., terms of order at least ϵ^2, can only affect the shape of the resulting periodic orbit but not its existence. In this first treatment of the nonlinear case, we consider only special cases of the other interesting questions—such as gravitational ionization—that are suggested by the electrodynamic analogy discussed in [10]. A treatment of the general problem on the basis of linear perturbation theory is contained in previous work [10]. Superposition of linear perturbations due to the Fourier components of a pulse of gravitational radiation permits a general analysis in that case; however, the validity of the treatment is restricted in time as temporal evolution leads to a breakdown of the linear perturbation theory. Therefore, the intrinsic nonlinearity of the system must in general be taken into account for applications in celestial mechanics. In this regard, we mention that the equations of motion of a binary influenced by the gravitational attraction of a massive distant third body can also be treated using the methods developed here; this constitutes a special limiting case of the three-body problem and is discussed in Appendix B.

We will be mainly concerned with the continuation of Keplerian orbits under perturbation by a resonant gravitational wave. In general, we show that all but a finite set of such resonant orbits are not continued to periodic orbits under the influence of the incident wave and that in general all elements of the finite exceptional set are, in fact, continued.

The plan of this paper is the following. In § 2, we transform the perturbation problem to Delaunay elements and obtain explicit expressions for the transformed perturbation in terms of Fourier series with coefficients that involve the Bessel functions. In § 3, we outline a continuation method based on the Lyapunov-Schmidt reduction that is adapted from [3]. In § 4, the results of the continuation theory are applied to the perturbation of a binary influenced by a normally incident wave. In particular, we find bifurcation equations for the problem, that is, a system of equations whose simple zeros correspond to the continuable periodic Keplerian orbits and we show that these equations indeed have simple zeros. In § 5, we consider the special case of circularly polarized gravitational waves, a case that is not covered by the results of § 4. In this case we show that there are periodic solutions. We also show for sufficiently weak perturbations of Keplerian ellipses that the semimajor axis of the osculating ellipse remains bounded for all time. The final section, § 6, contains a brief discussion of some additional speculative results on the ionization problem. Some standard formulas are relegated to Appendix A, and in Appendix B we consider a special case of the three body problem: A binary influenced by a distant massive third body.

2. Delaunay Elements and Fourier Series Expansion

In terms of the canonical variables $(p_r, p_\theta, r, \theta)$, that is

$$x = r\cos\theta, \quad y = r\sin\theta, \quad p_r = \dot r, \quad p_\theta = r^2\dot\theta,$$

the Hamiltonian for our perturbation problem, with perturbation parameter ϵ, may be expressed in the following general form

$$\mathcal{H}(p_r, p_\theta, r, \theta) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{k}{r} + \epsilon r^2(\phi(t)\cos 2\theta + \psi(t)\sin 2\theta), \tag{5}$$

where ϕ and ψ are periodic functions with a common period. We will continue to use this general form in order to show how our theory can be applied. However, for the sinusoidal monochromatic gravitational wave model we will consider only the case where

$$\phi(t) = \frac{1}{2}\alpha\Omega^2\cos(\Omega t), \qquad \psi(t) = \frac{1}{2}\beta\Omega^2\cos(\Omega t + \rho). \tag{6}$$

The unperturbed Hamiltonian

$$H(p_r, p_\theta, r, \theta) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{k}{r} \tag{7}$$

is called the Kepler Hamiltonian. We will consider only those motions corresponding to negative energy $E = H(p_r, p_\theta, r, \theta)$.

Following S. Sternberg [13, Vol. 2, pp. 234-247], we define

$$L := \Big(\frac{-k^2}{2E}\Big)^{1/2}, \qquad G := p_\theta,$$

and we let $a(1 \pm e)$ denote the roots of the quadratic polynomial

$$r^2 - \frac{2L^2}{k}r + \frac{G^2L^2}{k^2} = 0$$

so that

$$a = \frac{L^2}{k}, \qquad e = \frac{1}{L}(L^2 - G^2)^{1/2}. \tag{8}$$

Here, a is the semimajor axis and e is the eccentricity of the Keplerian ellipse with $0 \le e < 1$. However, we will only consider the case $e > 0$, that is noncircular orbits. With this restriction in force, we define $\widehat{u}$, the eccentric anomaly and v, the true anomaly, implicitly by the formulas

$$r = a(1 - e\cos\widehat{u}), \qquad r = \frac{a(1-e^2)}{1 + e\cos v}, \tag{9}$$

and new variables ℓ and g by

$$\ell = \widehat{u} - e \sin \widehat{u}, \qquad g = \theta - v.$$

As proved in [13], the change of coordinates

$$(p_r, p_\theta, r, \theta) \to (L, G, \ell, g)$$

is canonical. Here ℓ and g are "angle variables", defined modulo 2π, while L and G are "action variables". The new coordinates (L, G, ℓ, g) are called the Delaunay elements.

In Delaunay variables, our Hamiltonian (5) is transformed to

$$\mathcal{H} = -\frac{k^2}{2L^2} + \epsilon(\mathcal{C}(L, G, \ell, g)\phi(t) + \mathcal{S}(L, G, \ell, g)\psi(t)),$$

where $\mathcal{C}$ (resp. $\mathcal{S}$) is the function obtained by expressing $r^2 \cos 2\theta$ (resp. $r^2 \sin 2\theta$) in terms of the Delaunay elements. Using the fact that the change of coordinates is canonical, the differential equations of motion are given by the Hamiltonian system

$$\begin{aligned}
\dot{L} =& \ -\epsilon(\frac{\partial \mathcal{C}}{\partial \ell}(L, G, \ell, g)\phi(t) + \frac{\partial \mathcal{S}}{\partial \ell}(L, G, \ell, g)\psi(t)),\\
\dot{G} =& \ -\epsilon(\frac{\partial \mathcal{C}}{\partial g}(L, G, \ell, g)\phi(t) + \frac{\partial \mathcal{S}}{\partial g}(L, G, \ell, g)\psi(t)),\\
\dot{\ell} =& \omega + \epsilon(\frac{\partial \mathcal{C}}{\partial L}(L, G, \ell, g)\phi(t) + \frac{\partial \mathcal{S}}{\partial L}(L, G, \ell, g)\psi(t)),\\
\dot{g} =& \quad \epsilon(\frac{\partial \mathcal{C}}{\partial G}(L, G, \ell, g)\phi(t) + \frac{\partial \mathcal{S}}{\partial G}(L, G, \ell, g)\psi(t)),
\end{aligned} \tag{10}$$

where $\omega := k^2/L^3$ is the frequency of the elliptical Keplerian orbit.

In order to analyze system (10), we must find computable expressions for the partial derivatives of $\mathcal{C}$ and $\mathcal{S}$. This can be done in several ways; however, for our purposes, the most useful expressions are obtained from Fourier series expanded as functions of the angle variable ℓ. The determination of these series is outlined in Appendix A, and the result can be expressed as

$$\begin{aligned}
\mathcal{C}(L, G, \ell, g) =& \frac{5}{2}a^2 e^2 \cos 2g + a^2 \sum_{\nu=1}^{\infty}(A_\nu \cos 2g \cos \nu\ell - B_\nu \sin 2g \sin \nu\ell),\\
\mathcal{S}(L, G, \ell, g) =& \frac{5}{2}a^2 e^2 \sin 2g + a^2 \sum_{\nu=1}^{\infty}(A_\nu \sin 2g \cos \nu\ell + B_\nu \cos 2g \sin \nu\ell),
\end{aligned} \tag{11}$$

where

$$\begin{aligned} A_\nu =& \frac{4}{\nu^2 e^2}(2\nu e(1-e^2)J'_\nu(\nu e) - (2-e^2)J_\nu(\nu e)), \\ B_\nu =& -\frac{8}{\nu^2 e^2}\sqrt{1-e^2}\,(eJ'_\nu(\nu e) - \nu(1-e^2)J_\nu(\nu e)). \end{aligned} \tag{12}$$

3. Continuation Theory

In order to analyze the continuation (persistence) of periodic solutions of the Kepler system to system (10), we use a method proposed in [3]. Here, we outline the main ideas; the reader is referred to [3] for the details.

Consider a system of the form

$$\dot{u} = F(u) + \epsilon h(u,t), \tag{13}$$

where u is a coordinate on a manifold M consisting of a cross product of Euclidean spaces and tori, h is $2\pi/\Omega$ periodic in its second variable, and ϵ is a small parameter. Let $t \mapsto u(t,\xi,\epsilon)$ denote the solution of (13) with initial condition $u(0,\xi,\epsilon) = \xi$, $\xi \in M$. Also, we define the mth order Poincaré map by $\mathcal{P}^m(\xi,\epsilon) = u(2m\pi/\Omega,\xi,\epsilon)$; it corresponds to a strobe that illuminates the orbit after m cycles of the perturbation. Of course, a fixed point of $\xi \mapsto \mathcal{P}^m(\xi,\epsilon)$ corresponds to a periodic orbit of (13). If m is the smallest such integer for which ξ is a fixed point, then ξ is the initial point of a subharmonic of order m.

Suppose that there is a submanifold $\mathcal{Z} \subset M$ consisting entirely of fixed points of the unperturbed order m Poincaré map, defined by $p^m(\xi) := \mathcal{P}^m(\xi,0)$. Our continuation theory is a method, one among many, to decide if any of these fixed points survive after perturbation. More precisely, we say a point $z \in \mathcal{Z}$, and therefore the unperturbed periodic orbit of (13) with initial point z, is continuable (or that it persists) if there is a continuous curve $\epsilon \mapsto \gamma(\epsilon)$ in M such that $\gamma(0) = z$ and $\mathcal{P}^m(\gamma(\epsilon),\epsilon) \equiv \gamma(\epsilon)$. Here, $\gamma(\epsilon) \in M$ is the initial point of a periodic solution of (13).

In order to apply the method of [3], namely Lyapunov-Schmidt reduction to the Implicit Function Theorem, the fixed-point manifold (resonance manifold) $\mathcal{Z}$ must satisfy a nondegeneracy condition relative to the unperturbed Poincaré map. To specify this condition, consider $z \in \mathcal{Z}$ and the derivative $Dp^m(z)$ viewed as a linear transformation of the tangent space T_zM. The base point stays fixed because p^m is the identity on $\mathcal{Z}$. Moreover, every vector in T_zM that is tangent to the submanifold $\mathcal{Z}$ is fixed by $Dp^m(z)$, or, as we will say, every such vector is in the kernel of the infinitesimal displacement

$\mathcal{D}(z) = Dp^m(z) - I$. The manifold $\mathcal{Z}$ is called normally nondegenerate if the kernel of the infinitesimal displacement is exactly the tangent space $T_z\mathcal{Z} \subset M$. Equivalently, $\mathcal{Z}$ is normally nondegenerate, if for each $z \in \mathcal{Z}$, the dimension of the kernel of the infinitesimal displacement at z is equal to the dimension of the manifold $\mathcal{Z}$.

Suppose $\mathcal{Z}$ has dimension Δ and that it is a normally nondegenerate submanifold of M. In this case the range of the infinitesimal displacement at each point in $\mathcal{Z}$ has codimension Δ. Thus, for $z \in \mathcal{Z}$, there is a vector space complement $\tilde{\mathcal{S}}(z)$, to the range of $\mathcal{D}(z)$. We let $\tilde{s}(z)$ denote the projection of T_zM to $\tilde{\mathcal{S}}(z)$. By choosing local coordinates, we note that both $\mathcal{Z}$ and $\tilde{\mathcal{S}}(z)$ may be identified with $\mathbb{R}^\Delta$.

Let $z \in \mathcal{Z}$ and consider the curve in M given by $\epsilon \mapsto \mathcal{P}^m(z, \epsilon)$. This curve passes through z at $\epsilon = 0$. Its tangent vector at $\epsilon = 0$, which may be identified with the partial derivative $\mathcal{P}^m_\epsilon(z, 0)$, is in T_zM. We define the bifurcation function $\mathcal{B}$ to be the map, from $\mathcal{Z}$ to the complement $\tilde{\mathcal{S}}$ of the range of the infinitesimal displacement, given by

$$\mathcal{B}(z) = \tilde{s}(z)\mathcal{P}^m_\epsilon(z, 0).$$

In local coordinates $\mathcal{B} : \mathbb{R}^\Delta \to \mathbb{R}^\Delta$. We will say $z \in \mathcal{Z}$ is a simple zero of the bifurcation function provided $\mathcal{B}(z) = 0$ and the derivative $D\mathcal{B}(z)$ is invertible.

A result in [3] is the following continuation theorem:

Theorem 3.1. *If $\mathcal{Z}$ is a normally nondegenerate fixed-point submanifold of M for system* (13) *and if $z \in \mathcal{Z}$ is a simple zero of the associated bifurcation function, then the unperturbed periodic orbit of* (13) *with initial point z is continuable.*

To use Theorem 3.1 as a practical tool, we must be able to compute $\mathcal{P}^m_\epsilon(z, 0)$. Fortunately, this partial derivative can usually be computed. In fact, if $\Omega = \Omega(\epsilon)$, then

$$\mathcal{P}^m_\epsilon(z, 0) = -\frac{2m\pi}{\Omega(0)^2}\Omega'(0)\dot{u}(2m\pi/\Omega(0), z, 0) + u_\epsilon(2m\pi/\Omega(0), z, 0).$$

Thus, if $t \mapsto W(t)$ is the solution of the second variational initial value problem

$$\dot{W} = DF(u(t, z, 0))W + h(u(t, z, 0), t), \quad W(0) = 0,$$

then

$$\mathcal{P}^m_\epsilon(z, 0) = -\frac{2m\pi}{\Omega(0)^2}\Omega'(0)F(u(2m\pi/\Omega(0), z, 0)) + W(2m\pi/\Omega(0)).$$

In effect, $W(t) = u_\epsilon(t, z, 0)$ with $W(0) = 0$ because $u(0, z, \epsilon) = z$.

In our gravitational radiation model, it seems appropriate that the frequency of the gravitational wave is independent of the amplitude of the wave. Thus, we will assume below that Ω does not depend on ϵ. This simplifies the expression for $\mathcal{P}^m_\epsilon(z,0)$ by removing the "detuning"
.

4. Continuation of Kepler Orbits

To apply Theorem 3.1 to our perturbation problem (10), we must define a normally nondegenerate fixed-point manifold. For this, we consider the Kepler orbits that are in resonance with the periodic perturbation.

If there are fixed relatively prime positive integers m and n and a fixed value of ω, the frequency of the Keplerian orbit, such that

$$m\frac{2\pi}{\Omega} = n\frac{2\pi}{\omega},$$

then the unperturbed solution of (10) starting at (L, G, ℓ, g) is given by

$$t \mapsto (L, G, \omega\hat{t} + \ell, g),$$

where $\hat{t} = t - t_0$ and t_0 is an integration constant that denotes the starting instant of time. A detailed analysis shows that t_0 can be set equal to zero here without loss of generality. Since ℓ is defined modulo 2π, this solution is periodic of period $2\pi/\omega$. Moreover, the mth order unperturbed Poincaré map is defined by

$$p^m(L, G, \ell, g) = (L, G, 2\pi m\frac{\omega}{\Omega} + \ell, g).$$

If we define the three-dimensional manifold

$$\mathcal{Z}^L := \{(L, G, \ell, g) : m\omega = n\Omega\},$$

and recall that ℓ and g are defined modulo 2π, then it follows immediately that $\mathcal{Z}^L$ is fixed by p. To check that $\mathcal{Z}^L$ is normally nondegenerate, we compute

$$\mathcal{D}(L, G, \ell, g) = Dp^m(L, G, \ell, g) - I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6\pi n/L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we note that the infinitesimal displacement has a three-dimensional kernel that is spanned by the usual basis vectors

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Moreover, the range of the infinitesimal displacement is complemented by the span of the vectors

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

To compute the bifurcation function associated with (10), we must compute the partial derivative $\mathcal{P}^m_\epsilon(G, L, g, \ell, 0)$ on the manifold $\mathcal{Z}^L$ and then project the result into the complement of the range of the infinitesimal displacement. To do this, we simply solve the variational initial value problem

$$\begin{aligned}
\dot{L}_\epsilon &= -\frac{\partial \mathcal{C}}{\partial \ell}(L, G, \omega t + \ell, g)\phi(t) - \frac{\partial \mathcal{S}}{\partial \ell}(L, G, \omega t + \ell, g)\psi(t),\\
\dot{G}_\epsilon &= -\frac{\partial \mathcal{C}}{\partial g}(L, G, \omega t + \ell, g)\phi(t) - \frac{\partial \mathcal{S}}{\partial g}(L, G, \omega t + \ell, g)\psi(t),\\
\dot{\ell}_\epsilon &= -\frac{3k^2}{L^4}L_\epsilon + \frac{\partial \mathcal{C}}{\partial L}(L, G, \omega t + \ell, g)\phi(t) + \frac{\partial \mathcal{S}}{\partial L}(L, G, \omega t + \ell, g)\psi(t),\\
\dot{g}_\epsilon &= \frac{\partial \mathcal{C}}{\partial G}(L, G, \omega t + \ell, g)\phi(t) + \frac{\partial \mathcal{S}}{\partial G}(L, G, \omega t + \ell, g)\psi(t)
\end{aligned}$$

with zero initial values and then project the solution computed at $t = m2\pi/\Omega$ into the complement of the range of the infinitesimal displacement. From this procedure, we obtain the following bifurcation function

$$\mathcal{B}(G, \ell, g) = (B^L(G, \ell, g), B^G(G, \ell, g), B^g(G, \ell, g)),$$

where

$$B^L(G, \ell, g) := -\frac{\partial \mathcal{I}}{\partial \ell},\quad B^G(G, \ell, g) := -\frac{\partial \mathcal{I}}{\partial g},\quad B^g(G, \ell, g) := \frac{\partial \mathcal{I}}{\partial G}, \tag{14}$$

and

$$\mathcal{I} := \int_0^{m2\pi/\Omega} \Big[\mathcal{C}(L, G, \omega t + \ell, g)\phi(t) + \mathcal{S}(L, G, \omega t + \ell, g)\psi(t)\Big]\, dt.$$

Using the resonance relation, we have

$$\mathcal{I} = \int_0^{m2\pi/\Omega} \Big[\mathcal{C}(L, G, \frac{n}{m}\Omega t + \ell, g)\phi(t) + \mathcal{S}(L, G, \frac{n}{m}\Omega t + \ell, g)\psi(t)\Big]\, dt$$

and, after changing the variable to $\hat{\sigma} = \Omega t/m + \ell/n$, we obtain

$$\begin{aligned}\mathcal{I} =& \frac{m}{\Omega}\int_{\ell/n}^{2\pi+\ell/n} \Big[\mathcal{C}(L, G, n\hat{\sigma}, g)\phi(m(\hat{\sigma} - \ell/n)/\Omega) \\ &+ \mathcal{S}(L, G, n\hat{\sigma}, g)\psi(m(\hat{\sigma} - \ell/n)/\Omega)\Big]\, d\hat{\sigma}.\end{aligned}$$

Using the fact that the last integrand is periodic with period 2π as a function of $\hat{\sigma}$ and substituting ϕ and ψ given in (6), we find

$$\begin{aligned}\mathcal{I} =& \frac{m\Omega}{2}\int_0^{2\pi} \Big[\alpha\mathcal{C}(L, G, n\hat{\sigma}, g)\cos(m\hat{\sigma} - \frac{m\ell}{n}) \\ &+ \beta\mathcal{S}(L, G, n\hat{\sigma}, g)\cos(m\hat{\sigma} - \frac{m\ell}{n} + \rho)\Big]\, d\hat{\sigma}.\end{aligned}$$

To compute $\mathcal{I}$, we substitute the Fourier series (11) for $\mathcal{C}$ and $\mathcal{S}$ into the last expression for $\mathcal{I}$ and use trigonometric relations together with the fact that m and n are relatively prime, to conclude that $\mathcal{I} = 0$ unless $n = 1$. Of course, there may be continuable orbits for $n > 1$, but they are not detected by our first order method. In case $n = 1$, that is for the $(m : n) = (m : 1)$ resonance, we find that

$$\begin{aligned}\mathcal{I} = \frac{1}{2}\pi m a^2\Omega\Big(&\alpha(A_m \cos m\ell \cos 2g - B_m \sin m\ell \sin 2g) \\ &+\beta(A_m \cos(m\ell - \rho)\sin 2g + B_m \sin(m\ell - \rho)\cos 2g)\Big).\end{aligned}$$

This result can be rewritten as

$$\begin{aligned}\mathcal{I} =& \frac{1}{4}\pi m a^2\Omega\Big[(A_m + B_m)(\alpha - \beta\sin\rho)\cos(2g + m\ell) \\ &+ (A_m - B_m)(\alpha + \beta\sin\rho)\cos(2g - m\ell) + \beta(A_m + B_m)\cos\rho\sin(2g + m\ell) \\ &+ \beta(A_m - B_m)\cos\rho\sin(2g - m\ell)\Big]. \end{aligned} \tag{15}$$

It is possible to express equation (15) in a more compact form, in the usual manner, by defining

$$\begin{aligned}&\mathcal{E}\cos\sigma := \beta\cos\rho, \quad \mathcal{E}\sin\sigma := \alpha + \beta\sin\rho, \\ &\mathcal{F}\cos\tau := \beta\cos\rho, \quad \mathcal{F}\sin\tau := \alpha - \beta\sin\rho,\end{aligned} \tag{16}$$

so that

$$\mathcal{E} = (\alpha^2 + \beta^2 + 2\alpha\beta\sin\rho)^{1/2}, \quad \mathcal{F} = (\alpha^2 + \beta^2 - 2\alpha\beta\sin\rho)^{1/2},$$

and

$$\mathcal{I} = \frac{1}{4}\pi m a^2\Omega\Big[(A_m+B_m)\mathcal{F}\sin(2g+m\ell+\tau)+(A_m-B_m)\mathcal{E}\sin(2g-m\ell+\sigma)\Big].$$

The simple zeros of the bifurcation function are then the same as the simple zeros of

$$
\begin{aligned}
\mathcal{F}(A_m+B_m)\cos(2g+m\ell+\tau) &=0,\\
\mathcal{E}(A_m-B_m)\cos(2g-m\ell+\sigma) &=0,\\
\Big(\frac{\partial A_m}{\partial G}+\frac{\partial B_m}{\partial G}\Big)\mathcal{F}\sin(2g+m\ell+\tau)+\Big(\frac{\partial A_m}{\partial G}-\frac{\partial B_m}{\partial G}\Big)\mathcal{E}\sin(2g-m\ell+\sigma) &=0.
\end{aligned}
\tag{17}
$$

To obtain explicit formulas for the partial derivatives of the functions A_m and B_m with respect to G, we assume that $G>0$ so that

$$
\frac{\partial e}{\partial G}=-\sqrt{1-e^2}\,\frac{1}{Le}.
$$

In case $G<0$, the partial derivative has the opposite sign and the subsequent analysis is similar. We use (32),(33),(34) (from Appendix A), and (12) to obtain

$$
\begin{aligned}
\frac{\partial A_m}{\partial G} &=-\sqrt{1-e^2}\,\frac{4}{Lm^2e^4}\Big((2m^2(1-e^2)^2+4)J_m(me)-me(6-e^2)J_m'(me)\Big),\\
\frac{\partial B_m}{\partial G} &=-\frac{8}{Lm^2e^4}\Big(-3m(1-e^2)J_m(me)+e((2-e^2)(1-m^2e^2)+m^2)J_m'(me)\Big).
\end{aligned}
\tag{18}
$$

The simple zeros of the function $\mathcal{B}$ given by (14) correspond to the continuable periodic orbits. Equivalently, the continuable periodic orbits correspond to the simple solutions of the system of equations given by (17). In order to find the simple zeros of (17), we will use the following proposition:

Proposition 4.1. *If $\mathcal{E}\mathcal{F}\neq 0$ (that is, $|\alpha|\neq|\beta|$, or $|\alpha|=|\beta|$ but $|\sin\rho|<1$) and if the system of equations (17) has a solution, then*

$$
(A_m^2-B_m^2)\Big\{\Big(\frac{\partial A_m}{\partial G}+\frac{\partial B_m}{\partial G}\Big)^2\mathcal{F}^2-\Big(\frac{\partial A_m}{\partial G}-\frac{\partial B_m}{\partial G}\Big)^2\mathcal{E}^2\Big\}
\tag{19}
$$

is zero.

Proof. If (17) has a solution and equation (19) does not vanish, then, since the first factor of equation (19) is not zero, we have

$$
\cos(2g+m\ell+\tau)=0,\qquad \cos(2g-m\ell+\sigma)=0.
$$

This implies that

$$
\sin(2g+m\ell+\tau)=\pm1,\qquad \sin(2g-m\ell+\sigma)=\pm1,
$$

and, since the third equation of (17) is zero, that

$$
\Big(\frac{\partial A_m}{\partial G}+\frac{\partial B_m}{\partial G}\Big)\mathcal{F}\pm\Big(\frac{\partial A_m}{\partial G}-\frac{\partial B_m}{\partial G}\Big)\mathcal{E}=0,
$$

in contradiction to the fact that (19) is not zero. □

Proposition 4.1 reduces the search for solutions to several cases. Just note that as soon as one of the factors of (19) vanishes, the value of e and hence the values of

$$A_m - B_m, \quad A_m + B_m, \quad \frac{\partial A_m}{\partial G}, \quad \frac{\partial B_m}{\partial G}$$

are fixed. Thus, equations (17) reduce to solvable trigonometric equations. It is important to note that Proposition 4.1 does not cover the interesting case of circular polarization, which is therefore deferred to § 5.

To study the zeros of (19) we will use the following proposition.

Proposition 4.2. *For the* $(1:1)$ *resonance, the functions* A_1+B_1 *and* A_1-B_1 *appearing in equations* (17) *and viewed as functions of* e *are both negative on the interval* $0<e<1$. *The functions* $\partial A_1/\partial G$ *and* $\partial B_1/\partial G$, *viewed as functions of* e, *each have exactly one simple zero on the interval and their zeros are distinct. For the* $(m:1)$ *resonance with* $m>1$, *the range of the function* $(\partial A_m/\partial G)/(\partial B_m/\partial G)$, *viewed as a function of* e *on the interval* $0<e<1$, *contains the interval* $[-1,1]$.

Proof. We will outline the proof of the proposition. Some of the computations were checked using a computer algebra system.

Consider the case $m=1$. We will use the following elementary lemma [4, Lemma 3.5] to show that the function f defined by

$$\begin{aligned} f(e) :=& \frac{e^2}{4}(A_1 - B_1) \\ =& 2e((1-e^2)+\sqrt{1-e^2})J_1'(e) - ((2-e^2)+2(1-e^2)^{3/2})J_1(e) \end{aligned} \tag{20}$$

is negative on the interval $I_0 := \{e : 0 < e < 1\}$.

Lemma 4.3. *Suppose* f *is a smooth function defined on an interval* $[a,b)$ *with the additional property that there is a number* $\epsilon>0$ *such that* $f(x)f'(x)>0$ *for* $a<x<a+\epsilon$. *If there are smooth functions* p, q, *and* r *defined on* (a,b) *such that* $p(x)r(x)>0$ *and*

$$p(x)f''(x) = q(x)f'(x) + r(x)f(x) \tag{21}$$

on the interval (a,b), *then* f *is strictly monotone on* $[a,b)$. *In particular,* $f(x)$ *has the same sign on* (a,b) *that it does on* $(a,a+\epsilon)$.

The function f defined by (20) satisfies a differential equation of the form (21) with $x = e$, $w := \sqrt{1-e^2}$, and

$$\begin{aligned} p(e) :=& e^4 w^2 (5w+2), \\ q(e) :=& e^3 (15w^3 + 4w^2 + 2), \\ r(e) :=& e^2 (5w^5 - 8w^4 - 16w^3 + 26w^2 + 24w + 4). \end{aligned}$$

To test the sign of $p(e)r(e)$, we change the variable to w and note that $0 < w < 1$. Let

$$p^*(w) := p(\sqrt{1-w^2}), \quad r^*(w) := r(\sqrt{1-w^2}).$$

Clearly, p^* is positive on $0 < w < 1$. The second factor of r^* is easily shown to be positive on the same interval. For example, the second factor is positive at $w = 0$ and has no roots in the interval. The fact that there are no roots can be checked by computing a Sturm sequence (cf. [6]). This proves $p(e)r(e) > 0$ for $e \in I_0$.

To complete the proof of this case, it remains only to show that there is some $\epsilon > 0$ such that $f(e) < 0$ and $f'(e) < 0$ on the interval $0 < e < \epsilon$. To do this, we simply note that the Taylor series of f and f' at $e = 0$ are given by

$$f(e) = -\frac{7}{48}e^5 + O(e^7), \quad f'(e) = -\frac{35}{48}e^4 + O(e^6).$$

The fact that $A_1 + B_1$ is negative on the interval I_0 can be proved in a similar manner.

We must show that $\partial A_1/\partial G$ has exactly one simple root on the interval I_0. To do this it suffices to prove that the function given by

$$e \mapsto (6 - e^2)J_2(e) + e(2e^2 - 3)J_1(e)$$

has exactly one simple zero on I_0. Equivalently, using the fact that $J_2(e)$ does not vanish on I_0, it suffices to show that the function

$$f_0(e) := \frac{6 - e^2}{2e^2 - 3} + e\frac{J_1(e)}{J_2(e)}$$

has the same property. This fact follows from the expression (18) for $\partial A_1/\partial G$, the recurrence formula

$$\nu J_\nu(x) - xJ'_\nu(x) = xJ_{\nu+1}(x), \tag{22}$$

that is a simple consequence of equation (32) of Appendix A, and the connection between e and G.

The fact that f_0 has exactly one simple zero is a consequence of the following three propositions: (a) The function $f_1(e) := (6-e^2)/(2e^2-3)$ is monotone decreasing on I_0. (b) The function $f_2(e) := eJ_1(e)/J_2(e)$ is monotone decreasing on I_0. (c) The function f_0 has a zero in I_0.

Statement (a) is immediate: f_1' is negative on the interval. Statement (b) follows from the product representation of the Bessel function J_ν given by [1]

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \prod_{s=1}^{\infty} \Big(1 - \frac{x^2}{j_{\nu,s}^2}\Big),$$

where $j_{\nu,s}$ denotes the sth zero of J_ν and the fact that the zeros are interlaced as follows:

$$j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \cdots .$$

Statement (c) follows from two facts: $\lim_{e\to 0^+} f_0(e) = 2$ and $f_0(1) < 0$. The first fact is immediate from the above product representation or from the Taylor series for the Bessel functions at $e = 0$. To obtain the inequality, we use the product representation of the Bessel functions to deduce

$$f_0(1) = -5 + 4 \prod_{s=1}^{\infty} \Big(1 - \frac{1}{j_{1,s}^2}\Big) \Big/ \prod_{s=1}^{\infty} \Big(1 - \frac{1}{j_{2,s}^2}\Big).$$

Since $j_{2,s} > j_{1,s} > 1$ for all s, the quotient of the two products is less than unity, as required.

The fact that $\partial B_1/\partial G$ has a unique simple zero is proved using a similar analysis. Just as before, it suffices to show that the following function has exactly one simple zero on I_0:

$$\tilde{f}(e) := -e^3 \frac{J_1(e)}{J_2(e)} + e^4 - 3e^2 + 3.$$

Both terms are monotone decreasing on I_0 with $\tilde{f}(0) = 3$ and $\tilde{f}(1) < 0$.

We claim the zeros of $\partial A_1/\partial G$ and $\partial B_1/\partial G$ do not occur at the same point. After division by nonzero factors and the substitution $m = 1$ in (18), it suffices to show that the following functions do not have a common zero on I_0:

$$\begin{aligned}
\widehat{\alpha}(e) :=& (2(1-e^2)^2 + 4) J_1(e) - e(6 - e^2) J_1'(e), \\
\widehat{\beta}(e) :=& -3(1-e^2) J_1(e) + e((2-e^2)(1-e^2) + 1) J_1'(e).
\end{aligned}$$

If the functions do have a common zero, then the function

$$\widehat{f}(e) := \widehat{\alpha}(e) - e\widehat{\beta}(e)$$

has at least one zero on I_0. To prove the claim, we show that $\widehat{f}$ is negative on I_0.

Using the recurrence formulas for Bessel's functions, we find that

$$\widehat{f}(e) = (-e^6 + 3e^4 + e^3 - 3e^2 - 6e) J_0(e) + (e^5 + 2e^4 - 6e^3 - 5e^2 + 6e + 12) J_1(e).$$

To prove this function is negative on I_0 we will apply Lemma 4.3. We find that $\widehat{f}$ satisfies a differential equation of the specified form with

$$\begin{aligned}
\widehat{p}(e) :=& e^2(e^{10} - 7e^8 - 4e^7 + 31e^6 + 14e^5 - 55e^4 - 45e^3 + 84e^2 + 9e - 36),\\
\widehat{q}(e) :=& e(11e^{10} - 63e^8 - 32e^7 + 217e^6 + 84e^5 - 275e^4 - 180e^3 + 252e^2 + 18e - 36),\\
\widehat{r}(e) :=& - e^{12} - 27e^{10} + 12e^9 + 95e^8 + 11e^7 - 235e^6\\
&+ 273e^5 - 19e^4 - 468e^3 + 351e^2 + 90e - 108.
\end{aligned}$$

Using Sturm sequences, it can be proved that $\widehat{p}(e)\widehat{r}(e) > 0$ for $e \in I_0$. Moreover, we find that

$$\widehat{f}(e) = -\frac{3}{4}e^3 + O(e^4), \quad \widehat{f}(e)\widehat{f}'(e) = \frac{27}{16}e^5 + O(e^6).$$

This completes the proof of the claim.

In case $m > 1$, it suffices to consider the range of the function F_m given by $e \mapsto (\partial A_m/\partial e)/(\partial B_m/\partial e)$. A computation shows that the Taylor series of both the numerator and the denominator of F_m is given by $-5e + O(e^2)$ in case $m = 2$ and, in case $m > 2$, by

$$\frac{8m^m(m-1)(m-2)}{2^m m! m^2}e^{m-3} + O(e^{m-2}).$$

It follows that $\lim_{e\to 0^+} F_m(e) = 1$.

We claim that $e \mapsto \partial B_m/\partial e$ has at least one zero on I_0. If not, then B_m is a monotone function of e. A computation shows that B_m has a removable singularity at $e = 0$ and that

$$B_m(e) = \frac{8m^m(m-1)}{2^m m! m^2}e^{m-2} + O(e^{m-1}).$$

If $m > 2$, then B_m is increasing for $0 < e << 1$. But, from the definition of B_m, we have $\lim_{e\to 1^-} B_m(e) = 0$, in contradiction. For $m = 2$, we find that

$$B_2(e) = 1 - \frac{5}{2}e^2 + O(e^4)$$

and B_2 decreases for $0 < e << 1$. But, B_2 is negative for $0 << e < 1$. To see this just note that near $e = 1$, the sign of B_2 is determined by $-J_2'(2e)$. By standard properties of the Bessel functions (cf. [1]), $J_2(x)$ is positive on the interval $(0, j'_{2,1})$, where $j'_{\nu,s}$ denotes the sth zero of J'_ν. Since $\nu \le j'_{\nu,s}$, we have that $-J_2'(2e) < 0$ for $0 << e < 1$. Again, since $\lim_{e\to 1^-} B_2(e) = 0$, we have a contradiction. This proves the claim.

Suppose for the moment that $\partial B_m/\partial e > \partial A_m/\partial e$ on the interval $0 < e < 1$ and consider the first zero e_* of $e \mapsto \partial B_m/\partial e$. It follows that $\partial A_m/\partial e(e_*) < 0$ while $\partial B_m/\partial e > 0$ on the interval $0 < e < e_*$.

Thus, we have $\lim_{e\to e_*^-} F_m(e) = -\infty$ and the range of F_m contains the interval $(-\infty, 1]$, as required.

To complete the proof, it suffices to show that the function G_m given by

$$e \mapsto m^2 e^3 \Big(\partial B_m/\partial e - \partial A_m/\partial e\Big)$$

for $m > 1$ is positive on the interval $0 < e < 1$. This fact follows from Lemma 4.3. The Taylor series of G_m at $e = 0$ is given by

$$G_m(e) = \frac{(5m+2)m^m}{2^m m!(m+1)} e^{m+4} + O(e^{m+5}).$$

Thus, it follows that $G_m(e)G_m'(e) > 0$ for $0 < e << 1$. We also find that there are functions $p_m(e)$, $q_m(e)$ and $r_m(e)$ such that $p_m(e)r_m(e) > 0$ and

$$p_m(e)G_m''(e) = q_m(e)G_m'(e) + r_m(e)G_m(e)$$

on the interval $0 < e < 1$. In fact,

$$\begin{aligned}
p_m :=& e^3 w^6 \Big(5w^5 m^3 + (-8w^4 + 30w^2)m^2 + (4w^3 + 24w)m + 8\Big),\\
q_m :=& e^2 w^4 \Big((25w^7 - 10w^5)m^3 + (-32w^6 + 68w^4 + 30w^2)m^2\\
&+ (12w^5 + 24w^3 + 48w)m + 24\Big),\\
r_m :=& e w^3 m \Big(5w^{10} m^4 + (-18w^9 + 60w^7)m^3 + (-12w^8 - 52w^6 + 180w^4)m^2\\
&+ (30w^7 - 76w^5 + 166w^3)m - 12w^6 + 32w^4 - 12w^2 + 24\Big).
\end{aligned}$$

(To verify that G_m satisfies the second order differential equation with these coefficients, we compute the derivatives of G_m and then convert all the expressions to the variable w.) Finally, to show $p_m(e)r_m(e) > 0$, it suffices to show that the inequality holds for $0 < w < 1$. To do this, view p_m and r_m as polynomials in m and note that all their coefficients are positive functions of w. □

4.1. **The** $(1:1)$ **Resonance.** The fundamental physical result of this section is the following proposition: Among the periodic Keplerian orbits in $(1:1)$ resonance with an incident gravitational wave, there are generally a (nonzero) finite number of continuable periodic motions. In fact, the frequency of the gravitational wave fixes the semimajor axis, a, while the amplitudes and the phase shift, α, β, ρ, of the wave fix the eccentricity of the unperturbed Keplerian orbits that are excited by the perturbation. The inclination of the major axis and the angular position on the ellipse that complete the set of initial conditions for a continuable orbit on the excited ellipse are given by formulas presented below. However, two facts complicate the mathematical analysis: there are exceptional choices of the wave amplitudes α and β such that none of the periodic orbits in $(1:1)$ resonance with the incident gravitational wave are continuable and there are zeros of the bifurcation function that are not simple.

The precise mathematical result that we will prove requires a genericity assumption. For this we will say that a property of the zero set of (17) is generic relative to the parameter vector $(\alpha, \beta, \rho) \in \mathbb{R}^3$, if it holds for an open and dense subset of $\mathbb{R}^3$.

Proposition 4.4. *If $m = 1$, then, generically relative to the parameters (α, β, ρ), the zero set of system* (17) *is a nonempty finite set consisting entirely of simple zeros. If $m = 1$, $\alpha^2+\beta^2 \neq 0$, and $\alpha\beta \sin\rho = 0$, then system* (17) *has a nonzero finite number of zeros which are all simple.*

Proof. The first generic assumption is $\mathcal{E}\mathcal{F} \neq 0$, the second generic assumption is that $\alpha^2+\beta^2 \neq 0$. (Of course, if $\alpha^2+\beta^2 = 0$, then there is no perturbation of the Keplerian orbits.) According to Proposition 4.2, we have $A_1^2 - B_1^2 \neq 0$ for $0 < e < 1$. Thus, if system (17) has a solution, then, according to Proposition 4.1, we must have

$$\Big(\frac{\partial A_1}{\partial G} + \frac{\partial B_1}{\partial G}\Big)^2 \mathcal{F}^2 - \Big(\frac{\partial A_1}{\partial G} - \frac{\partial B_1}{\partial G}\Big)^2 \mathcal{E}^2 = 0.$$

Define

$$\kappa := \frac{\mathcal{E} - \mathcal{F}}{\mathcal{E} + \mathcal{F}}$$

and note that, after a simple algebraic manipulation and after taking into account the obvious fact that the partial derivatives with respect to G can be replaced with no loss of generality by partial derivatives with respect to the eccentricity e, the last condition is equivalent to the requirement that

$$\Big(\frac{\partial A_1}{\partial e} - \kappa\frac{\partial B_1}{\partial e}\Big)\Big(\frac{\partial B_1}{\partial e} - \kappa\frac{\partial A_1}{\partial e}\Big) = 0.$$

Also, taking into account the fact that $\mathcal{E} \geq 0$ and $\mathcal{F} \geq 0$, our assumption that $\alpha^2 + \beta^2 \neq 0$ implies $0 < |\kappa| \leq 1$.

To find the solutions of system (17), suppose for the moment that the equation

$$\frac{\partial A_1}{\partial e} - \kappa\frac{\partial B_1}{\partial e} = 0 \tag{23}$$

has a solution. For this value of e the third equation of system (17) vanishes provided $\sin(2g + \ell + \tau)$ and $\sin(2g - \ell + \sigma)$ are both equal to one or both equal to minus one. In either case, $\cos(2g + \ell + \tau)$ and

$\cos(2g-\ell+\sigma)$ both vanish. Thus, for all integers $\mathcal{M}$ and $\mathcal{N}$ such that

$$2g+\ell+\tau=\frac{\pi}{2}+2\pi\mathcal{M},\quad 2g-\ell+\sigma=\frac{\pi}{2}+2\pi\mathcal{N},$$

or such that

$$2g+\ell+\tau=-\frac{\pi}{2}+2\pi\mathcal{M},\quad 2g-\ell+\sigma=-\frac{\pi}{2}+2\pi\mathcal{N},$$

the fixed value of e together with the (nonzero) finite number of simultaneous solutions of these last equations with the property that $0\le g,\ell<2\pi$ give a set of solutions of system (17). A similar result is valid in case

$$\frac{\partial B_1}{\partial e}-\kappa\frac{\partial A_1}{\partial e}=0. \tag{24}$$

To determine the simplicity of these solutions, we must compute the Jacobian of system (17) with respect to the variables (G,ℓ,g) at the given solution. This Jacobian is easily computed by expanding along the first column of the Jacobian matrix. Up to a nonzero constant multiple, we find the value of the Jacobian to be

$$(A_1^2-B_1^2)\Big(\big(\frac{\partial^2 A_1}{\partial G^2}+\frac{\partial^2 B_1}{\partial G^2}\big)\mathcal{F}\sin(2g+\ell+\tau)+\big(\frac{\partial^2 A_1}{\partial G^2}-\frac{\partial^2 B_1}{\partial G^2}\big)\mathcal{E}\sin(2g-\ell+\sigma)\Big).$$

In particular, using the fact that e is a monotone function of G, it follows that the solution (G,ℓ,g) of system (17) is simple provided the corresponding solution e of equation (23), respectively (24), is simple.

To finish the proof, we must determine the existence and simplicity of the solutions of equations (23) and (24).

If either $\alpha=0$, $\beta=0$, or $\sin\rho=0$, then $\kappa=0$ and both equations (23) and (24) have unique simple solutions by Proposition 4.2. This proves the second statement of the proposition.

Since the left-hand sides of equations (23) and (24) are both analytic functions of e, there are at most a finite number of solutions on the interval $0<e<1$. Moreover, since the map $(\alpha,\beta,\rho)\mapsto\kappa(\alpha,\beta,\rho)$ is regular on an open and dense subset of $\mathbb{R}^3$, if some of the solutions of one of the equations are not simple, then there is an arbitrarily small perturbation of the triplet (α,β,ρ) such that the perturbed equations have a finite number of simple zeros.

The existence part of the first statement of the proposition is a consequence of the following facts. If $\kappa\ne 1$ (a generic assumption), then equation (24) has at least one zero. To see this, we note that the function $e\mapsto\partial B_1/\partial e$ has value -3 at $e=0$ and has limit ∞ as $e\to 1^-$ while the function $e\mapsto\partial A_1/\partial e$ has value -3 at $e=0$ and has a finite value at $e = 1$. As long as $\kappa \neq 1$, then the function

$$e \mapsto \frac{\partial B_1}{\partial e} - \kappa \frac{\partial A_1}{\partial e}$$

has a negative value at $e = 0$ and has limit ∞ as $e \to 1^-$. $\square$

The proposition does not give a complete description of the zero set of system (17). However, since this description is reduced to an investigation of equations (23) and (24) that are algebraic combinations of Bessel's functions, numerical approximations suggest the following scenario. If $\kappa \neq 1$, then the function

$$e \mapsto \frac{\partial B_1}{\partial e} - \kappa \frac{\partial A_1}{\partial e} \tag{25}$$

has exactly one simple zero on the interval $0 < e < 1$. If $\kappa = 1$, then (25) vanishes at $e = 0$ and increases monotonically to ∞ as $e \to 1^-$. If $\kappa \leq 0$, then the function

$$e \mapsto \frac{\partial A_1}{\partial e} - \kappa \frac{\partial B_1}{\partial e} \tag{26}$$

has exactly one simple zero on the interval $0 < e < 1$. There is a number $\kappa_* \approx 0.036$ such that if $0 < \kappa < \kappa_*$, then (26) has exactly two simple zeros, while if $\kappa > \kappa_*$, then (26) has no zeros. If $\kappa = \kappa_*$, then (26) has exactly one zero which is not simple.

Remark 4.5. *In case $\beta = 0$, that is the wave is plane polarized in a very special direction, it appears that the zeros of* (19) *are all near $e = 1$. The smallest occurs for the $(m : 1) = (1 : 1)$ resonance in case $\partial A_1/\partial G = 0$. Its root is larger than $e = 0.68$. The root is larger for the higher order resonances. This suggests that only some "comets" could remain periodic after perturbation by a gravitational wave with this particular polarization.*

4.2. **The** $(m : 1)$ **Resonance,** $m > 1$**.** For the $(m : 1)$ resonance we will prove that there are perturbed periodic solutions for the generic α, β and ρ. This is the content of the next proposition.

Proposition 4.6. *If $m > 1$, then generically, relative to the parameters α, β and ρ, system* (17) *has at least one simple zero.*

Proof. It suffices to consider α, β and ρ such that $\mathcal{E}\mathcal{F} \neq 0$. Let g and ℓ denote a solution of the equations

$$2g + m\ell + \tau = \pi/2, \qquad 2g - m\ell + \sigma = \pi/2,$$

and note that with this choice of g and ℓ, system (17) has a solution provided G , equivalently e, is chosen such that

$$\Big(\frac{\partial A_m}{\partial G}+\frac{\partial B_m}{\partial G}\Big)\mathcal{F}+\Big(\frac{\partial A_m}{\partial G}-\frac{\partial B_m}{\partial G}\Big)\mathcal{E}=0.$$

Equivalently, as in Proposition (4.4), there is a solution provided

$$\frac{\partial A_m}{\partial G}-\kappa\frac{\partial B_m}{\partial G}=0. \tag{27}$$

Since $|\kappa|<1$, an application of Proposition (4.2) shows that (27) has at least one solution. Moreover, since $A_m^2-B_m^2$ is not the zero function, it has only a finite number of zeros for $0<e<1$. Also, its zeros do not depend on the choice of the parameters α, β and ρ. Thus, if necessary, after a perturbation of the parameters we can be sure that our solution of (27) is not a zero of $A_m^2-B_m^2$ and that it is a simple zero of the left hand side of (27). As in Proposition (4.4), it follows that the corresponding choice of (G,ℓ,g) is a simple zero of system (17). □

5. Circularly Polarized Waves

In this section we consider the equations of motion (2) for the case of a circularly polarized incident wave. This corresponds to the special case where, in the components of the tidal matrix K in (3), we take $\alpha=\beta$ and $\rho=\pm\pi/2$. The minus sign corresponds to a right circularly polarized wave, while the plus sign corresponds to a left circularly polarized wave. We note that this is the main case excluded from the analysis of the previous section. There, the bifurcation function does not have simple zeros for the bifurcation problem corresponding to circular polarization.

The equations of motion for the right circularly polarized wave have the form

$$\begin{aligned}\frac{d^2x}{dt^2}+\frac{kx}{r^3}&=-\,\epsilon\alpha\Omega^2(x\cos\Omega t+y\sin\Omega t),\\ \frac{d^2y}{dt^2}+\frac{ky}{r^3}&=\epsilon\alpha\Omega^2(-x\sin\Omega t+y\cos\Omega t).\end{aligned}$$

This system can be treated in a similar manner as the analogous system that arises in Hill's lunar theory (cf. [8] [12] [13]). The key idea is to view the system in a new Cartesian coordinate system that rotates relative to the inertial system with half the frequency of the gravitational wave. This factor of 1/2 is due to the fact that the wave has helicity +2. In these rotating coordinates, that we again call x and y,

FIGURE 1. Orbits of a Poincaré map for the Hamiltonian system with Hamiltonian (29). The parameters are $\alpha = 1$, $\Omega = 1$, $k = 1$, $\epsilon = 0$ (left hand panel), $\epsilon = .026$ (right hand panel), and the energy is $H(p_r, p_\theta, r, \theta) = H(0, 1, 1, 0)$. For post script figures contact carmenchicone.cs.missouri.edu

we obtain the following system

$$\begin{aligned}
&\frac{d^2x}{dt^2} - \Omega\frac{dy}{dt} - (\frac{1}{4} - \epsilon\alpha)\Omega^2 x + \frac{kx}{r^3} = 0,\\
&\frac{d^2y}{dt^2} + \Omega\frac{dx}{dt} - (\frac{1}{4} + \epsilon\alpha)\Omega^2 y + \frac{ky}{r^3} = 0.
\end{aligned} \tag{28}$$

We note that the replacements $t \to -t$ and $\Omega \to -\Omega$ leave the system invariant. Thus, it suffices to consider the equations of motion for either state of circular polarization.

A remarkable fact, also utilized by Hill, is that system (28) is equivalent to a Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}(X^2 + Y^2) + \frac{\Omega}{2}(yX - xY) - \frac{k}{r} + \frac{\epsilon}{2}\alpha\Omega^2(x^2 - y^2),$$

where $X := \dot{x} - \Omega y/2$, and $Y := \dot{y} + \Omega x/2$. This Hamiltonian is given in polar coordinates by

$$H = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{k}{r} - \frac{\Omega}{2}p_\theta + \frac{\epsilon}{2}\alpha\Omega^2 r^2 \cos 2\theta, \tag{29}$$

where $p_r = (xX + yY)/r$ and $p_\theta = xY - yX$, and in Delaunay elements by

$$H = -\frac{k^2}{2L^2} - \frac{\Omega}{2}G + \frac{\epsilon}{2}\alpha\Omega^2\mathcal{C}(L, G, \ell, g).$$

The Delaunay form of the Hamiltonian is expressed in action-angle variables and is in the correct form to apply the Kolmogorov-Arnold-Moser (KAM) theory (see for example [2] [5]). Here, the Hamiltonian is degenerate. But, as pointed out in Sternberg [13, Vol. 2, p. 257], the Hamiltonian system with Hamiltonian H^2 has the same trajectories as the original system, only the speed along trajectories is changed. Moreover, the unperturbed part of H^2 satisfies the nondegeneracy assumption for the KAM theorem—its Hessian, with respect to the actions, has a nonzero determinant. Thus, the perturbed trajectory remains bounded in time, being trapped between two-dimensional invariant tori in the three-dimensional energy surfaces of our *two-degree* of freedom Hamiltonian. Thus, sufficiently weak circularly polarized gravitational waves do not "ionize" the Keplerian ellipses; that is, the osculating semimajor axes do not become unbounded. This is illustrated in Fig. 1, where "phase portraits" for a typical Poincaré map for the Hamiltonian system corresponding to (29) is depicted. After an energy H_0 is fixed, each orbit on the graph is produced by first choosing an initial point (p_r, r) in the depicted plane and then by marking the position of the (p_r, r) coordinates of the Hamiltonian orbit, with initial condition (p_r, r), $\theta = 0$, and with p_θ the implicit solution of $H(p_r, p_\theta, r, 0) = H_0$, at each time when $\theta(t)$ is a multiple of 2π and $\dot{\theta}(t)\dot{\theta}(0) > 0$. In the actual computation, θ is reset to zero each time a crossing is marked. The figure contrasts the foliation by invariant tori for the unperturbed system, where there appears to be an incidental resonance of order two and one of order three, with the existence of several invariant tori, as well as a strongly stochastic layer, for the perturbed Poincaré map.

We mention that the existence of periodic solutions of the equations of motion in the rotating coordinate system (these correspond to periodic or quasiperiodic motions of the original system) can be proved along the lines of Poincaré's periodic solutions of the first and the second kind for the restricted three-body problem.

The continuation theory for the periodic solutions of the first kind does not depend on the perturbation terms, only on the Floquet multipliers of the "circular" periodic orbits of the Kepler problem in the rotating coordinate system (see [11] [13]). The unperturbed orbits that continue are given by $p_\theta = C$, $p_r = 0$, $r = C^2/k$ for a fixed constant C.

The continuation theory for the elliptical orbits, periodic orbits of the second kind, can be completed along the lines attributed to Poincaré and subsequent authors as outlined in [13, Vol 2, p. 274]. However, these can also be found using the continuation theory of § 3. In the following brief outline of the procedure, we will use the Delaunay formulation of the equations of motion.

After isoenergetic reduction, by implicitly solving for the angular momentum G in the perturbed Hamiltonian as a power series in ϵ, and reformulation of the reduced system as a system of differential equations with the timelike independent variable g, one obtains a system of two differential equations that are π-periodic in g. The continuation theory of § 3 can be applied to this reduced system.

Here, the Poincaré section is given by the submanifold defined by $g = 0$, and the return map is an iterate of the strobe after each g interval of length π. An $(m : n) = (m : 1)$ resonant unperturbed orbit corresponds to an invariant one dimensional torus in the Poincaré section. All such tori are normally nondegenerate due to the fact that the periods of the unperturbed orbits, in the reduced unperturbed system, change monotonically with L (this is equivalent to the twist condition for the Poincaré map). The corresponding bifurcation function maps the angular variable ℓ along the unperturbed orbit to the average of the first order part of the reduced differential equation for the action L over the unperturbed resonant orbit with initial value ℓ. In fact, the function is given (up to a nonzero constant multiple) by $\ell \mapsto (A_m(e) - B_m(e)) \sin m\ell$ where A_m and B_m are defined in (12). This function has simple zeros (for almost all values of the eccentricity). Hence, the unperturbed resonant orbits have continuations. In particular, our method produces a periodic orbit of the form $g \mapsto (L(g,\epsilon), \ell(g,\epsilon))$ for the reduced system with independent variable g.

Using the fact that G is implicitly given as a function of the form

$$G := G(L(g,\epsilon), \ell(g,\epsilon), g, \epsilon),$$

we see that G is also periodic in g. Finally, to obtain g as a function of time, we use the first order differential equation

$$\frac{dg}{dt} = -\frac{\Omega}{2} + \epsilon\frac{1}{2}\alpha\Omega^2\frac{\partial\mathcal{C}}{\partial G}(L(g,\epsilon), G(g,\epsilon), \ell(g,\epsilon), g).$$

This last equation, at least for sufficiently small ϵ, has solutions $g(t)$ such that, for some period $T(\epsilon) > 0$, its solution satisfies $g(t + T(\epsilon)) = g(t) - 2\pi$. Since g is an angular variable, the corresponding function

$$t \mapsto (L(g(t),\epsilon), G(g(t),\epsilon), \ell(g(t),\epsilon), g(t))$$

produces a periodic solution of the original perturbation problem in the rotating coordinate system. These solutions are analogous to Poincaré's periodic solutions of the second kind.

6. Speculations, Conjectures and Numerics

Will a Keplerian binary perturbed by an incident gravitational wave ionize? To make this question precise, we consider the unperturbed system to be a Keplerian ellipse, that is, the eccentricity e of the unperturbed orbit satisfies $0 \leq e < 1$; equivalently, the energy of the unperturbed system defined by Hamiltonian (7) is negative. The corresponding perturbed orbit (the Hamiltonian trajectory given by (5)) generally does not lie on an ellipse. However, we define its osculating conic section at the instant the perturbed motion reaches the state $(p_r, p_\theta, r, \theta)$ to be the conic obtained as the Keplerian orbit with this initial data, that is the Keplerian motion that would be obtained if the perturbation were "turned off" at this instant of time. To ionize, the flow of energy between the binary and the wave must turn unidirectional in a time averaged sense in the course of the perturbation such that the binary would steadily absorb energy; in time, the binary system would be permanently disrupted and the two bodies would eventually be infinitely far apart from each other. On the other hand, the basic equation of motion (2) breaks down once the semimajor axis of the osculating ellipse becomes comparable to the (reduced) wavelength of the incident gravitational wave. To study the route to ionization, we therefore introduce the notion of dissociation. We say the Keplerian ellipse determined by the initial data $(p_r, p_\theta, r, \theta)$ at time t_0 *dissociates* under the influence of the perturbation if at some later time the osculating conic along the perturbed orbit is a hyperbola. Equivalently, if one wishes to remove the geometric language of this definition, the requirement for dissociation may be recast as follows: the Keplerian energy $H(p_r(t), p_\theta(t), r(t), \theta(t))$, where H is given by (7), defined along the perturbed orbit becomes positive in the course of time.

The ionization question probably does not have a simple answer. However, two facts are clear. If the strength of the perturbation is sufficiently small, there are Keplerian binaries that do not ionize. Independent of the strength of the perturbation, there are Keplerian binaries that do dissociate. The first fact is proved in this paper: some of the resonant Keplerian orbits continue to periodic orbits of the perturbed system. We also recall that in the case of sufficiently weak circularly polarized incident gravitational waves, as discussed in § 5, none of the Keplerian orbits ionize. On the other hand, to see that dissociation is possible and to speculate on the fate of all orbits, we must review the geometry of our problem.

Recall that Hamiltonian (5) defines a $2\frac{1}{2}$-degree of freedom Hamiltonian system. This system is equivalent to the three-degree of freedom Hamiltonian system given by

$$\mathcal{H}^*(J, p_r, p_\theta, s, r, \theta) = J + \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{k}{r} + \epsilon r^2(\phi(s)\cos 2\theta + \psi(s)\sin 2\theta), \tag{30}$$

where J is a "fictitious" action variable conjugate to the "time" s. Note here that the phase space of (30) is six dimensional and that the five dimensional submanifold $\mathcal{P}$ given by

$$H(p_r, p_\theta, \theta, r) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{k}{r} = 0$$

separates the phase space. This submanifold corresponds to the parabolic Kepler orbits while a Keplerian binary corresponds to an initial point in the region of the six dimensional phase space given by $H(p_r, p_\theta, \theta, r) < 0$. Dissociation occurs provided the perturbed orbit of the Keplerian binary eventually crosses the manifold $\mathcal{P}$.

To determine that some perturbed orbits do in fact cross the manifold $\mathcal{P}$, in both directions, we simply compute the derivative of H along the perturbed orbit to obtain

$$\dot{H} = -2\epsilon\Big(rp_r(\phi(s)\cos 2\theta + \psi(s)\sin 2\theta) + p_\theta(\psi(s)\cos 2\theta - \phi(s)\sin 2\theta)\Big).$$

This derivative may be interpreted as a measure of the cosine of the angle between the perturbed Hamiltonian velocity field and the normal to the submanifold $\mathcal{P}$. It is apparent that there are open sets on $\mathcal{P}$ where $\dot{H} > 0$. Thus, there are open subsets (obtained by reversing the flow on the boundary set) of the region $H < 0$ such that every point of the subset corresponds to a Keplerian ellipse that dissociates. However, we emphasize the fact that time, represented by s in our three-degree of freedom system, is one of the variables under consideration when these open sets are determined. Thus, the initial data for a Keplerian ellipse that dissociates include an initial time $t = t_0$. We do not know, from this analysis, how far back in time the reversed orbits remain in the region where $H < 0$.

A more sophisticated analysis might be based on the diffusion properties of the orbits of $\mathcal{H}^*$. The geometric picture that is believed to hold for the dynamics of a nearly integrable Hamiltonian system with at least three degrees of freedom is easy to describe informally: there is a dense set of invariant tori coexisting with a dense set of orbits some of which are dense in their respective energy surfaces (see [2] [5]).

For the Hamiltonian (30), it is easy to see that each five dimensional energy surface intersects $\mathcal{P}$. In fact, each energy surface intersects the subsets of $\mathcal{P}$ defined by $\dot{H} > 0$ and $\dot{H} < 0$. Thus, in the situation of the conjectured dynamics, for a sufficiently small perturbation strength, some of the orbits not on invariant tori ionize while a large subset of the orbits on invariant tori do not. Of course, some of the invariant tori of the perturbed system might intersect $\mathcal{P}$; under our definition, the corresponding orbits will dissociate even though these same orbits will repeatedly return to the region where their osculating conics are ellipses.

We note that the usual theory that is used to prove the existence of invariant tori for nearly integrable Hamiltonian systems, namely the KAM theory, is not directly applicable to the Hamiltonian given by (30) because the unperturbed system does not meet the required nondegeneracy conditions. In fact, this Hamiltonian is degenerate and isoenergetically degenerate (cf. [2, p. 408]). In particular, these facts are evident from the Delaunay action-angle coordinates for (30), where we see that the three-dimensional unperturbed invariant tori given by fixing J, L, and G do not even have dense trajectories because $\dot{g} = 0$.

To obtain "nondegenerate" tori, one must consider the two dimensional tori given by fixing J, L, G, and g while leaving ℓ and s free. Some of these tori may survive perturbation and, given their dimensions, it is possible that some of the perturbed tori are "whiskered": they have stable and unstable manifolds (each with dimension at most three). The existence of these invariant manifolds—together with the stable and unstable manifolds associated with periodic solutions (one dimensional invariant tori) and the intersections among them—is likely responsible for the diffusion of some of the orbits not in the union of these invariant sets and the invariant tori.

FIGURE 2. Projections into (p_r, r) plane of part of one orbit, approximately 5000 iterates in each panel, of the Poincaré map for the Hamiltonian system with Hamiltonian (5). The parameters are $\alpha = \beta = 2$, $k = 1$, $\rho = 0$, and ϵ in the panels from left to right is 0.0, 0.001, 0.002 and 0.0025. The initial values are $(p_r, p_\theta, r, \theta) = (.5, 1, 1, 0)$. In this case, Ω is chosen ($\Omega \approx 3.897$) so that the unperturbed Keplerian ellipse has frequency approximately 1/6th the frequency of the incident gravitational wave. The region bounded by the branches of the curve given by $rp_r^2 = 2k$ shown in the panels corresponds to elliptical motion. To obtain the .ps files for this figure or hard copies of the figure,
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We end this section with a short description of some of the numerical experiments performed on the Hamiltonian system (5) given by

$$\mathcal{H}(p_r, p_\theta, r, \theta) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - \frac{k}{r} + \epsilon r^2(\phi(t)\cos 2\theta + \psi(t)\sin 2\theta),$$

where ϕ and ψ are given by equation (6). The results of a typical experiment that suggests the possibility of dissociation for an elliptical Keplerian orbit with eccentricity $e = 0.5$ are depicted in Fig. 2. To obtain the figure, the above $2\frac{1}{2}$-degree of freedom Hamiltonian system is integrated numerically, and the values of the solution are projected into the (p_r, r)-plane after each time interval of length $2\pi/\Omega$. The figure suggests that dissociation will occur for values of ϵ that exceed $\epsilon \approx 0.002$.

To gain some insight into the absorption of gravitational waves by a Newtonian binary, we have also tested the "rate of dissociation", defined to be inversely proportional to the number of iterates of the

Poincaré map required before the osculating conic of the perturbed ellipse becomes a hyperbola, by numerical integration of the Hamiltonian system for various values of the frequency Ω and phase shift ρ of the incident wave. We assume here that $\alpha = \beta$; moreover, the initial elliptical motion is counterclockwise. Although these experiments are somewhat difficult to interpret, one fact seems to emerge regarding the sensitivity of the rate of dissociation to the polarization of the wave. For fixed Ω, the maximal dissociation rate is in the vicinity of $\rho = -\pi/2$ while the minimal dissociation rate is in the vicinity of $\rho = \pi/2$. This rate also depends on Ω, but in a seemingly unpredictable manner.

7. Appendix A: $\mathcal{C}$ and $\mathcal{S}$ in terms of Delaunay elements

The purpose of this appendix is to express $\mathcal{C} = r^2 \cos 2\theta$ and $\mathcal{S} = r^2 \sin 2\theta$ in terms of Delaunay elements. It follows from the relation $\theta = v + g$ that

$$\begin{aligned}\mathcal{C} =& r^2 \cos(2g + 2v) = r^2 \cos 2v \cos 2g - r^2 \sin 2v \sin 2g,\\ \mathcal{S} =& r^2 \sin(2g + 2v) = r^2 \sin 2v \cos 2g + r^2 \cos 2v \sin 2g.\end{aligned}$$

Moreover,

$$\cos v = \frac{\cos \widehat{u} - e}{1 - e\cos\widehat{u}}, \quad \sin v = \sqrt{1-e^2}\,\frac{\sin\widehat{u}}{1 - e\cos\widehat{u}};$$

these relations follow from (9), and the fact that by definition $v \to \widehat{u}$ as $e \to 0$ (cf. [13, Vol. 1, p. 100]). Therefore,

$$\begin{aligned}r^2 \cos 2v =& a^2 (\frac{3}{2}e^2 - 2e\cos\widehat{u} + \frac{1}{2}(2 - e^2)\cos 2\widehat{u}),\\ r^2 \sin 2v =& a^2 \sqrt{1-e^2}\,(\sin 2\widehat{u} - 2e \sin\widehat{u}).\end{aligned}$$

There are classical expansions for $\cos j\widehat{u}$ and $\sin j\widehat{u}$ in Fourier series whose coefficients are expressible in terms of the Bessel function J_ν of order ν. Here, this Bessel function is most conveniently defined by

$$J_\nu(x) := \frac{1}{2\pi}\int_0^{2\pi} \cos(\nu t - x\sin t)\, dt.$$

Following, for example, J. Kovalevsky [8, p. 49], one finds that

$$\begin{aligned}\cos\widehat{u} =& -\frac{e}{2} + \sum_{\nu=1}^{\infty} \frac{1}{\nu}[J_{\nu-1}(\nu e) - J_{\nu+1}(\nu e)]\cos\nu\ell,\\ e\sin\widehat{u} =& 2\sum_{\nu=1}^{\infty}\frac{1}{\nu}J_\nu(\nu e)\sin\nu\ell,\end{aligned}$$

and, for $j > 1$,

$$\cos j\hat{u} = \sum_{\nu=1}^{\infty} \frac{j}{\nu}[J_{\nu-j}(\nu e) - J_{\nu+j}(\nu e)] \cos \nu\ell,$$
$$\sin j\hat{u} = \sum_{\nu=1}^{\infty} \frac{j}{\nu}[J_{\nu-j}(\nu e) + J_{\nu+j}(\nu e)] \sin \nu\ell.$$

Using these expansions, we obtain the Fourier series given in (11) where

$$\begin{aligned} A_\nu =& \frac{1}{\nu}\Big((2-e^2)[J_{\nu-2}(\nu e) - J_{\nu+2}(\nu e)] - 2e[J_{\nu-1}(\nu e) - J_{\nu+1}(\nu e)]\Big), \\ B_\nu =& \frac{2}{\nu}\sqrt{1-e^2}\,\Big([J_{\nu-2}(\nu e) + J_{\nu+2}(\nu e)] - 2J_\nu(\nu e)\Big). \end{aligned} \tag{31}$$

We note that the functions $\mathcal{C}$ and $\mathcal{S}$ are analytic and 2π periodic in the angle variables ℓ and g. Moreover, partial derivatives with respect to the Delaunay elements can be obtained by differentiation of their Fourier series term by term.

To simplify the expressions for the Fourier coefficients computed above, we will use the following elementary identities for the Bessel functions [8, p. 48]:

$$J_\nu(x) = \frac{x}{2\nu}[J_{\nu-1}(x) + J_{\nu+1}(x)],$$
$$J'_\nu(x) = \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)], \tag{32}$$
$$J''_\nu(x) = \frac{1}{4}[J_{\nu-2}(x) - 2J_\nu(x) + J_{\nu+2}(x)], \tag{33}$$

and Bessel's equation

$$x^2 J''_\nu(x) + xJ'_\nu(x) + (x^2 - \nu^2)J_\nu(x) = 0. \tag{34}$$

The final expressions for A_ν and B_ν given in (12) are obtained from (31) using (32)- (34); in fact, the formula for A_ν is obtained by standard methods using (32), and the formula for B_ν is derived from the original expressions (31) and (33) after noticing that B_ν is proportional to $J''_\nu(\nu e)$ and using Bessel's equation (34).

8. Appendix B: A binary influenced by a distant massive third body

The purpose of this appendix is to explore the possibility of applying our results to the three body problem. For a binary system influenced by a distant massive third body, our "tidal" approach results in a limiting case of the celebrated problem of three bodies and the question is whether the continuation theory of § 3 would be applicable in this case. The existence of periodic orbits in the three-body problem has been established in the classical work of Poincaré [12].

We study the effect of a massive body, metaphorically the Sun, on a binary, metaphorically the Earth-Moon system, where the Sun is viewed as giving rise to a periodic perturbation of the Earth-Moon orbit by tidal forces. To derive the equations of motion that will be considered in this appendix, we will model the Earth-Moon-Sun system according to the following scenario. The motion of the Sun is neglected due to its great mass, its gravitational attraction brings about the motion of the Earth-Moon system as a whole on an almost Keplerian orbit about the Sun and its tidal influence perturbs the orbit of the Moon about the Earth. It is this latter motion that constitutes the lunar problem under investigation here.

To obtain the mathematical model, let us consider the equations of motion of m_1 and m_2—the Earth and Moon in our approximation, respectively—as given by (1), with a single perturbing body, namely the Sun, with potential

$$\Phi(\mathbf{X}) = -\frac{\mathbf{G_0 M_\odot}}{|\mathbf{X} - \mathbf{X}_\odot|}. \tag{35}$$

It is useful to transform the equations of motion of m_1 and m_2 from $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ to the relative coordinates $\mathbf{r} = \mathbf{X}_1 - \mathbf{X}_2$ and the center-of-mass coordinates $\mathbf{X}_{\rm cm} = (m_1\mathbf{X}_1 + m_2\mathbf{X}_2)/(m_1 + m_2)$. In terms of these new coordinates, the solar gravitational attraction involves the relative coordinates $\mathbf{r}$ and $\mathbf{R} = \mathbf{X}_{\rm cm} - \mathbf{X}_\odot$. Let us further assume that $\mu = r/R \ll 1$, where r and R denote the magnitudes of the corresponding vectors, and consider the expansion of the solar influence in these equations in terms of μ. Using the fact that

$$\frac{\mathbf{R} + \eta\mathbf{r}}{|\mathbf{R} + \eta\mathbf{r}|^3} = \frac{\mathbf{R}}{R^3} + \frac{\eta}{R^3}\Big(\mathbf{r} - 3\frac{\mathbf{R}\cdot\mathbf{r}}{R^2}\mathbf{R}\Big) + O(\mu^2),$$

where η is a constant parameter with $|\eta| < 1$, the equations of motion reduce to

$$\frac{d^2\mathbf{X}_{\rm cm}}{\mathbf{dt^2}} = -\, G_0 M_\odot \frac{\mathbf{R}}{R^3} + O(\mu^2), \tag{36}$$

$$\frac{d^2 r^i}{dt^2} + \frac{k r^i}{r^3} = -\, K_{ij} r^j + O(\mu^2), \tag{37}$$

where we have introduced the "tidal matrix" K_{ij} such that

$$K_{ij} = \frac{G_0 M_\odot}{R^3}\Big(\delta_{ij} - 3\frac{R^i R^j}{R^2}\Big). \tag{38}$$

We consider the external body (the Sun) to be so massive ($M_\odot >> m_1, m_2$) that it is essentially unaffected by the presence of m_1 and m_2 (the Earth and Moon, respectively). Thus, we can take $\mathbf{X}_\odot = \mathbf{0}$ and therefore the Sun remains fixed at the origin of the inertial coordinate system under consideration. Neglecting terms of order r^2/R^2 in (36), this equation reduces to the Newtonian two-body equation for the relative motion of the center-of-mass about the Sun. We take this orbit to be slightly elliptic (for example, for the Earth-Moon orbit about the Sun, the eccentricity e_1 is approximately 0.017). The resulting expression for $\mathbf{R}$ can be substituted into (37) to give the equations describing the dynamics of the Earth-Moon system in the presence of the Sun. We further assume that the relative orbit as well as the center-of-mass motion occurs in the equatorial plane of the Sun. This should be a reasonable approximation as the Earth-Moon orbital plane makes an angle of approximately 5° with the ecliptic (the ecliptic is essentially the plane of the Earth's orbit around the Sun) while the ecliptic makes an angle of approximately 7° with the equatorial plane of the Sun. It is clear that this "tidal" approach to the three-body problem is somewhat different from the standard "restricted" approach; in the latter case, the mass of the Moon is effectively set equal to zero.

The Earth-Moon orbit about the Sun has a small eccentricity; therefore, the tidal matrix in (37) will be determined to first order in the eccentricity. To this end, let $\Omega^2 = G_0 M_\odot / a_\odot^3$ (with $a_\odot$ being the semimajor axis of the Earth-Moon orbit around the Sun) and note that the eccentric anomaly is $\hat{u} \approx \Omega t + e_1 \sin \Omega t$, the true anomaly is $v \approx \Omega t + 2e_1 \sin \Omega t$, and $R \approx a_\odot(1 - e_1 \cos \Omega t)$. Using $R^1 = R \cos v$ and $R^2 = R \sin v$, the Cartesian components of the tidal matrix are given by

$$
\begin{aligned}
K_{11} =& -\Omega^2\Big(\frac{1}{2} + \frac{3}{2}\cos 2\Omega t - \frac{3}{2}e_1 \cos \Omega t(3 - 7\cos 2\Omega t)\Big),\\
K_{12} =& -\frac{3}{2}\Omega^2(\sin 2\Omega t + e_1 \sin \Omega t(3 + 7\cos 2\Omega t)),\\
K_{22} =& -\Omega^2\Big(\frac{1}{2} - \frac{3}{2}\cos 2\Omega t + \frac{3}{2}e_1 \cos \Omega t(5 - 7\cos 2\Omega t)\Big),\\
K_{13} =& K_{23} = 0.
\end{aligned}
\tag{39}
$$

As the tidal matrix is traceless and symmetric, the above equations determine all of its elements.

Using (37) and (39) we can write the associated Hamiltonian for this system. Because the motion is taken to be in the equatorial plane, polar coordinates are convenient. In these coordinates the Hamiltonian is

$$\begin{aligned}\mathcal{H} =&\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)-\frac{k}{r}-\frac{\Omega^2 r^2}{4}\{1+3e_1\cos\Omega t\\&+3\cos 2\theta[\cos 2\Omega t-e_1\cos\Omega t(4-7\cos 2\Omega t)]\\&+3\sin 2\theta[\sin 2\Omega t+e_1\sin\Omega t(3+7\cos 2\Omega t)]\}.\end{aligned}\tag{40}$$

Upon expressing the Hamiltonian equations in terms of intrinsic dimensionless quantities, it becomes clear that the strength of the interaction between the binary and the third body is $\Omega^2/\omega^2 \ll 1$, but the square root of this perturbation parameter also occurs in the harmonic terms that render the Hamiltonian (40) explicitly time-dependent. In particular, the period of the harmonic terms becomes unbounded as the perturbation parameter goes to zero. Therefore, the continuation method of § 3 is not directly applicable in this case; in fact, the resolution of this problem is due to Hill (cf. [8] [12]). In Hill's approach, the equation of relative motion (37) is referred to a Cartesian system of coordinates $\mathbf{r}'$ that rotates with frequency Ω with respect to the inertial system. Let $r^i = S_{ij}r'^j$, where the nonzero elements of the orthogonal matrix S are given by

$$S_{11}=S_{22}=\cos\Omega t,\quad -S_{12}=S_{21}=\sin\Omega t,\quad S_{33}=1;$$

then, the equations of motion in the new system are $(r'=r)$

$$\begin{aligned}\frac{d^2x'}{dt^2}-2\Omega\frac{dy'}{dt}-\Omega^2x'+\frac{kx'}{r'^3}&=-\left(K'_{11}x'+K'_{12}y'\right),\\\frac{d^2y'}{dt^2}+2\Omega\frac{dx'}{dt}-\Omega^2y'+\frac{ky'}{r'^3}&=-\left(K'_{12}x'+K'_{22}y'\right),\end{aligned}\tag{41}$$

where $K'=S^TKS$, i.e.,

$$K'_{11}=-2\Omega^2(1+3e_1\cos\Omega t),\quad K'_{12}=-6\Omega^2e_1\sin\Omega t,\quad K'_{22}=-\frac{1}{2}K'_{11},\tag{42}$$

and $K'_{13}=K'_{23}=0$. The system (41) is autonomous for $e_1=0$. In this case, periodic solutions exist as originally demonstrated by Hill and Poincaré (cf. [12]). The continuation of such solutions using e_1, $0\le e_1\ll 1$, as the expansion parameter can be proved, using the Implicit Function Theorem, as originally conceived by Poincaré (cf. [11] [8] [12]) for the restricted three-body problem. Of course, the method of § 3 is also applicable by following the ideas for isoenergetic reduction as discussed in § 5.

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