Sharp Estimates in Ruelle Theorems for Matrix Transfer Operators

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Abstract

A matrix coefficient transfer operator $(\mathcal{L}\Phi)(x) = \sum \phi(y)\Phi(y)$, $y \in f^{-1}(x)$ on the space of C^r-sections of an m-dimensional vector bundle over n-dimensional compact manifold is considered. The spectral radius of $\mathcal{L}$ is estimated by $\exp\sup\{h_\nu + \lambda_\nu : \nu \in \mathcal{M}\}$ and the essential spectral radius by

$$\exp\sup\{h_\nu + \lambda_\nu - r \cdot \chi_\nu : \nu \in \mathcal{M}\}.$$

Here $\mathcal{M}$ is the set of ergodic f-invariant measures, and for $\nu \in \mathcal{M}$, h_ν is the measure-theoretic entropy of f, λ_ν is the largest Lyapunov exponent of the cocycle over f generated by ϕ, and χ_ν is the smallest Lyapunov exponent of the differential of f.

Keywords. Transfer operator, Lyapunov exponents, multiplicative ergodic theorem, weighted translation operators

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1 Introduction

In the present paper we employ a technique from [24] to study spectral properties of a matrix coefficient transfer operator $\mathcal{L}$, the Ruelle operator. The operator $\mathcal{L}$ acts on the space $C^r = C^r_E(X)$, $r = 0, 1, \dots$ of smooth sections of an m-dimensional real vector bundle E over an n-dimensional compact manifold X by the rule

$$(\mathcal{L}\Phi)(x) = \sum_{y \in f^{-1}(x)} \phi(y)\Phi(y), \quad \Phi \in C^r, x \in X, \tag{1}$$

where $f : X \to X$ is a smooth covering and ϕ is a smooth bundle map, $\phi(x) : E_x \to E_{f(x)} \simeq \mathbb{R}^m$, with $\det \phi(x) > 0$, $x \in X$.

We sharpen the following results by D. Ruelle from his celebrated paper [31]. A map f is called *expanding* if $d(f(x), f(y)) \geq \theta d(x, y)$ whenever $d(x, y) < \varepsilon$, for a positive ε and some $\theta > 1$. Let $\mathcal{M}$ be the set of ergodic f-invariant Borel probability measures on X, h_ν be the entropy of f with respect to $\nu \in \mathcal{M}$, and $\|\phi(x)\|$ be the operator norm.

Ruelle Theorem 1 *Assume f is expanding. The spectral radius of $\mathcal{L}$ on C^r, $r = 0, 1, \dots$ is less than or equal to*

$$\exp \sup_{\nu \in \mathcal{M}} \{h_\nu + \int_X \log \|\phi(x)\| \, d\nu\}. \tag{2}$$

Ruelle Theorem 2 *Assume f is expanding. The essential spectral radius of $\mathcal{L}$ on C^r, $r = 0, 1, \dots$ is less than or equal to*

$$\exp \sup_{\nu \in \mathcal{M}} \{h_\nu + \int_X \log \|\phi(x)\| \, d\nu - r \cdot \log \theta\}.$$

To formulate our results, let λ_ν denote the *largest* Lyapunov-Oseledec exponent of the cocycle $\phi_k(x) = \phi(f^{k-1}x) \cdot \dots \cdot \phi(x)$ over f and χ_ν denote the *smallest* Lyapunov-Oseledec exponent of the differential $Df^k(x)$, $k \in \mathbb{N}$, given for each $\nu \in \mathcal{M}$ by the multiplicative ergodic theorem [27].

Theorem 1 *Assume f is any smooth covering. The spectral radius of $\mathcal{L}$ on C^r, $r = 0, 1, \dots$ is less than or equal to* $\exp \sup_{\nu \in \mathcal{M}} \{h_\nu + \lambda_\nu\}$.

Theorem 2 *Assume f is any smooth covering. The essential spectral radius of $\mathcal{L}$ on C^r, $r = 0, 1, \ldots$ is less than or equal to* $\exp \sup_{\nu\in\mathcal{M}}\{h_\nu + \lambda_\nu - r \cdot \chi_\nu\}$.

Theorem 3 *Assume f is expanding. The spectral radius of $\mathcal{L}$ on C^0 is equal to* $\exp \sup_{\nu\in\mathcal{M}}\{h_\nu + \lambda_\nu\}$.

Theorem 4 *Assume f is expanding. The approximate point spectrum of $\mathcal{L}$ on C^0 contains the annulus*

$$\left\{ z \in \mathbb{C} : \exp \sup_{\nu\in\mathcal{M}} \lambda_\nu \le |z| \le \exp \sup_{\nu\in\mathcal{M}} \{h_\nu + \lambda_\nu\} \right\}. \tag{3}$$

The spectral theory for the Ruelle operator $\mathcal{L}$ is very well developed in the scalar case $m = 1$. We refer the interested reader to the books [33] and [28], an excellent review [6], papers [7, 8, 9, 13, 16, 18, 29, 32] and the references therein. For *invertible* f the operator $\mathcal{L}$ becomes a so-called *weighted translation* operator T defined as follows:

$$(T\Phi)(x) = \phi(f^{-1}x)\Phi(f^{-1}x),\ \Phi \in C^r, \quad f \quad \text{is invertible.} \tag{4}$$

The spectral properties of the weighted translation operators on C^0 for $m \ge 1$ are well-understood, see [1, 2, 3, 19, 24] and references there. These operators were intensively used to study the hyperbolic dynamical systems starting from the paper by J. Mather [26], see [3, 24] and also [22, 23] for infinite-dimensional bundles. It is known, for instance, that the spectrum of T on C^0 for $m \ge 1$ is rotationally invariant provided f has a dense set of aperiodic points [1, 3]. Also [24, 25], the spectral radius of T on C^0 is equal to $\exp \sup_{\nu\in\mathcal{M}} \lambda_\nu$. In the present paper we use some methods from the theory of the weighted translation operators to study the Ruelle operator $\mathcal{L}$.

We make the following remarks concerning Theorems 1–4.

1. The estimates in Theorems 1–2 are clearly sharper than in Ruelle Theorems 1–2. In general, $\lambda_\nu \le \int_X \log \| \phi \| \, d\nu$ with strict inequality possible, and $\chi_\nu \ge \log \theta$ for any $\nu \in \mathcal{M}$. Also, the estimate in Theorem 2 is *local*, unlike the estimate in Ruelle

Theorem 2. In particular, for the scalar case $m = 1$ the essential spectral radius of $\mathcal{L}$ on C^r is less than or equal to

$$\exp \sup_{\nu \in \mathcal{M}} \{h_\nu + \int_X \log \phi \, d\nu - r \cdot \chi_\nu\}.$$

2. Theorem 3 shows that the estimate in Theorem 1 cannot be improved. Theorem 1 can be derived from Ruelle Theorem 1 and Remark 3.4 in [31], as follows. By Ruelle Theorem 1 the n-th root of the spectral radius of $\mathcal{L}^n$ is at most

$$\exp(\frac{1}{n} P(\log \parallel \phi_n \parallel)).$$

Letting $n \to \infty$,

$$\frac{1}{n} P(\log \parallel \phi_n \parallel) = \frac{1}{n} \sup_\nu \{h_\nu(f^n) + \int\limits_X \log \parallel \phi_n \parallel d\nu\}$$

converges to $\sup\limits_\nu \{h_\nu(f) + \lambda_\nu\}$, by the Oseledec multiplicative ergodic theorem. This argument was suggested to the authors by A. Katok, and, of course, depends upon the Ruelle estimate, while our proof is derived independently.

3. For the multiplicative matrix-valued cocycles ϕ_k, $m > 1$ the construction

$$\sup\{h_\nu + \lambda_\nu : \nu \in \mathcal{M}\}, \quad \text{with} \quad \lambda_\nu = \lim_{k \to \infty} k^{-1} \log \|\phi_k(x)\|, \quad \nu - \text{ a.a. } x \in X$$

is a natural replacement for the pressure (see, e.g., [35])

$$P(f, \log \phi) = \sup\{h_\nu + \int_X \log \phi(x) d\nu : \nu \in \mathcal{M}\}$$

for the scalar case $m = 1$.

4. In Theorems 3–4, instead of an expanding map on a manifold, one can assume that (X, f) is a topologically mixing one-sided subshift of finite type.

5. For the scalar case $m = 1$ and expanding f the spectrum of $\mathcal{L}$ on C^0 is a disc centered at the origin ([10, 30]). We do not know under which conditions for $m > 1$ the spectrum of $\mathcal{L}$ on C^0 is also the disc with the radius given in Theorem 3. Theorem 4 is a result in this direction.

We conclude the Introduction with an informal discussion of the ideas of the proofs of Theorems 1–4. To avoid technicalities, we assume $X \subset \mathbb{R}^n$, $E = X \times \mathbb{R}^m$ and $r = 0$ or $r = 1$. The detailed proofs of Theorems 1 and 3 are given in Section 2, Theorem 4 in Section 3, and Theorem 2 in Section 4.

The main idea in the proof of Theorem 1, as in [24, Theorem 4.5], is to use the following scalarization of the cocycle ϕ_k (cf. the proof of the multiplicative ergodic theorem in [14]). Define $\widetilde{X} = X \times S^{m-1}$, $S^{m-1} \equiv \{v \in \mathbb{R}^m : \|v\|_{\mathbb{R}^m} = 1\}$ and $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ and $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ as follows:

$$\widetilde{f} : (x, v) \mapsto \left(f(x), \frac{\phi(x)v}{\|\phi(x)v\|} \right), \quad \widetilde{\phi}\,(x, v) = \|\phi(x)v\|,\ (x, v) \in \widetilde{X}\,. \tag{5}$$

Note that in C^0,

$$(\mathcal{L}^k \Phi)(x) = \sum_{y \in f^{-k}(x)} \phi_k(y) \Phi(y), \quad \|\mathcal{L}^k\| \leq \max_{x \in X} \sum_{y \in f^{-k}(x)} \|\phi_k(y)\|. \tag{6}$$

Using (5), one proves that the logarithm of the spectral radius of $\mathcal{L}$ in C^0 is dominated by the pressure $P(\widetilde{f}, \log \widetilde{\phi})$, and a short argument with projections of $\tilde{f}$-invariant measures from $\widetilde{X}$ to X gives Theorem 1 for $r = 0$. For $r = 1$ an estimate similar to (6) holds, and the proof goes through as before.

To prove Theorem 3, as in [24, Theorem 4.5], we follow the original proof of the multiplicative ergodic theorem [27] and triangularize the cocycle ϕ_k. Let $SO(m)$ denote the set of orthogonal $(m \times m)$-matrices with the unit determinant. Define $\overline{X} = X \times SO(m)$ and $\overline{f} : \overline{X} \to \overline{X}$ and $\overline{\phi} : \overline{X} \to \mathbb{R}$ as follows:

$$\overline{f} : (x, V) \mapsto (f(x), \tilde{V}(x, V)), \quad \overline{\phi}(x, V) = \tilde{V}^T(x, V)\phi(x)V, \quad (x, V) \in \overline{X}, \tag{7}$$

where "T" means transposition, and for each $(x, V) \in \overline{X}$ we denote by $\tilde{V}(x, V)$ the matrix in $SO(m)$ such that $\overline{\phi}(x, V)$ is lower triangular with positive diagonal entries. Then $\overline{\phi}$ and $\overline{f}$ define on $C^0_{\overline{X}}(\overline{X} \times \mathbb{R}^m)$ a transfer operator $\overline{\mathcal{L}}$ with the spectral radius less or equal then the spectral radius of $\mathcal{L}$ on $C^0_X(E)$. Since $\overline{\phi}$ in $\mathcal{L}$ is triangular, it is possible to apply the scalar $(m = 1)$ result for the spectral radius of transfer operator from, e.g, [10, 33] to estimate the spectral radius of $\overline{\mathcal{L}}$ in terms of $\overline{f}$-pressure for the diagonal elements of $\overline{\phi}$. By projecting the $\overline{f}$-invariant measures from $\overline{X}$ to X one derives Theorem 3.

The main idea of the proof of Theorem 2, cf. [19] and [13], is to projectivize the operator $\mathcal{L}$, acting on $C^1_X(E)$, as follows. Define

$$\mathcal{X} = \{(x, v) : x \in X, v \in \mathbb{R}^n, \|v\|_{\mathbb{R}^n} = 1\}, \quad \mathcal{E} = \mathcal{X} \times \mathbb{R}^m$$

and

$$F : \mathcal{X} \to \mathcal{X} : (x, v) \mapsto \left(f(x), \frac{Df(x)}{\|Df(x)\|}\right), \quad \psi(x, v) = \frac{\phi(x)}{\|Df(x)v\|}, \tag{8}$$

where Df is differential of f. Introduce an operator $\mathcal{D} : C^1_X(E) \to C^0_{\mathcal{X}}(\mathcal{E})$ by the rule $(\mathcal{D}\Phi)(x, v) = D\Phi(x)v$, and consider on $C^0_{\mathcal{X}}(\mathcal{E})$ the transfer operator $\mathcal{K}$, induced by F and ψ as follows:

$$(\mathcal{K}\Psi)(x, v) = \sum_{(y,u)\in F^{-1}(x,v)} \psi(y, u)\Psi(y, u), \quad \Psi \in C^0_{\mathcal{X}}(\mathcal{E}).$$

A short calculation shows that $\mathcal{DL}\Phi = \mathcal{KD}\Phi + R\Phi$, $\Phi \in C^1_X(E)$ where $R : C^1_X(E) \to C^0_{\mathcal{X}}(\mathcal{E})$ is a compact operator that contains terms with no derivative of Φ. General facts on the stability of Fredholm spectrum under compact perturbations imply that the essential spectral radius of $\mathcal{L}$ on $C^1_X(E)$ is dominated by the spectral radius of $\mathcal{K}$ on $C^0_{\mathcal{X}}(\mathcal{E})$. Now we can apply Theorem 1 to $\mathcal{K}$ to prove the estimate in Theorem 2.

We note that the operator $\mathcal{D}$ and the identity $\mathcal{DL} = \mathcal{KD} + R$ were used, in fact, in [13] for $m = n = 1$. In the case $n = 1$ the operator $\mathcal{D}$ is just the differentiation. We remark that for the weighted translation operator T, the same idea as in [13] was used in [2, 4, 20] to describe the spectrum of T on C^r. Using the same approach, the spectrum of the weighted translation operator T on C^r was described for $m = 1$, $n \geq 1$ in [19].

The idea of the proof of Theorem 4 is related to the principle of localization of almost-eigenfunctions for the weighted translation operator T. This property of the almost-eigenfunctions was noticed first in [26], see also [23], and means that T has almost-eigenfunctions supported in an arbitrarily small neighborhood along an orbit of a point in X. In the proof of Theorem 4 for the operator $\mathcal{L}$, by rescaling, we may assume that $1 = \exp\sup_{\nu\in\mathcal{M}}\{h_\nu + \lambda_\nu\}$ belongs to the approximate point spectrum of $\mathcal{L}$. Then for every $N \in \mathbb{N}$ there exists a norm-one $\Phi \in C^0$ such that $\|\mathcal{L}^k\Phi\|$ is bounded for $k = 0, 1, \dots, 2N$, and $\|\mathcal{L}^N\Phi - \Phi\|$ is small. Take an aperiodic point $x_0 \in X$ where $\|(\mathcal{L}^N\Phi)(x_0)\|$ is of order one, a small ball $B \ni x_0$, and a "bump"-function β supported in B with $\beta(x_0) = 1$. For z from the annulus (3) define

$$\Gamma(x) = \sum_{j=0}^{2N} z^{N-j} \cdot \gamma(j) \cdot \mathcal{L}^j(\beta \circ \phi^N \cdot \Phi)(x), \qquad \operatorname{supp}\Gamma \subset \cup_{j=0}^{2N} f^{j-N}(B), \tag{9}$$

where $\gamma : \mathbb{N} \to [0, 1]$ is supported on $[0, 2N]$ such that $\gamma(N) = 1$ and $|\gamma(j) - \gamma(j-1)|$ is of order $1/N$. One checks that $\|z\Gamma - \mathcal{L}\Gamma\| = O(1/N)\|\Gamma\|$ for z as in (3), that proves the theorem.

Operator Notation. For an operator A on a Banach space $\mathcal{Y}$ we denote: $\sigma(A) = \sigma(A, \mathcal{Y})$, the spectrum, $\sigma_{\text{ap}}(A) = \sigma_{\text{ap}}(A, \mathcal{Y})$ the approximate point spectrum, $\sigma_F(A) = \sigma_F(A, \mathcal{Y})$, the Fredholm spectrum, $\operatorname{rsp}(A) = \operatorname{rsp}(A, \mathcal{Y})$ the spectral radius, $\operatorname{ress}(A) = \operatorname{ress}(A, \mathcal{Y})$ the essential spectral radius. Also, $\mathcal{B}(\mathcal{Y})$ denotes the algebra of bounded linear operators on a Banach space $\mathcal{Y}$.

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2 Spectral Radius

In this section we prove Theorems 1 and 3. These proofs are similar to the proof of Theorem 4.13 from [24], where another type of transfer operator was considered.

To fix notation, let $E = (E, X, p)$ be an m-dimensional vector bundle with the projection $p : E \to X$, fibers $E_x \simeq \mathbb{R}^m$ and base X. Here X is an n-dimensional smooth compact manifold with the tangent bundle TX. For fixed $r = 0, 1, \dots$ we assume that operator $\mathcal{L}$ is defined as in (1) on $C^r_X(E)$ with a C^r-smooth covering $f : X \to X$ and a C^r-smooth bundle automorphism $\phi : E \to E$, such that $\phi(x) : E_x \to E_{f(x)}$ is linear and $\det \phi(x) > 0$ for each $x \in X$. The main case in this section is $r = 0$, the estimate in Theorem 1 for $r \geq 1$ goes similarly.

To put Theorems 1 and 3 into a broader picture, we make the following remark.

Remark 1 Consider the scalar case $m = 1$ first. For the weighted translation operator T in (4) with a homeomorphism f it was proved in [5, 21] that

$$\mathrm{rsp}(T; C^0) = \exp \sup \{ \int_X \log \phi(x) \, d\nu : \nu \in \mathcal{M} \}.$$

For the transfer operator $\mathcal{L}$ induced by an expanding endomorphism f it was proved (see, e.g., [10, 30]) that $\mathrm{rsp}(\mathcal{L}; C^0) = \exp P(f, \log \phi)$, where $P(f, \log \phi)$ is the topological pressure, that, by the variational principle, see, e.g., [17, 35], is equal to $\sup\{h_\nu + \int_X \log \phi(x) \, d\nu : \nu \in \mathcal{M}\}$. For the matrix case $m > 1$ and weighted translation operator (4) it was proved in [24] that $\mathrm{rsp}(T; C^0) = \exp \sup\{\lambda_\nu : \nu \in \mathcal{M}\}$. For the transfer operator $\mathcal{L}$ Ruelle Theorem 1 gives the estimate $\mathrm{rsp}(\mathcal{L}; C^0) \leq \exp P(f, \log \|\phi\|)$ or as in (2). Therefore, our Theorems 1 and 3 complete the calculations for $\mathrm{rsp}(\mathcal{L}; C^0)$.

Throughout the paper we will need a part of the multiplicative ergodic theorem (see [27] or [35]) in the following setting. Let $f : X \to X$ be continuous and $\phi : E \to E$ be a continuous bundle automorphism. Form the cocycle

$$\phi_k(x) = \phi(f^{k-1}(x)) \cdot \ldots \cdot \phi(x) \in GL(\mathbb{R}^m), \quad x \in X, k \in \mathbb{N}.$$

For every f-ergodic Borel probability measure $\nu \in \mathcal{M} = \mathcal{M}(f)$ on X there exists a full ν-measure subset $X_\nu \subset X$ such that for every $x \in X_\nu$ and every $v \in E_x$ the following exact Lyapunov-Oseledec exponent exists:

$$\lambda(v, \nu) = \lim_{k \to \infty} \frac{1}{k} \log \|\phi_k(x) v\|.$$

There are $m' = m'(\nu) \le m$ different exponents $\lambda(v, \nu)$, they do not depend on $x \in X_\nu$, and we order them as follows: $\lambda_\nu \equiv \lambda_\nu^1 > \lambda_\nu^2 > \ldots > \lambda_\nu^{m'}$. Also,

$$\lambda_\nu = \lim_{k \to \infty} \frac{1}{k} \log \|\phi_k(x)\|, \quad x \in X_\nu.$$

If $\phi(x)$ is lower triangular with the positive entries $g_i(x)$, $i = 1, \ldots, m$ on the main diagonal, then the set of the Lyapunov-Oseledec exponents is just

$$\{\int_X \log g_i(x)\, d\nu : i = 1, \ldots, m,\ \nu \in \mathcal{M}\}.$$

The multiplicative ergodic theorem, of course, holds for the cocycle $Df^k(x)$, where Df is the differential. We will denote the Lyapunov-Oseledec exponents for this cocycle as $\chi_\nu^1 > \chi_\nu^2 > \ldots > \chi_\nu^{n'} \equiv \chi_\nu$.

To fix notation, recall the definition of the topological pressure, see, e.g., [35]. A set $\mathcal{F} \subset X$ is called (k, ϵ)-separated if $\max_{0 \le i \le k-1} d(f^i x, f^i y) > \epsilon$ for each $x \ne y$ in X. A set $\mathcal{G} \subset X$ is called (k, ϵ)-spanning if for each $y \in X$ there exists $x \in \mathcal{G}$ such that $\max_{0 \le i \le k-1} d(f^i x, f^i y) \le \epsilon$. For a continuous $\rho : X \to \mathbb{R}$, $k \in \mathbb{N}$ and $\epsilon > 0$ denote:

$$\begin{aligned} P_k(f, \rho, \epsilon) &= \sup \left\{ \sum_{x \in \mathcal{F}} \exp \left(\sum_{i=0}^{n-1} \rho(f^i x) \right) : \mathcal{F} \text{ is a } (k, \epsilon)- \text{ separated set} \right\}, \\ Q_k(f, \rho, \epsilon) &= \inf \left\{ \sum_{x \in \mathcal{G}} \exp \left(\sum_{i=0}^{n-1} \rho(f^i x) \right) : \mathcal{G} \text{ is a } (k, \epsilon)- \text{ spanning set} \right\}. \end{aligned}$$

The topological pressure then is

$$P(f, \rho) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{k} \log P_k(f, \rho, \epsilon) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{k} \log Q_k(f, \rho, \epsilon).$$

Proof of Theorem 1. For $r = 0$ we use (5) and (6) to estimate $\|\mathcal{L}^k\|_{\mathcal{B}(C^0)}$, $k \in \mathbb{N}$ as follows:

$$\|\mathcal{L}^k\|_{\mathcal{B}(C^0)} \leq \sum_{y \in f^{-k}(x_k)} \|\phi_k(y)\| = \sum_{y \in f^{-k}(x_k)} \|\phi_k(y)v_y\| = \sum_{y \in f^{-k}(x_k)} \widetilde{\phi}_k(y, v_y) \tag{10}$$

for appropriate, by compactness, x_k, where the max in (6) attains, and v_y chosen to satisfy $\| v_y \|= 1$ and $\| \phi_k(y) \|=\| \phi_k(y)v_y \|$. Here the *scalar* cocycle $\widetilde{\phi}_k$ over $\widetilde{f}$ is defined by (5) as follows:

$$\widetilde{\phi}_k : \widetilde{X} \to \mathbb{R}_+, \quad \widetilde{\phi}_k(x, v) = \widetilde{\phi}(\widetilde{f}^{k-1}(x, v)) \cdot \ldots \cdot \widetilde{\phi}(x, v), \quad (x, v) \in \widetilde{X}.$$

Because f is a covering, the last sum in (10) may be estimated using the topological pressure for $\widetilde{f}$ as follows. For $k \in \mathbb{N}$ and

$$F_k(x) = \left\{(y, v_y) : y \in f^{-k}x\right\} \subseteq \widetilde{X},$$

for sufficiently small ϵ and each $x \in X$, the set $F_k(x)$ is (k, ϵ)-separated for $\widetilde{f}$ since f is a covering in X. Then

$$\sum_{y \in f^{-k}(x)} \widetilde{\phi}_k(y, v_y) \leq P_k(\widetilde{f}, \log \widetilde{\phi}, \epsilon). \tag{11}$$

Hence, for $r = 0$,

$$\log \operatorname{rsp}(\mathcal{L}; C^r) = \lim_{k \to \infty} \frac{1}{k} \log \|\mathcal{L}^k\| \leq P(\widetilde{f}, \log \widetilde{\phi}). \tag{12}$$

The same estimate (12) also holds for $r = 1, 2, \ldots$. Indeed, as in the proof of Theorem 3.1 in [31, p. 248], by taking derivatives in (6), one gets from (10)–(11) the inequality $\|\mathcal{L}^k \Phi\|_{C^r} \leq p(k) P_k(\widetilde{f}, \log \widetilde{\phi}, \epsilon) \|\Phi\|_{C^r}$ for a polynomial $p(k)$, and (12) follows.

Now we use the variational principle [35] and arguments from [24, Theorem 4.7] and [24, Lemma 4.14] to see that

$$P(\widetilde{f}, \log \widetilde{\phi}) = \sup_{\mu \in \mathcal{M}(\widetilde{f})} \left\{h_\mu(\widetilde{f}) + \int_{\widetilde{X}} \log \widetilde{\phi} \, d\mu\right\} \leq \sup_{\nu \in \mathcal{M}(f)} \{h_\nu(f) + \lambda_\nu\}. \tag{13}$$

Indeed, for a measure $\mu \in \mathcal{M}(\tilde{f})$ on $\tilde{X}$ its projection $\nu = \mathrm{proj}\mu$ on X, $\nu(e) = \mu(e \times S)$, $e \subset X$, belongs to $\mathcal{M}(f)$ and satisfies $h_\mu(\tilde{f}) = h_\nu(f)$ (see [24, Lemma 4.14]) and $\int_{\tilde{X}} \log \tilde{\phi}\, d\mu \leq \lambda_\nu$ (see [24, Theorem 4.7] or [14]). This completes the proof of Theorem 1. ∎

Proof of Theorem 3. We use the notation in (7). Also, for the bundle $E = (E, X, p)$ let $\overline{E} = (\overline{E}, \overline{X}, \overline{p})$ be the bundle $\overline{E} = E \times SO(m)$ over $\overline{X} = X \times SO(m)$ with projection $\overline{p} : (x, V, u) \mapsto (x, V)$, $u \in E_x$. Consider natural projections

$$\pi_1 : \overline{X} \to X : (x, V) \mapsto x, \quad \pi_2 : \overline{X} \to SO(m) : (x, V) \mapsto V.$$

On the space $\overline{C}^0 = C^0_{\overline{X}}(\overline{E})$ consider two transfer operators, induced by the map $\overline{f}$:

$$(\overline{\mathcal{L}}\Psi)(x, V) = \sum_{(y,U) \in \overline{f}^{-1}(x,V)} \overline{\phi}(y, U)\Psi(y, U), \tag{14}$$

and

$$(L\Psi)(x, V) = \sum_{(y,U) \in \overline{f}^{-1}(x,V)} \phi \circ \pi_1(y, U)\Psi(y, U).$$

Since $\overline{\phi}$ is lower triangular, we will be able to estimate $\log \mathrm{rsp}(\overline{\mathcal{L}}; \overline{C}^0)$ from below in terms of the pressure of the diagonal elements of $\overline{\phi}$. To use this to obtain the desired estimate for $\mathrm{rsp}(\mathcal{L}; C^0)$, we will need the following fact.

Lemma 1

$$rsp(\mathcal{L}; C^0) \geq rsp(L; \overline{C}^0) = rsp(\overline{\mathcal{L}}; \overline{C}^0). \tag{15}$$

Proof of Lemma 1. By our definitions,

$$\overline{\phi}(x, V) = [\pi_2(x, V)]^T \cdot \phi \circ \pi_1(x, V) \cdot [\pi_2(x, V)].$$

If the operator Π_2 is defined on $\overline{C}^0$ as $(\Pi_2\Psi)(x, V) = V \cdot \Psi(x, V)$ then one has $\Pi_2^{-1} L \Pi_2 = \overline{\mathcal{L}}$, and the right-hand equality in (15) is proved.

To finish the proof we show that $\|L\| \le \|\mathcal{L}\|$. Let $\epsilon > 0$ be given and choose $\Psi \in \overline{C}^0$ satisfying $\|\Psi\| \le 1$ and $\|L\Psi\| \ge \|L\| - \epsilon$. Then

$$\begin{aligned} \| L\Psi \| &= \max_{(x,U)} \| \sum_{(y,W)\in \overline{f}^{-1}(x,U)} \phi \circ \Pi_1(y,W)\Psi(y,W) \| \\ &= \| \sum_{(y,W)\in \overline{f}^{-1}(x_0,U_0)} \phi \circ \Pi_1(y,W) \cdot \Psi(y,W) \|, \end{aligned}$$

for fortuitously chosen (x_0, U_0), by compactness. For each $y \in f^{-1}(x_0)$, choose W_y to satisfy $(y, W_y) \in \overline{f}^{-1}(x_0, U_0)$ and define $\Phi(y) = \Psi(y, W_y)$. Observe that for each $y \in f^{-1}(x_0)$, $\| \Phi(y) \| \le 1$. For each such y, find an open neighborhood B_y so that if y, y' are distinct points in $f^{-1}(x_0)$ then $B_y \cap B_{y'} = \emptyset$. Using a partition of unity subordinate to the collection $\{B_y\}_{y\in f^{-1}(x_0)}$, extend Φ to all of X so that $\Phi \in C^0$ and $\| \Phi \| \le 1$. Then

$$\|L\| - \epsilon \le \|L\Psi\| = \left\| \sum_{y\in f^{-1}(x_0)} \phi(y)\Phi(y) \right\| \le \|\mathcal{L}\Phi\|_{C^0} \le \|\mathcal{L}\|_{B(C^0)},$$

and the lemma is proved. ∎

Going back to the proof of the theorem, let us set $\xi = (x, V)$, $\eta = (y, U)$ and re-write (14) as

$$(\overline{\mathcal{L}}\Psi)(\xi) = \sum_{\eta\in \overline{f}^{-1}(\xi)} \overline{\phi}(\eta)\Psi(\eta), \ \Psi \in C^0_{\overline{X}}(\overline{E}). \tag{16}$$

Denote the diagonal entries of $\overline{\phi}(\xi)$ as $g_j(\xi)$, $1 \le j \le m$. For each ξ and k, the value of the cocycle $\overline{\phi}_k(\xi) \equiv \overline{\phi}(\overline{f}^{k-1}(\xi)) \cdot \ldots \cdot \overline{\phi}(\xi)$ is a lower triangular matrix with positive diagonal entries. Then

$$\| \overline{\mathcal{L}}^k \| = \sup_{\|\Psi\|=1} \max_{\xi\epsilon\overline{X}} \| \sum_{\eta\in\overline{f}^{-1}(\xi)} \overline{\phi}_k(\eta)\Psi(\eta) \|_{\mathbb{R}^m} .$$

For $j = 1, \ldots, m$ we may choose $\Psi(\xi) = \Psi_j(\xi) \equiv e_j$ (where $e_1, \ldots, e_m$ is the standard basis for $\mathbb{R}^m$). Then

$$\| \overline{\mathcal{L}}^k \| \ge \max_{1\le j\le m} \max_{\xi\in\overline{X}} \| \sum_{\eta\in\overline{f}^{-k}(\xi)} \overline{\phi}_{k,j}(\eta) \|_{\mathbb{R}^m} \ge \max_{1\le j\le m} \max_{\xi\in\overline{X}} \sum_{\eta\in\overline{f}^{-k}(\xi)} d_j(\eta, k).$$

where $\overline{\phi}_{k,j}(\eta)$ is the j-th column of $\overline{\phi}_k(\eta)$, with the j-th entry $d_j(\eta,k) = \prod_{i=0}^{k-1} g_j(\overline{f}^i(\eta))$. As a result,

$$\frac{1}{k}\log\|\overline{\mathcal{L}}^k\| \geq \frac{1}{k}\log\max_{\xi\in\overline{X}}\sum_{\eta\in\overline{f}^{-k}(\xi)} d_j(\eta,k), \quad j=1,\ldots,m. \tag{17}$$

Fix $\epsilon > 0$. Since f is expanding there exists $k_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ and any $\xi \in \overline{X}$ the set $\mathcal{G}_k(\xi) \equiv \left\{\overline{f}^{-(k+k_0)}(\xi)\right\}$ is (k,ϵ)-spanning for $\overline{f}$. Then, for $n = k + k_0$ and each $j = 1,\ldots,m$,

$$\max_{\xi}\sum_{\eta\in\overline{f}^{-k}(\xi)} d_j(\eta,k) \geq Q_n(\overline{f},\log g_j,\epsilon).$$

We use (17), the definition of the pressure and the variational principle to obtain for each $j = 1,\ldots,m$:

$$\log\mathrm{rsp}(\mathcal{L}) \geq \log\mathrm{rsp}(\overline{\mathcal{L}}) \geq P(\overline{f},\log g_j) = \sup_{\mu\in\mathcal{M}(\overline{f})}\left\{h_\mu(\overline{f}) + \int_{\overline{X}}\log g_j\,d\mu\right\}. \tag{18}$$

The cocycles $\phi_k(x)$ and $\overline{\phi}_k(\xi)$ are homologous and have the same Lyapunov-Oseledec exponents. Since $\overline{\phi}_k(\xi)$ is diagonal, its Lyapunov-Oseledec exponents are

$$\int_{\overline{X}}\log g_j\,d\mu, \quad j=1,\ldots,m.$$

As in the last paragraph of the proof of Theorem 4.15 in [24], we conclude that the right-hand side of (18) is greater than or equal to $\sup_{\nu\in\mathcal{M}(f)}\{h_\nu(f) + \lambda_\nu\}$. ∎

3 Annulus in the Spectrum

In this section we will prove Theorem 4. Our proof is based on the localization property for almost-eigenfunctions of $\mathcal{L}$ in C^0, cf. [26] and [23]. Recall, that for an operator A on a Banach space $\mathcal{Y}$ the approximate point spectrum $\sigma_{\mathrm{ap}}(A)$ is the set of $z \in \mathbb{C}$ such that for every $\epsilon > 0$ there exists an $h \in \mathcal{Y}$ such that $\|zh - Ah\| \leq \epsilon\|h\|$. There always exists an $z \in \sigma_{\mathrm{ap}}(A)$ with $|z| = \mathrm{rsp}(A;\mathcal{Y})$.

We will need the following auxiliary result (cf. Remark 1 above).

Lemma 2

$$\lim_{n\to\infty} \max_{x\in X} \|\phi_n(x)\|^{1/n} \le \exp \sup_{\nu\in\mathcal{M}} \lambda_\nu.$$

Proof of Lemma 2 goes exactly as in [25, Theorem 3.3]. We briefly indicate the arguments. Define the scalarization $\widetilde{f}$ and $\widetilde{\phi}$ as in (5). Then

$$\max_{x\in X} \|\phi_n(x)\|^{1/n} = \exp \max_{(x,v)\in\widetilde{X}} \frac{1}{n}\sum_{j=0}^{n-1} \log\widetilde{\phi}\,(\widetilde{f}^{\,j}(x,v)),$$

and the last maximum attains at some $(x_n, v_n) \in \widetilde{X}$. Take on $\widetilde{X}$ the Borel probability measures

$$\mu_n(h) = \frac{1}{n}\sum_{j=0}^{n-1} \log h(\widetilde{f}^{\,j}(x_n, v_n)),\ h \in C^0_{\mathbb{R}}(\widetilde{X}),\ n \in \mathbb{N},$$

and a w^*-limit point μ of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. Then

$$\lim_{n\to\infty} \max_{(x,v)\in\widetilde{X}} \frac{1}{n}\sum_{j=0}^{n-1} \log\widetilde{\phi}\,(\widetilde{f}^{\,j}(x,v)) = \int_{\widetilde{X}} \log\widetilde{\phi}\,(x,v)\,d\mu.$$

Let $\nu \in \mathcal{M}(f)$ be the projection of μ on X. By the Birkhoff-Khinchine ergodic theorem for μ and by the multiplicative ergodic theorem for ν there exists $(x,v) \in \widetilde{X}$ so that for every $\epsilon > 0$

$$\begin{aligned}\int_{\widetilde{X}} \log\widetilde{\phi}\,(x,v)\,d\mu - \epsilon &\le \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\widetilde{\phi}\,(\widetilde{f}^{\,j}(x,v)) \\ &= \lim_{n\to\infty} \frac{1}{n}\log\|\phi_n(x)v\| \le \lambda_\nu \le \sup_{\nu\in\mathcal{M}} \lambda_\nu,\end{aligned}$$

and the lemma follows. ∎

Proof of Theorem 4. Consider the operator $\mathcal{L}$ on $C^0 = C^0_E(X)$. Assume f is expanding. Then, by Theorem 3, $\mathrm{rsp}\mathcal{L} = \exp\sup\{h_\nu + \lambda_\nu : \nu \in \mathcal{M}\}$. By rescaling, we may assume that $1 = \exp\sup\{h_\nu + \lambda_\nu : \nu \in \mathcal{M}\}$, and $1 \in \sigma_{\mathrm{ap}}(\mathcal{L})$.

Fix a positive integer N. Since $1 \in \sigma_{\mathrm{ap}}(\mathcal{L})$, with the appropriate choice of $\Phi \in C^0$, one can have $(\mathcal{L} - I)\Phi$ arbitrary small. Thus, we may find $\Phi \in C^0$, depending on N, but with $\|\Phi\|_{C^0} = 1$ and satisfying

$$\| \, (\mathcal{L}^N - I)\Phi \, \| \;\; \leq \;\; 1/8; \tag{19}$$

$$\| \, \mathcal{L}^j \Phi \, \| \;\; \leq \;\; 2, \text{ for } \quad j = 0, 1, \ldots, 2N. \tag{20}$$

(See a proof of this simple fact in [23, Lemma 3.1]). Choose a point $x_0 \in X$ so that x_0 is aperiodic for f and $\| \, \Phi(x_0) \, \|_{\mathbb{R}^m} \geq 3/4$. Recall that f is a covering, and for a small enough neighborhood B of x_0 the set $f^{-1}(B)$ has disjoint components. With this in mind, choose a neighborhood B of x_0, and a bump-function $\beta : B \to \mathbb{R}$ satisfying:

i) all components of the set $\cup_{j=0}^{2N} f^{j-N}(B)$ are disjoint;

ii) an appropriate branch of f^{N-j}, $j = 0, \ldots, 2N$ maps each of these components on B homeomorphically.

iii) β is continuous, supported on $B, 0 \leq \beta(x) \leq 1$, and $\beta(x_0) = 1$.

Then $\alpha = \beta \circ f^N$ is supported on $f^{-N}(B)$. Define a damping function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t) = \frac{t \cdot (2N - t)}{N^2}, \qquad \text{if } t \in [0, 2N], \qquad \text{and } \; \gamma(t) = 0 \; \text{ otherwise.}$$

Then $0 \leq \gamma \leq 1$ with $\gamma(0) = \gamma(2N) = 0$ and $\gamma(N) = 1$. Also,

$$|\gamma(t+1) - \gamma(t)| \leq 4/N \qquad \text{for all } \; t \in \mathbb{R}. \tag{21}$$

Now fix z in the annulus (3), which, of course, may degenerate to a circle. We define a function $\Gamma \in C^0$ by

$$\Gamma(x) = \sum_{j=0}^{2N} z^{N-j} \cdot \gamma(j) \cdot \mathcal{L}^j(\alpha \cdot \Phi)(x), \quad x \in X. \tag{22}$$

Eventually we will show that Γ has large supremum norm (at least 1/2) and that $\|\mathcal{L}\Gamma - z\Gamma\|$ is $O(1/N)$.

A calculation shows that

$$z\Gamma(x) - \mathcal{L}\Gamma(x) = \sum_{j=1}^{2N} z^{N-j+1}[\gamma(j) - \gamma(j-1)]\mathcal{L}^j(\alpha\Phi)(x). \tag{23}$$

We note that the summands in (23) have disjoint support:

$$\text{supp}\mathcal{L}^j(\alpha\Phi) \subset f^{j-N}(B) \quad \text{for } j = 0, 1, \dots, 2N.$$

For any $j = 0, 1, \dots, 2N$ fix $x \in f^{j-N}(B)$. Note that $\alpha(y) \neq 0$ for $y \in f^{-j}(x)$ only provided $y \in f^{-j}(x) \cap f^{-N}(B)$. Fix any $y \in f^{-j}(x) \cap f^{-N}(B)$. Then $f^N(y) \in f^{N-j}(x) \cap B$. Thus, for each $x \in f^{j-N}(B)$ there exists $\tilde{x} \in B$, independent of y, such that $f^N(y) = \tilde{x}$. Hence, $\alpha(y) = \beta(f^N(y)) = \beta(\tilde{x})$. As a result, for $j = 0, 1, \dots, N$ one has $\mathcal{L}^j(\alpha\Phi)(x) = \beta(\tilde{x})(\mathcal{L}^j\Phi)(x)$. For $j = N+1, \dots, 2N$ one has $\phi_j(y) = \phi_{j-N}(\tilde{x})\phi_N(y)$, and, as a result, $(\mathcal{L}^j\alpha\Phi)(x) = \beta(\tilde{x})\phi_{j-N}(\tilde{x})(\mathcal{L}^N\Phi)(\tilde{x})$.

Apply (21) to estimate the norm of (23). We see, that

$$\|z\Gamma - \mathcal{L}\Gamma\|_{C^0} \leq \frac{4}{N}M, \tag{24}$$

where M is the maximum of two numbers:

$$\max_{1\leq j\leq N} \max_{x\in f^{j-N}(B)} \|z^{N-j+1}\beta(\tilde{x})\mathcal{L}^j\Phi(x)\|, \tag{25}$$

$$\max_{N+1\leq j\leq 2N} \max_{x\in f^{j-N}(B)} \|z^{N-j+1}\beta(\tilde{x})\phi_{j-N}(\tilde{x})\mathcal{L}^N\Phi(\tilde{x})\|. \tag{26}$$

Since $|z| \leq 1$ and $\max_x |\beta(x)| \leq 1$, we can use (20) for $j = 1, \dots, N$ to see that (25) is less than or equal to 2. Also, (20) with $j = N$ and Lemma 2 show that (26) is less than or equal to

$$\begin{aligned} &\max_{N+1\leq j\leq 2N} 2|z|^{N-j+1} \max_{x\in X} \|\phi_{j-N}(x)\| \\ &\leq \max_{N+1\leq j\leq 2N} 2|z|^{N-j} \cdot C\exp\left\{(j-N)\sup_{\nu\in\mathcal{M}} \lambda_\nu\right\} \end{aligned} \tag{27}$$

with a constant C independent of N. Since $|z|^{-1}\exp\sup\lambda_\nu \leq 1$ by (3), we conclude that (27) is bounded by a constant, independent on N. Thus, M in (24) does not depend on N.

To estimate $\|\Gamma\|_{C^0}$ from below, we recall that $x_0 \in B$. Then, from (23) and (19):

$$\|\Gamma\|_{C^0} \geq \|\Gamma(x_0)\| = \|(\mathcal{L}^N\alpha\Phi)(x_0)\| = \|\mathcal{L}^N\Phi(x_0)\| \geq \|\Phi(x_0)\| - \frac{1}{8} \geq \frac{1}{2}.$$

We combine this with (24) to see that $z \in \sigma_{\text{ap}}(\mathcal{L})$. ∎

4 Essential Spectral Radius

In this section we prove Theorem 2 for the essential spectral radius of the transfer operator $\mathcal{L}$ acting on $C^r_X(E)$, $r = 1, 2, \ldots$. The idea is to extend X to a wider compact $\mathcal{X}$ and f to a map $F : \mathcal{X} \to \mathcal{X}$. Then, up to a compact operator, we construct an intertwining $\mathcal{D}$ of the transfer operator $\mathcal{L}$ on *smooth* sections over X with a transfer operator $\mathcal{K}$ on *continuous* sections over $\mathcal{X}$. This allows us to estimate the essential spectral radius of $\mathcal{L}$ via the spectral radius of $\mathcal{K}$. Next, we apply Theorem 1 to estimate the spectral radius of $\mathcal{K}$. To get a better intuition one can think that $X \subset \mathbb{R}^n$ and E is a trivial bundle $X \times \mathbb{R}^m$. More details on global definitions of the operators $\mathcal{D}$ and $\mathcal{K}$, involving connections on vector bundles, etc., will be given elsewhere [11].

To fix notation, recall that $E = (E, X, p)$ is an m-dimensional vector bundle with the projection $p : E \to X$, fibers $E_x \simeq \mathbb{R}^m$ and base X. Here X is an n-dimensional smooth compact manifold with the tangent bundle $TX = (TX, X, \pi)$ with the natural projection $\pi : TX \to X$. Let $T^1X = (T^1X, X, \pi)$ be the unit tangent bundle with the fibers T^1_xX and projection $\pi : w \to x$, $w = (x, v)$, $v \in T^1_xX$, $x \in X$. For $r = 1, 2, \ldots$ we use bold face to denote r-tuples, e.g., $\mathbf{w} = (w_1, \ldots, w_r)$, $\mathbf{v} = (v_1, \ldots, v_r)$, etc.

Fix $r = 1, 2, \ldots$ and define the bundle $T^1X^{(r)} = (T^1X^{(r)}, X, \overline{\pi})$ as

$$\begin{aligned} T^1X^{(r)} &= \{\mathbf{w} = (w_1, \ldots, w_r) \in T^1X \times \ldots \times T^1X : \\ &\qquad w_i \in T^1X,\ i = 1, \ldots, r,\ \pi(w_1) = \ldots = \pi(w_r)\} \end{aligned}$$

with the fibers $T^1X^{(r)}_x = T^1_xX \times \ldots \times T^1_xX$ and the projection $\overline{\pi} : \mathbf{w} = (x, \mathbf{v}) \mapsto x = \pi(w_i)$. Here $\mathbf{w} = (w_1, \ldots, w_r)$, $w_i = (x, v_i)$, and $\mathbf{v} \in T^1X^{(r)}_x$, that is $\mathbf{v} = (v_1, \ldots, v_r)$, $v_i \in T^1_xX$ for $i = 1, \ldots, r$. We identify $T^1X^{(r)}$ with the compact manifold

$$\mathcal{X} = \{(x, \mathbf{v}) : x \in X,\ \mathbf{v} = (v_1, \ldots, v_r),\ v_i \in T^1_xX,\ i = 1, \ldots, r\}$$

by the map $q : T^1X^{(r)} \to \mathcal{X} : \mathbf{w} \mapsto (\overline{\pi}(\mathbf{w}), \mathbf{v})$ with $\mathbf{w} = (x, \mathbf{v})$, $w_i = (x, v_i)$, $i = 1, \ldots, r$, $\mathbf{v} \in T^1X^{(r)}_x$.

Consider the bundle

$$\mathcal{E} = \{(e, \mathbf{w}) \in E \times T^1X^{(r)} : p(e) = \overline{\pi}(\mathbf{w})\} \quad \text{over} \quad \mathcal{X}$$

with projection $\mathcal{E} \to \mathcal{X} : (e, \mathbf{w}) \mapsto q(\mathbf{w})$ and fibers $\mathcal{E}_{(x,\mathbf{v})} \simeq \mathbb{R}^m$. Locally,

$$\mathcal{E} = \{(x, \mathbf{v}, u) : x \in X, \mathbf{v} \in T^1X_x^{(r)},\ u \in \mathbb{R}^m\}.$$

Define a map $F : \mathcal{X} \to \mathcal{X}$ and a bundle automorphism $\psi : \mathcal{E} \to \mathcal{E}$ as follows:

$$\begin{aligned} F : (x, \mathbf{v}) &\mapsto \left(f(x), \frac{Df(x)v_1}{\|Df(x)v_1\|}, \ldots, \frac{Df(x)v_r}{\|Df(x)v_r\|} \right), \\ \psi(x, \mathbf{v}) &= \phi(x) \cdot \prod_{i=1}^{r} \|Df(x)v_i\|^{-1}, \end{aligned}$$

where $Df(x)$ is differential of f and $\mathbf{v} \in T^1X_x^{(r)}$. These F and ψ generate in the space $\mathcal{C}^0 = C^0_{\mathcal{X}}(\mathcal{E})$ of continuous sections of $\mathcal{E}$ a transfer operator, $\mathcal{K}$, by the rule

$$(\mathcal{K}\Psi)(x, \mathbf{v}) = \sum_{(y,\mathbf{u}) \in F^{-1}(x,\mathbf{v})} \psi(y, \mathbf{u})\Psi(y, \mathbf{u}), \quad (x, \mathbf{v}) \in \mathcal{X}, \Psi \in C^0_{\mathcal{X}}(\mathcal{E}). \tag{28}$$

Recall, that the space $C^r = C^r_X(E)$ of smooth sections of E is equipped with the usual norm

$$\|\Phi\|_{C^r} = \max_{0 \le k \le r} \|D^k\Phi\|_k = \max_{0 \le k \le r} \max_{x \in X} \max_{\mathbf{v}} \| \left(D^k\Phi\right)(x)(v_1, \ldots, v_k)\|_{\mathbb{R}^m},$$

where $\mathbf{v} = (v_1, \ldots, v_k)$, $v_i \in T^1_xX$, $i = 1, \ldots, k$ and the k-th differential $D^k\Phi(x)$ is a k-linear operator from the k-fold product $T^1_xX \times \ldots \times T^1_xX$ to $\mathbb{R}^m$, $x \in X$.

Consider an operator $\mathcal{D} : C^r_X(E) \to C^0_{\mathcal{X}}(\mathcal{E})$ defined as follows:

$$(\mathcal{D}\Phi)(x, \mathbf{v}) = D^r\Phi(x)(\mathbf{v}), \quad \mathbf{v} = (v_1, \ldots, v_r),\ v_i \in T^1_xX,\ i = 1, \ldots, r.$$

Note, that $\|D^r\Phi\|_r = \|\mathcal{D}\Phi\|_{\mathcal{C}^0}$. Clearly, $\mathcal{D}$ has dense range and $\dim \operatorname{Ker} \mathcal{D} = m$. We will need the following lemma.

Lemma 3

$$\mathcal{D}\mathcal{L}\Phi = \mathcal{K}\mathcal{D}\Phi + R\Phi, \quad \Phi \in C^r_X(E),$$

where R is a compact operator from $C^r_X(E)$ to $C^0_{\mathcal{X}}(\mathcal{E})$.

Proof of Lemma 3. Take a smooth finite partition of unity $\{\rho_i\}$ for X,

$$\sum_i \rho_i(x) = 1,\ x \in X,\ \mathrm{supp}\rho_i \setminus (\cup_{j\neq i}\mathrm{supp}\rho_j) \neq \emptyset.$$

By writing $\Phi = \sum_i \rho_i \Phi$ one can assume that support of Φ in the lemma is arbitrary small, and make local (that is, in coordinates) calculations.

We will proceed for $r = 1$ first. In this case, using Leibniz and chain rules, one has:

$$(D\mathcal{L}\Phi)(x,v) = \sum_{y\in f^{-1}(x)} \phi(y)(D\Phi)(y)[Df(y)]^{-1}(v) + (R\Phi)(x,v), \tag{29}$$

where

$$(R\Phi)(x,v) = \sum_{y\in f^{-1}(x)} A(y,x,v)\Phi(y),$$

and $A(y,x,v)$ are some $(m\times m)$-matrices.

The second term, $R\Phi$, in (29) does not contain differentiation. As a result, R in the lemma is a compact operator. Indeed, the imbedding $\mathrm{Id} : C^1_X(E) \to C^0_X(E)$ is compact. Define on $C^0_{\mathcal{X}}(\mathcal{E})$ a bounded operator $\mathcal{A}$ as $(\mathcal{A}\Psi)(x,v) = \sum_y A(y,x,v)\Psi(y,v)$, and a bounded imbedding $I : C^0_X(E) \to C^0_{\mathcal{X}}(\mathcal{E})$ as $(I\Phi)(x,v) \equiv \Phi(x)$. Then $R = \mathcal{A}\circ I\circ \mathrm{Id}$ is compact.

To handle $\mathcal{K}$, take the first term in (29) and re-write it as

$$\sum_{y\in f^{-1}(x)} \phi(y)\cdot \|[Df(y)]^{-1}(v)\|\cdot \left\|(D\Phi)(y)\left(\frac{[Df(y)]^{-1}(v)}{\|[Df(y)]^{-1}(v)\|}\right)\right\|$$

$$= \sum_{(y,u)\in F^{-1}(x,v)} \psi(y,u)(\mathcal{D}\Phi)(y,u) = (\mathcal{K}\mathcal{D}\Phi)(x,v),$$

by the definition of ψ and since $(y,u) = [Df(y)]^{-1}(v)/\|[Df(y)]^{-1}(v)\|$ for $(y,u) \in F^{-1}(x,v)$ by the chain rule and the definition of F.

We omit similar, but tedious calculations for $r > 1$. ∎

Since the construction of the operators $\mathcal{K}$ and $\mathcal{D}$ in general case are rather cumbersome, we specialize it for the following particular examples.

Example 1. For $r = 1$, $n = 1$, $X \simeq [0, 1]$ and the scalar case $m = 1$ the operator $\mathcal{D}$ is just the differentiation $\mathcal{D}\Phi = \Phi'$. Clearly, $\mathcal{X} = [0, 1]$ and $F = f$. Also,

$$(\mathcal{L}\Phi)' = \mathcal{K}\Phi' + \sum_y [\phi(y)]'\Phi(y), \qquad \text{where} \qquad \mathcal{K}\Phi' = \sum_y \frac{\phi(y)}{|f'(y)|}\Phi'(y).$$

Since $\mathcal{D}$ is onto and has one-dimensional kernel, it is clear (see remarks before Lemma 4) that the Fredholm spectrum $\sigma_F(\mathcal{L}; C^1) = \sigma(\mathcal{K}; C^0)$. This approach was used in [13] for a description of the essential spectrum $\sigma_{\text{ess}}(\mathcal{L}, C^1)$ in the case $m = n = 1$. See also [2, 4, 20] for the description of $\sigma(T, C^1)$ for the weighted translation operator (4) and $m = n = 1$. ∎

Example 2. For $r = 1$, $n > 1$, $X \subset \mathbb{R}^n$ and $m = 1$ one has $(\mathcal{D}\Phi)(x, v) = (\bigtriangledown\Phi)(x) \cdot v$ for the gradient $\bigtriangledown$ and $v \in S^{n-1} = \{v \in \mathbb{R}^n : \|v\|_{\mathbb{R}^n} = 1\}$. We think about v as an $(n \times 1)$-vector and $(\bigtriangledown\Phi)(x)$ as a $(1 \times n)$-covector. Also,

$$\mathcal{X} = X \times S^{n-1} \qquad \text{and} \qquad F(x, v) = (f(x), \text{Jac}f(x) \cdot v/\|\text{Jac}f(x) \cdot v\|)$$

for the Jacoby matrix Jac$f$. Take gradient from $\mathcal{L}\Phi$ to obtain Lemma 3 by collecting terms with no derivatives of Φ as $R\Phi$. For a fixed $y \in f^{-1}(x)$ by the chain rule one has

$$\bigtriangledown[\Phi(y)] = (\bigtriangledown\Phi)(y) \cdot [\text{Jac}f(y)]^{-1}$$

and

$$(\mathcal{K}\Phi)(x, v) = \sum_{y \in f^{-1}(x)} \phi(y)(\bigtriangledown\Phi)(y) \cdot [\text{Jac}f(y)]^{-1}v.$$

We divide and multiply the last term by $\|[\text{Jac}f(y)]^{-1}v\|$. Also,

$$F^{-1}(x, v) = \{(y, u) : x = f(y) \text{ and } v = \text{Jac}f(y)u/\|\text{Jac}f(y)u\|\}.$$

If $\psi(x, v)$ is defined as

$$\psi(x, v) = \phi(x)/\|\text{Jac}f(x)v\|,$$

then, for $(y, u) \in F^{-1}(x, v)$, one has

$$\phi(y)\|[\text{Jac}f(y)]^{-1}v\| = \psi(y, u)$$

and

$$(\mathcal{K}\Phi)(x,v) = \sum_{(y,u)\in F^{-1}(x,v)} \psi(y,u)(\nabla\Phi)(y)\cdot u.$$

See [19] for an application of this idea to the description of $\sigma(T;C^1)$ for the weighted translation operator (4) in the case $n>1$ and $m=1$. ∎

We need a lemma connecting the essential spectral radius $\mathrm{ress}(\mathcal{L};C^r_X(E))$ and $\mathrm{rsp}(\mathcal{K};C^0_{\mathcal{X}}(\mathcal{E})$, for which we use the following elementary facts from operator theory.

Recall (see, e.g., [28]), that the essential spectrum for an operator A on a Banach space $\mathcal{Y}$ is the set $\sigma_{\mathrm{ess}}(A)$ consisting of $z\in\mathbb{C}$ so that one of the following is true:

(i) Range $z-A$ is not closed;

(ii) z is an accumulation point of the spectrum of A;

(iii) $\cup_{i=1}^{\infty}$ Ker $(z-A)$ is infinite dimensional.

Also (see, e.g., [28]), the Nussbaum formula for the essential spectral radius is given by

$$\mathrm{ress}(A) = \lim_{n\to\infty}\inf_{R\in\mathcal{J}} \|A^n - R\|^{1/n}, \tag{30}$$

where $\mathcal{J}$ is the ideal of compact operators in $\mathcal{B}(\mathcal{Y})$. An operator A is called *Fredholm* if its range is closed and both $\dim\mathrm{Ker}A$ and $\dim\mathrm{Coker}A$ are finite. The Fredholm spectrum $\sigma_F(A)$ of A is the set of $z\in\mathbb{C}$ so that $z-A$ is not Fredholm. In other words, the image $i(z-A)$ of $z-A$ under the factor-projection i of $\mathcal{B}(\mathcal{Y})$ to the Calkin factor-algebra $\mathcal{B}(\mathcal{Y})/\mathcal{J}$ is not invertible in this factor-algebra. Since (30) is exactly the formula for the spectral radius of $i(A)$ in $\mathcal{B}(\mathcal{Y})/\mathcal{J}$, the Fredholm spectral radius $\sup\{|z| : z\in\sigma_F(A)\}$ is equal to $\mathrm{ress}(A)$.

We say that an operator $A\in\mathcal{B}(\mathcal{Y})$ is a Φ_+*-operator* if its range is closed and $\dim\mathrm{Ker}A$ is finite. For a Φ_+-operator A the index $\mathrm{Ind}A$ is given by $\dim\mathrm{Ker}A - \dim\mathrm{Coker}A$, and may take value $-\infty$. We will need four simple facts about Φ_+-operators (see, e.g., [12, Section I.11]).

(1). If A is a Φ_+-operator and R is compact, then $A+R$ is a Φ_+-operator.

(2). If A is a Φ_+-operator, then $A+B$ is a Φ_+-operator with $\mathrm{Ind}(A+B) = \mathrm{Ind}A$ for every $B \in \mathcal{B}(\mathcal{Y})$ with small enough $\|B\|$.

(3). For any A, $B \in \mathcal{B}(\mathcal{Y})$, if BA is a Φ_+-operator, then A is a Φ_+-operator.

(4). If A is a Φ_+-operator and B is invertible, then BA is a Φ_+-operator.

We will need the following lemma.

Lemma 4 *For $\mathcal{L}$ and $\mathcal{K}$ as in Lemma 3 one has:*

$$ress(\mathcal{L}; C^r_X(E)) \leq rsp(\mathcal{K}; C^0_{\mathcal{X}}(\mathcal{E})).$$

Proof of Lemma 4. Take $z \in \sigma_F(\mathcal{L}; C^r_X(E))$ such that $|z| = \mathrm{ress}(\mathcal{L}; C^r_X(E))$. Suppose, by contradiction, that $|z| > \mathrm{rsp}(\mathcal{K}; C^0_{\mathcal{X}}(\mathcal{E}))$. Then $z - \mathcal{K}$ is invertible. Since $\mathcal{D}$ has closed range and finite dimensional kernel, that is, is a Φ_+-operator, $(z-\mathcal{K})\mathcal{D}$ is a Φ_+-operator by (4). Since, by Lemma 3, R is a compact operator, (1) implies that $\mathcal{D}(z-\mathcal{L}) = (z-\mathcal{K})\mathcal{D} + R$ is also a Φ_+-operator. Then $z - \mathcal{L}$ is a Φ_+-operator by (3). Since z belongs to the boundary of $\sigma_{\mathrm{ess}}(\mathcal{L})$, there is a sequence $z_n \to z$ such that $z_n - \mathcal{L}$ are invertible. In particular, $\mathrm{Ind}(z_n - \mathcal{L}) = 0$. Hence, by (2), $\mathrm{Ind}(z-\mathcal{L}) = 0$, and $z - \mathcal{L}$ is Fredholm, in contradiction to $z \in \sigma_F(\mathcal{L}; C^r_X(E))$. ∎

Proof of Theorem 2. By Theorem 1 for $\mathcal{K}$ on $C^0_{\mathcal{X}}(\mathcal{E})$ we have the estimate

$$\log \mathrm{rsp}(\mathcal{K}; C^0_{\mathcal{X}}(\mathcal{E})) \leq \sup_{\mu \in \mathcal{M}(F)} \{h_\mu(F) + \Lambda_\mu\},$$

where Λ_μ is the largest Lyapunov-Oseledec exponent for the cocycle

$$\begin{aligned} \psi_k(x, \mathbf{v}) &= \psi(F^{k-1}(x,\mathbf{v})) \cdot \ldots \cdot \psi(x,\mathbf{v}) \\ &= \phi_k(x) \prod_{i=1}^r \|Df^k(x) v_i\|^{-1}, \ \mathbf{v} = (v_1, \ldots, v_r) \end{aligned}$$

over F. By the multiplicative ergodic theorem for the cocycle $\psi_k(x,\mathbf{v})$ there exists a full μ-measure set $\mathcal{X}_\mu \subset \mathcal{X}$ such that

$$\Lambda_\mu = \lim_{k\to\infty} \frac{1}{k} \log \left(\|\phi_k(x)\| \prod_{i=1}^r \|Df^k(x) v_i\|^{-1} \right) \quad \text{for all} \quad (x, \mathbf{v}) \in \mathcal{X}_\mu. \tag{31}$$

Denote $\nu = \text{proj}\mu \in \mathcal{M}(f)$, the projection of $\mu \in \mathcal{M}(F)$ on X. As in [24, Lemma 4.14], $h_\mu(F) = h_\nu(f)$, and it remains to estimate Λ_μ.

Use the multiplicative ergodic theorem for the cocycle $Df^k(x)$ over f to take a full ν-measure set $X_\nu \subset X$ such that for all $x \in X_\nu$ and all $v \in T^1_x X$ there exist limits

$$\lim_{k\to\infty} \frac{1}{k} \log \|Df^k(x)v\| \geq \chi_\nu,$$

with χ_ν to be the smallest Lyapunov-Oseledec exponent for the cocycle $Df^k(x)$. By the multiplicative ergodic theorem for the cocycle $\phi_k(x)$ over f we may also assume that

$$\lambda_\nu = \lim_{k\to\infty} \frac{1}{k} \log \|\phi_k(x)\| \quad \text{for all} \quad x \in X\nu$$

with λ_ν to be the largest Lyapunov-Oseledec exponent for the cocycle $\phi_k(x)$. Fix $(x, \mathbf{v}) \in \mathcal{X}_\mu$ so that $x \in X_\nu$. Then (31) implies

$$\Lambda_\mu = \lim_{k\to\infty} \frac{1}{k} \log \|\phi_k(x)\| - \sum_{i=1}^{r} \lim_{k\to\infty} \frac{1}{k} \log \|Df^k(x)v_i\| \leq \lambda_\nu - r \cdot \chi_\nu,$$

as required. ∎

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