Bound States for Schrödinger Hamiltonians: Phase Space Methods and Applications

Ph. Blanchard
Theoretische Physik and BiBoS, Universität Bielefeld
D-33615 Bielefeld

J. Stubbe
Dept. de Physique Théorique, Université de Genève
CH-1211 Geneve
and
Theory Division, CERN, CH-1211 Genève 23

Abstract

Properties of bound states for Schrödinger operators are reviewed. These include: bounds on the number of bound states and on the moments of the energy levels, existence and nonexistence of bound states, phase space bounds and semi-classical results, the special case of central potentials, and applications of these bounds in quantum mechanics of many particle systems and dynamical systems. For the phase space bounds relevant to these applications we improve the explicit constants.

Dedicated to André Martin,
an old friend, who "retired" recently
in order to accomplish even more!

1 Introduction

Consider the Schrödinger operator

$$H = -\Delta + V(x) \tag{1}$$

on $L^2(\mathbf{R}^d)$. It describes either the motion of a quantum particle of mass 1/2 in an external potential V or the relative motion of two particles having reduced mass 1/2 interacting via the potential V. Units have been chosen such that Planck's constant $\hbar$ equals to one. In the present review we want to report about estimates for the number of bound states and their energies $E_j \leq 0$. A bound state is a stationary state with a negative eigenvalue for the Schrödinger operator H; the bound states are the quantum states corresponding to the closed orbits of the classical theory. Such estimates are of obvious importance in quantum mechanical problems and they play a fundamental role in the proof of the stability of matter. This involves a problem first raised by L. Onsager in the 1930's. A basic fact of astrophysics claims that bulk matter in the absence of nuclear forces undergoes gravitational collapse. Onsager asked why bulk matter does not undergo electrostatic collapse. The Pauli exclusion principle for the electrons is the basic mechanism which prevents bulk matter from collapse. The first rigorous proof of this fact was given by F. Dyson and A. Lenard [DL67, DL68] in the sixties. Pioneering work was done ten years after the Dyson-Lenard result by E. Lieb and W. Thirring who gave a rather elementary proof of the stability of bulk matter using new principles which provided much better physical insights. Lieb and Thirring relate stability problems for the quantum many particle system to the Thomas-Fermi model [LT75]. Certain properties of the Thomas-Fermi model are then used to prove the stability of the quantum system. As a byproduct one obtains quite accurate bounds on the energy of atoms and molecules. The basic new idea is a Sobolev type inequality for fermions, i.e. a lower bound on the kinetic energy of N-fermions in terms of an appropriate functional of the corresponding one particle density. The proof of this new principle relies on the methods and results which will be reviewed in the present article.

A beautiful presentation of the main physical and mathematical aspects of this problems has been given by E. Lieb [L76b, L89, L90a]. However, such estimates are also useful in different fields of mathematical physics as in the theory of infinite dimensional dynamical systems where by means of such bounds one can estimate the Hausdorff dimension of attractors [Ru82, L84].

Physically, the number of bound states $N_0(V)$ of H is given by the number of cells of size h^d, where h denotes Planck's constant, in the phase space where the classical Hamiltonian $H(x,p) = p^2 + V(x)$ is negative. Since $h = 2\pi$ in our units one should have

$$N_0(V) \sim (2\pi)^{-d} \iint_{H(x,p)\leq 0} dx\, dp = (2\pi)^{-d} B_d \int_{\mathbf{R}^d} |V_-(x)|^{d/2} dx \ . \tag{2}$$

Here $B_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ denotes the volume of the unit ball in $\mathbf{R}^d$, $\Gamma(\alpha)$ for $\alpha > 0$ being the Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ and $V_-(x)$ denotes the negative part of

$V(x)$, i.e. $V_-(x) = \sup(-V(x), 0)$. Mathematically, $N_0(V)$ is the dimension of the spectral projection $P_{(-\infty,0)}$ for the operator H.

It turns out that (1.2) is true in the strong coupling limit, or alternatively in the semiclassical limit. This was first proven by A. Martin [M72] in 1972 for Hölder continuous potentials of compact support and later by H. Tamura [Ta] under milder restrictions on $V(x)$. However, there have been earlier attempts. K. Chadan informed us about a paper by J.S. De Wet and F. Mandl from 1949 where the asymptotic validity of (1.2) was proven for dimensions 2 and 3 for a restrictive class of potentials going to infinity at large distances [WM50].

To be more precise, (1.2) is valid in the following sense [RSIV, Theorem XIII.79].

Theorem 1.1. Let $V : \mathbf{R}^d \to \mathbf{R}$ be a continuous function of compact support. For any $\lambda > 0$, $N_0(\lambda V)$ denotes the dimension of the spectral projection $P_{(-\infty,0)}$ of $-\Delta + \lambda V$. Then

$$\lim_{\lambda\to\infty} \frac{N_0(\lambda V)}{\lambda^{d/2}} = (2\pi)^{-d} B_d \int_{\mathbf{R}^d} |V_-(x)|^{d/2} dx \ . \ \square \tag{3}$$

The proof is rather elementary and uses the Dirichlet-Neumann bracketing method, a tool which can be already found in the book by Courant and Hilbert [CH].

The above asymptotic result suggests to look for bounds of $N_0(V)$ in terms of the classical phase space volume given by the r.h.s. of (1.2) and (1.3), respectively. We should remark that on the other hand such bounds will allow us to extend the asymptotic formula (1.3) to all $V \in L^{d/2}(\mathbf{R}^d)$. This will be the case if $d \geq 3$. [RSIV, Theorem XIII.80]

These considerations will be generalized in the following way. For $\gamma \geq 0$ let

$$S_\gamma(V) \equiv \sum_{E_j \leq 0} |E_j|^\gamma \tag{4}$$

where E_j denote the (negative) eigenvalues of H. Obviously $S_0(V) = N_0(V)$. Then we should look for bounds on $S_\gamma(V)$ in terms of the corresponding classical phase space expression, i.e. for bounds of the form

$$S_\gamma(V) \leq \mathbf{C}_{\gamma,d} (2\pi)^{-d} \iint_{\{H(x,p)\leq 0\}} |H(x,p)|^\gamma dx dp \tag{5}$$

or after performing the p-integration

$$S_\gamma(V) \leq L_{\gamma,d} \int_{\mathbf{R}^d} |V_-(x)|^{d/2+\gamma} dx \tag{6}$$

In fact, using the techniques of Theorem 1.1., one can prove the following

Corollary 1.1. Let $V : \mathbf{R}^d \to \mathbf{R}$ be a continuous function of compact support. Then for $\gamma \geq 0$,

$$\lim_{\lambda\to\infty} \frac{S_\gamma(\lambda V)}{\lambda^{d/2+\gamma}} = L^{class}_{\gamma,d} \int_{\mathbf{R}^d} |V_-(x)|^{d/2+\gamma} dx \tag{7}$$

where $L^{class}_{\gamma,d}$ denotes the "classical constant" given by $L^{class}_{\gamma,d} \equiv (4\pi)^{-d/2}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\frac{d}{2})}$ $\square$.

The integrating theme of this article is the study of the bound (1.5). In the next section we shall review some technical tools and some general properties of the function $N_0(V)$ and its generalisation $N_E(V)$ for $E \leq 0$. The main technical tool will be the so-called Birman-Schwinger principle [Bir61, Schw61], which relates the problem of counting eigenvalues of H to the problem of counting eigenvalues of a compact operator associated to H. The solution of boundary and eigenvalue problems by variational methods are essentially based on the existence of suitable continuous and compact embeddings of Sobolev spaces. We wish therefore to discuss briefly the most important resulting inequalities. In section 3 we discuss conditions of the form (1.5) which imply the nonexistence of bound states, i.e. $N_0(V) = 0$, and estimates for the ground state eigenvalue of a Schrödinger operator. We also show that such conditions cannot exist in dimensions 1 and 2 by proving that any attractive potential has at least one bound state. We present some generalisations to conditions guaranting $N_E(V) = 0$ for $E < 0$.

Since central potentials are much simpler to treat some bounds on $N_0(V)$ existed already since the early fifties when R. Jost and A. Pais [JP51] and V. Bargmann [Ba52] proved a bound on the number of s-states in a central potential. Since the partial differential equation decomposes into an infinite family of one-dimensional problems we may apply well-known results about ordinary differential equations. In particular, there is a strong relation between the nonexistence conditions of section 2 and the bounds on the number of bound states in a given sector. The major problem in this approach is then to sum over all sectors. This will be discussed in Section 4.

Finally, in Section 5 we discuss the general case for phase space estimates and present bounds on $S_\gamma(V)$ which are of the form (1.5). For $\gamma = 0$ we separately review the results and techniques by Cwickel [Cw77], Lieb [L76a] and Li and Yau [LY83]. These bounds imply bounds on $S_\gamma(V)$ for all $\gamma \geq 0$. The first bounds on $S_\gamma(V)$ for $\gamma > 0$, however, were obtained by Lieb and Thirring [LT75, LT76]. By combining these ideas with Lieb's method [L76a] we are able to improve the constants in the estimates for $S_\gamma(V)$ [BS94]. The particular case of dimensions 1 and 2 will be considered in Section 5.5. In Section 5.6 we present some general properties of phase space bounds leading to some interesting conjectures and open problems. In Section 5.7 we mention recent results about Schrödinger operators with magnetic fields.

In Section 6 we present some applications of phase space bounds. We sketch the Lieb-Thirring proof [LT75] of Sobolev's inequality for fermions leading to the proof of stability of matter and to estimate on the number of characteristic exponents for the Navier Stokes equations. A second application we use the nonexistence condition of Section 3 to obtain nonbinding results for many particle systems. This example is due to [GGMT76].

In an Appendix we discuss the best numerical values for different relevant constants available in the literature so far and give the improvements detailled in the present paper which are relevant for the applications.

2 General Methods and Principles

In this section we give few proofs, our purpose being merely to list and recall some general results and technical tools which will be needed to derive the bounds on $S_\gamma(V)$. We do not discuss the problem of selfadjointness of $H = -\Delta + V$ where we refer to the wide literature on the subject [RSI-IV]. Generally, we suppose that V is a measurable function on $\mathbf{R}^d$ which belongs to a suitable L^p space.

As we shall see later almost all bounds on $S_\gamma(V)$ for $\gamma > 0$ are derived from bounds for the counting function $N_E(V)$. Therefore we start with discussing some properties of this function.

2.1 Properties of $N_E(V)$

For $E \leq 0$ we denote by $N_E(V)$ the dimension of the spectral projection $P_{(-\infty,E)}$ for H.

Lemma 2.1. Let $V \leq W$ pointwise. Then

$$N_E(W) \leq N_E(V) \tag{8}$$

for all $E \leq 0$. In particular,

$$N_E(V) \leq N_E(-V_-) \ .\square \tag{9}$$

The proof follows immediately from the comparison Theorem for operators [RSIV].

Evidently, $N_E(V)$ is also a monotone function of E. The next result is a little bit more subtle, although its proof is still elementary.

Lemma 2.2. [LT75,76] For all $\alpha \in [0,1]$, and $E \leq 0$

$$N_E(V) \leq N_{\alpha E}(-|V(x) - (1-\alpha)E|_-) \tag{10}$$

where $|V(x)-(1-\alpha)E|_-$ denotes the negative part of the potential $V(x)-(1-\alpha)E$.$\square$

Looking at the graph of $N_E(V)$ one sees that integrating $N_E(V)$ with respect to E yields (minus) the sum of the negative eigenvalues. More generally, since

$$\frac{\partial}{\partial E} N_E = \sum_j \delta(E - E_j)$$

where $\{E_j\}_j$ denote the eigenvalues of H, we have

Lemma 2.3. [LT75,76] Let $S_\gamma(V) = \sum_{E_j<0} |E_j|^\gamma$. Then

$$S_\gamma(V) = \gamma \int_0^\infty d|E||E|^{\gamma-1} N_{-|E|}(V) \ . \tag{11}$$

2.2 The Birman-Schwinger principle

In 1961, M. Birman [Bir61] and J. Schwinger [Schw61], independently proved a bound on $N_E(V)$ by introducing a bounded integral operator whose eigenvalue are related to the bound states of the Schrödinger operator H. Here we follow the presentation given in the review article by B. Simon [Si76a].

First of all we note the following

Lemma 2.4. Consider the operator $-\Delta + \lambda V$ with $V \leq 0$, and $\lambda > 0$. E is an eigenvalue, $E < 0$, if and only if λ^{-1} is an eigenvalue of

$$K_E = |V|^{1/2}(-\Delta + E)^{-1}|V|^{1/2} \tag{12}$$

and their multiplicities are equal. □

The proof of the Lemma 2.4. is nothing but a careful analysis of the formal correspondance that $(-\Delta + \lambda V)\psi = E\psi$ holds if and only if $K_E\phi = \lambda^{-1}\phi$ where $\phi = |V|^{1/2}\psi$.

Remark: We shall call K_E given by (2.5) the Birman-Schwinger operator.

The second step is

Lemma 2.5. The number $N_E(V)$ of eigenvalues of $-\Delta + V$ less than $E < 0$ is equal to the number of $\lambda \in (0, 1)$ for which E is an eigenvalue of $-\Delta + \lambda V$. □.

The proof follows from the fact that the eigenvalues of $-\Delta + \lambda V$ are monotone and continuous in λ and converge to zero as λ tends to zero.

These two lemmas imply the Birman-Schwinger principle, which is of considerable importance.

Theorem 2.1.
Let $V \leq 0, E < 0$. The number of eigenvalues of $-\Delta+V$ in $(-\infty, E)$, $N_E(V)$, is the same as the number of eigenvalues of the Birman-Schwinger operator K_E in $(1, \infty)$ counting the multiplicities. □

On $\mathbf{R}^3$, for suitable V, K_E is a Hilbert-Schmidt operator. Since $(-\Delta+E)^{-1}$ has an integral kernel given by $(4\pi|x-y|)^{-1}\exp(-\sqrt{-E}|x-y|)$ one has

$$\begin{aligned} N_E(V) &\leq Tr(K_E K_E^*) \\ &= \frac{1}{(4\pi)^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} |V(x)||V(y)||x-y|^{-2}\exp(-2\sqrt{-E}|x-y|)dxdy\ . \end{aligned} \tag{13}$$

As E approaches 0 we obtain the Birman-Schwinger bound. The theorem has been proved for $V \in \mathbf{R}$, the set of Rollnik potentials. We recall that a measurable function V on $\mathbf{R}^3$ is called a Rollnik potential if

$$\|V\|_R^2 \equiv \int_{\mathbf{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2} dx\, dy < +\infty \ .$$

We next present a more general version of the Birman-Schwinger principle by introducing a one-parameter family of bounded operators whose number of eigenvalues bigger than one yields a bound on $N_E(V)$. To our knowledge this generalisation was introduced by Lieb and Thirring [LT76] to obtain the celebrated bound on the sum $S_1(V)$ over the eigenvalues of $-\Delta + V$. We shall see in Section 5 that by means of this generalisation we will obtain the currently best numerical constants in bounds of the form (1.5).

For $E \leq 0, \alpha \in [0,1]$ we define the generalised Birman-Schwinger operator

$$\begin{aligned} K_{\alpha E} &\equiv |V(x) - (1-\alpha)E|_-^{1/2}(-\Delta - \alpha E)^{-1}|V(x) - (1-\alpha)E|_-^{1/2} \\ &\equiv |V_\alpha|^{1/2}(-\Delta + \alpha|E|)^{-1}|V_\alpha|^{1/2} \end{aligned} \tag{14}$$

where $V_\alpha \equiv |V(x) - (1-\alpha)E|_-$ denotes the negative part of the function $V(x) - (1-\alpha)E$.

Using Lemma 2.2. we obtain the following

Theorem 2.2. The number $N_E(V)$ of eigenvalues of $-\Delta + V$ in $(-\infty, E)$, is bounded above by the number of eigenvalues of the generalised Birman-Schwinger operator $K_{\alpha E}$ in $(1, \infty)$ counting multiplicities. $\square$

Alternatively, we can formulate the Birman-Schwinger principle in momentum space [Si71]. For a function $f(x)$ let us define its Fourier transform $\hat{f}(p)$ by

$$\hat{f}(p) = \int_{\mathbf{R}^d} e^{ipx} f(x)\ dx\ . \tag{15}$$

The eigenvalue problem for the Schrödinger equation in momentum space reads

$$T_E(p)\hat{\psi}(p) + (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{V}(p-p')\hat{\psi}(p')dp' = 0 \tag{16}$$

with $T_E(p) = p^2 - E,\ E \leq 0$. Defining $\phi(p) = T_E(p)^{-1/2}\hat{\psi}(p)$ we see that the operator $\hat{K}_E$ given by

$$(\hat{K}_E\phi)(p) \equiv -(2\pi)^{-d} \int_{\mathbf{R}^d} T_E(p)^{-1/2}\hat{V}(p-p')T_E^{-1/2}(p')\phi(p')\ dp' \tag{17}$$

has an eigenvalue 1.

By mimicking the x-space proof we obtain the Birman-Schwinger principle in its p-space version.

Theorem 2.3. The number of eigenvalues $N_E(V)$ of $-\Delta + V$ in $(-\infty, E)$, $E \leq 0$ equals the number of eigenvalues of the Birman- Schwinger operator $\hat{K}_E$ in $[1, \infty)$ counting the multiplicities. $\square$

It is interesting to remark that for $d = 3$ the bound corresponding to (2.6) will be

$$N_E(V) \leq tr(\hat{K}_E^* \hat{K}_E) \; . \tag{18}$$

It is elementary to check that

$$\begin{aligned} tr(K_E^* K_E) &= tr(\hat{K}_E^* \hat{K}_E) \\ &= \frac{1}{32}\pi^{-4} \int_{\mathbf{R}^3} \frac{|\hat{V}(p)|^2}{|p|} \arctan \frac{|p|}{2\sqrt{|E|}} \; . \end{aligned} \tag{19}$$

As in configuration space, one may introduce a one parameter family of operators $\hat{K}_{\alpha E}$, $\alpha \in [0,1]$, by using $T_{\alpha E}(p)$ and the Fouriertransform of $V_\alpha = |V(x) + (1 - \alpha)E|_-$.

Remark: While in this article only stated for Schrödinger-type operator the Birman-Schwinger principle holds in general for an operator of the form $H = A + \lambda B$ for A, B selfadjoint and B relatively A-compact. For general results in this abstract situation and some results about existence of eigenvalues we refer to [Kl82].

2.3 Some classical inequalities

To make this review accessible to as wide a readership as possible we recall now classical inequalities playing a major role in various areas of applied mathematics and mathematical physics. This short section is devoted to such questions. Beside local inequalities we will recall also integrated ones. The term "Sobolev inequality" properly applies to convolution inequalities of the form $| < R_\alpha f, g > | \leq C \|f\|_p \|g\|_{q'}$, where $R_\alpha = \frac{a}{|x|^{d-\alpha}}$ with $0 < \alpha < d$ is the Riesz kernel in $\mathbf{R}^d$. By extension however any inequality that a function is in a certain class if its derivatives are in another class is called a Sobolev inequality. The constants given below will be the best possible ones for the particular inequality, in the sense that it cannot be replaced by a smaller one.

A useful result is the convolution inequality by Young.

Young's inequality [Be75, BL76]

Let $p, q, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Let $\|f\|_p = [\int_{\mathbf{R}^d} |f(x)|^p dx]^{1/p}$ and $\|f\|_\infty =$ ess $\sup_{x \in \int_{\mathbf{R}^d}} |f(x)|$. By p' we denote the dual exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Then

$$| \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(x) g(x-y) k(y) dx \; dy | \leq [C_p C_q C_r]^d \|f\|_p \|g\|_q \|h\|_r \tag{20}$$

with

$$C_p^2 = p^{1/p}\ (p')^{-1/p'}\ . \tag{21}$$

We note that $C_pC_qC_r \le 1$.

Sobolev's inequality and its generalisations [So38, GGMT76, L83, O, Ta76]

In 1938 Sobolev found an inequality relating the L^q norm of a function in $\mathbf{R}^d$ to the L^p-norms of the functions and its derivations up to order k if $\frac{1}{q} = \frac{1}{p} - \frac{k}{d}$.

In the present paper we shall extensively use the following generalised Sobolev inquality [GGMT76, L83].

Let $d \ge 3,\ 0 \le b \le 1$, and $p = 2d/(2b + d - 2)$. Then the following inequality holds

$$K_{n,p}\|\nabla f\|_2 \ge \||x|^{-b} f\|_p \tag{22}$$

with

$$\begin{aligned} K_{n,p} &= \omega_{d-1}^{-1/2r}(d-2)^{1/2r-1}\left(\frac{r-1}{r}\right)^{\frac{r-1}{2r}} \\ &\quad \cdot \left(\frac{\Gamma(2r)}{\Gamma(r+1)\Gamma(r)}\right)^{1/2r} \end{aligned} \tag{23}$$

where $r = \frac{p}{p-2} = \frac{d}{2}\frac{1}{1-b}$ and $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of the unit sphere in $\mathbf{R}^d$.

In particular, if $b = 0$, then $p = \frac{2d}{d-2} = 2^*$ (the dual Sobolev exponent) and

$$K_{d,2^*} = [\pi d(d-2)]^{-1/2}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{1/d} . \tag{24}$$

If $1 - d/2 \le b < 0$, the above inequality holds only for spherically symmetric functions. $K_{n,p}$ is given by the same expression. $\square$

The first calculation of the sharp constant $K_{3,2^*}$ is due to Rosen [Ro71].

In [St90] the following generalisation was obtained. Let $0 \le \alpha < 1$, then

$$K_{n,p}^2(\alpha)(\|\nabla f\|_2^2 - \alpha(\frac{d-2}{2})^2\|\frac{1}{|x|}f\|_2^2) \ge \||x|^{-b}f\|_2^2 \tag{25}$$

with p and b as above and

$$K_{n,p}(\alpha) = (1-\alpha)^{-\frac{d-1}{2d}}K_{n,p}\ . \tag{26}$$

If one considers only spherically symmetric function the inequality holds for all $\alpha < 1$.

By using a series of reduction procedures (see [L83] for the details) the above inequalities can be related to the following one-dimensional Sobolev inequality:

For $2 < p < \infty$

$$M_p(\|f'\|_2^2 + \|f\|_2^2) \geq \|f\|_p^2 \tag{27}$$

with

$$M_p = 2^{\frac{1}{r}-2} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} \left\{\frac{\Gamma(2r)}{\Gamma(r)\Gamma(r+1)}\right\}^{1/r} \tag{28}$$

where $r = \frac{p}{p-2}$.

Hardy-Littlewood-Sobolev inequalities and the weak Young inequality [HL28, HL30, L83, So38]

Let $1 < p, t < \infty,\ 0 < \lambda < d$ and $\frac{1}{p} + \frac{1}{t} + \frac{\lambda}{d} = 2$. Then the following inequality holds

$$|\int\int f(x)|x-y|^{-\lambda} g(y) dx\ dy| \leq N_{p,\lambda,d} \|f\|_p \|g\|_t\ . \tag{29}$$

If $p = t = \frac{2d}{2d-\lambda}$, the constant $N_{p,\lambda,d}$ is explicitly given by

$$N_{p,\lambda,d} = \pi^{\lambda/2} \frac{\Gamma(d/2 - 1/2)}{\Gamma(d-\lambda/2)} \left\{\frac{\Gamma(d/2)}{\Gamma(d)}\right\}^{\lambda/d-1}\ . \tag{30}$$

For $p = 2$ or $p = \frac{2d}{2d-\lambda}$ (i.e. $t = 2$), and $d/2 < \lambda < d$, one has

$$N_{p,\lambda,d} = \pi^{\lambda/2} \frac{\Gamma(\frac{d}{2} - \frac{\lambda}{2})}{\Gamma(\lambda/2)} \left\{\frac{\Gamma(\lambda - \frac{d}{2})}{\Gamma(\frac{3d}{2} - \lambda)}\right\}^{1/2} \left\{\frac{\Gamma(\frac{d}{2})}{\Gamma(d)}\right\}^{\frac{\lambda}{d}-1}\ . \tag{31}$$

The sharp constants in (2.23) and (2.24) are due to [L83]. A very nice alternative derivation was found in [CL90].

The generalisation of the Hardy-Littlewood-Sobolev inequality is the weak Young inequality:

$$|\int\int f(x)h(x-y)g(y)dx\ dy| \leq N_{p,\lambda,d} \|f\|_p \|g\|_t \|h\|_{d/\lambda,\omega} \tag{32}$$

where as before $\frac{1}{p} + \frac{1}{t} + \frac{\lambda}{d} = 2$ and $N_{p,\lambda,d}$ denotes the same constant as in (2.17) and $\|h\|_{q,\omega}$ denotes one of the two following definitions of the weak L^q-norm:

$$\|h\|_{q,\omega}^* = \left(\frac{1}{B_d}\right)^{1/q} \sup_{\alpha>0} \mathrm{Vol}\{x \in \mathbf{R}^d | |h(x)| > \alpha\}^{1/q} \tag{33}$$

or

$$\|h\|_{q,\omega}^{**} = \frac{q-1}{q} \left(\frac{1}{B_d}\right)^{1/q} \sup_{\mathrm{Vol}\ A<\infty} (\mathrm{Vol}\ A)^{1/q-1} \int_A |h(x)|\ dx\ . \tag{34}$$

In both definitions the weak-L^q-norm of $|x|^{-\lambda}$ equals 1, for $q = d/\lambda$.

Hausdorff-Young inequality [Be75]

Let $p' \geq 2$, then

$$\|\hat{f}\|_p \leq (2\pi)^{d/p'} C_p^d \|f\|_{p'} \tag{35}$$

with $C_p^2 = p^{1/p}(p')^{-1/p}$ as in Young's inequality. p' and p are dual exponents, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Lieb has proved that equality in (2.28) occurs only for Gaussians [L90b].

(2.28) expresses the fact that the Fourier transformation is a continuous linear transformation for $L^{p'}(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$.

3 Nonexistence of Bound States

In this section we discuss conditions on a potential which imply that the operator $-\Delta + V$ has no bound state, ore more generally no bound state below a given energy $E \leq 0$.

These conditions will then imply lower bounds on $\inf \operatorname{spec}(H)$ which is in fact the principal application of these results. For $d = 3$ several conditions guaranting the absence of bound states have been known for a long time. For instance, Jost and Pais [JP51], Bargmann [Ba52], Birman [Bir61] and Schwinger [Schw61] have shown that a spherically symmetric potential $V(x) = v(r), r = |x|$, which satisfies

$$\int_0^\infty v_-(r) r \; dr < 1 \tag{36}$$

has no bound states. Another example is that if for a potential $V(x)$ the following inequality holds

$$\sup_{x \in \mathbf{R}^3} |x|^2 V_-(x) < 1/4 \tag{37}$$

then $N_0(V) = 0$ [CH].

Finally, making use of the Birman-Schwinger bound (2.6) or (2.11) we see that there is no bound state below $E < 0$ if

$$\frac{1}{(4\pi)^2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{V(x) \; V(y)}{|x-y|^2} e^{-2\sqrt{-E}|x-y|} dx \; dy < 1 \; . \tag{38}$$

In particular, by letting $E \to 0$, this will be the case if

$$\frac{1}{(4\pi)^2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{V(x) \; V(y)}{|x-y|^2} dx \; dy \; < 1 \tag{39}$$

or equivalently

$$\frac{1}{(4\pi)^3} \int_{\mathbf{R}^3} \frac{|\hat{V}(p)|^2}{|p|} dp < 1 \; . \tag{40}$$

On the other hand, in dimensions one and two any "reasonable" purely attractive potential has at least one bound state. We shall give a more precise statement below. However, for any $E < 0$ there will be similar bounds as in higher dimensions.

The most important result on the absence of bound states is due to V. Glaser, H. Grosse, A. Martin and W. Thirring [GGMT76]. For $d = 3$ they computed the sharp constants $K_{3,p}$ in the Sobolev inequalities (2.14) and by means of these inequalities they obtained a family of optimal integral conditions on the potential V which guarantee the nonexistence of bound states. Its generalisation to all space dimension $d \geq 3$ is straight-forward and will be presented in the following subsection.

Remark: The occurence of a bound state at $E = 0$ has nothing exceptional. Although eigenvalues at $E > 0$ are classical unexpected, explicit examples of potentials having positive energy bound states are known. These potentials are characterized by the fact that they fall off slowly as $\frac{1}{r}$ and oscillate rapidly at infinity [RSIV XIII.13].

3.1 The GGMT-Bounds

We consider $H = -\Delta + V(x)$ on $L^2(\mathbf{R}^d), d \geq 3$, where $V : \mathbf{R}^d \to \mathbf{R}$. For simplicity, we assume that V is C_0^∞. For more general potentials such that one of the integrals over V given below is finite the bounds will follow by a standard limiting argument.

Theorem 3.1. Let b, r, p satisfy the conditions of the Sobolev inequalitites (2.14) i.e. $r \geq 1$, $p \geq 2$, $1 - d/2 \leq b \leq 1$ and $p = \frac{2d}{d-2+2b}$, $r = \frac{p}{p-2}$. Let $K_{d,p}$ be the constant given in (2.14b).

(a) If $r \geq d/2$, then H has no bound state if

$$K_{d,p}^{2r} \inf_{x \in \mathbf{R}^d} \int_{\mathbf{R}^d} |x - y|^{2r-d} |V_-(y)|^r dy < 1 \ . \tag{41}$$

(b) If $1 \leq r < d/2$, then for a spherically symmetric potential $V(x) = v(r)$ there is no bound state if

$$K_{d,p}^{2r} \int_{\mathbf{R}^d} |x|^{2r-d} v_-(|x|) dx < 1 \ . \qquad \square(42)$$

Remark: In the two limiting cases $r = 1$ and $r \to \infty$ we obtain the d-dimensional versions of conditions (3.2) and (3.1), respectively. More precisely there is no bound state, if

$$\frac{1}{d-2} \int_0^\infty v_-(r) r \ dr < 1 \tag{43}$$

which is the generalisation of Bargmann's bound and

$$\inf_{y \in \mathbf{R}^d} \sup_{x \in \mathbf{R}^d} |x - y|^2 V_-(x) < \left(\frac{d-2}{2} \right)^2 \ . \tag{44}$$

For $r = d/2$ the left hand side of (3.6) is proportional to the corresponding classical phase space expression.

Remark: The conditions (3.5) are optimal in the following sense. If $1 < r < \infty$ there are potentials having a zero energy bound state solution and the left hand side of (3.6) is equal to one. This follows from the fact that in Sobolev's inequality there is an optimizing function. For $r = 1$ and $r \to \infty$ we conclude by a limiting argument as for Sobolev's inequality. Said more briefly, Sobolev-type inequalities imply sharp conditions on the absence of bound states (and vice versa!).

Proof of Theorem 3.1.
If we can show that the quadratic form associated with H

$$< H\psi, \psi >= \int_{\mathbf{R}^d} |\nabla\psi|^2 dy + \int_{\mathbf{R}^d} V|\psi|^2 dy \tag{45}$$

is nonnegative for all $\psi \in C_0^\infty(\mathbf{R}^d)$, ψ real valued, then it will follow that H has no bound state.
The first step is to prove part (a). Let $r \geq d/2$. Then $0 \leq b \leq 1$.
Obviously $< H\psi, \psi >\geq \int_{\mathbf{R}^d} |\nabla\psi|^2 dy - \int_{\mathbf{R}^d} V_-(y)|\psi|^2 dy$.
For any $x \in \mathbf{R}^d$ we rewrite the potential term as

$$\int_{\mathbf{R}^d} |x-y|^{2b} V_-(y)|x-y|^{-2b}|\psi|^2 dy \ .$$

Applying Hölder's inequality with exponents r and $p/2$ and Sobolev's inequality (2.14a), respectively we obtain

$$< H\psi, \psi >\geq \|\nabla\psi\|_2^2 (1 - K_{n,p}^2 \| |x-y|^{2b} V_- \|_r) \ . \tag{46}$$

With this we have established that the right hand side of (3.9) is positive for some $x \in \mathbf{R}^d$ if V satisfies (3.5a).
To prove part (b) we note that it is sufficient to consider only spherically symmetric functions ψ since the ground state, if it exists, has necessarily spherical symmetry. Then we follow the same lines except that we have to choose $x = 0$. $\square$
For spherically symmetric potentials GGMT also obtained conditions for the absence of bound states of a given angular momentum ℓ. In [St90] these bounds were put into a more general context in connection with the generalised Sobolev inequality (2.15).
Consider the Schrödinger operator defined as follows

$$\begin{aligned} H(\alpha) &= -\Delta - \alpha \left(\frac{d-2}{2}\right)^2 \frac{1}{|x|^2} + V(x) \\ &= H - \alpha \left(\frac{d-2}{2}\right)^2 \frac{1}{|x|^2} \end{aligned} \tag{47}$$

where $0 \leq \alpha < 1$ in the general case and $\alpha < 1$ for spherically symmetric potentials $v(r)$.

The problem of finding a condition for the absence of bound states for given angular momentum ℓ of the operator $-\Delta + v(r)$ is then equivalent to the question how to find a condition such that $H(\alpha)$ has no bound state for

$$\alpha = -\left(\frac{2}{d-2}\right)^2 \ell(\ell + d - 2) \ .$$

By using the generalised Sobolev inequality (2.15) we obtain the following conditions implying the nonexistence of bound states for $H(\alpha)$, [St90].

Theorem 3.2. Let b, r, p, d and $K_{d,p}$ be as in Theorem 3.1.

(a) If $r \geq d/2$ and $0 \leq \alpha < 1$, then $H(\alpha)$ has no bound state if

$$K_{d,p}^{2r}(1-\alpha)^{\frac{1}{2}(1-2r)} \inf_{x \in \mathbf{R}^d} \int_{\mathbf{R}^d} |x-y|^{2r-d} |V_-(y)|^r dy < 1 \ . \tag{48}$$

(b) If $V(x) = v(r)$ and $\alpha < 1$, then $H(\alpha)$ has no bound state if

$$K_{d,p}^{2r}(1-\alpha)^{\frac{1}{2}(1-2r)} \int_{\mathbf{R}^d} v_-(|x|)^r |x|^{2r-d} dx < 1 \ . \qquad \square(49)$$

The following corollary gives sufficient conditions for the absence of bound states of a given angular momentum ℓ.

Corollary 3.1. Let V be spherically symmetric, i.e. $V(x) = v(r)$. Then H has no bound state of angular momentum ℓ if

$$K_{d,p}^{2r}\left(\frac{d-2+2\ell}{d-2}\right)^{1-2r} \int_{\mathbf{R}^d} |v_-(|x|)|^r |x|^{2r-d} dx < 1 \ . \qquad \square(50)$$

To obtain a condition such that $N_E(V) = 0$ for some $E < 0$ we may replace V_- by $|V(x) - E|_-$. However, such conditions are no more optimal. We study improved conditions for $E < 0$ in the next subsection.

We conclude this subsection by discussing a conjecture by A. Martin (private communication) about a possible extension of Theorem 3.1 for non-spherically symmetric potentials.

The conjecture is that part (b), in the case $1 \leq r < d/2$ should extend to non-spherically symmetric potentials in the following sense. If, for
$1 \leq r < d/2$ the following inequality holds

$$K_{d,p}^{2r} \sup_{x \in \mathbf{R}^d} \int_{\mathbf{R}^d} |x-y|^{2r-d} |V_-(y)|^r dy < 1 \ . \tag{51}$$

Then $H = -\Delta + V$ has no bound state.

It turns out from a paper by Hunziker [Hu61] on the convergence of the Born series, that H has no bound state for $d = 3$, if

$$\sup_{x\in\mathbf{R}^3} \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{V_-(y)}{|x-y|}dy < 1 \tag{52}$$

which is the natural extension of the Bargmann bound (3.1) to nonradial potentials. To prove (3.17) we consider the integral equation

$$\psi(x) = -\int_{\mathbf{R}^3} G_E(x-y)V(y)\psi(y)\ dy \tag{53}$$

with $G_E(x-y) = \frac{1}{4\pi}\frac{1}{|x-y|}e^{-\sqrt{-E}|x-y|}$.

Taking the sup-norm on both sides of (3.16) yields that if (3.16) has a bounded solution then $\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3} G_E(x-y)|V_-(y)|dy \geq 1$.

For $r \in (1, d/2)$ Martin and Sabatier [MS77] could prove the following. If

$$\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3} |x-y|^{r-2}|x|^{r-1}V_-(x)^r < \frac{1}{4\pi}\left(\frac{\pi^2}{4}\right)^{1-r} \tag{54}$$

then H has no bound state.

3.2 Absence of bound states and lower bounds on Hamiltonians

This subsection is concerned with some lower bounds on $H = -\Delta + V$ in terms of integral expression of V and conditions which guarantee $N_E(V) = 0$ for negative energies.

We start from the nonlinear functional

$$G_p(u) = \frac{\|\nabla u\|_2^2 + \|u\|_2^2}{\|u\|_p^2}\ , \tag{55}$$

defined on $H^1(\mathbf{R}^d)$, where $p \in (2, 2^*)$ if $d \geq 3$ and $p \in (2,\infty)$ if $d = 1, 2$. We define

$$g_p = \inf_{u\in H^1(\mathbf{R}^d)} G_p(u)\ . \tag{56}$$

It is now of interest to introduce the related functional $F_p(u)$ given by

$$F_p(u) = \frac{\|\nabla u\|_2^{\alpha}\|u\|_2^{2-\alpha}}{\|u\|_p^2} \tag{57}$$

with $\alpha = d(1-\frac{2}{p})$. Similarly, we define

$$f_p = \inf_{u\in H^1(\mathbf{R}^d)} F_p(u)\ . \tag{58}$$

We need one more prepatory well known result, e.g. [W83, BB]:

Proposition 3.1. The minima g_p and f_p are attained by some function $u \in H^1(\mathbf{R}^d)$. By scaling we have

$$g_p = (\frac{\alpha}{2})^{-\alpha/2}(\frac{2-\alpha}{2})^{-\frac{2-\alpha}{2}} f_p \ . \tag{59}$$

Remark: If $d \geq 3$, we see by using Hölder's and Sobolev's inequality, respectively

$$f_p > K_{d,2^*}^{-\alpha} \qquad \text{if} \quad d \geq 3 \tag{60}$$

and for $d = 1$ we see from (3.16)

$$g_p = \frac{1}{M_p} \qquad \text{if} \quad d = 1 \tag{61}$$

where M_p is the constant given in (3.16b). $\square$

We prove the following:

Theorem 3.3. Let $p > 2$ and $p < 2^*$, if $d \geq 3$. Let $r = \frac{p}{p-2}$ and $\gamma = \frac{p}{p-2} - \frac{d}{2}$. Then for $\gamma > 0$ we have

$$H \geq -g_p^{-r/\gamma} \|V_-\|_r^{r/\gamma} \ . \tag{62}$$

Equivalently, if for $E < 0$

$$g_p^{-r} \int_{\mathbf{R}^d} V_-(x)^{\frac{d}{2}+\gamma} dx < |E|^\gamma \tag{63}$$

then $N_E(V) = 0$. $\square$

Remark: From Theorem 3.2. one concludes that the Sobolev-type inequality $G_p(u) \geq g_p$ (or $F_p(u) \geq f_p$) implies the lower bound (3.22) on the Hamiltonian H. On the other hand (3.22) implies

$$< Hu, u > \geq -g_r^{-r/\gamma} \|V_-\|_r^{r/\gamma} \|u\|_2^2$$

for all suitable u and V_-. Taking $V_- = \lambda |u|^{p/r} = \lambda |u|^{p-2}$ we obtain

$$\|\nabla u\|_2^2 - \lambda \|u\|_p^p + g_p^{-r/\gamma} \lambda^{r/\gamma} \|u\|_p^{p/\gamma} \|u\|_2^2 \ .$$

Optimizing with respect to λ yields $F_p(u) \geq f_p$. So the lower bound (3.22) implies a Sobolev-type inequality.

Proof of Theorem 3.3.

For all $\psi \in C_0^\infty(\mathbf{R}^d)$ we obtain by using Hölder's inequality and the Sobolev-type inequality $F_p(u) \geq f_p$, respectively

$$\begin{aligned} < H\psi, \psi > \ &\geq \ \|\nabla \psi\|_2^2 - \|V_-\|_r \|\psi\|_p^2 \\ &\geq \ \|\nabla \psi\|_2^2 - f_p^{-1} \|V_-\|_r \|\nabla \psi\|_2^\alpha \|\psi\|_2^{2-\alpha} \end{aligned} \tag{64}$$

with $\alpha = d(1-\frac{2}{p}) = d/r$.
Optimizing with respect to $\|\nabla\psi\|_2$ and using relation (3.20) between f_p and g_p we deduce the desired result. $\square$

We turn now to similar lower bounds in momentum space. For an eigenvalue $E<0$, the Schrödinger equation in p-space is given by (2.9) and the quadratic form associated to $H-E$ for any $E<0$, can be written as

$$
\begin{aligned}
&< (H-E)\psi, \psi > \\
&= \int T_E(p)|\hat\psi(p)|^2 + (2\pi)^{-d} \iint \hat\psi(p)^* \hat V(p-p')\hat\psi(p')dp\, dp' \qquad (65)
\end{aligned}
$$

with $T_E(p) = p^2 - E$.

The first result is derived using Hardy-Littlewood Sobolev's inequality (2.18)

Theorem 3.4. Let $2 \geq q > \max(1, \frac{2d}{d+2})$ and r' given by $\frac{1}{r'} = 2(1-\frac{1}{q}) = \frac{2}{q'}, \frac{1}{r}+\frac{1}{r'} = 1$ and $r = \frac{d}{2}+\gamma$. If $\hat V$ satisfies

$$
(2\pi)^{-d} N_{q,\frac{d}{r'},d} \left(\int_{\mathbf{R}^d} T_{-1}(p)^{-\frac{q}{2-q}}\right)^{\frac{2-q}{q}} \|\hat V\|_{r',w} \leq |E|^{\gamma/r} \qquad (66)
$$

for $E<0$, where $N_{q,\frac{d}{r'},d}$ is the constant in the Hardy-Littlewood-Sobolev inequality (2.18) and $T_{-1}(p) = p^2+1$, then $N_E(V) = 0$.

In particular

$$
H \geq -(\text{l.h.s. of } (3.31))^{r/\gamma} \;. \qquad \square(67)
$$

This statement is reinforced by the fact that these estimates are sharp and saturated for Coulomb potentials $V(x) = -\frac{Z\alpha}{|x|}$ for $d \geq 2$.

Remark: Note that

$$
\left(\int_{\mathbf{R}^d} T_{-1}(p)^{-\frac{d}{2-d}} dp\right)^{\frac{2-q}{q}} = \left[\frac{\pi^{d/2}\Gamma(\frac{q}{2-q}-\frac{d}{2})}{\Gamma(\frac{q}{2-q})}\right]^{\frac{2-q}{q}} \;. \qquad (68)
$$

Replacing Hardy-Littlewood-Sobolev's inequality by Young's inequality we obtain

Corollary 3.2. Let q, r', r as in Theorem 3.4. If $\hat V$ satisfies for some $E<0$

$$
(2\pi)^{-d}[C_q^2 C_{r'}]^d \|T_{-1}^{-1}\|_{\frac{q}{2-q}} \|\hat V\|_{r'} \leq |E|^{\gamma/r} \qquad (69)
$$

then $N_E(V) = 0$.

If $r' \geq 2$ (i.e. $q \leq 4/3$ which gives a non-empty condition if $d \leq 3$) we may apply the Hausdorff-Young inequality (2.20) which gives a condition of the form (3.23) which, however, is not sharp, but has an explicit constant for $d=2$.

3.3 About dimensions one and two

We want to give a precise version of the statement that any reasonable potential in one or two-dimensions has a bound state.

It is easy to see that, if $V \in C_0^\infty(\mathbf{R}^d), d = 1,2, V \leq 0, V \not\equiv 0$, then the Birman-Schwinger operator K_E or $\hat{K}_E$ given in (2.5) and (2.10) respectively, is Hilbert-Schmidt for all $E < 0$ but that its operator norm diverges as $E \to 0$. Hence K_E has at least one eigenvalue bigger than one for all E sufficiently close to zero. By the Birman-Schwinger principle we conclude that H has a bound state.

For $d = 1$, a more general result is proven in [AHS]. We can do more and extend this to $d = 2$.

Proposition 3.2. Let $V \in L^1(\mathbf{R}^d) + (L^\infty(\mathbf{R}^d))_\epsilon,\ d = 1,2$. Then $H = -\Delta + V$ has a negative eigenvalue if any of the following conditions holds:

(i) $V \leq 0 \qquad V \not\equiv 0$

(ii) $\int |V| < \infty \qquad \int V < 0$

(iii) $\int V_+ < \infty \qquad \int V_- = \infty \qquad \textit{where } V_\pm = \sup(0, \pm V)$

Proof: With the help of $L^1(\mathbf{R}^d) + L^\infty(\mathbf{R}^d)_\epsilon$-assumption we find immediately that the essential spectrum of H is $[0,\infty)$. Therefore it is sufficient to find a trial function ψ in $L^2(\mathbf{R}^d)$ such that $< H\psi, \psi > < 0$.

Note that by increasing V, we can suppose that we are in the second case.

d=1: Taking $\psi_a = e^{-a|x|}, a > 0$, then

$$\int_{\mathbf{R}} |\nabla\psi_a|^2 = a^2 \int_{\mathbf{R}^d} e^{-2a|x|} = a$$

and

$$\lim_{a\to 0} \int_{\mathbf{R}} V(x)|\psi_a(x)|^2 = \int_{\mathbf{R}} V(x) < 0$$

by the dominated convergence theorem.

d=2: Taking $\psi_a = e^{-\frac{1}{2}|x|^a},\ a > 0$, then

$$\int_{\mathbf{R}^2} \|\nabla\psi_a\|^2 = \frac{\pi}{2}a$$

and

$$\lim_{a\to 0} \int_{\mathbf{R}^2} V(x)|\psi_a(x)|^2 = \int_{\mathbf{R}^2} V(x) < 0\ . \qquad \square$$

4 Central Potentials - Bounds on $N_0(V)$

If $V(x) = v(r), r = |x|$ then $H = -\Delta + v(r)$ splits into an infinite family of radial Hamiltonians h_ℓ which read in its reduced form

$$h_\ell = -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + \frac{(\ell + \frac{d-2}{2})^2}{r^2} + v(r) \tag{70}$$

$\ell = 0, 1, \ldots$.

Indeed, if ψ is an eigenfunction of H, the eigenvalue problem reduces to an eigenvalue problem for h_ℓ by introducing the reduced wave function

$$\psi(x) = r^{\frac{1-d}{2}} u_\ell(r) Y_\ell(x/r). \tag{71}$$

The degeneracy of each level is determined by the number H_ℓ of harmonic polynomials of degree ℓ in d dimensions

$$D_\ell = H_\ell - H_{\ell-2} \qquad \text{with} \quad H_\ell = \binom{d+\ell-1}{\ell} . \tag{72}$$

We quote the following examples as an illustration:
For $d = 2, D_\ell = 2$ if $\ell \geq 1$, if $d = 3, D_\ell = 2\ell + 1$ and if $d = 4, D_\ell = (\ell + 1)^2$.

By $n_\ell(v)$ we denote the dimension of the spectral projection $P_{(-\infty,0)}$, of h_ℓ. Obviously n_ℓ satisfies Lemma 2.1, and, in particular n_ℓ is monotone in ℓ. The total number of bound states $N_0(v)$ is related to the $n_\ell' s$ by

$$N_0(v) = \sum_{\ell=0}^{\infty} D_\ell n_\ell(v) \ . \tag{73}$$

For our purposes the following two results are important

Theorem 4.1. Let $V \in C_0^\infty(\mathbf{R}^d)$ be the spherically symmetric, i.e. $V(x) = v(r)$. Let u_ℓ be the zero energy solution for h_ℓ satisfying Dirichlet boundary conditions at zero, i.e.

$$h_\ell u_\ell = 0 \qquad\qquad u_\ell(0) = 0 \ .$$

Then $n_\ell(v)$ is the number of zeros of u_ℓ in $(0, \infty)$.

Theorem 4.2. Let $I_\ell(v) = \int_0^\infty \rho_\ell(r)|v(r)|^p dr,\ l \geq 0$, for $\rho_\ell \geq 0$. If $I_\ell(v) < 1$ implies $n_\ell(0) = 0$ then

$$n_\ell(v) \leq I_\ell(v) \ . \tag{74}$$

The proof of Theorem 4.1 follows from the mini-max principle and the Sturm- Liouville oscillation theorem for ordinary differential equations [RSIV].

Theorem 4.2. is a consequence of Theorem 4.1.
Indeed let u_ℓ be the zero energy solution for h_ℓ and let $r_0 = 0$, $r_1 \dots r_{n_\ell(v)}$ be its zeros.

Then on each interval $(r_{i-1}, r_i)(i = 1, \dots, n_\ell(v)$, the operator h_ℓ restricted to (r_{i-1}, r_i) with Dirichlet boundary conditions has a zero energy eigenstate given by $u_\ell \chi_{(r_{i-1},r_i)}$. Hence $I_\ell(V\chi_{(r_{i-1},r_i)}) \geq 1$. Summing over all intervals yields $I_\ell(v) \geq n_\ell(v)$ [GGMT76, Si76a].

Hence we may take the conditions of Corollary 3.1. about the nonexistence of bound states in the sector ℓ to obtain

Corollary 4.1. Let $d \geq 3$, $p \geq 2$, $1 \in r \leq \infty$ with $r = \frac{p}{p-2}$. If $K_{d,p}$ denotes the Sobolev constant in (2.14 b) then

$$n_\ell(v) \leq \left(\frac{d-2}{2\ell + d - 2}\right)^{2r-1} K_{d,p}^{2r} \int_{\mathbf{R}^d} |v_-(|x|)^r |x|^{2r-d} dx \ . \tag{75}$$

Corollary 4.1. extends to $d = 2$ by noting that the constant on the r.h.s. of (4.6) has a finite limit for $l > 0$ as $d \to 2$. For $\ell = 0$ the bound is divergent reflecting the fact that a negative potential always has a bound state in $d = 2$.

Remark: We note in passing that for $r = 1$ we recover Bargmann's bound in d dimensions

$$n_\ell(v) \leq \frac{1}{2\ell + d - 2} \int_0^\infty v_-(r) r \ dr \ . \tag{76}$$

However, the original proof given by Bargmann [Ba52] is different.

To obtain a bound on the total number of bound states we have, according to (4.4), to sum over all sectors up to a maximal angular momentum $\ell_{\max}$ which is the biggest integer ℓ such that the r.h.s. of (4.6) is less or equal than one.

We define $I_r(V) = (d-2)^{2r-1} K_{d,p}^{2r} \int_{\mathbf{R}^d} V_-(|x|)^r |x|^{2r-d} dx$.

The interesting case $r = d/2$ which corresponds to the classical phase space volume expression gives a bound which always contains a logarithm of I.

For simplicity, we consider only the case $d = 3$. Then we obtain, replacing sums by integrals,

$$N_0(V) \leq I_{3/2}(V)[1 + \frac{1}{4} \ell n I_{\frac{3}{2}}(V)] \tag{77}$$

for $I_{\frac{3}{2}}(V) \geq 1$.

It is worth noting that although in each sector ℓ the bound is optimal, it is saturated for different potentials. Hence suming over various values of ℓ we overestimate the number of bound states. On the other hand the first term in (4.7) is optimal since in view of the nonexistence condition (Theorem 3.1) the best one can hope to prove is $N_0(v) \leq I_{3/2}(V)$.

The problem of overestimating the number of bound states by the summation procedure was partially overcome in [GGM78] by making a different use of Theorem 4.1. and relating bounds on $N_0(V)$ in d dimensions for a spherically symmetric potential $v(r)$ to estimates on $S_\gamma(v)$ in one dimension.

For $d = 3$ they obtained $N_0(V) \leq C I_{\frac{3}{2}}(V)$ with $C > 1$ while for $d = 4$ they proved $N_0(V) \leq I_2(V)$ for V spherically symmetric.

There is another bound on $n_\ell(V)$ due to Calogero which requires a careful analysis of the zero energy solutions u_ℓ [Ca65]. More precisely, if $v(r) \leq 0, v'(r) \geq 0$ then

$$n_\ell(v) \leq \frac{2}{\pi} \int_0^\infty |v(r)|^{1/2} dr \ . \tag{78}$$

The bound is optimal in the sector $\ell = 0$ and saturated by a square well potential.

Recently, a sharper bound for $\ell > 0$ was found [CMS95a]

$$n_\ell(v) \leq \frac{2}{\pi} \int_0^\infty |v(r)|^{\frac{1}{2}} dr + 1 - \sqrt{1 + (\frac{2}{\pi})^2 \ell(\ell+1)} \ . \tag{79}$$

The bound (4.9) looks like an extension of the bounds (4.6) to $r = \frac{1}{2}$. This suggested to look for bounds containing integrals of powers of the potential between 1/2 and 1. [GGMT76]

Recently, this gap was closed in [CMS95b] where it was shown that if $v(r) \leq 0$ and $\frac{d}{dr} r^{1-2p}(-v)^{1-p} \leq 0$ for some $p \in [\frac{1}{2}, 1)$ then

$$n_\ell(v) \leq p(1-p)^{p-1}(2\ell+1)^{1-2p} \int_0^\infty (-r^2 v)^p \frac{dr}{r} \ . \tag{80}$$

It should be stressed that the constants in (4.11) except in the limit $p \to 1$ are not sharp.

Finally, Chadan and Kobayashi extended Calogero's bound to a larger class of potentials. [CK95, CKMS95]. More precisely, if $v(r) \leq 0$ and $\frac{d}{dr} r^{1-2p}(-v)^{1-p} \leq 0$ for some $p \in [\frac{1}{2}, 1)$ then

$$n_\ell(v) \leq \frac{1}{2\sqrt{1-p}} \int_0^\infty (-v)^{1/2} dr \ . \tag{81}$$

5 Phase Space Bounds: The General Case

In this section we present bounds on $S_\gamma(V) = \sum |E_j|^\gamma$ in terms of the corresponding classical phase space expression, i.e. bounds of the form (1.5) and (1.6), respectively. For $\gamma > 0$, the first bounds have been obtained by Lieb and Thirring in 1975 by using the one parameter family of generalised Birman-Schwinger operators $K_{\alpha E}$ defined in (2.7), and the Birman-Schwinger principle as formulated in Theorem 2.2. In particular, they were interested in the case $\gamma = 1, d = 3$ since this estimate played an important role in a simple proof of the "stability of matter" (see also Section 6 of this review).
The case $\gamma = 0$ was still an open problem. The first bounds on $N_0(V)$ having the correct large coupling behaviour are due to Simon [Si69] and Martin [M75], however they are not in terms of the classical phase space expression. The break through was independently achieved in 1976 by Cwickel [Cw77], Lieb [L76a] and Rosenbljum [Ro72]. We shall discuss these methods seperately. In 1983 a different, simple method has been proposed by Li and Yau [LY83] and later be generalised by Blanchard, Rezende and Stubbe [BRS87] yielding an elementary proof of the bound on $N_o(V)$. Another approach has been proposed by Conlon [Co85].

Once such bounds have been established the question for best constants in these estimates was raised [LT76, L89]. To be more precise we would like to know what is

$$R(\gamma, d) \equiv \sup_V \left\{ \frac{\sum_{E_j \leq 0} |E_j|^\gamma}{(2\pi)^{-d} \int\int_{H(x,p)\leq 0} |H(x,p)|^\gamma dx\ dp} \right\}, \tag{82}$$

or equivalently

$$R(\gamma, d) \equiv \sup_V \left\{ \frac{S\gamma(V)}{L^{class}_{\gamma,d} \int_{\mathbf{R}^d} |V_-|^{d/2+\gamma} dx} \right\} . \tag{83}$$

In view of the asymptotic results presented in Theorem 1.1, and Corollary 1.1, respectively, we always have $R(\gamma, d) \geq 1$. We shall discuss these problems together with some general properties of $R(\gamma, d)$ in Section 5.5.

In this article we shall obtain the currently best constants in the case $\gamma = 1$ by combining Lieb's method on Wiener integrals and the generalised Birman-Schwinger operators $K_{\alpha E}$. [BS93]

5.1 Cwickel's bound

Cwickel proved the following result:
Let A be an integral operator defined by

$$(A\phi)(x) = \int_{\mathbf{R}^d} f(x-y) g(y) \phi(y) dx$$

with $f \in L^{p'}_{weak}(\mathbf{R}^d)$ and $g \in L^p(\mathbf{R}^d)$ where $2 < p < \infty$.

By μ_k we denote its singular values.

Theorem 5.1. Under the above assumptions A is a bounded operator on $L^2(\mathbf{R}^d)$ with singular values μ_k obeying

$$|\mu_k| \leq c_{p,d} \|f\|^*_{p'} \|g\|_p k^{-1/p}$$

where $c_{p,d}$ is a constant depending only on p and d.

In particular

$$c_{p,d} \leq \frac{p}{2} \left(\frac{4}{\frac{p}{2} - 1} \right)^{1-2/p} (B_d)^{1/p} \pi^{d/p}. \Box$$

The Theorem is applied to the operator given by $(-\Delta)^{-\frac{1}{2}} |V_-|^{\frac{1}{2}},\ d \geq 3$ and $p = d$.

For $f(x) = \frac{1}{2}\pi^{-\frac{d+1}{2}} \Gamma\left(\frac{d-1}{2}\right) |x|^{1-d}$ with $\|f\|^*_{d-1} = \frac{1}{2}\pi^{-\frac{d+1}{2}} \Gamma\left(\frac{d-1}{2}\right)$ and $g(x) = V_-(x)$ we find by the Birman-Schwinger principle

$$N_0(V) \leq c^d_{d,d} 2^{-d} \pi^{-\frac{d+1}{2}d} \Gamma \left(\frac{d-1}{2} \right)^d \int_{\mathbf{R}^d} V_-^{\frac{d}{2}} dx \tag{84}$$

which yields the constant $\frac{27}{2\pi^2}$ if $d = 3$ which is about 17 times bigger than the best possible value $K^3_{3,2^*} = \frac{4}{3\sqrt{2}\pi^2}$.

5.2 Lieb's method

Lieb's derivation of the phase space bound on $N_E(V)$ makes use of the Wiener Integral representation of functions F of the Birman-Schwinger operator κ_E given in (2.5).

Let $d\mu_{x,y,t}$ be the conditional Wiener measure on paths $\omega(t)$ with $\omega(0) = x$ and $\omega(t) = y$. This measure allows a representation of the Green's function for the semigroup $e^{t\Delta}$ by

$$\int d\mu_{x,y,t}(\omega) = (4\pi t)^{-d/2} \exp \left(-\frac{|x-y|^2}{4t} \right) . \tag{85}$$

By means of the Feynman-Kac formula Lieb proved the following

Theorem 5.2. [Lieb's trace formula] Let $V \leq 0$ and $V \in L^p(\mathbf{R}^d) + L^q(\mathbf{R}^d)$, with $p = \frac{d}{2}$ (if $d \geq 3$), $p > 1 (d = 2), p = 1 (d = 1)$, and $p < q < \infty$. Let f be a nonnegative lower semicontinuous function on $[0, \infty)$ satisfying $f(0) = 0$ and $x^r f(x) \to 0$ as $x \to \infty$ for some $r < \infty$. Define F by

$$F(x) = \int_0^\infty f(x,y) e^{-y} \frac{dy}{y} . \tag{86}$$

Then

$$TrF(K_E) = \int_0^\infty \frac{dt}{d} e^{-|E|t} \int_{\mathbf{R}^d} \int d\mu_{x,x,t} f(\int_0^t V_-(\omega(s))ds) \ . \tag{87}$$

If f is in addition a convex function we may apply Jensen's inequality to the last term in Lieb's trace formula. Then we have

$$f(\int_0^t V_-(\omega(s))ds) \leq \frac{1}{t}\int_0^t f(tV_-(\omega(s)))ds \ . \tag{88}$$

By Fubini's Theorem we may change the order of integration ds and $d\mu$ with dx and thereby obtain the inequality

$$T_r f(K_E) \leq \int_0^\infty \frac{dt}{t} e^{-|E|t} \int d\mu_{0,0;t} \int_0^t \frac{ds}{t} \int_{\mathbf{R}^d} dx \ f(V_-(\omega(s)+x) \ . \tag{89}$$

Since dx is translation invariant we may drop the $\omega-$term for any field path $t \to \omega(t)$ and we are now in position to perform the s-integration and the integration with respect to $d\mu_{0,0,t}$ using the representation (5.4).

We find [L76a, L80]

Corollary 5.1. Let f be convex and satisfy the conditions of Theorem 5.2. Then

$$TrF(K_E) \leq (4\pi)^{-d/2} \int_0^\infty \frac{dt}{t} t^{-d/2} e^{-|E|t} \int_{\mathbf{R}^d} dx f(tV_-(x)). \tag{90}$$

Letting $E \nearrow 0$ and doing a change of the t variable we obtain using the Birman-Schwinger principle.

Corollary 5.2. Let f be convex and satisfy the conditions of Theorem 5.2. Then

$$N_0(V) \leq \frac{\int_0^\infty \frac{ds}{s} s^{-d/2} f(x)}{\int_0^\infty \frac{ds}{s} e^{-s} f(s)} (4\pi)^{-d/2} \int_{\mathbf{R}^d} |V_-(x)|^{d/2} dx \tag{91}$$

or

$$R(0,d) \leq \Gamma(\frac{d}{2}+1) \frac{\int_0^\infty \frac{ds}{s} s^{-d/2} f(s)}{\int_0^\infty \frac{ds}{s} e^{-s} f(s)} \ . \tag{92}$$

¿From this we get that the best constant is given by

$$\begin{aligned} \ell_d \ \equiv \ \inf &\left\{ \Gamma(\frac{d}{2}+1) \int_0^\infty \frac{ds}{s} s^{-d/2} f(s) / \int_0^\infty \frac{ds}{s} e^{-s} f(s) \right. \\ & \left. f \not\equiv 0, f(0)=0, f \text{ nonnegative and convex} \right\}. \end{aligned} \tag{93}$$

To make use of this inequality we choose

$$f(s)=\left\{\begin{array}{ll} 0 & b\le x \\ s-x & b\ge x \end{array}\right.$$

for some $x>0$ and for $d>2$ we arrive at

$$\ell_d \le \min_{x>0} \frac{\Gamma(\frac{d-2}{2})}{x^{d/2}\Gamma(-1,x)} \tag{94}$$

where $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ denotes the incomplete Γ-function.

If $d=3$ we obtain $\ell_3\le 0.116$.

We note that the above choice of f is the best possible as can be seen from the following formal calculation.

Rewriting the right hand side of (5.11) as

$$I_{a,b}(f)=\frac{\int_0^\infty a''(s)f(s)}{\int_0^\infty b''(s)f(s)}$$

with $a(s)=\Gamma(\frac{d-2}{2})s^{+1-d/2}$ and

$$b(s)=\int_s^\infty\int_t^\infty \frac{e^{-t'}}{t'}dt'dt=s\Gamma(-1,s)$$

we find by integration by parts

$$I_{a,b}(f)=\frac{\int_0^\infty a(s)f''(s)}{\int_0^\infty b(s)f''(s)}\ge \min\frac{a(s)}{b(s)}$$

which is attained if $f''(s)\simeq\delta(s-x)$ where x is the position of the minimum of $\frac{a(s)}{b(s)}$.

Bounds on $R(\gamma,d)$ can be obtained in the following way [L84].

Since $N_E(V)\le\frac{TrF(K_E)}{F(1)}$ Corollary 5.1 yields a bound on $N_E(V)$. Using the integration formula (2.4) we do the E-integration. Dividing both sides by the classical constant $L^{\text{class}}_{\gamma,d}$ given in (1.7) we obtain

Corollary 5.3. Let $\gamma\ge 0$ and $d+2\gamma>2$. Then

$$R(\gamma,d)\le\ell_{d+2\gamma} \tag{95}$$

with ℓ_d defined in (5.12). In particular

$$R(\gamma,d)\le\min_{x>0}\frac{\Gamma(\frac{d+2\gamma-2}{2})}{x^{d/2+\gamma}\Gamma(-1,x)}. \qquad \square(96)$$

Evidently, we also have

$$R(\gamma,d)\le\ell_d \tag{97}$$

provided $\gamma \geq 0$ and $d > 2$. Note that the range of γ and d in Corollary 5.3 is bigger and we shall see later that this condition is optimal since there is no such bound for $\gamma = 0$ and $d = 2$ (by the result of Section 3) and for $0 \leq \gamma < \frac{1}{2}$ and $d = 1$.

In Section 5.6 we shall prove a result by Aizenman and Lieb [AL78] which states that $R(\gamma, d)$ is nonincreasing in γ. (Theorem 5.5)

Therefore we have

Corollary 5.4. Let $\gamma \geq 0$ and $d + 2\gamma > 2$.

Then

$$R(\gamma, d) \leq \min_{0 \leq \epsilon \leq \gamma} \ell_{d+2\epsilon} \; . \qquad \square(98)$$

Setting $y = \frac{d}{2} + \gamma$ we consider the function

$$F(x, y) = \frac{\Gamma(y-1)}{x^y \Gamma(-1, x)} \; .$$

Numerically, we find its minimum at $x_0 = 0.8984, y_0 = 2.3563$ and $F(x_o, y_o) = 5.95884$ which shows that for all $d \leq 4$ and γ such that $d/2 + \gamma \geq y_0$

$$R(\gamma, d) \leq 5.95884 \; . \tag{99}$$

Remark: An extension of Lieb's method to more general kinetic energy terms $T(p)$, including in particular the relativistic expression $T(p) = \sqrt{p^2 + m^2} - m$ where m denotes the mass of the particle can be found in [D83].

5.3 The method of Li and Yau

A different way of exploiting the Birman-Schwinger principle and connecting it with Sobolev inequalities was presented in a beautiful paper by Li and Yau [LY]. The problem of counting the bound states of a Schrödinger operator will be related to the eigenvalues of the following Dirichlet boundary value problem

$$\begin{aligned} -\frac{1}{q(x)} \Delta f &= \lambda f \qquad\qquad \text{on} \quad \Omega \subset \mathbf{R}^d \\ f|_{\partial\Omega} &= 0 \end{aligned} \tag{100}$$

in the space $L^2(\Omega, q(x)dx)$ where Ω denotes some bounded domain in $\mathbf{R}^d$ and $q(x)$ is a strictly positive function on $\bar{\Omega}$. If one takes

$$q(x) = V_-(x)$$

then the number of eigenvalues below 1 of the boundary value problem is equal to the number of bound states of the Schrödinger operator $-\Delta_D - q(x)$ where $-\Delta_D$

denotes the Dirichlet Laplacian on Ω. By some technical steps, which we shall describe below, we may perform the limit "$\Omega \to \mathbf{R}^d$" to the operator $-\Delta - q(x)$ on $\mathbf{R}^d$.

The main result is the following.

Theorem 5.3. Let $d \geq 3$. Let λ_k denote the k-th eigenvalue of the Dirichlet boundary value problem (5.17). Then

$$\lambda_k^{d/2} \int_\Omega q_{(x)}^{d/2} dx \geq k K_{d,2^*}^{-d} \dot{e}^{1-d/2} \ . \tag{101}$$

We recall that $K_{d,2^*}$ denotes the best constant of the Sobolev inequality (2.15) given by

$$K_{d,2^*} = [\pi d(d-2)]^{-1/2} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{1/d} \ .$$

Remark: The constant given by Li and Yau differs apart from the numerical error in [LY83] by a factor of e. The improvement of this constant is due to a refinement of the Li-Yau method given in a paper by Blanchard, Stubbe and Rezende [BSR87]. To prove the Theorem we therefore follow the proof given in [BSR87].

Proof of Theorem 5.3.

Let $\{f_i\}_{i\in\mathbf{N}}$ denote the set of orthonormal eigenfunctions (in $L^2(\Omega, q(x)dx)$ with eigenvalues $\{\lambda_i\}_{i=1}^\infty$. The key idea is to study the associated heat kernel

$$H_0(x,y,t) = \sum_{i=1}^\infty e^{-\lambda_i t} f_i(x) f_i(y) \tag{102}$$

and the "weighted" heat kernel

$$\begin{aligned} H_{-1}(x,y,t) &\equiv \frac{1}{2}\int_t^\infty H_0(x,y,s)ds \\ &\equiv \sum_{i=1}^\infty (2\lambda_i)^{-1} e^{-\lambda_i t} f_i(x) f_i(y) \ . \end{aligned} \tag{103}$$

The function H_{-1} was introduced in [BSR87] and yields the improvement of the Li-Yau method. We note that both H_0 and H_{-1} are positive in the interior of $\Omega \times \Omega$ and vanish on $\partial\Omega \times \partial\Omega$ for all $t > 0$. Furthermore they satisfy the heat equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{q(y)} \Delta_y \right) H_\alpha(x,y,t) = 0 \ , \qquad \alpha = 0, -1 \ . \tag{104}$$

Next starting from

$$\begin{aligned} h_{-1}(t) &\equiv \sum_{i=1}^\infty (2\lambda_i)^{-1} e^{-2\lambda_i t} \\ &\equiv \int_\Omega \int_\Omega H_{-1}(x,y,t) H_0(x,y,t) q(x) q(y) dx\, dy \end{aligned} \tag{105}$$

we arrive, after some simple steps involving Hölder's inequality and Sobolev's inequality at the following differential inequality

$$h_{-1}(t) \leq K_{d,2^*}^2 (\int_\Omega q(x)^{d/2})^{2/d} \left(-\frac{dh_{-1}(t)}{dt}\right)^{\frac{d-2}{d}} \quad . \tag{106}$$

To show (5.25), we rewrite $h_{-1}(t)$ as

$$\begin{aligned} h_{-1}(t) &= \int_\Omega \int_\Omega H_{-1}(x,y,t) \cdot [H_0(x,y,t) q(y)^{\frac{d+2}{4}}]^{2/d} \\ &\quad \cdot [H_0(x,y,t)^2 q(y)]^{\frac{d-2}{2d}} dy \; q(x) \; dx \end{aligned} \tag{107}$$

and apply Hölder's inequality with exponents $\frac{2d}{d-2}, \frac{d}{2}, \frac{2d}{d-2}$ for the y-integration and then with exponents $2, d, \frac{2d}{d-2}$ for the x-integration. We obtain therefore

$$\begin{aligned} h_{-1}(t) &\leq \left[\int_\Omega q(x) \left(\int_\Omega H_{-1}(x,y,t)^{\frac{2d}{d-2}} \; dy\right)^{\frac{d-2}{d}} dx\right]^{1/2} \\ &\cdot \left[\int_\Omega q(x) \left(\int_\Omega H_0(x,y,t) q(y)^{\frac{d+2}{4}} \; dy\right)^2 dx\right]^{1/d} \\ &\cdot \left[\int_\Omega q(x) \int_\Omega H_0(x,y,t)^2 q(y) dy \; dx\right]^{\frac{d-2}{2d}} \quad . \end{aligned} \tag{108}$$

At this stage it may be remarked that the last term equals $\left(-\frac{dh_{-1}}{dt}\right)^{\frac{d-2}{2d}}$. To estimate the second term we observe that the function $Q(x,t) = \int_\Omega H_0(x,y,t)$ $q(y)^{\frac{d+2}{2d}} \; dy$ also satisfies the heat equation with $Q(x,t) = 0$ on $\partial\Omega$ for $t > 0$ and initial value $Q(x,0) = q(x)^{\frac{d-2}{4}}$. We compute

$$\begin{aligned} \frac{\partial}{\partial t} \int_\Omega Q^2(x,t) \; q(x) \; dx &= -2 \int_\Omega |\nabla_x Q(x,t)|^2 dx \\ &\leq 0 \; . \end{aligned} \tag{109}$$

Hence

$$\begin{aligned} \int_\Omega Q^2(x,t) \; q(x) \; dx &\leq \int_\Omega Q^2(x,0) q(x) dx \\ &= \int_\Omega q(x)^{d/2} dx \; . \end{aligned} \tag{110}$$

There remains the question of estimating the first term. Applying Sobolev's inequality (2.14) and the identity

$$\begin{aligned} \frac{1}{2} h_{-1}(t) &= -\int_\Omega \int_\Omega H_{-1}(x,y,t) \frac{\partial}{\partial t} H_{-1}(x,y,t) \; q(x) \; q(y) \; dx \; dy \\ &= -\int_\Omega \int_\Omega H_{-1}(x,y,t) \Delta_y H_{-1}(x,y,t) \; q(x) \; dx \; dy \\ &= \int_\Omega q(x) \int_\Omega |\nabla_y H_{-1}(x,y,t)|^2 dy \; dx \end{aligned} \tag{111}$$

where we have used the definition of the functions $h_{-1}(t)$ and $H_{-1}(x,y,t)$ and the heat equation (5.21) we conclude that the first term is bounded by

$$K_{d,2^*} \cdot (\frac{1}{2} h_{-1}(t))^{1/2} \ .$$

Combining these estimates we obtain inequality (5.25). Integrating with respect to t yields

$$h_{-1}(t) \leq \left(\frac{d-2}{2}\right)^{\frac{d-2}{2}} \left(\frac{1}{2} K_{d,2^*}^2\right)^{d/2} \int_\Omega q(x)^{d/2} dx \ t^{\frac{2-d}{2}} \ . \tag{112}$$

In the following we derive an estimate for the sum of the first k eigenvalues which is slightly stronger than the inequality presented in the theorem.

Using the fact that $\frac{1}{2\lambda} e^{-2\lambda t}$ is strictly convex in λ we may replace in (5.29) $h_{-1}(t)$ by

$$k \frac{1}{\frac{1}{k}\sum_{l=1}^k (2\lambda_i)} \exp\left(-\frac{1}{k}\sum_{l=1}^k (2\lambda_i) t\right) \ .$$

Choosing $t = \frac{d-2}{2} \frac{1}{\frac{1}{k}\sum_{l=1}^k (2\lambda_i)}$ we obtain

$$\left(\frac{1}{k}\sum_{l=1}^k \lambda_i\right)^{d/2} \int_\Omega q(x)^{d/2} \ dx \geq k \ K_{d,2^*}^{-d} e^{1-d/2} \tag{113}$$

which implies the estimate stated in the Theorem. □

We will now apply the Birman-Schwinger principle and the Theorem to estimate the number of bound states of the Schrödinger operator $H = -\Delta + V(x)$.

Corollary 5.4. Let $N_0(V)$ denote the number of bound states of the Schrödinger operator $H = -\Delta + V$ on $\mathbf{R}^d$, $d \geq 3$, where the negative part $V_-(x)$ of the potential is an element of $L^{d/2}(\mathbf{R}^d)$. $N_0(V)$ satisfies the estimate

$$N_0(V) \leq e^{\frac{d-2}{2}} K_{d,2^*}^d \int_{\mathbf{R}^d} |V_-(x)|^{d/2} dx \tag{114}$$

where $K_{d,2^*} = [\pi d(d-2)]^{-1/2} \left(\frac{\Gamma(d)}{\Gamma(\frac{d}{2})}\right)^{1/d}$. □

The proof of the Corollary consists of a series of reduction procedures to apply the Theorem.

(i) We use first the fact that $N_0(V) \leq N_0(-V_-)$.

(ii) We may next suppose that $V_-(x) > 0$ for all $x \in \mathbf{R}^d$. Otherwise we approximate $V_-(x)$ by a sequence of strictly positive functions in $L^{d/2}(\mathbf{R}^d)$. To apply the theorem we then take $q(x) = V_-(x)$.

(iii) Evidently the number k of negative eigenvalues of $-\Delta_D - q(x)$ on $\Omega \subset \mathbf{R}^d$ bounded, is nothing else as the number of eigenvalues of $-\frac{1}{q(x)}\Delta_D$ less than one from which it follows that

$$\begin{aligned} k &\leq e^{\frac{d-2}{2}} K_{d,2^*}^d \int_\Omega q(x)^{d/2} dx \\ &\leq e^{\frac{d-2}{2}} K_{d,2^*}^d \int_{\mathbf{R}^d} V_-(x)^{d/2} dx \ . \end{aligned}$$

(iv) Finally we exhaust $\mathbf{R}^d$ by bounded subdomains $\Omega_n \to \mathbf{R}^d$ and use the fact that the negative eigenvalues of $-\Delta_D - q(x)$ converges towards the negative eigenvalues of $-\Delta - q(x)$. □

There are several extensions of these results:

In [BSR87] the generalised Sobolev inequalities were used to derive the following estimate on the number of bound states:

Let $\gamma \geq d/2$ with $d \geq 3$. Then for any $x_0 \in \mathbf{R}^d$

$$N_0(V) \leq e^{\gamma-1} K_{d,p}^{2\gamma} \int_{\mathbf{R}^d} V_-(x)^\gamma |x - x_0|^{2\gamma-d} dx \tag{115}$$

where $p = \frac{2\gamma}{\gamma-1}$ and $K_{d,p}$ is the Sobolev constant given in (2.17).

In [St90] potentials with so called critical decay were considered. Suppose, for example that

$$\lim_{|x|\to\infty} V_-(x)|x|^2 = c \tag{116}$$

with $0 \leq c < (\frac{d-2}{2})^2$. Then

$$\begin{aligned} N_0(V) \ \leq \ & e^{\frac{d-2}{2}} K_{d,2^*}^d (1 - \frac{4c}{(d-2)^2})^{\frac{d-1}{2}} \\ & \int_{\mathbf{R}^d} (V(x) + \frac{c}{|x|^2})_-^{d/2} dx \ . \end{aligned} \tag{117}$$

Consequently, a potential may have only a finite number of bound states although its classical phase space volume is infinite.

5.4 Improved bounds on $R(\gamma, d)$

We briefly describe a slightly more sophisticated method to obtain better numerical constants on $R(\gamma, d)$ for $\gamma > 0$. Basically it relies on the use of the one parameter family of generalised Birman-Schwinger operators $K_{\alpha E}, \alpha \in [0, 1)$ given in (2.7). This method was introduced by Lieb and Thirring [LT75,76] when they derived the celebrated bound on the sum of the eigenvalues of a Schrödinger operator, i.e. $R(1, d)$, without having a bound on $R(0, d)$!

In order to illustrate the basic idea we sketch the original proof of Lieb and Thirring for $R(1, 3)$.

By Theorem 2.2 and (2.6) we have for all $\alpha \in [0,1)$

$$
\begin{aligned}
N_E(V) &\leq trK^*_{\alpha E}K_{\alpha E} \qquad (118)\\
&= \frac{1}{(4\pi)^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} |V_\alpha(x)||V_\alpha(y)||x-y|^{-2}\exp(-2\sqrt{\alpha E}|x-y|)dxdy
\end{aligned}
$$

where $V_\alpha(x) = |V(x) - (1-\alpha)E|_-$.

We estimate the integral by Young's inequality (2.13) using exponents $p = r = 2$ and $q = 1$. This yields the bound

$$
N_E(V) \leq \frac{1}{8\pi}|\alpha E|^{-\frac{1}{2}}\int_{\mathbf{R}^3}|V_\alpha(x)|^2 dx \ . \tag{119}
$$

We integrate (5.36) with respect to E. Using the change of variables $|E| \to \frac{|V(x)|_-}{1-\alpha}\beta$ we surprisingly obtain the correct phase space term $\int_{\mathbf{R}^3}|V(x)|_-^{5/2}dx$. Optimizing with respect to α (take $\alpha = \frac{1}{2}$) we find

$$
S_1(V) \leq \frac{1}{4\pi}\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}\int_{\mathbf{R}^3}|V(x)|_-^{5/2}dx \tag{120}
$$

or equivalently

$$
R(1,3) \leq 4\pi \ . \tag{121}
$$

In the following we use the same trick in Lieb's trace formula (Theorem 5.2). We obtain for f convex and satisfying the conditions of Theorem 5.2

$$
N_E(V) \leq \frac{1}{F(1)}(4\pi)^{-d/2}\int_0^\infty dt\ t^{-1-d/2}e^{-\alpha|E|t}\int_{\mathbf{R}^d} dx f(tV_\alpha(x)) \tag{122}
$$

where F and f are related by the transformation $F(z) = \int_0^\infty \frac{ds}{s}f(sz)e^{-s}$.

Integrating (5.39) with respect to the energy and doing the following changes of variables

$$
|E| \to \frac{|V(x)|_-}{1-\alpha}\beta \text{ and } t \to \frac{1-\alpha}{\alpha}\frac{1}{\beta|V(x)|_-}s \tag{123}
$$

we find

$$
R(1,d) \leq \frac{\Gamma(2+\frac{d}{2})}{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{1/2}\frac{\int_0^\infty ds s^{-1-d/2}e^{-s}\int_0^\infty d\beta\ \beta^{d/2}f(\frac{1-\alpha}{\alpha}\frac{1-\beta}{\beta}s)}{\int_0^\infty ds\ s^{-1}e^{-s}f(s)} \ . \tag{124}
$$

Choosing f as a piecewise linear function as in Section 5.3 we can do the β-integration.

Finally, defining a new parameter y by $y = \frac{\alpha}{1-\alpha}x$ where $x > 0$ we obtain

$$
R(1,d) \leq F_d(x,y) \qquad \text{for all } x,y \geq 0 \tag{125}
$$

Dimension d	Min $F_d(x,y)$	x	y
1	5.81029	0.037191	0.060372
2	5.17690	0.23019	0.207166
3	5.21803	0.533354	0.356744
4	5.39902	0.896431	0.491124
5	5.62201	1.29481	0.608821
6	5.85781	1.71579	0.711530

where

$$F_d(x,y) = \Gamma\left(\frac{d}{2}\right)\left(\frac{y}{x}\right)^{d/2}\left(\frac{1}{x}+\frac{1}{y}\right)\frac{\Gamma(\frac{2-d}{2},y)}{\Gamma(-1,x)} \tag{126}$$

and $\Gamma(d,x) = \int_x^\infty dt t^{d-1} e^{-t}$ denotes the incomplete $\Gamma-$function.

The limit $y \to 0$ corresponds to $\alpha \to 0$ and the limit $y \to \infty$ corresponds to $\alpha \to 1$.

We have

$$\lim_{y\to 0} F_d(x,y) = \frac{\Gamma(\frac{d-2}{2})}{x^{d/2}\Gamma(-1,x)} \quad \text{for } d > 2 \tag{127}$$

$$\textit{and}$$

$$\lim_{y\to\infty} F_d(x,y) = \frac{\Gamma(d/2)}{x^{d/2+1}\Gamma(-1,x)}, \tag{128}$$

which corresponds to the bounds (5.16) and (5.15) respectively.

In the following table we list the minimum value of $F_d(x,y)$, Min $F_d(x,y)$ and the position (x,y) where it is attained.

Generalizing to arbitrary γ we obtain

$$\begin{aligned} R(\gamma,d) \leq \frac{\Gamma(\gamma+\frac{d}{2}-1)}{\Gamma(-1,x)} \cdot (x+y)^\gamma \\ \cdot x^{-\gamma-\frac{d}{2}} \cdot U(\gamma, 2-\frac{d}{2}, y) \end{aligned} \tag{129}$$

for all $x, y \geq 0$ where $U(\gamma, 2-\frac{d}{2}, y) = \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-yt}t^{\gamma-1}(1+t)^{1-\gamma-d/2}$ is the Kummer function.

(5.46) yields some improvements for small d and large γ, for example we find $R(20,2) \leq 4.0099$ or $R(100,2) \leq 3.9601$.

5.5 Bounds for one-dimensional Schrödinger operators

The only case where it can be shown that $R(\gamma,d) = 1$ is for $d = 1$ and $\gamma \geq 3/2$ by a technique completely different from those considered before. This was first done in [LT76].

Let $V \in C_0^\infty$ and consider the initial value problem for the Korteweg-de Vries equation

$$\begin{aligned} W_t + W_{xxx} - 6WW_x &= 0 \\ W(x,0) &= V(x) \ . \end{aligned} \tag{130}$$

¿From the theory of this equation the following results are well known:

1. The eigenvalues of $-\frac{d^2}{dx^2} + W(x,t)$ remain time independent.

2. $\int_{\mathbf{R}} W^2(x,t)dx = \int_{\mathbf{R}} V^2(x)\ dx$.

3. As $t \to \infty, W(x,t)$ evolves into a train of solitons of the form $f_a(\xi) =$
   $= -2a^2 \cosh^{-2}(a\xi)$ where

   $$\xi = x - ct \ , \qquad c = 4a^2$$

   plus a dispersive part going to zero in the sup-norm. These solitons are separated as $t \to \infty$ since they have different velocities. Any soliton potential has exactly one bound state. Its energy is $-a^2$.

As a consequence we have

$$S_{3/2}(V) = \sum |E_j|^{3/2} = \sum_{\text{solitons}} a^3 \tag{131}$$

and

$$\int V^2(x) \geq \sum_{\text{solitons}} \int f_a^2(x)\ dx = \frac{16}{3}(\sum_{\text{solitons}} a^3) \tag{132}$$

which implies

Theorem 5.4.

$$R(\frac{3}{2}, 1) = 1 \ . \tag{133}$$

As we shall see in the next Section general properties of $R(\gamma, d)$ will then imply

$$R(\gamma, 1) = 1 \tag{134}$$

for all $\gamma \geq 3/2$. (see Theorem 5.5.)

5.6 General Properties of $R(\gamma, d)$ and some Conjectures

In this Section we derive some properties of $R(\gamma, d)$. The first result is due to Aizenman and Lieb [AL78] stating that for d fixed $R(\gamma, d)$ is a non-increasing function in γ.

Theorem 5.5. For any $\epsilon \geq 0$

$$R(\gamma + \epsilon, d) \leq R(\gamma, d) \ . \tag{135}$$

Proof: We fix the potential V. Let $\epsilon > 0$. Then

$$S_{\gamma+\epsilon}(V) = \frac{\Gamma(\epsilon + \gamma + 1)}{\Gamma(\epsilon)\Gamma(\gamma + 1)} \int_0^\infty da a^{\epsilon-1} \sum_j |E_j(V) + a|_-^\gamma \ . \tag{136}$$

However, $(E_j(V) + a)$ are the eigenvalues of the potential $V(x) + a$.
Hence

$$\begin{aligned} S_{\gamma+\epsilon}(V) \ &\leq \ \frac{\Gamma(\epsilon)\Gamma(\gamma+1)}{\Gamma(\epsilon+\gamma+1)} \\ &\cdot \ R(\gamma, d) L^{class}_{\gamma+\epsilon,d} \int_0^\infty da a^{\epsilon-1} \int_{\mathbf{R}^d} dx |V(x) + a|_-^\gamma \\ &= \ R(\gamma, d) L^{class}_{\gamma,d} \int_{\mathbf{R}^d} |V(x)|^{\gamma+\epsilon} dx \end{aligned}$$

which implies the desired result. □

Remark: As stressed in the preceding Section Theorem 5.5 implies that $R(\gamma, 1) = 1$ for all $\gamma \geq 3/2$.

The second result is due to Martin [M90] and relates different dimensions d.

Theorem 5.6.

$$R(\gamma + \frac{1}{2}, d) \leq R(\gamma, d + 1) \ . \tag{137}$$

Proof: By the definition of $R(\gamma + \frac{1}{2}, d)$ for any $\epsilon > 0$, there is a potential V, $V \leq 0$, such that

$$\frac{\sum |E_j(V)|^{\gamma+\frac{1}{2}}}{L^{class}_{\gamma+\frac{1}{2},d} \int_{\mathbf{R}^d} |V|^{\gamma+\frac{1}{2}+d} dx} \geq R(\gamma, d) - \epsilon \ . \tag{138}$$

In the following we take ϵ and V fixed, satisfying condition (5.57).

We consider the following Schrödinger operator on $\mathbf{R}^{d+1}$ which is a direct sum of the operator $H = -\Delta + V(x)$ acting on $L^2(\mathbf{R}^d)$ and an one dimensional infinite square well (or Dirichlet Laplacian on an interval)

$$\tilde{H} = -\Delta - \partial_t^2 + V(x) + W_R(t) \tag{139}$$

with

$$W_R(t) = \begin{cases} 0 & |t| \leq R \\ \infty & |t| \geq R \end{cases}$$

Its eigenvalues are given by

$$\tilde{E}_{j,n} = E_j(V) + \frac{\pi^2}{4R^2} n^2 \ . \tag{140}$$

Since

$$\begin{aligned}
&\int_{\mathbf{R}^{d+1}} \int |V(x) + W(t)|^{\gamma + \frac{d+1}{2}} dx \ dt \\
&\quad = 2R \int_{\mathbf{R}^d} |V(x)|^{\gamma + \frac{1}{2} + d/2} dx
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{R\to\infty} \frac{1}{2R} \sum_j \sum_n |E_j(V) + \frac{\pi^2}{(2R)^2} n^2|_-^\gamma \\
&\quad = \sum_{E_j<0} \int_0^\infty |E_j(V) + \pi^2 \gamma^2|_-^\gamma ds \\
&\quad = \frac{1}{\pi} (\sum_{E_j<0} |E_j(V)|^{\gamma+\frac{1}{2}}) \int_0^1 (1-\sigma^2)^\gamma d\sigma \\
&\quad = (4\pi)^{-\frac{1}{2}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)} \sum_{E_j<0} |E_j(v)|^{\gamma+\frac{1}{2}}
\end{aligned}$$

we have

$$\begin{aligned}
R(\gamma, d+1) &\geq \lim_{R\to\infty} \frac{\sum_j \sum_{n_s} |E_j(V) + \frac{\pi^2}{(2R)^2} n^2|_-^\gamma}{L^{class}_{\gamma,d+1} \int_{\mathbf{R}^{d+2}} \int |V(x) + W(t)|^{\gamma+\frac{d+1}{2}} dx \ dt}_- \\
&= (4\pi)^{-\frac{1}{2}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{3}{2})} \frac{\sum_j |E_j(V)|^{\gamma+\frac{1}{2}}}{L^{class}_{\gamma,d+1} \int_{\mathbf{R}^d} |V(x)|_-^{\gamma+\frac{1}{2}+\frac{d}{2}} dx} \\
&= \frac{\sum |E_j(V)|^{\gamma+\frac{1}{2}}}{L^{class}_{\gamma+\frac{1}{2},d} \int_{\mathbf{R}^d} |V(x)|^{\gamma+\frac{1}{2}+d} dx} \\
&\geq R(\gamma + \frac{1}{2}, d) - \epsilon
\end{aligned}$$

which proves the Theorem.

The following facts about $R(\gamma,d)$ are known. Comparing the zero or one bound state constant $K^d_{d,2^*}$ with $L^{class}_{0,d}$ one sees that $R(0,d)>1$ if $d<7$. A more sophisticated argument by Glaser, Grosse and Martin [GGM78] yields $R(0,d)>1$ for $d\geq 7$.

It is conjectured that for $3\leq d\leq 6$,

$$R(0,d)=\frac{K^d_{d,2^*}}{L^{class}_{0,d}}\ .$$

$R(\gamma,d)$ is finite if $\gamma+d/2>1$. The two boundary points are $(\gamma,d)=\{(\frac{1}{2},1),(0,2)\}$.

It was claimed in [LT76] that $R(\frac{1}{2},1)<\infty$. This was recently proved by Weidel [Weid95]. On the other hand $R(\gamma,1)$ is infinite if $0\leq\gamma<\frac{1}{2}$. This can be seen by taking a potential well $V_R(x)=-\frac{1}{R}\chi(0,R)$. As $R\to 0$, $V_R(x)$ converges towards $-\delta(x)$ which has a finite energy bound state while $\lim_{R\to 0}\int|V_R(x)|^{\frac{1}{2}+\gamma}=0$.

¿From Theorem 4.2 we know that $R(0,2)$ is infinite.

¿From Theorem 3.3 together with (3.26) we deduce that if the operator $-\frac{d^2}{dx^2}+V$ has a bound state then its eigenvalue satisfies

$$|E|^\gamma<M_p^r\int_{\mathbf{R}}|V(x)|^{\frac{1}{2}+\gamma}\,dx$$

with M_p given in (2.20).

As pointed out in Section 3.2 this condition is optimal. Comparing M_p^σ with $L^{class}_{\gamma,1}$ we see that $R(\gamma,1)>1$ for $1/2\leq\gamma<3/2$.

These considerations lead to conjecture that for each $d\leq 7$ there is a $\gamma_d>0$ such that $R(\gamma,d)>1$ if $\gamma<\gamma_d$ and γ_d is defined by the point where the one bound state constant and the classical constant coincide. This conjecture was originally formulated by Lieb and Thirring for any d, but the argument by Glaser et al. [GGM] mentioned above shows that this cannot hold for $d\geq 8$. The conjecture is true for $d=1$.

An important conjecture, related to the nonrelativistic stability of matter is that $R(1,d)=1$ for $d\geq 3$.

Glaser et al. conjectured also that

$$\lim_{d\to\infty}R(0,d)=1\ .$$

5.7 Schrödinger operators with magnetic fields

Recently phase space bounds have been generalized to Schrödinger operators with magnetic fields [LSY94,LLS95]. As in [LSY94] we only consider the case of a constant magnetic field. Some generalizations to nonconstant fields have been obtained in [Er95]. The two dimensional case has already been presented in [LSY95].

We consider the Schrödinger operator for a spin $\frac{1}{2}$ particle

$$\begin{aligned} H &= (p+\mathbf{A}(x))^2+\sigma\cdot\mathbf{B}+V(x) \\ &= H_{\mathbf{A}}+V(x) \end{aligned} \tag{141}$$

where $\mathbf{B} = (0, 0, B), \mathbf{A} = \frac{1}{2}\mathbf{B} \wedge x$ and σ denotes the Pauli matrices. We suppose $V_- \in L^{3/2}(\mathbf{R}^3) \cap L^{5/2}(\mathbf{R}^2)$.

The spectrum of $H_{\mathbf{A}}$ is described by the Landau bounds

$$E_{p,\nu} = 2\nu B + p^2 \tag{142}$$

where $\nu = 0, 1, \ldots$ and $p \in \mathbf{R}$ is the momentum along the field B.

The spin-dependent term $\sigma \cdot \mathbf{B}$ makes the problem interesting.

Without the $\sigma \cdot \mathbf{B}$ term, by the diamagnetic inequality [Si79] we can apply all phase space bounds for the $B = 0$ case.

With the spin-term, however, the bottom of the essential spectrum is shifted by $-B$ down to zero so that one estimates all the spectrum below σ_{ess}. A further complication comes from the observation that H may have infinitely many eigenvalues even when V has compact support [AHS78]. However, the phase space bounds obtained in [LSY94] show that these infinitely many eigenvalues are summable.

In fact there is the following bound on $S_1(V, B) = \sum_{E_j<0} |E_j(V, B)|$

$$S_1(V, B) \leq L_1 B \int_{\mathbf{R}^3} V_-(x)^{5/2} dx + L_2 \int_{\mathbf{R}^3} V_-^{3/2}(x) dx \tag{143}$$

with $L_1 = \frac{4}{3\pi}$ and $L_2 = \frac{8\sqrt{6}}{5\pi}$.

(5.62) is proved by using the generalized Birman Schwinger operators $K_{\alpha E}, \alpha \in [0, 1)$ corresponding to (2.7) and a splitting of the operators into a projection onto the lowest Landau bound and onto higher bounds.

The bound (5.62) does not correspond to the classical phase space expression which is given by

$$S_1(V, B) \simeq \frac{B}{3\pi^2} \int |V_-(x)|^{3/2} + 2 \sum_{v=1}^{\infty} |V(x) + 2\nu B|_-^{3/2} dx \ . \tag{144}$$

However, the two terms in the bound (5.62) correspond, respectively, to the $B \to \infty$ (first term) and the $B \to 0$ (second term) asymptotics of (5.63).

To obtain a bound in terms of the r.h.s. of (5.63) we use the inequality

$$\frac{|V(x)|^{5/2}}{5B} \equiv \int_0^\infty |V(x) + 2B\sigma|_-^{3/2} d\sigma \leq \sum_{v=0}^{\infty} |V(x) + 2B\nu|_-^{3/2} \ . \tag{145}$$

6 Applications

The most important applications of the phase space bounds presented in this review are related to the problem of the stability of matter and to the theory of dynamical systems. We outline them in the sequel.

6.1 A Sobolev inequality for Fermions

The bound on the sum of the bound state energies of a Schrödinger operator with potential $V \leq 0$ given by

$$\sum_{E_j \leq 0} |E_j| \leq L_{1,d} \int_{\mathbf{R}^d} |V|^{d/2+1} dx \tag{146}$$

will imply a Sobolev-type inequality for the kinetic energy of N Fermions. The constant $L_{1,d}$ is related to $R(1,d)$ by $L_{1,d} = R(1,d) \quad L_{1,d}^{class}$ with $L_{1,d}^{class} = (4\pi)^{-d/2} \frac{1}{\Gamma(2+d/2)}$.

For simplicity we do not take into account the spin of the particles.

Theorem 6.1.[LT75,76]: Let $\psi : R^{dN} \to \mathbf{C}$, $\psi = \psi(x_1, \ldots, x_N)$, $x_j \in \mathbf{R}^d$, $j = 1, \ldots, N$ be an antisymmetric L^2-wave function and let $\rho_\psi(x) = N \int_{\mathbf{R}^{d(N-1)}} |\psi(x_1, x_2, \ldots, x_N)|^2 dx_2 \ldots dx_N$ the corresponding one-particle density. Then

$$\begin{aligned}\sum_{i=1}^{N} & \int_{\mathbf{R}^{dN}} |\nabla_{x_i} \psi(x_1, \ldots, x_N)|^2 dx_1, \ldots, dx_N \\ & \geq \tfrac{d}{d+2} K_d \int_{\mathbf{R}^d} \rho_\psi(x)^{1+2/d} dx\end{aligned} \tag{147}$$

with

$$\begin{aligned} K_d &= \left(\frac{d+2}{2} L_{1,d}\right)^{-2/d} \\ &= 4\pi \left[\frac{\Gamma(\frac{d+2}{2})}{R(1,d)}\right]^{2/d} . \end{aligned} \tag{148}$$

Proof: Consider the N-particle Hamiltonian

$$H_N = \sum_{L=1}^{N} -\Delta_{x_i} + V(x_i) \quad , \qquad V \leq 0 \tag{149}$$

By (6.1) we have for any antisymmetric N-particle wave function

$$\begin{aligned} & < H_N \psi, \psi >_{L^2(R^{dN})} \\ & = \sum_{L=1}^{N} \int_{\mathbf{R}^{dN}} |\nabla_i \psi(x_1, \ldots, x_N)|^2 dx_1 \ldots dx_N + \int_{\mathbf{R}^d} V(x) \rho(x) dx \\ & \geq -L_{1,d} \int_{\mathbf{R}^d} |V|^{1+d/2} dx \ . \end{aligned} \tag{150}$$

By standard Sobolev's inequality one shows $\rho \in L^{1+2/d}(\mathbf{R}^d)$ if the kinetic energy of ψ is finite. Hence $\rho \in L^1 \cap L^{1+2/d}$. Choosing $V = -\lambda \rho^{2/d}(x)$ we have $V \in L^{d/2}(\mathbf{R}^d) \cap L^{1+d/2}(\mathbf{R}^d)$ and consequently V has only a finite number of bound states and the sum over the bound state energies is finite. Optimizing with respect to λ yields the desired inequality. $\square$

The corresponding classical value (or Thomas-Fermi value) K_d^{cl} in the kinetic energy inequality (6.2) is given by

$$K_d^{cl} = 4\pi(\Gamma(1 + d/2))^{2/d} \ . \tag{151}$$

The case $d = 3$ is relevant in the proof of the stability of matter. Using our estimates in Section 5 we have the following numerical values

$$\begin{aligned} K_3^{cl} &= (6\pi^2)^{2/3} = 15.1927 & (152) \\ K_3 \geq (5.21)^{-2/3} K_3^{cl} &= 0.3324 K_3^{cl} = 5.0501 \ . & (153) \end{aligned}$$

The Lieb-Thirring bound on the ground state of N identical fermions with charge -1 and mass m having q different spin states and k point nuclei with charges $Z_1, \ldots, Z_k$ has the form

$$E(N, q) > -4.42 (K_3)^{-1} q^{-1} N \left[1 + \sqrt{\frac{\sum_{j=1}^k Z_j^{7/3}}{N}} \right] m\alpha \tag{154}$$

where α denotes the fine structure constant. For N electrons ($q = 2$) we obtain the constant 2.316 which is only a minor improvement of the constant 2.52 of [M90]. The best possible value is 0.770.

In view of recent interests in two dimensional quantum systems we give the values for $d = 2$.

$$\begin{aligned} K_2^{cl} &= 4\pi & (155) \\ K_2 &\geq (5.1769)^{-1} \quad K_2^{cl} = 2.4273 \ . & (156) \end{aligned}$$

The problem of stability of matter in magnetic fields was recently solved in [LLS95]. The second application is related to the Navier-Stokes equations on a bounded domain D in $\mathbf{R}^d$. As in [L84] we consider the Hamiltonian $H = \nu\Delta - W(x)$ on $L^2(D)$ with Dirichlet boundary conditions; ν is the kinematic viscosity and the nonnegative potential W is related to the average rate of energy dissipation per unit mass in the flow governed by the Navier Stokes equations. The number N^+ defined by

$$N^+ = \text{ smallest } N \text{ such that } \quad \sum_{j=1}^{N} E_j(V) \geq 0, \tag{157}$$

is related to the number of nonnegative characteristic exponents of the Navier Stokes equations. In the following we sketch a simplified derivation of a bound on N^+ yielding improved constants which is due to [L84]. However, we improve this derivation and obtain a better estimate (see (6.17) and (6.18)).

As in [L84] we use a result by Li and Yau [LY83] on the energy of N fermions in a bounded domain with impenetrable walls

$$\sum_{i=1}^{N} \int_{D^N} |\text{grad}\psi(x_1, x_2, \ldots, x_N)|^2 dx_1 \ldots dx_N > \frac{d}{d+2} K_d^{cl} (N/|D|)^{2/d} N \tag{158}$$

where $|D|$ denotes the volume of D.

Let $(\phi_1, \ldots, \phi_N)$ be a set of eigenfunctions corresponding to the eigenvalues $E_1, \ldots, E_N$. Then

$$\sum_{j=1}^{N} E_j \geq \nu \sum_{i=1}^{N} \int_D |\mathrm{grad}\phi_i|^2 dx - \int_D W\rho dx \tag{159}$$

where $\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2$ is the corresponding one particle density.

Hölder's inequality yields

$$\sum_{j=1}^{N} E_j \geq \nu \sum_{i=1}^{N} \int_D |\mathrm{grad}\ \phi_i|^2 dx - \|W\|_{1+d/2} \|\rho\|_{1+2/d}\ . \tag{160}$$

The following trick improves the result of [L84].

Writing $\sum_{i=1}^{N} \int_D |\mathrm{grad}\phi_i|^2 dx$ as $\left(\sum_{i=1}^{N} \int_D |\mathrm{grad}\ \phi_i|^2 dx\right)^{\frac{d}{d+2}}$ $\left(\sum_{i=1}^{N} \int_D |\mathrm{grad}\ \phi_i|^2 dx\right)^{\frac{d}{d+2}}$ and applying (3.1) and (3.7), respectively, we obtain

$$\sum_{j=1}^{N} E_j \geq \left[\nu \frac{d}{d+2}(K_d) \frac{d}{d+2}(K^{cl}) \frac{2}{d+2} N^{\frac{2}{d}} |D|^{\frac{-4}{d(d+2)}} - \|W\|_{1+d/2}\right] \|\rho\|_{1+2/d}\ . \tag{161}$$

Therefore $\sum_{j=1}^{N} E_j \geq 0$ if the square bracket in the above inequality is nonnegative. Hence N^+ is bounded above by

$$N^+ \leq \nu^{-d/2} A_d |D| \left[\frac{\int_D W(x)^{1+d/2} dx}{|D|}\right]^{\frac{d}{d+2}} \tag{162}$$

where the constant A_d is given by

$$\begin{aligned} (A_d)^{2/d} &= \frac{d+2}{d} (K_d^{cl})^{-2/(d+2)} (K_d)^{-d/(d+2)} \\ &= \left(\frac{d}{d+2} K_d^{cl}\right)^{-1} R(1,d)^{\frac{2}{d+2}}\ . \end{aligned} \tag{163}$$

Notice that (6.18) improves eq. (41) in [L84]. In dimensions $d = 2$ and $d = 3$ we have the following bounds on A_d

$$0.15915 \leq A_2 \leq\ 0.3621 \tag{164}$$

$$0.03633 \leq A_3 \leq 0.09791 \tag{165}$$

where the left hand side is the corresponding classical value.

6.2 No binding results and lower bounds for N-body Hamiltonians

Our first example deals with muonic atoms and is taken from [GGMT76]. We consider an atom with a μ^- and N electrons and a nucleus of charge Z as described by the Schrödinger equation.

We write the Hamiltonian as

$$H = H_{N,Z,m} + h_{Z\mu} \tag{166}$$

where H_N denotes the N-electron Hamiltonians given by

$$H_{N,Z,m} = \sum_{i=1}^{N} \frac{p^2}{2m} - \frac{Z}{|x_i|} + \sum\sum_{i<j} \frac{1}{|x_i - x_j|} \tag{167}$$

and $h_{Z\mu}$ denotes the muonic Hamiltonian

$$h_{Z,\mu} = \frac{p^2}{2\mu} - \frac{Z}{|x|} + \sum_{i=1}^{N} \frac{1}{|x - x_i|} \ . \tag{168}$$

Let $E = \inf \text{spec } (H)$ be the ground state energy of the muonic atom and $E(N,Z) = \inf \text{spec } (H_{N,Z})$ the ground state energy of N electrons in the potential of a nucleus of charge Z.

An upper bound on E can be easily obtained by taking a trial function of the form $\phi(x)\psi(x_1,\dots,x_N)$ with $\phi(x)$ the ground state of $\frac{p^2}{2\mu} - \frac{Z}{|x|}$ and $\psi(x_1,\dots,x_N)$ a N-electron wave function (with a spherically symmetric one particle density) such that $< H_{N,Z-1,m}\psi, \psi > \to E(N, Z-1) < \psi, \psi >$.

Then we find

$$E \le -\frac{Z^2\mu}{2} + E(N, Z-1) \ . \tag{169}$$

The upper bound corresponds to the energy one obtains from the naive assumption that the electrons just feel an effective charge $Z - 1$.

To obtain lower bounds we employ the projection method

$$\begin{aligned} H \ \ \ge \ & \frac{p^2}{2\mu} - \frac{Z}{|x|} + H_{N,Z,m} \\ & + \sum_{i=1}^{N} P(P|x - x_i|P)^{-1}P \end{aligned}$$

where P is the projector onto the ground state of $\frac{p^2}{2\mu} - \frac{Z}{|x|}$. Then

$$P(P|x - x_i|P)^{-1}P = \frac{1}{|x_i|} - \frac{1}{|x_i|}U(Z\mu|x_i|)\frac{1}{|x_i|} - \frac{1}{|x_i|}U(Z\mu|x_i|) \tag{170}$$

where

$$U(r) = \left[1 + \frac{r^2}{1 - e^{-2r}(1 + \frac{r}{2})}\right]^{-1} \ . \tag{171}$$

For any $m_1 > m$ we have the operator inequality

$$\begin{aligned} H &\geq \left[-\frac{Z^2\mu}{2} + H_{N,Z-1,m'}\right] P \\ &+(1-P)\left[-\frac{Z^2\mu}{8} + E(N,Z)\right] \\ &+P\left\{\sum_{i=1}^{N} \frac{P_i^2}{2m_1} - \frac{1}{|x_i|} U(Z\mu|x_i|)\right\} \end{aligned}$$

with $m' = \frac{mm_1}{m_1 - m}$.

According to Theorem 3.1 the bracket $\{\}$ is nonnegative if

$$\begin{aligned} \frac{2m_1}{Z\mu} &\leq \frac{r}{r-1}\left[\frac{(r-1)\Gamma^2(r)}{\Gamma(2r)}\right]^{1/r} \\ &\cdot \left[\int_0^\infty U^r(t) t^{r-1} dt\right]^{-1/r} \end{aligned}$$

for some $r \geq 1$.

The numerical result of [GGMT] for the best choice of r is $r = 1.8242$ giving

$$\frac{2m_1}{Z\mu} \leq 1.2706 \; . \tag{172}$$

Using the scaling properties of Coulomb Hamiltonians we find

$$\begin{aligned} E &\geq \left\{(1 - \frac{1.574m}{Z\mu})^{-1} E(N, Z-1) - \frac{Z^2\mu}{2} \right. \\ &\left. - \frac{Z^2\mu}{8} + E(N,Z)\right\} \end{aligned} \tag{173}$$

provided $\frac{Z\mu}{m} \geq 1.574$.

Again from scaling we infer that

$$E(N,Z) > (\frac{Z}{Z-1})^2 \cdot E(N, Z-1) \; . \tag{174}$$

Therefore

$$E \geq -\frac{Z^2\mu}{2} + (1 - \frac{1.574m}{Z\mu})^{-1} E(N, Z-1) \tag{175}$$

if

$$|E(N,Z-1)| \left[\left(\frac{Z}{Z-1}\right)^2 - \left(1 - \frac{1.574m}{Z\mu}\right)^{-1}\right] < \frac{3}{8} Z^2\mu \tag{176}$$

which is satisfied for all $\mu \geq m$.

This argument also shows that a system composed of a proton, an electron and a negative particle of mass $\mu \geq 1.574m$ is not bounded.

Our second example concerns the N-body Hamiltonian with a general pair interaction V

$$H_N = \sum_{i=1}^{N} \frac{1}{2m} p_i^2 + \sum\sum_{i<j} V(x_i - x_j) \ . \tag{177}$$

First of all we derive a lower bound on H_N in terms of a suitable two-body Hamiltonian. We follow the presentation of [BMR90a,b] where the best general lower bound so far is obtained.

The starting point is the Lagrange identity

$$\sum_{i=1}^{N} p_i^2 = \frac{1}{N}(\sum_{i=1}^{N} p_i)^2 + \frac{1}{N}\sum\sum_{i<j}(p_i - p_j)^2 \ . \tag{178}$$

Since the conjugate momentum to the relative distance $x_i - x_j$ is $\pi_{ij} = \frac{1}{2}(p_i - p_j)$ we have

$$\begin{aligned} H_N &\geq \sum\sum_{i<j} \frac{4\pi_{ij}^2}{2Nm} + V(x_i - x_j) \\ &\geq \frac{N(N-1)}{2} E_2(\frac{N}{2}m, V) \end{aligned} \tag{179}$$

where

$$E_2(\frac{N}{2}m, V) = \inf \operatorname{spec} \left(\frac{p^2}{\frac{N}{2}m} + V(x) \right) . \tag{180}$$

Therefore we may apply the conditions for the nonexistence of bound states obtained in Section 3 also to H_N:
If $\frac{p^2}{\frac{N}{2}m} + V(x)$ has no bound state then H_N has no bound state, too.

Acknowledgements

We are grateful to E. Lieb for many useful suggestions during the preparation of this work. We also thank K. Chadan, H. Grosse and A. Martin for stimulating discussions and comments.

J. Stubbe acknowledges the support of the Theoretical Physics Division at CERN and the support of the Fonds National Suisse.

Appendix

Some numerical constants in phase space bounds and related inequalities

In the following we list some explicit constants in phase space bounds and related inequalities reviewed in this article. We give the best numerical values in the literature so far and its improvements obtained in the present review, which are relevant for the applications.

1. Bounds on the number of bound states for $d = 3$

For the bound

$$N_0(V) \leq L_{0,3} \int_{\mathbf{R}^3} |V_-|^{3/2} dx \tag{A.1}$$

there are the following bounds on $L_{0,3}$.

$$L_{0,3} \leq 0.116 \tag{A.2}$$

(compare also (5.13)) which is due to [L76a, L80].

The following lower bound on $L_{0,3}$ has been found in [GGMT76] by considering the one bound state problem (see Theorem 3.1.),

$$L_{0,3} \geq 0.078 \; . \tag{A.3}$$

Equivalently, we have the following bounds on $R(0,3)$

$$4.6189 \leq R(0,3) \leq 6.869 \; . \tag{A.4}$$

2. Bounds on $\mathbf{R}(\mathbf{1}, \mathbf{d})$

So far there have been the following bounds on $R(1,d)$ [L84] (actually, in [L84] the bounds are given for $L_{1,d}$)

$$\begin{aligned} R(1,1) &\leq 6.28319 (= 2\pi) \\ R(1,2) &\leq 6.03388 \\ R(1,3) &\leq 5.96677 \; . \end{aligned} \tag{A.5}$$

Our results are given in the table in Chapter 5.4. More precisely

$$\begin{aligned} R(1,1) &\leq 5.81029 \\ R(1,2) &\leq 5.17690 \\ R(1,3) &\leq 5.21803 \; . \end{aligned} \tag{A.6}$$

Due to these improvements the constants discussed below will be improved, too.

3. The constant in Sobolev's inequality for fermions

The Sobolev inequality for fermions (without spin) reads (see also (6.21)

$$\begin{aligned} (A.7) \qquad \sum_{i=1}^{N} \int_{\mathbf{R}^d} & |\nabla_{x_i}\psi(x_1,\ldots,x_N)|^2 dx_1,\ldots,dx_N \\ & \geq \frac{d}{d+2} K_d \int_{\mathbf{R}^d} \rho_\psi(x)^{1+2/d} dx \ . \end{aligned}$$

We stress that our K_d differs from the one in [L84]. In our notation the estimates presented in [L84] read

$$(A.8) \qquad \begin{aligned} K_1 &\geq 0.2500 \\ K_2 &\geq 2.0813 \\ K_3 &\geq 4.6180 \ . \end{aligned}$$

By means of (6.3) where K_d is related to $R(1,d)$ we find

$$(A.9) \qquad \begin{aligned} K_1 &\geq 0.2923 \\ K_2 &\geq 2.4274 \\ K_3 &\geq 5.0500 \ . \end{aligned}$$

4. Constants in the bounds on the number of positive Lyapunov exponents for the Navier Stokes equations

$$N^+ \leq \nu^{-\frac{d}{2}} A_d |D| \left[\frac{\int_D W(x)^{1+\frac{d}{2}} dx}{|D|} \right]^{\frac{d}{d+2}}$$

(see (6.17)).

Equation (6.18) which relates A_d to K_d already implies that the estimates improve those presented in [L84] independently of the quality of the bounds on K_d. We have the following estimates

$$(A.10) \qquad \begin{aligned} A_1 &\leq 0.99116 \\ A_2 &\leq 0.36212 \\ A_3 &\leq 0.09791 \ . \end{aligned}$$

References

[AHS78] Avron, J., Herbst, J., and Simon, B., Schrödinger operators with magnetic fields, I, Duke Math. J. 45, 847–883 (1978)

[AL78] Aizenmann, M., and Lieb, E., On semiclassical bounds for eigenvalues of Schroedinger operators, Phys. Lett. 66A, 427-429 (1978)

[BB] Blanchard, Ph., and Brüning, E., Variational Methods in Mathematical Physics. A Unified Approach, Texts and Monographs in Physics, Springer (1992)

[Be75] Beckner, W., Inequalities in Fourier analysis, Ann. Math. 102, 159-182 (1975)

[Bir61] Birman, M., The spectrum of singular boundary problems, Math. Sb. 55, 124–174 (1961); Amer. Math. Soc. Trans. 53, 23–80 (1966)

[Bl30] Bliss, G.A., An integral inequality, J. London Math. Soc. 5, 40-46 (1930)

[BL76] Brascamp, H.J., and Lieb, E.H., Best constants in Young's inequality, its converse, and its generalization to more than three functions, Adv. in Math. 20, 151-173 (1976)

[BLL74] Brascamp, H.J., Lieb, E.H., and Luttinger, J.M., A general rearrangement inequality for multiple integrals, J. Funct. Anal. 17, 227-237 (1974)

[BMR90a] Basdevant, J.L., Martin, A., and Richard, J.M., Improved bounds on many-body Hamiltonian I. Selfgravitating bosons, Nucl. Phys. B 343, 60–68 (1990).

[BMR90b] Basdevant, J.L., Martin, A., and Richard, J.M., Improved bounds on many-body Hamiltonian II. Baryons from mesons in the quark model, Nucl. Phys. B 343, 69–85 (1990).

[BS93] Blanchard, Ph., and Stubbe, J., Phase Space Bounds for Schrödinger Hamiltonians and Applications, BiBoS-preprint No. 596-11 (1993), unpublished

[BSR87] Blanchard, Ph., Stubbe, J., and Rezende, J., New Estimates on the Number of Bound States of Schroedinger Operators, Lett. Math. Phys. 14, 215-225 (1987)

[Ca65] Calogero, F., Upper and lower limits for the number of bound states in a given central potential, Comm. Math. Phys. 1, 80–88 (1965)

[CL90] Carlen, E. and Loss, M., Extremals of functionals with competing symmetries, J. Funct. Anal. 99, 437–456 (1990)

[Co85] Conlon, J., A new proof of the Cwickel-Lieb-Rosenbljum bound, Rocky Mountain J. Math. 15, 117–122 (1985)

[CH] Courant, R., Hilbert, D., Methods of mathematical physics, Vol. 1, Interscience,N.Y., 1953

[CK95] Chadan, K., Koboyaski, R., Generalisation de la borne de Calogero Cohn sur le nombre des états liès, C.R.Ac. Sci. Paris, II b, 320, 339–345 (1995)

[CKMS95] Chadan, K., Koboyaski, R., Martin, A., Stubbe, J., Generalization of the Calogero Cohn bound on the number of bound states, LPTHE Orsay, 95/32 Preprint

[CMS95a] Chadan, K., Martin, A. and Stubbe, J., The Calogero Bound for Nonzero Angular Momentum, J. Math. Phys. 36, 1616–1624 (1995)

[CMS95b] Chadan, K., Martin, A. and Stubbe, J., New bounds on the number of bound states for Schrödinger operators, Lett. Math. Phys., to appear

[Cw77] Cwickel, M., Weak type estimates for singular values and the number of bound states of Schroedinger operators, Ann. Math. 106, 93-100 (1977)

[D83] Daubechies, J., An uncertainty principle for Fermions with generalized kinetic Energy, Comm. Math. Phys. 90, 511–520 (1983)

[DL67] Dyson, F. and Lenard, A., Stability of Matter I, J. Math. Phys. 8, 423–434 (1967)

[DL68] Dyson, F. and Lenard, A., Stability of Matter II, J. Math. Phys. 9, 698–711

[Er95] Erdösz, L., Magnetic Lieb-Thirring inequalities, to appear in Comm. Math. Physics

[Er94] Erdösz, L., Magnetic Lieb-Thirring inequalities and estimates on stochastic oscillating integrals, PH. Theory, Princeton University 1994

[G80] Grosse, H., Quasiclassical estimates on moments of energy levels, Acta Phys. Austr. 52, 89-105 (1980)

[GGMT76] Glaser, V., Grosse, H., Martin, A., and Thirring, W., A family of optimal conditions for the absence of bound states in a potential, in E. Lieb, B. Simon, and A.S. Wightman (eds.), Studies in Mathematical Physics, Princeton University Press, 1976, pp. 169-194

[GGM78] Glaser, V. Grosse, H., and Martin, A., Bounds on the number of eigenvalues of the Schroedinger operator, Comm. Math. Phys. 59, 197-212 (1978)

[HL28] Hardy, G.H., and Littlewood, J.E., Some properties of fractional integrals, Math. Zeitschr. 27, 565–609 (1928)

[HL30] Hardy, G.H., and Littlewood, J.E., On certain inequalities connected with the calculus of variations, J. London Math. Soc. 5, 34–39 (1930)

[Hu61] Hunziker, W., Regularitätseigenschaften der Streuamplitude im Fall der Potentialstreuung, Helv. Phys. Acta 34, 593–620 (1961)

[JP51] Jost, R., and Pais, A., On the scattering of a particle by a static potential, Phys. Rev. 82, 840-850 (1951)

[Kl82] Klaus, M., Some applications of the Birman-Schwinger principle, Helv. Phys. Acta 55, 49–68 (1982)

[L76a] Lieb, E., Bounds on the eigenvalues of the Laplace and Schrödinger operators, Bull. AMS 82, 751-753 (1976)

[L76b] Lieb, E., The stability of matter, Rev. Mod. Phys. 48, 553–569 (1976)

[L80] Lieb, E., The number of bound states of one-body Schrödinger operators and the Weyl problem, Proc. Am. Math. Soc. Symp. in Pure Math., Ossermann, R., Weinstein, A. (eds.), Vol. 36, pp. 241-252 (1980)

[L83] Lieb, E., Sharp constants in Hardy-Littlewood-Sobolev and related inequalities, Ann. Math. 118, 349-374 (1983)

[L84] Lieb, E., On characteristic exponents in turbulence, Commun. Math. Phys. 92, 473-480 (1984)

[L89] Lieb, E., Kinetic Energy Bounds and their applications to the stability of matter, in: Schrödinger Operators, Proceedings Sönderborg Denmark 1988, H. Holden and A. Jensen, eds., Springer Lecture Notes in Physics 345, 371–382 (1989)

[L90a] Lieb, E., The stability of matter: From Atoms to Stars, 1989 Gibbs Lectures, Bull. Amer. Math. Soc. 22, 1–49 (1990)

[L90b] Lieb, E., Gaussian Kernels have only Gaussian Maximizers, Invent. Math. 102,. 179–208 (1990)

[LLS95] Lieb, E., Loss, M., and Solovej, J. Stability of matter in magnetic fields, Phys. Rev. Lett. to appear

[LSY94] Lieb, E., Solovej, J.P., and Yngvason, J., Asymptotic of Heavy Atoms in High Magnetic Fields: II. Semiclassical Regions, Comm. Math. Phys. 161, 77–124 (1994)

[LSY95] Lieb, E., Solovej, J.P., and Yngvason, J., The Ground State of Large Quantum Dots in Magnetic Fields, Phys. Rev. B51 10646–10665 (1995)

[LT76] Lieb, E., and Thirring, W., Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, in: Studies in mathematical physics: essays in honor of Valentine Bargmann, Lieb, E., Simon, B., Wightman, A. (eds.), pp. 269-303, Princeton, NJ: Princeton University Press 1976

[LT75] Lieb, E., and Thirring, W., Bound for the kinetic energy of fermions which proves the stability of matter. Phys. Rev. Lett. 35, 687-689 (1975); 35, 1116 (1975) (Erratum)

[LY83] Li, P., Yau, S.-T., On the Schrödinger equation and the eigenvalue problem, Commun. Math. Phys. 88, 309-318 (1983)

[M75] Martin, A., A bound on the total number of bound states in a potential, CERN-Preprint TH 2085, 17, October 1975, unpublished

[M72] Martin, A., Bound States in the Strong Coupling Limit, Helv. Phys. Acta 45, 140-148 (1972)

[M90] Martin, A., New Results on the Moments of the Eigenvalues of the Schrödinger Hamiltonian and Applications, Commun. Math. Phys. 129, 161-168 (1990)

[MS77] Martin, A., and Sabatier, P., Impedance, zero energy wave function, and bound states, J. Math. Phys. 18, 1623–1626 (1977)

[O] Okikiolu, G.O., Aspects of the Theory of bounded integral operators in L^p-spaces, Academic Press, N.Y. 1971

[RS I] Reed, M., and Simon, B., Methods of Modern Mathematical Physics I, Academic Press, New York, 1975

[RS II] Reed, M., and Simon, B., Methods of Modern Mathematical Physics II, Fourier Analysis, Selfadjointness, Academic Press, New York, 1975

[RS III] Reed, M., and Simon, B., Methods of Modern Mathematical Physics III, Scattering Theory, Academic Press, New York, 1978

[RS IV] Reed, M., and Simon, B., Methods of Modern Mathematical Physics IV, Analysis of Operators, Academic Press, New York 1978

[Ro71] Rosen, G., Minimum value for c in the Sovolev inequality $\| \varphi \|_6 \leq c \| \nabla\varphi \|_2$, SIAM J. Appl. Math. 21, 30–32 (1971)

[Ro72] Rosenbljum, G.V., Distribution of the discrete spectrum of singular operators, Dokl. Akad. Nauk. SSR, 202, 1012-1015 (1972)

[Ru82] Ruelle, D., Large volume limit of the distribution of characteristic exponents in turbulence, Commun. Math. Phys. 87, 287-302 (1982)

[Schw61] Schwinger, J., On the bound states of a given potential, Proc. Nat. Acad. Sci. U.S.A. 47, 122–129 (1961)

[Si] Simon, B., Functional Integration and Quantum Physics, Academic Press, New York, 1979

[Si69] Simon, B., On the growth of the number of bound states which increase in potential strength, J. Math. Phys. 10, 1123-1126 (1969)

[Si71] Simon, B., Quantum Mechanics for Hamiltonians Defined as Quadratic Forms, Princeton University Press (1971)

[Si76] Simon, B., Weak trace ideals and the number of bound states of Schrödinger operators, Trans. Am. Math. Soc. 224, 367–380 (1976)

[So63] Sobolev, S.L., On a theorem of functional analysis, Math.Sb. 4, 471–479 (1938); A.M.S. Transl. Ser. 2, 34, 39–68 (1963)

[St90] Stubbe, J., Bounds on the Number of Bound States with critical Decay at Infinity, J. Math. Phys. 31, 1177-1180 (1990)

[Ta76] Talenti, G., Best constant in Sobolev inequality, Ann. di Matem. Pura et Appl. 110, 353–372, (1976)

[T] Temam, R., Infinite-dimensional dynamical systems in mechanics and physics, Springer, Berlin-Heidelberg-New York (1988)

[Weid95] Weidl, T., On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq \frac{1}{2}$, submitted to Comm. Math. Phys.

[W83] Weinstein, M., Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87, 567-576 (1983)

[WM50] Wet, J.S. de, and Mandl, R., On the asymptotic distribution of eigenvalues, Proc. Roy. Soc. Lond. A 200, 572–580 (1950)