July 5, 1995

Quantum Dynamics and Decompositions of Singular Continuous Spectra

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Abstract. We study relations between quantum dynamics and spectral properties, concentrating on spectral decompositions which arise from decomposing measures with respect to dimensional Hausdorff measures.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space, $H : \mathcal{H} \to \mathcal{H}$ a self adjoint operator, and $\psi \in \mathcal{H}$ (with $\|\psi\| = 1$). The spectral measure μ_ψ of ψ (and H) is uniquely defined by [24]:

$$\langle \psi \, , \, f(H)\psi \rangle \; = \; \int_{\sigma(H)} f(x) \, d\mu_\psi(x) \; , \tag{1.1}$$

for any measurable (Borel) function f. The time evolution of the state ψ, in the Schrödinger picture of quantum mechanics, is given by

$$\psi(t) \; = \; e^{-iHt}\psi \; . \tag{1.2}$$

The relations between various properties of the spectral measure μ_ψ (with an emphasis on "fractal" properties) and the nature of the time evolution have been the subject of several recent papers [7,13,15–18,20,22,33,36,39]. Our purpose in this paper is twofold: First, we use a theory, due to Rogers and Taylor [28,29], of decomposing singular continuous measures with respect to Hausdorff measures to introduce a corresponding family of spectral decompositions of $\mathcal{H}$. These decompositions, which always fall within the singular continuous subspace, extend the usual spectral decomposition into pure point, singular continuous, and absolutely continuous subspaces, and reflect the possibly rich nature of singular continuous spectra. Second, we show that the above decompositions provide a natural framework for the discussion and formulation of some recent results which relate quantum dynamics to spectral properties. Moreover, the decomposition theory allows us to extend some of these results, and, in particular, we get a strengthened version of the Guarneri-Combes theorem [7,15].

While most of our discussion of quantum dynamics applies to any self-adjoint Hamiltonian, and is therefore rather general; the primary example we have in mind is that of a single electron in an external potential. That is, $H = -\Delta + V$ on $\ell^2(Z^d)$ or $L^2(R^d)$. An initial state $\psi(0) = \psi$, which is localized in space, will usually spread under the time evolution (1.2). It is the *rate* of this spreading and its relations with properties of the spectral measure μ_ψ, which is of major interest to us in this paper. Unlike propagation of classical point particles, the time evolution of wave packets, which describe quantum mechanical particles, is ,in general, a complex phenomena, involving infinitely many degrees of freedom. Various quantities can be introduced for characterizing this time evolution, and their behavior need not be coherent. A very elementary quantity, in this context, is the *survival probability*, which gives the probability of finding the particle at time t in its initial state ψ. It is given by:

$$|\langle \psi(0)\,,\, \psi(t)\rangle|^2 \;=\; |\langle \psi\,,\, e^{-iHt}\psi\rangle|^2 \;=\; \left| \int_{\sigma(H)} e^{-ixt}\, d\mu_\psi(x) \right|^2 \;=\; |\hat{\mu}_\psi(t)|^2 \,. \tag{1.3}$$

As is seen from (1.3), the survival probability coincides with the squared absolute value of the Fourier transform of the spectral measure μ_ψ. Another family of interesting quantities comes from looking at the expectations for various operators. Namely,

$$\langle A\rangle \;\equiv\; \langle A\rangle(t) \;\equiv\; \langle \psi(t)\,,\, A\psi(t)\rangle \,, \tag{1.4}$$

where A is a (usually self-adjoint) operator. Some special cases of interest are compact A's, such as finite dimensional projections, and certain unbounded A's, such as moments of the position operator in $\ell^2(Z^d)$:

$$|X|^m \;\equiv\; \sum_{n\in Z^d} |n|^m \langle \delta_n\,,\, \cdot\rangle \delta_n \,, \tag{1.5}$$

where $\delta_n(k) = \delta_{nk}$ and $m > 0$. Our discussion will mainly involve *time averaged* quantities, whose behavior seems to be naturally related to *continuity* properties of spectral measures. For any function f of time, we denote its Cesàro time average by:

$$\langle f\rangle_T \;\equiv\; \langle f(t)\rangle_T \;\equiv\; \frac{1}{T}\int_0^T f(t)\, dt \,. \tag{1.6}$$

We also remark that by "measure," we always mean a positive measure. We shall not consider any other kind of measures in this paper.

The rest of this paper is organized as follows: In Section 2, we give a brief introduction to spectral decompositions, and describe some central results which relate spectral decompositions to dynamics. This includes some classical results, as well as some of the recent results which motivated our study. In Section 3, we derive some central results which characterize the dynamics for vectors with uniformly Hölder continuous spectral measures. In Section 4, we describe parts of the rich theory, due to to Rogers and Taylor [28,29], of decomposing Borel measures with respect to Hausdorff measures. In particular, for each $\alpha \in [0, 1]$, a finite Borel measure μ can be decomposed as $d\mu = d\mu_{\alpha s} + d\mu_{\alpha c}$, where $\mu_{\alpha s}$ is singular with respect to the α-dimensional Hausdorff measure (namely, supported on a set with zero α-dimensional Hausdorff measure), and $\mu_{\alpha c}$ is continuous with respect to the α-dimensional Hausdorff measure (namely, it does not give weight to sets with zero α-dimensional Hausdorff measure). Measures which are continuous with respect to Hausdorff measures must be limits of uniformly Hölder continuous measures, and this link is of major importance in this paper. In Section 5, we use the decomposition theory of Section 4 to introduce corresponding decompositions of $\mathcal{H}$ into closed, invariant, mutually orthogonal subspaces. In Section 6, we combine the decomposition theory with the results of Section 3 to obtain a strengthened version of the Guarneri-Combes theorem, which is a lower bound on the growth rate of $\langle\langle |X|^m \rangle\rangle_T$. The main point in this strengthening is that the growth rate of $\langle\langle |X|^m \rangle\rangle_T$ is shown to be governed by the "most continuous" component of ψ, and unaffected by the existence of more singular components. In Section 7, we study an interesting example of the Almost Mathieu operator at large coupling and certain irrational frequencies which are extremely well approximated by rationals. The spectrum in this case is long known [3] to be purely singular continuous, and we show that it is, in fact, purely zero-dimensional, in the sense that all spectral measures are supported on sets of zero Hausdorff dimension. Nevertheless, we show that the growth rate of $\langle\langle |X|^2 \rangle\rangle_T$ can be arbitrarily close to ballistic (namely, to T^2), for some time scales. This indicates that the Guarneri-Combes theorem is a strictly one-sided inequality, as far as singularity or continuity with respect to Hausdorff measures is concerned. In Section 8, we discuss ergodic Schrödinger operators, and show that the spectra, arising from the various spectral decompositions introduced earlier, are the same for almost all the realizations of the potential. This generalizes known results about the decomposition into pure point, singular continuous, and absolutely continuous spectra. Finally, in Section 9, we discuss some results beyond this paper and some open problems.

2. Spectral decompositions and quantum dynamics: A brief introduction

By Lebesgue's decomposition theorem, any finite Borel measure on R can be decomposed into absolutely continuous, singular continuous, and pure point parts. Namely,

$$d\mu \; = \; d\mu_{ac} + d\mu_{sc} + d\mu_{pp} \; . \tag{2.1}$$

Absolutely continuous here, means with respect to Lebesgue measure, such that $d\mu_{ac}(x) = f(x)\,dx$ for some measurable function f. The pure point part, $d\mu_{pp}$, is a countable sum of atomic (Dirac) measures. The singular continuous part, $d\mu_{sc}$, is supported on some set of zero Lebesgue measure, and does not give weight to any individual points ($\mu(\{x\}) = 0\,,\ \forall\, x \in R$). In classical spectral theory, one uses this measure decomposition to establish a corresponding decomposition of the Hilbert space [24]:

$$\mathcal{H} \;=\; \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp} \;, \tag{2.2}$$

where:

$$\begin{aligned} \mathcal{H}_{ac} &= \{\psi \,|\, d\mu_\psi \text{ is purely absolutely continuous}\} \\ \mathcal{H}_{sc} &= \{\psi \,|\, d\mu_\psi \text{ is purely singular continuous}\} \\ \mathcal{H}_{pp} &= \{\psi \,|\, d\mu_\psi \text{ is purely point}\} \;. \end{aligned} \tag{2.3}$$

$\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$, and $\mathcal{H}_{pp}$ are closed (in norm), mutually orthogonal subspaces, which are *invariant* under H. The corresponding spectra σ_{ac}, σ_{sc}, and σ_{pp} are defined as the spectra of H restricted to the corresponding subspaces. σ, the spectrum of H, obeys

$$\sigma \;=\; \sigma_{ac} \cup \sigma_{sc} \cup \sigma_{pp} \;. \tag{2.4}$$

The continuous subspace is $\mathcal{H}_c \equiv \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$, and the singular subspace is $\mathcal{H}_s \equiv \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}$. The corresponding spectra obey $\sigma_c = \sigma_{ac} \cup \sigma_{sc}$, $\sigma_s = \sigma_{sc} \cup \sigma_{pp}$. Note that, while the various subspaces are mutually orthogonal, the corresponding spectra need not be disjoint.

There is also another standard decomposition of the spectrum into a discrete spectrum σ_{disc}, and an essential spectrum σ_{ess}. σ_{disc} is the union of all isolated eigenvalues of finite multiplicity. σ_{ess} contains the rest of the spectrum, namely, the continuous spectrum, eigenvalues of infinite multiplicity, and limit points of eigenvalues. One has $\sigma_{disc} \cup \sigma_{ess} = \sigma$ and $\sigma_{disc} \cap \sigma_{ess} = \emptyset$. (*Remark.* Some authors, e.g. [24], use slightly different conventions, such as calling our $\mathcal{H}_{sc}$, $\mathcal{H}_{sing}$, and defining σ_{pp} as the set of eigenvalues. Our definition of σ_{pp} coincides with the closure of the set of eigenvalues.) In the words of Reed and Simon [26]: "Spectral analysis of an operator A concentrates on identifying the five sets $\sigma_{ess}(A)$, $\sigma_{disc}(A)$, $\sigma_{ac}(A)$, $\sigma_{sc}(A)$, $\sigma_{pp}(A)$." Nevertheless, a further decomposition of the spectrum was suggested by Avron and Simon [2], who decomposed $\mathcal{H}$ into a transient subspace $\mathcal{H}_{tac}$ and a recurrent subspace $\mathcal{H}_{rec} \equiv \mathcal{H}_{tac}^{\perp}$. $\mathcal{H}_{tac}$ is a subspace of $\mathcal{H}_{ac}$, which, in some sense, extracts its smoothest component. It is given by $\mathcal{H}_{tac} = P_{ei}\mathcal{H}_{ac}$, where P_{ei} is the spectral projection on the essential interior of the essential support of the absolutely continuous part of the spectral measure class of H. The spectra $\sigma_{tac} \cup \sigma_{rec} = \sigma$ are defined accordingly, and $\sigma_{tac} \subseteq \sigma_{ac}$. It should be pointed out, however, that this decomposition behaves badly under taking direct sums, and it is therefore somewhat less fundamental than the previous decompositions we have discussed.

There are some classical results relating the dynamics to the above spectral decompositions (where by "classical" we mean, roughly, that they are over ten years old and have appeared in books). The Riemann-Lebesgue lemma, dating back to 1903 (see [25]), states:

Theorem 2.1. (Riemann-Lebesgue lemma) *If μ is a finite absolutely continuous measure, then its Fourier transform $\hat{\mu}(t)$ obeys $\hat{\mu}(t) \to 0$ as $t \to \infty$.*

So, for any $\psi \in \mathcal{H}_{ac}$ the survival probability vanishes as $t \to \infty$. Wiener's theorem, dating back to 1935 (see [26]), states:

Theorem 2.2. (Wiener's theorem) $\lim_{T \to \infty} \langle |\hat{\mu}(t)|^2 \rangle_T = \sum_{x \in R} |\mu(\{x\})|^2$.

In particular, it implies:

Corollary 2.2.1. $\lim_{T \to \infty} \langle |\hat{\mu}_\psi(t)|^2 \rangle_T = 0$ *if and only if $P_{pp}\psi = 0$, where P_{pp} is the orthogonal projection on $\mathcal{H}_{pp}$.*

It should be pointed out that Theorem 2.1 does not have an inverse, namely, the Fourier transform of singular continuous measures (which by Theorem 2.2 vanishes on the average) may or may not tend to zero in the limit $t \to \infty$. As shown by Amrein [1], the set $\mathcal{H}_w$ defined by

$$\mathcal{H}_w = \{\psi \mid \hat{\mu}_\psi(t) \to 0 \text{ as } t \to \infty\} \tag{2.4}$$

is a closed invariant subspace, obeying $\mathcal{H}_{ac} \subseteq \mathcal{H}_w \subseteq \mathcal{H}_c$. It is an open question whether or not $\mathcal{H}_w$ can be obtained from a spectral decomposition, namely, from the spectral projection on some definite subset of σ. A more modern result, from the late 70's, is the celebrated RAGE theorem, named after Ruelle, Amrein, Georgescu, and Enss (see [26]):

Theorem 2.3. (RAGE theorem) $\lim_{T \to \infty} \langle\langle A \rangle\rangle_T = 0$ *for any compact operator A if and only if μ_ψ is purely continuous.*

Theorem 2.3 has the corollary:

Corollary 2.3.1. *Let $\mathcal{H} = \ell^2(Z^d)$. If $P_c\psi \neq 0$, where P_c is the projection on $\mathcal{H}_c$, then, for any positive m,* $\lim_{T\to\infty} \langle\langle |X|^m \rangle\rangle_T = \infty$.

Proof. Let $\psi_1 = P_c\psi$, $\psi_2 = (1-P_c)\psi$, and denote $\psi_j(t) \equiv e^{-iHt}\psi_j$, for $j = 1, 2$. Since ψ_1 and ψ_2 belong to mutually orthogonal (invariant) subspaces, and since e^{-iHt} is unitary, $\psi_1(t)$ and $\psi_2(t)$ are orthogonal for any t, $\|\psi_1(t)\|^2 = \|\psi_1\|^2$, $\|\psi_2(t)\|^2 = \|\psi_2\|^2$, and $\|\psi_1\|^2 + \|\psi_2\|^2 = \|\psi\|^2 = 1$. Let P_N denote the projection on a sphere of radius N, namely,

$$P_N \equiv \sum_{|n|\leq N} \langle \delta_n\,,\,\cdot\rangle \delta_n\ . \tag{2.5}$$

We have $\|P_N\psi_1(t)\|^2 = \langle \psi_1(t),\, P_N\psi_1(t)\rangle$, and thus, by Theorem 2.3, $\langle \|P_N\psi_1(t)\|^2\rangle_T \to 0$ as $T\to\infty$. Since $\langle \|P_N\psi(t)\|^2\rangle_T \leq \langle (\|P_N\psi_1(t)\| + \|P_N\psi_2(t)\|)^2\rangle_T$, this implies

$$\limsup_{T\to\infty} \langle \|P_N\psi(t)\|^2\rangle_T \leq \limsup_{T\to\infty} \langle \|P_N\psi_2(t)\|^2\rangle_T \leq \|\psi_2\|^2\ . \tag{2.6}$$

Since also $1 = \langle \|\psi(t)\|^2\rangle_T = \langle \|P_N\psi(t)\|^2\rangle_T + \langle \|(1-P_N)\psi(t)\|^2\rangle_T$, we obtain

$$\liminf_{T\to\infty} \langle \|(1-P_N)\psi(t)\|^2\rangle_T \geq 1 - \|\psi_2\|^2 = \|\psi_1\|^2\ . \tag{2.7}$$

This implies

$$\liminf_{T\to\infty} \langle\langle |X|^m\rangle\rangle_T \geq \|\psi_1\|^2 N^m\ , \tag{2.8}$$

and since N is arbitrary we obtain

$$\lim_{T\to\infty} \langle\langle |X|^m\rangle\rangle_T = \infty\ . \quad \bigcirc \tag{2.9}$$

A slightly different type of results characterize certain subspaces as *closures* of sets of vectors which have certain dynamical characteristics. A result going back at least to Kato (see [26]) states:

Theorem 2.4. $\mathcal{H}_{ac} = \overline{\{\psi \mid \hat{\mu}_\psi(t) \in L^2\}}$.

Similarly, Avron and Simon [2] have shown that $\mathcal{H}_{tac} = \overline{\{\psi \mid \hat{\mu}_\psi(t) \in L^1\}}$.

More recently, several additional results in this spirit have been published. Simon, in a 1990 paper [33], has shown that, for Schrödinger operators, pure point spectrum implies the absence of ballistic motion, namely, for $H = -\Delta + V$ on $\ell^2(Z^d)$ which has pure point spectrum, and for any initially localized state ψ (ψ in the domain of X suffices), one has $\lim_{t\to\infty} t^{-2}\langle |X|^2\rangle(t) = 0$. His result also extends to continuous Schrödinger operators on $L^2(R^d)$ with some mild restrictions on V and ψ. Another set of results, which gives the main motivation to our current paper, involves relations between power-law rates associated to some dynamical quantities and what appears to be "fractal" (or dimension related) spectral properties. In order to describe some of these results, let us start with a definition:

Definition 2.1. *Let μ be a Borel measure on R, $\alpha \in [0, 1]$, and let $|\cdot|$ denote Lebesgue measure.*

(i) *We say that μ is uniformly α-Hölder continuous (denoted* UαH*) if there exists a constant C such that for every interval I with $|I| < 1$, $\mu(I) < C|I|^\alpha$.*

(ii) *We say that μ is uniformly strongly α-Hölder continuous (denoted* USαH*) if for any $\epsilon > 0$ there exists $\delta > 0$ such that for every interval I with $|I| < \delta$, $\mu(I) < \epsilon|I|^\alpha$.*

Strichartz, in a 1990 paper [36], has proven the following:

Theorem 2.5. *Let μ be a finite* UαH *measure, and for each $f \in L^2(R, d\mu)$ denote*

$$\widehat{f\mu} \equiv \widehat{f\mu}(t) \equiv \int e^{-ixt} f(x)\, d\mu(x) ,$$

then there exists a constant C, depending only on μ, such that for any $f \in L^2(R, d\mu)$ and $T > 0$

$$\langle |\widehat{f\mu}|^2\rangle_T < C\|f\|^2 T^{-\alpha} ,$$

where $\|f\|$ is the L^2 norm of f.

Remark. Strichartz's result is actually more general, involving σ-finite measures on R^d.

The special case $f = 1$ in Theorem 2.5 yields:

Corollary 2.5.1. *If μ is a finite* UαH *measure, then there exists a constant C such that, for any $T > 0$, $\langle |\hat{\mu}|^2\rangle_T < CT^{-\alpha}$.*

Guarneri, in a 1989 paper [15], used a weak version of Theorem 2.5 to show that, for any self-adjoint H on $\ell^2(Z^d)$, if μ_ψ is UαH, then $\langle\langle |X|^2\rangle\rangle_T > CT^{2\alpha/d}/\ln^2 T$, where C is a constant depending only on ψ. In 1993, Combes [7] used Theorem 2.5, as proven by Strichartz, to slightly improve this estimate and show:

Theorem 2.6. (Guarneri-Combes theorem) *If H is self-adjoint on $\ell^2(Z^d)$ and μ_ψ is* UαH, *then, for each $m > 0$, there exists a constant $C_{\psi,m}$, depending on ψ and m, such that for every $T > 0$*

$$\langle\langle |X|^m\rangle\rangle_T > C_{\psi,m}T^{m\alpha/d} .$$

Remarks.

(i) The formulations of Theorem 2.6 in the various papers by Guarneri [15,16] and by Combes [7] vary between considering only the second moment $m = 2$ and considering more general monotonely increasing functions of $|X|$. Our formulation in terms of any positive moment is a somewhat arbitrary compromise between generality and transparency. The full nature of the underlying estimates should become clear from our discussion in Sections 3 and 5.

(ii) Combes has also proven a version of Theorem 2.6 for continuous Schrödinger operators on $L^2(R^d)$ with potentials bounded from below. The requirement of ψ, in this case, is that it lies in the domain of e^H and that $\mu_{e^H\psi}$ is UαH.

A somewhat different variant of Theorem 2.6 has been developed by Guarneri in a 1993 paper [16]. It removes the requirement for uniform α-Hölder continuity, and replaces it by a requirement for a certain α scaling behavior, that only needs to hold for a.e. x with respect to μ_ψ. Another related result was introduced in a 1992 paper [22] by Ketzmerick, Petschel, and Geisel, who gave a formal calculation showing that if $\langle|\hat{\mu}(t)|^2\rangle_T \sim T^{-\alpha}$, then α is the correlation dimension of μ. A rigorous version of their result has been obtained by Holschneider [20], who has shown that the upper and lower power decay rates which can be associated with $\langle|\hat{\mu}(t)|^2\rangle_T$ are the same as the upper and lower correlation dimensions of μ.

3. Quantum dynamics for UαH measures

In this Section we summerize some central results which characterize the dynamics for vectors with UαH spectral measures. For the reader's convenience we include complete proofs, and, in particular, we give a proof of Theorem 2.5.

Theorem 3.1. *Let μ be a finite Borel measure on R and $\alpha \in [0, 1]$.*

(i) *If μ is UαH, then there exists a constant C such that, for any $T > 0$, $\langle |\hat{\mu}|^2 \rangle_T < CT^{-\alpha}$.*

(ii) *If there exists a constant C such that $\langle |\hat{\mu}|^2 \rangle_T < CT^{-\alpha}$ for any $T > 0$, then μ is U$\frac{\alpha}{2}$H.*

Remark. (i) coincides with Corollary 2.5.1 and is included here for completeness.

Lemma 3.1. *If μ is a finite Borel measure on R, then*

$$\langle |\hat{\mu}|^2 \rangle_T \;\geq\; \frac{1}{2} \int d\mu(x)\, d\mu(y)\, \frac{\sin^2((x-y)T/2)}{((x-y)T/2)^2} \;.$$

Proof. We have

$$\begin{aligned}
\langle |\hat{\mu}|^2 \rangle_T &= \frac{1}{T} \int_0^T dt \left| \int d\mu(x)\, e^{-ixt} \right|^2 \\
&= \frac{1}{T} \int_0^T dt \int d\mu(x)\, d\mu(y)\, e^{-i(x-y)t} \\
&= \frac{1}{T} \int d\mu(x)\, d\mu(y) \int_0^T dt\, \cos((x-y)t) \\
&= \int d\mu(x)\, d\mu(y)\, \frac{\sin((x-y)T)}{(x-y)T} \;,
\end{aligned} \tag{3.1}$$

and similarly,

$$
\begin{aligned}
\frac{1}{T^2}\int_0^T dt\, t\, |\hat{\mu}(t)|^2 &= \frac{1}{T^2}\int d\mu(x)\, d\mu(y) \int_0^T dt\, t\, \cos((x-y)t) \\
&= \frac{1}{T^2}\int d\mu(x)\, d\mu(y) \left(\frac{T\sin((x-y)T)}{(x-y)} + \frac{\cos((x-y)T)-1}{(x-y)^2}\right) \\
&= \int d\mu(x)\, d\mu(y) \left(\frac{\sin((x-y)T)}{(x-y)T} - \frac{2\sin^2((x-y)T/2)}{((x-y)T)^2}\right) \qquad (3.2)\\
&= \int d\mu(x)\, d\mu(y)\, \frac{\sin((x-y)T)}{(x-y)T} \\
&\qquad - \frac{1}{2}\int d\mu(x)\, d\mu(y)\, \frac{\sin^2((x-y)T/2)}{((x-y)T/2)^2}\ .
\end{aligned}
$$

Since the left-hand side of (3.2) is clearly positive, (3.1) and (3.2) imply the lemma. ○

Proof of Theorem 3.1. (i) will follow from proving Theorem 2.5 below. It remains to prove (ii). Suppose that μ is not $\mathrm{U}\frac{\alpha}{2}\mathrm{H}$, then there exists a sequence of intervals $\{I_n\}_{n=1}^{\infty}$, such that $|I_n| \to 0$ as $n \to \infty$ and $\mu(I_n) > n|I_n|^{\alpha/2}$. Let $T_n = \pi/(2|I_n|)$, then for every $x, y \in I_n$, we have $|(x-y)T_n| \le |I_n|T_n \le \pi/2$, and thus

$$
\frac{\sin^2((x-y)T_n/2)}{((x-y)T_n/2)^2} \ \ge\ \frac{\sin^2(\pi/4)}{(\pi/4)^2} \ =\ \frac{8}{\pi^2}\ . \qquad (3.3)
$$

By Lemma 3.1 we have

$$
\langle|\hat{\mu}|^2\rangle_{T_n} \ \ge\ \frac{1}{2}\int_{I_n\times I_n} d\mu(x)\, d\mu(y)\, \frac{\sin^2((x-y)T/2)}{((x-y)T/2)^2}\ , \qquad (3.4)
$$

and thus (3.3) implies

$$
\langle|\hat{\mu}|^2\rangle_{T_n} \ \ge\ \frac{4(\mu(I_n))^2}{\pi^2} \ \ge\ \frac{4n^2}{\pi^2}|I_n|^{\alpha} \ =\ \frac{4n^2\pi^{\alpha}}{\pi^2 2^{\alpha}}T_n^{-\alpha}\ , \qquad (3.5)
$$

which proves (ii). ○

Theorem 3.1 is optimal, in the sense that $\mathrm{U}\alpha\mathrm{H}$ does not imply more than $\langle|\hat{\mu}|^2\rangle_T < CT^{-\alpha}$, which in turn does not imply more than $\mathrm{U}\frac{\alpha}{2}\mathrm{H}$. Examples of $\mathrm{U}\alpha\mathrm{H}$ measures for which $\langle|\hat{\mu}|^2\rangle_T \sim T^{-\alpha}$ are provided by uniformly distributed measures on "nice," self similar, Cantor sets. In fact, Strichartz [36] has shown that this must be the case for a large class of such measures. An example of a measure for which $\langle|\hat{\mu}|^2\rangle_T < CT^{-\alpha}$, and yet μ is no more than U$\frac{\alpha}{2}$H, is given by the following:

Example 3.1. *Consider* $\beta \in (1/2, 1)$, $d\mu(x) = x^{-\beta}dx$ *on* $(0, 1]$, *and let* $\alpha = 2(1-\beta)$; *then* $|\hat{\mu}(t)|^2 < Ct^{-\alpha}$ *and* μ *is no more than* U$\frac{\alpha}{2}$H.

Proof. We have

$$\hat{\mu}(t) \;=\; \int_0^1 e^{-ixt}x^{-\beta}\,dx \;=\; t^{\beta-1}\int_0^1 e^{-ixt}(xt)^{-\beta}t\,dx \;=\; t^{-\alpha/2}\int_0^t e^{-iu}u^{-\beta}\,du\,. \tag{3.6}$$

Since $\int_0^\infty e^{-iu}u^{-\beta}\,du$ converges, we conclude that $t^\alpha|\hat{\mu}(t)|^2 \to Const$ as $t \to \infty$. Consider now $I_\epsilon = (0, \epsilon)$. We have

$$\mu(I_\epsilon) \;=\; \int_0^\epsilon x^{-\beta}\,dx \;=\; \frac{\epsilon^{1-\beta}}{1-\beta} \;=\; \frac{1}{1-\beta}|I_\epsilon|^{\alpha/2}\,, \tag{3.7}$$

so μ is U$\frac{\alpha}{2}$H, but does not have any stronger uniform Hölder continuity. ○

We now turn to proving Strichartz's theorem.

Proof of Theorem 2.5. Suppose that μ is UαH, and let $f \in L^2(R, d\mu)$, then

$$\begin{aligned}
\langle|\widehat{f\mu}|^2\rangle_T &= \frac{1}{T}\int_0^T dt\,|\widehat{f\mu}(t)|^2 \\
&\leq \frac{e}{T}\int_{-\infty}^\infty dt\,e^{-t^2/T^2}\,|\widehat{f\mu}(t)|^2 \\
&= \frac{e}{T}\int_{-\infty}^\infty dt\,e^{-t^2/T^2}\int d\mu(x)\,d\mu(y)\,f(x)\,\overline{f(y)}\,e^{-i(x-y)t} \\
&= \frac{e}{T}\int d\mu(x)\,d\mu(y)\,f(x)\,\overline{f(y)}\int_{-\infty}^\infty dt\,e^{-t^2/T^2-i(x-y)t} \\
&= e\sqrt{\pi}\int d\mu(x)\,d\mu(y)\,f(x)\,\overline{f(y)}\,e^{-(x-y)^2T^2/4} \\
&\leq e\sqrt{\pi}\int d\mu(x)\,d\mu(y)\,(|f(x)|e^{-(x-y)^2T^2/2})(|f(y)|e^{-(x-y)^2T^2/2})\,.
\end{aligned} \tag{3.8}$$

By the Cauchy-Schwartz inequality, (3.8) implies

$$\langle|\widehat{f\mu}|^2\rangle_T \;\le\; e\sqrt{\pi}\int d\mu(x)\,|f(x)|^2\int d\mu(y)\,e^{-(x-y)^2T^2/4}\,. \tag{3.9}$$

Since μ is UαH, there exists C such that $\mu(I) < C|I|^\alpha$ for $|I| < 1$. Without loss, we can assume $T > 1$, and thus, for every x,

$$\int d\mu(y)\,e^{-(x-y)^2T^2/4} \;=\; \sum_{n=0}^{\infty}\int_{\frac{n}{T}\le|x-y|<\frac{n+1}{T}} d\mu(y)\,e^{-(x-y)^2T^2/4} \;\le\; \sum_{n=0}^{\infty} 2CT^{-\alpha}e^{-n^2/4}\,. \tag{3.10}$$

Let $C_1 = 2C\sum_{n=0}^{\infty} e^{-n^2/4}$, then we obtain from (3.9) and (3.10)

$$\langle|\widehat{f\mu}|^2\rangle_T \;\le\; e\sqrt{\pi}\int d\mu(x)\,|f(x)|^2\,C_1T^{-\alpha} \;=\; e\sqrt{\pi}C_1\|f\|^2T^{-\alpha}\,. \quad \bigcirc$$

Our next theorem can be described as a RAGE-like Hilbert space adaptation of Strichartz's theorem.

Theorem 3.2. *If μ_ψ is* UαH, *then there exists a constant C_ψ such that for any compact operator A, $p \in N$, and $T > 0$:*

$$\langle|\langle A\rangle|\rangle_T \;<\; C_\psi^{1/p}\,\|A\|_p\,T^{-\alpha/p}\;,$$

where $\|A\|_p \equiv (\mathrm{Tr}(|A|^p))^{1/p}$ denotes the p-th Schatten norm of A.

Remarks.

(i) $\|\cdot\|_1$ is the trace norm; $\|\cdot\|_2$ is the Hilbert-Schmidt norm.

(ii) A compact operator A may have $\|A\|_p = \infty$ for all p's, in which case Theorem 3.2 is empty.

Lemma 3.2. *If μ_ψ is* UαH, *then there exists a constant C_ψ such that for any $\varphi \in \mathcal{H}$ with $\|\varphi\| \le 1$:*

$$\langle|\langle\varphi,\,\psi(t)\rangle|^2\rangle_T \;<\; C_\psi T^{-\alpha}\,.$$

Proof. By the spectral theorem [24], H restricted to the cyclic subspace spanned by ψ (and H) is unitarily equivalent to multiplication by x on $L^2(R,\, d\mu_\psi)$. In particular, for each $\varphi \in \mathcal{H}$ there exists $f_\varphi \in L^2(R,\, d\mu_\psi)$ with $\|f_\varphi\| \leq \|\varphi\|$, such that

$$\langle \varphi,\, \psi(t)\rangle \;=\; \langle \varphi,\, e^{-iHt}\psi\rangle \;=\; \int e^{-ixt} f_\varphi(x)\, d\mu_\psi(x) \;\equiv\; \widehat{f_\varphi \mu_\psi}\,. \tag{3.11}$$

Thus, Theorem 2.5 implies the Lemma. ○

Proof of Theorem 3.2. Since A is compact, there exist orthonormal bases $\{\psi_n\}_{n=1}^\infty$ and $\{\varphi_n\}_{n=1}^\infty$ of $\mathcal{H}$, and a monotonely decreasing sequence $\{E_n\}_{n=1}^\infty$, $E_n \geq 0$, such that A is given by the (norm-convergent) sum [30]

$$A \;=\; \sum_{n=1}^\infty E_n \langle \varphi_n,\, \cdot\,\rangle \psi_n\,. \tag{3.12}$$

Moreover, $\|A\|_p = (\sum_{n=1}^\infty E_n^p)^{1/p}$. Thus, we have

$$\begin{aligned}
\langle |\langle A\rangle|\rangle_T \;&=\; \left\langle \left| \sum_{n=1}^\infty E_n \langle \varphi_n,\, \psi(t)\rangle\langle \psi(t),\, \psi_n\rangle \right| \right\rangle_T \\
&\leq \sum_{n=1}^\infty E_n \,\langle |\langle \varphi_n,\, \psi(t)\rangle\langle \psi(t),\, \psi_n\rangle|\rangle_T \\
&\leq \sum_{n=1}^\infty E_n \left(\langle |\langle \varphi_n,\, \psi(t)\rangle|^2\rangle_T\right)^{1/2} \left(\langle |\langle \psi(t),\, \psi_n\rangle|^2\rangle_T\right)^{1/2}\,.
\end{aligned} \tag{3.13}$$

Let $p, q \in N$ obey $1/p + 1/q = 1$, then, by Hölder's inequality, we obtain from (3.13)

$$\begin{aligned}
\langle |\langle A\rangle|\rangle_T \;&\leq\; \left(\sum_{n=1}^\infty E_n^p\right)^{1/p} \left[\sum_{n=1}^\infty \left(\langle |\langle \varphi_n,\, \psi(t)\rangle|^2\rangle_T\right)^{q/2} \left(\langle |\langle \psi(t),\, \psi_n\rangle|^2\rangle_T\right)^{q/2}\right]^{1/q} \\
&\leq \|A\|_p \left[\left(\sum_{n=1}^\infty \left(\langle |\langle \varphi_n,\, \psi(t)\rangle|^2\rangle_T\right)^q\right) \left(\sum_{n=1}^\infty \left(\langle |\langle \psi(t),\, \psi_n\rangle|^2\rangle_T\right)^q\right)\right]^{1/2q}\,.
\end{aligned} \tag{3.14}$$

By Lemma 3.2, we have for every n

$$\langle |\langle \varphi_n,\, \psi(t)\rangle|^2\rangle_T \;<\; C_\psi T^{-\alpha}\,, \qquad \langle |\langle \psi_n,\, \psi(t)\rangle|^2\rangle_T \;<\; C_\psi T^{-\alpha}\,. \tag{3.15}$$

Moreover, since the ψ_n's and φ_n's are orthonormal bases and since e^{-iHt} is unitary, we have:

$$\sum_{n=1}^{\infty}\langle|\langle\varphi_n,\,\psi(t)\rangle|^2\rangle_T \;=\; \sum_{n=1}^{\infty}\langle|\langle\psi_n,\,\psi(t)\rangle|^2\rangle_T \;=\; \|\psi\|^2 \;=\; 1\,. \tag{3.16}$$

Thus,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\big(\langle|\langle\varphi_n,\,\psi(t)\rangle|^2\rangle_T\big)^q \;<(C_\psi T^{-\alpha})^{q-1}\,,\\
&\sum_{n=1}^{\infty}\big(\langle|\langle\psi_n,\,\psi(t)\rangle|^2\rangle_T\big)^q \;<(C_\psi T^{-\alpha})^{q-1}\,,
\end{aligned} \tag{3.17}$$

and from (3.14) we obtain

$$\begin{aligned}
\langle|\langle A\rangle|\rangle_T \;&< \|A\|_p\left[(C_\psi T^{-\alpha})^{q-1}(C_\psi T^{-\alpha})^{q-1}\right]^{1/2q} \;=\; \|A\|_p(C_\psi T^{-\alpha})^{(q-1)/q}\\
&= \|A\|_p(C_\psi T^{-\alpha})^{1/p}\,.\quad \bigcirc
\end{aligned} \tag{3.18}$$

4. Hausdorff measures and Rogers-Taylor decomposition theory

Our purpose in this section is to describe the theory of decomposing finite Borel measures with respect to Hausdorff measures and dimensions. This theory has been extensively studied by Rogers and Taylor [28,29] in the late 50's and early 60's. For our relatively restricted purposes, we shall rely only on results which can be found in the last chapter of Rogers' book [27], and derive whatever we need beyond that. It should be pointed out, however, that all the results we describe in this section, and much more, can be found in the original Rogers-Taylor papers [28,29].

We start by recalling some standard terminology:

Definition 4.1. *Let μ be a Borel measure on R.*

(i) *μ is called σ-finite if $R=\bigcup_{j=1}^{\infty}S_j$ and $\mu(S_j)<\infty$ for each j. Similarly, we say that $S\subseteq R$ has σ-finite μ measure, if $S=\bigcup_{j=1}^{\infty}S_j$ and $\mu(S_j)<\infty$ for each j.*

(ii) *We say that μ is supported on S, $S\subseteq R$, if $\mu(R\setminus S)=0$.*

Definition 4.2. *Let $S \subseteq R$. A countable collection of intervals $\{b_\nu\}_{\nu=1}^\infty$ is called a $\delta-$cover of S if $S \subset \bigcup_{\nu=1}^\infty b_\nu$ and $|b_\nu| < \delta$ for all ν's.*

Definition 4.3. *Let $\alpha \in [0, 1]$. For any subset $S \subseteq R$, the α-dimensional Hausdorff measure of S is*

$$h^\alpha(S) \equiv \lim_{\delta \to 0} \inf_{\delta-covers} \sum_{\nu=1}^\infty |b_\nu|^\alpha .$$

Remarks.

(i) It is well known (see, e.g., [12]), and not hard to verify, that the above limit exists for any $S \subseteq R$ (possibly being ∞), and that h^α is an outer measure on R. Moreover, h^α restricted to Borel subsets of R is a Borel measure. Below, we shall be mostly interested in Borel subsets, and we shall relate to h^α as a Borel measure on R.

(ii) h^1 coincides with Lebesgue measure; h^0 is the counting measure.

(iii) h^α can be defined also for $\alpha < 0$ or $\alpha > 1$. For $\alpha < 0$, however, $h^\alpha(S) = \infty$ for any $S \neq \emptyset$; while for $\alpha > 1$, $h^\alpha(R) = 0$. Thus, there is no real interest in such h^α's. For the purpose of our discussion below (namely, for the strict correctness of some of our statements), however, it is useful to think of h^α as being defined also for α outside $[0, 1]$.

(iv) Given any right-continuous monotonely increasing function $h : [0, \infty) \to [0, \infty)$, which obeys $h(x) > 0$ for $x > 0$, one can define the corresponding h-Hausdorff measure on R, via

$$h^h(S) \equiv \lim_{\delta \to 0} \inf_{\delta-covers} \sum_{\nu=1}^\infty h(|b_\nu|) .$$

Moreover, essentially all of the decomposition theory that we describe below can be extended to such general Hausdorff measures and to suitable families of such measures. We refer to Rogers [27] and Rogers-Taylor [28,29] for such more general treatments.

(v) For $\alpha < 1$, h^α is *not* σ-finite. Moreover, it is not regular in the usual sense of being outer regular by open sets and inner regular by compacts. h^α is, however, G_δ outer regular and F_σ inner regular in the following sense [27]: For any Borel set $S \subseteq R$ there exists a G_δ set S_1 such that $S \subseteq S_1$ and $h^\alpha(S) = h^\alpha(S_1)$. Moreover, if $h^\alpha(S) < \infty$, then there exists an F_σ set S_2 such that $S_2 \subseteq S$ and $h^\alpha(S) = h^\alpha(S_2)$.

Given any $\emptyset \neq S \subseteq R$, there exists a unique $\alpha(S) \in [0, 1]$ such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$. $h^{\alpha(S)}(S)$ may be 0, finite, or infinite. This unique $\alpha(S)$ is called the *Hausdorff dimension* of S, and is denoted by $\dim_{\mathrm{H}}(S)$.

The above notions of Hausdorff measures and dimensions lead to a rich collection of notions of continuity and singularity, as given by the following definitions:

Definition 4.4. *Let μ be a Borel measure on R and $\alpha \in [0, 1]$.*

(i) *μ is called α-continuous (denoted αc) if $\mu(S) = 0$ for any set S with $h^\alpha(S) = 0$.*

(ii) *μ is called strongly α-continuous (denoted $s\alpha c$) if $\mu(S) = 0$ for any set S which has σ-finite h^α measure.*

(iii) *μ is called α-singular (denoted αs) if it is supported on a set S with $h^\alpha(S) = 0$.*

(iv) *μ is called almost α-singular (denoted $a\alpha s$) if it is supported on a set S which has σ-finite h^α measure.*

(v) *μ is called absolutely continuous with respect to h^α (denoted αac) if $d\mu = f(x)\, dh^\alpha$ for some Borel function f.*

Remark. If μ is σ-finite, then it follows from the Radon-Nikodym theorem that μ is absolutely continuous with respect to h^α if and only if it is both α-continuous and almost α-singular.

Definition 4.5. *Let μ be a Borel measure on R and $\alpha \in [0, 1]$.*

(i) *μ is called α-dimension continuous (denoted αdc) if $\mu(S) = 0$ for any set S with $\dim_{\mathrm{H}}(S) < \alpha$.*

(ii) *μ is called strongly α-dimension continuous (denoted $s\alpha dc$) if $\mu(S) = 0$ for any set S with $\dim_{\mathrm{H}}(S) \leq \alpha$.*

(iii) *μ is called α-dimension singular (denoted αds) if it is supported on $S = \bigcup_{j=1}^\infty S_j$, where $\dim_{\mathrm{H}}(S_j) < \alpha$ for each j.*

(iv) *μ is called almost α-dimension singular (denoted $a\alpha ds$) if it is supported on a set S with $\dim_{\mathrm{H}}(S) \leq \alpha$.*

(v) *We say that μ has exact dimension α (denoted $ed\alpha$) if μ is both α-dimension continuous and almost α-dimension singular.*

Given a finite Borel measure μ and $\alpha \in [0,\, 1]$, we define the upper α-derivative of μ by

$$D_\mu^\alpha(x) \;\equiv\; \limsup_{\epsilon \to 0} \frac{\mu((x-\epsilon,\, x+\epsilon))}{(2\epsilon)^\alpha}\;, \tag{4.1}$$

and denote

$$\begin{aligned} T_0 \;&\equiv T_0(\alpha,\, \mu) \;\equiv\; \{x \mid D_\mu^\alpha(x) = 0\,\} \\ T_+ \;&\equiv T_+(\alpha,\, \mu) \;\equiv\; \{x \mid 0 < D_\mu^\alpha(x) < \infty\,\} \\ T_\infty \;&\equiv T_\infty(\alpha,\, \mu) \;\equiv\; \{x \mid D_\mu^\alpha(x) = \infty\,\}\,. \end{aligned} \tag{4.2}$$

Theorem 4.1. (Rogers and Taylor [27,28]) *If μ is a finite Borel measure on R, then T_0, T_+, and T_∞ are Borel sets, and*

(i) $h^\alpha(T_\infty) = 0$.
(ii) *T_+ has σ-finite h^α measure.*
(iii) *$\mu(S \cap T_+) = 0$ for any S with $h^\alpha(S) = 0$.*
(iv) *$\mu(S \cap T_0) = 0$ for any S which has σ-finite h^α measure.*

Since the sets T_0, T_+, and T_∞ are disjoint and $T_0 \cup T_+ \cup T_\infty = R$, one can decompose μ as $d\mu = \chi_{T_\infty} d\mu + \chi_{T_+} d\mu + \chi_{T_0} d\mu$, where $\chi_{(\cdot)}$ denotes characteristic function. Thus, Theorem 4.1 implies the following:

Corollary 4.1.1. *Let μ be a finite Borel measure on R, then for every $\alpha \in [0,\, 1]$, μ has a unique decomposition*

$$d\mu \;=\; d\mu_{\alpha s} + d\mu_{\alpha ac} + d\mu_{s\alpha c}\;,$$

where $d\mu_{\alpha s}$ is α-singular, $d\mu_{\alpha ac}$ is absolutely continuous with respect to h^α (on a set of σ-finite h^α measure), and $d\mu_{s\alpha c}$ is strongly α-continuous.

Remarks.

(i) If $\alpha = 0$, then $d\mu_{\alpha s} = 0$, and the decomposition $d\mu = d\mu_{\alpha ac} + d\mu_{s\alpha c}$ coincides with the decomposition of μ into a pure point part and a continuous part. If $\alpha = 1$, then $d\mu_{s\alpha c} = 0$, and the decomposition $d\mu = d\mu_{\alpha s} + d\mu_{\alpha ac}$ coincides with the decomposition of μ into a singular part and an absolutely continuous (with respect to Lebesgue measure) part.

(ii) $d\mu_{\alpha c} \equiv d\mu_{\alpha ac} + d\mu_{s\alpha c}$ is the α-continuous part of $d\mu$; $d\mu_{a\alpha s} \equiv d\mu_{\alpha s} + d\mu_{\alpha ac}$ is the almost α-singular part of $d\mu$.

The decomposition in Corollary 4.1.1 is with respect to the Hausdorff measure h^α. One can obtain further decompositions by also considering Hausdorff dimensions. For a given $\alpha \in [0, 1]$, define

$$
\begin{aligned}
T_{d\infty} &\equiv T_{d\infty}(\alpha,\, \mu) \;\equiv\; \bigcup_{n=1}^{\infty} T_\infty(\alpha - n^{-1},\, \mu)\ , \\
T_{d\infty+} &\equiv T_{d\infty+}(\alpha,\, \mu) \;\equiv\; T_\infty(\alpha,\, \mu) \setminus T_{d\infty}(\alpha,\, \mu)\ , \\
T_{d0} &\equiv T_{d0}(\alpha,\, \mu) \;\equiv\; \bigcup_{n=1}^{\infty} T_0(\alpha + n^{-1},\, \mu)\ , \\
T_{d+0} &\equiv T_{d+0}(\alpha,\, \mu) \;\equiv\; T_0(\alpha,\, \mu) \setminus T_{d0}(\alpha,\, \mu)\ , \\
T_{d+} &\equiv T_{d+}(\alpha,\, \mu) \;\equiv\; T_{d\infty+}(\alpha,\, \mu) \cup T_+(\alpha,\, \mu) \cup T_{d+0}(\alpha,\, \mu)\ .
\end{aligned}
\tag{4.3}
$$

Lemma 4.1. *If μ is a finite Borel measure on R, then $T_{d\infty}$, $T_{d\infty+}$, T_{d+0}, T_{d0}, and T_{d+} are Borel sets, and*

(i) $\dim_{\mathrm{H}}(T_{d+0}) \leq \alpha$.

(ii) $\dim_{\mathrm{H}}(T_{d+}) \leq \alpha$.

(iii) $\mu(S \cap T_{d0}) = 0$ *for any* S *with* $\dim_{\mathrm{H}}(S) \leq \alpha$.

(iv) $\mu(S \cap T_{d\infty+}) = 0$ *for any* S *with* $\dim_{\mathrm{H}}(S) < \alpha$.

(v) $\mu(S \cap T_{d+}) = 0$ *for any* S *with* $\dim_{\mathrm{H}}(S) < \alpha$.

Proof. That the various T's in (4.3) are Borel sets is clear from their construction. Assertions (i)–(v) follow very easily from Theorem 4.1. It is helpful to note that the dimension of a countable union of sets is the supremum of their dimensions, and that the sets $T_{d\infty+}$ and T_{d+0} obey

$$
\begin{aligned}
T_{d\infty+} &= T_\infty(\alpha,\, \mu) \cap \left(\bigcap_{n=1}^{\infty} T_0(\alpha - n^{-1},\, \mu) \right)\ , \\
T_{d+0} &= T_0(\alpha,\, \mu) \cap \left(\bigcap_{n=1}^{\infty} T_\infty(\alpha + n^{-1},\, \mu) \right)\ . \quad \bigcirc
\end{aligned}
\tag{4.4}
$$

The sets T_{d0}, T_{d+}, and $T_{d\infty}$ are easily seen to be disjoint, and obey $T_{d0} \cup T_{d+} \cup T_{d\infty} = R$. Thus, one can also decompose μ as $d\mu = \chi_{T_{d\infty}} d\mu + \chi_{T_{d+}} d\mu + \chi_{T_{d0}} d\mu$ to obtain the following:

Corollary 4.1.2. *Let μ be a finite Borel measure on R, then for every $\alpha \in [0, 1]$, μ has a unique decomposition*

$$d\mu = d\mu_{\alpha ds} + d\mu_{ed\alpha} + d\mu_{s\alpha dc} ,$$

where $d\mu_{\alpha ds}$ is α-dimension singular, $d\mu_{ed\alpha}$ has exact dimension α, and $d\mu_{s\alpha dc}$ is strongly α-dimension continuous.

Remark. $d\mu_{\alpha dc} \equiv d\mu_{ed\alpha} + d\mu_{s\alpha dc}$ is the α-dimension continuous part of $d\mu$; $d\mu_{a\alpha ds} \equiv d\mu_{\alpha ds} + d\mu_{ed\alpha}$ is the almost α-dimension singular part of $d\mu$.

So, while Corollary 4.1.1 decomposes μ with respect to a Hausdorff measure, Corollary 4.1.2 gives an analogous decomposition with respect to the corresponding Hausdorff dimension. Note that at the edges $\alpha = 0$ and $\alpha = 1$, where the decomposition of Corollary 4.1.1 coincides with the decompositions into pure point+continuous and singular+absolutely continuous; the decomposition of Corollary 4.1.2 gives something quite different. For $\alpha = 1$, we have $d\mu = d\mu_{1ds} + d\mu_{ed1}$, where μ_{ed1} combines the absolutely continuous part of μ along with some of its singular part. Similarly, for $\alpha = 0$ we have $d\mu = d\mu_{ed0} + d\mu_{s0dc}$, where μ_{ed0} combines the pure point part of μ along with some of its continuous part.

One can also combine the two different decompositions to obtain a 5-part decomposition for each α. Namely,

Corollary 4.1.3. *Let μ be a finite Borel measure on R, then for every $\alpha \in [0, 1]$, μ has a unique decomposition*

$$d\mu = d\mu_{\alpha ds} + d\mu_{ed\alpha/\alpha s} + d\mu_{\alpha ac} + d\mu_{ed\alpha/s\alpha c} + d\mu_{s\alpha dc} ,$$

where $d\mu_{\alpha ds}$ is α-dimension singular, $d\mu_{ed\alpha/\alpha s}$ is α-singular and has exact dimension α, $d\mu_{\alpha ac}$ is absolutely continuous with respect to h^α (on a set of σ-finite h^α measure), $d\mu_{ed\alpha/s\alpha c}$ is strongly α-continuous and has exact dimension α, and $d\mu_{s\alpha dc}$ is strongly α-dimension continuous.

Remarks.

(i) Note that the extremal parts of the decomposition in Corollary 4.1.3 are the same as those of Corollary 4.1.2.

(ii) Corollary 4.1.3 can be obtained by employing Corollary 4.1.1 to decompose $d\mu_{ed\alpha}$. One can also use the decompositions $d\mu = d\mu_{\alpha s} + d\mu_{\alpha c}$ and $d\mu = d\mu_{a\alpha s} + d\mu_{s\alpha c}$ to decompose $d\mu_{ed\alpha}$, such that $d\mu_{ed\alpha} = d\mu_{ed\alpha/\alpha s} + d\mu_{ed\alpha/\alpha c}$ and $d\mu_{ed\alpha} = d\mu_{ed\alpha/a\alpha s} + d\mu_{ed\alpha/s\alpha c}$.

Continuity properties with respect to h^α are related to the Hölder continuity, discussed in previous sections (and defined in Definition 2.1), by the following:

Theorem 4.2. (Rogers and Taylor [27,29]) *Let μ be a finite Borel measure on R and let $\alpha \in [0, 1]$; then*

(i) *μ is α-continuous if and only if for each $\epsilon > 0$ there are mutually singular Borel measures μ_1^ϵ, μ_2^ϵ, such that $d\mu = d\mu_1^\epsilon + d\mu_2^\epsilon$, μ_1^ϵ is* UαH, *and $\mu_2^\epsilon(R) < \epsilon$.*

(ii) *μ is strongly α-continuous if and only if for each $\epsilon > 0$ there are mutually singular Borel measures μ_1^ϵ, μ_2^ϵ, such that $d\mu = d\mu_1^\epsilon + d\mu_2^\epsilon$, μ_1^ϵ is* USαH, *and $\mu_2^\epsilon(R) < \epsilon$.*

Remark. Theorem 4.2 says that α-continuous measures are precisely limits of UαH measures in an appropriate topology. Similarly, strongly α-continuous measures are precisely limits of USαH measures in the same topology.

The richness of possible decompositions, described in Corollaries 4.1.1, 4.1.2, and 4.1.3 above, seems too great for most practical purposes; and suggests that it may be useful to concentrate on some partial set of decompositions. Theorem 4.2 suggests that the decomposition $d\mu = d\mu_{\alpha s} + d\mu_{\alpha c}$ (which decomposes μ into an α-singular part and an α-continuous part) should be of particular interest in our context. Indeed, we shall adopt it below as "the canonical decomposition within dimension α," in the sense that, for $\alpha \in (0, 1)$, we shall concentrate most of our attention on this particular decomposition. The situation is somewhat different at the edges $\alpha = 0$ and $\alpha = 1$. For $\alpha = 0$, the α-singular part always vanishes, while for $\alpha = 1$, $d\mu = d\mu_{\alpha s} + d\mu_{\alpha c}$ coincides with the decomposition into a singular part and an absolutely continuous part. Thus, at these edges, we shall also be very much interested in the dimensional decompositions of Corollary 4.1.2. In particular, it seems useful to have compact terminology for the dimensional continuity and singularity properties at these edges. For this reason we define:

Definition 4.6. *Let μ be a Borel measure on R.*

(i) *μ is called zero-dimensional (denoted zd) if it is supported on a set S with* $\dim_{\mathrm{H}}(S) = 0$.

(ii) *μ is called positive-dimensional (denoted pd) if $\mu(S) = 0$ for any set S with* $\dim_{\mathrm{H}}(S) = 0$.

(iii) *μ is called one-dimensional (denoted od) if $\mu(S) = 0$ for any set S with* $\dim_{\mathrm{H}}(S) < 1$.

(iv) *μ is called sub-one-dimensional (denoted sod) if it is supported on $S = \bigcup_{j=1}^{\infty} S_j$, where $\dim_{\mathrm{H}}(S_j) < 1$ for each j.*

Remark. The terminology of Definition 4.6 overlaps Definition 4.5, such that a zero-dimensional measure is the same as a measure of exact dimension 0, a positive-dimensional measure is the same as a strongly 0-dimension continuous measure, etc. This should not lead, however, to any real confusion.

In the language of Definition 4.6, Corollary 4.1.2 implies

Corollary 4.1.4. *Let μ be a finite Borel measure on R, then μ has unique decompositions:*

(i) *$d\mu = d\mu_{zd} + d\mu_{pd}$, where $d\mu_{zd}$ is zero-dimensional and $d\mu_{pd}$ is positive-dimensional.*

(ii) *$d\mu = d\mu_{sod} + d\mu_{od}$, where $d\mu_{sod}$ is sub-one-dimensional and $d\mu_{od}$ is one-dimensional.*

5. Decompositions of $\mathcal{H}$

Given any $\alpha \in [0, 1]$, we define:

$$
\begin{aligned}
\mathcal{H}_{\alpha c} &\equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-continuous}\} , \\
\mathcal{H}_{\alpha s} &\equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-singular}\} .
\end{aligned}
\tag{5.1}
$$

Theorem 5.1. *$\mathcal{H}_{\alpha c}$ and $\mathcal{H}_{\alpha s}$ are closed (in norm), mutually orthogonal subspaces, which are invariant under H, and $\mathcal{H} = \mathcal{H}_{\alpha s} \oplus \mathcal{H}_{\alpha c}$.*

Remarks.

(i) We call $\mathcal{H}_{\alpha s}$ the α-singular subspace and $\mathcal{H}_{\alpha c}$ the α-continuous subspace. The spectra $\sigma_{\alpha s}$ and $\sigma_{\alpha c}$ are defined as the spectra of H restricted to the corresponding subspaces, and they are called α-singular spectrum and α-continuous spectrum accordingly. Clearly, $\sigma = \sigma_{\alpha s} \cup \sigma_{\alpha c}$.

(ii) For $\alpha = 0$, $\mathcal{H}_{\alpha s} = 0$ and $\mathcal{H}_{\alpha c} = \mathcal{H}$. For $\alpha > 0$, $\mathcal{H}_{\alpha c}$ is a subspace of $\mathcal{H}_c$, and $\mathcal{H}_{pp}$ is a subspace of $\mathcal{H}_{\alpha s}$.

Proof. Let $\{\psi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Denote

$$T_{\infty}^{\mathcal{H}} \;=\; \bigcup_{n=1}^{\infty} T_{\infty}(\alpha,\, \mu_{\psi_n}) \;, \tag{5.2}$$

and let $P_{T_{\infty}^{\mathcal{H}}}$ be the spectral projection on $T_{\infty}^{\mathcal{H}}$. One easily sees that $\mathcal{H}_{\alpha s} = P_{T_{\infty}^{\mathcal{H}}}\mathcal{H}$ and $\mathcal{H}_{\alpha c} = (1 - P_{T_{\infty}^{\mathcal{H}}})\mathcal{H}$, so Theorem 5.1 follows from known facts about spectral projections [24]. ○

We denote

$$\mathcal{H}_{uh}(\alpha) \;\equiv\; \{\psi \,|\, \mu_{\psi} \text{ is U}\alpha\text{H}\} \,. \tag{5.3}$$

Theorem 5.2. *For every $\alpha \in [0,\, 1]$, $\mathcal{H}_{uh}(\alpha)$ is a vector space and*

$$\overline{\mathcal{H}_{uh}(\alpha)} \;=\; \mathcal{H}_{\alpha c} \;,$$

where $\overline{\cdot}$ denotes norm closure in $\mathcal{H}$.

Proof. Let $\psi_1,\, \psi_2 \in \mathcal{H}_{uh}(\alpha)$ and let $\varphi = a\psi_1 + b\psi_2$. By assumption, there are constants $C_1,\, C_2$ such that $\mu_{\psi_1}(I) < C_1|I|^{\alpha}$ and $\mu_{\psi_2}(I) < C_2|I|^{\alpha}$, for any interval I with $|I| \le 1$. For each such I let P_I be the spectral projection on I, then

$$\begin{aligned} \mu_{\varphi}(I) \;&= \langle \varphi,\, P_I\varphi\rangle \;=\; \langle a\psi_1 + b\psi_2,\, aP_I\psi_1 + bP_I\psi_2\rangle \\ &\le |a|^2\langle \psi_1,\, P_I\psi_1\rangle + |b|^2\langle \psi_2,\, P_I\psi_2\rangle + 2|a||b||\langle \psi_1,\, P_I\psi_2\rangle| \,. \end{aligned} \tag{5.4}$$

Since

$$|\langle \psi_1,\, P_I\psi_2\rangle| \;\le\; \sqrt{\langle \psi_1,\, P_I\psi_1\rangle\langle \psi_2,\, P_I\psi_2\rangle} \;\le\; \frac{1}{2}\left(\langle \psi_1,\, P_I\psi_1\rangle + \langle \psi_2,\, P_I\psi_2\rangle\right) \;, \tag{5.5}$$

we obtain from (5.4)

$$\begin{aligned}\mu_\varphi(I) &\le (|a|^2+|a||b|)\langle\psi_1,\, P_I\psi_1\rangle + (|b|^2+|a||b|)\langle\psi_2,\, P_I\psi_2\rangle \\ &= (|a|^2+|a||b|)\mu_{\psi_1}(I) + (|b|^2+|a||b|)\mu_{\psi_2}(I) \\ &\le (|a|+|b|)^2 \sup\{C_1,\, C_2\}\, |I|^\alpha\,. \end{aligned} \tag{5.6}$$

Thus, $\mathcal{H}_{uh}(\alpha)$ is clearly a vector space. Since, by Theorem 4.2, $\mathcal{H}_{uh}(\alpha) \subseteq \mathcal{H}_{\alpha c}$, and since $\mathcal{H}_{\alpha c}$ is closed, we clearly have $\overline{\mathcal{H}_{uh}(\alpha)} \subseteq \mathcal{H}_{\alpha c}$. Let $\varphi \in \mathcal{H}_{\alpha c}$ and let $\epsilon > 0$ be given. By Theorem 4.2 we have $d\mu_\varphi = d\mu^\epsilon_{\varphi,1} + d\mu^\epsilon_{\varphi,2}$ such that $\mu^\epsilon_{\varphi,1}$ is UαH, $\mu^\epsilon_{\varphi,2}(R) < \epsilon$, and $\mu^\epsilon_{\varphi,1}$, $\mu^\epsilon_{\varphi,2}$ are mutually singular. Let $S_{\varphi,\epsilon} \subset R$ be such that $\mu^\epsilon_{\varphi,2}$ is supported on $S_{\varphi,\epsilon}$, $\mu^\epsilon_{\varphi,1}(S_{\varphi,\epsilon}) = 0$, and let $P_{S_{\varphi,\epsilon}}$ be the spectral projection on $S_{\varphi,\epsilon}$. We have $\varphi = P_{S_{\varphi,\epsilon}}\varphi + (1-P_{S_{\varphi,\epsilon}})\varphi$, $(1-P_{S_{\varphi,\epsilon}})\varphi \in \mathcal{H}_{uh}(\alpha)$, and $\|P_{S_{\varphi,\epsilon}}\varphi\|^2 < \epsilon$. Thus, φ is a norm limit of vectors in $\mathcal{H}_{uh}(\alpha)$, and we obtain $\mathcal{H}_{\alpha c} \subseteq \overline{\mathcal{H}_{uh}(\alpha)}$, which completes the proof. ○

Theorem 5.2 and the dynamical results of Section 3 about UαH measures are the main reasons that we consider the decomposition $\mathcal{H} = \mathcal{H}_{\alpha s} \oplus \mathcal{H}_{\alpha c}$ to be of special interest. Nevertheless, it's easy to verify that each of the measure decompositions discussed in Section 4 leads to a corresponding decomposition of $\mathcal{H}$. For each $\alpha \in [0,\, 1]$, the subspaces $\mathcal{H}_{\alpha ac}$, $\mathcal{H}_{s\alpha c}$, $\mathcal{H}_{a\alpha s}$, $\mathcal{H}_{\alpha ds}$, $\mathcal{H}_{ed\alpha}$, $\mathcal{H}_{s\alpha dc}$, $\mathcal{H}_{a\alpha ds}$, $\mathcal{H}_{\alpha dc}$, $\mathcal{H}_{ed\alpha/\alpha s}$, $\mathcal{H}_{ed\alpha/\alpha c}$, $\mathcal{H}_{ed\alpha/s\alpha c}$, and $\mathcal{H}_{ed\alpha/a\alpha s}$ are all well defined analogously to (5.1), and they are all closed and invariant under H. They give rise to a total of 14 different possible decompositions of $\mathcal{H}$, such as $\mathcal{H} = \mathcal{H}_{\alpha ds} \oplus \mathcal{H}_{\alpha dc}$, $\mathcal{H} = \mathcal{H}_{\alpha ds} \oplus \mathcal{H}_{ed\alpha} \oplus \mathcal{H}_{s\alpha dc}$, $\mathcal{H} = \mathcal{H}_{\alpha s} \oplus \mathcal{H}_{\alpha ac} \oplus \mathcal{H}_{ed\alpha/s\alpha c} \oplus \mathcal{H}_{s\alpha dc}$, etc. For each of these subspaces there is an associated spectrum, defined as the spectrum of H restricted to the corresponding subspace. At the edges $\alpha = 0$ and $\alpha = 1$, we shall use the terminology introduced in Definition 4.6 and the dimensional decomposition of Corollary 4.1.4. We have $\mathcal{H} = \mathcal{H}_{zd} \oplus \mathcal{H}_{pd}$, where we call $\mathcal{H}_{zd}$ the zero-dimensional subspace and $\mathcal{H}_{pd}$ the positive-dimensional subspace. They are defined in the obvious way, analogously to (5.1). The corresponding spectra obey $\sigma = \sigma_{zd} \cup \sigma_{pd}$. Similarly, we have $\mathcal{H} = \mathcal{H}_{sod} \oplus \mathcal{H}_{od}$, where we call $\mathcal{H}_{sod}$ the sub-one-dimensional subspace and $\mathcal{H}_{od}$ the one-dimensional subspace. The corresponding spectra obey $\sigma = \sigma_{sod} \cup \sigma_{od}$.

For $\alpha > 0$, Theorem 5.2 along with Theorem 3.1 relate $\mathcal{H}_{\alpha c}$ to vectors with power-law decaying survival probabilities, in the sense that $\mathcal{H}_{\alpha c}$ must have a dense subset of vectors for which $\sup_T T^\alpha \langle|\hat{\mu}_\psi|^2\rangle_T < \infty$. The relation is actually more extensive: Given a finite measure μ and $\alpha \in (0,\, 1)$, consider the integral

$$J_\alpha(\mu) \;\equiv\; \int_0^\infty |\hat{\mu}(t)|^2 \, t^{\alpha-1} \, dt \;. \tag{5.7}$$

An elementary calculation (also see, e.g., [12]) shows that

$$\begin{aligned} J_\alpha(\mu) \;&=\; \int_0^\infty dt\, t^{\alpha-1} \int d\mu(x)\, d\mu(y)\, e^{-i(x-y)t} \\ &= \int d\mu(x)\, d\mu(y) \int_0^\infty dt\, t^{\alpha-1}\, e^{-i(x-y)t} \\ &= \int d\mu(x)\, d\mu(y) \int_0^\infty dt\, t^{\alpha-1}\, \cos(|x-y|t) \\ &= \int \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} \int_0^\infty du\, u^{\alpha-1}\, \cos(u) \\ &= C \int \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} \;, \end{aligned} \tag{5.8}$$

where $C = \int_0^\infty du\, u^{\alpha-1}\, \cos(u)$. While (5.8) is only a formal calculation, it can be easily justified rigorously by considering the integration over t from 0 to T and then taking $T \to \infty$. The following lemma is classical:

Lemma 5.1. (See, e.g., Falconer [12]) *If $J_\alpha(\mu) < \infty$, then μ is α-continuous.*

Proof. If $J_\alpha(\mu) < \infty$, then, by (5.8), $\int d\mu(y)/|x-y|^\alpha < \infty$ for a.e. x with respect to μ. Since, for every x and $\epsilon > 0$,

$$\mu((x-\epsilon,\, x+\epsilon)) \;\le\; \epsilon^\alpha \int \frac{d\mu(y)}{|x-y|^\alpha} \;, \tag{5.9}$$

we have $D^\alpha_\mu(x) < \infty$ for a.e. x with respect to μ. ○

We also have:

Lemma 5.2. *If $\sup_T T^\alpha \langle|\hat{\mu}|^2\rangle_T < \infty$, then, for any $0 < \epsilon < \alpha$, $J_{\alpha-\epsilon}(\mu) < \infty$.*

Remark. Note that for every $T > 0$, $T^\alpha \langle|\hat{\mu}|^2\rangle_T \le J_\alpha(\mu)$, and so $J_\alpha(\mu) < \infty$ implies $\sup_T T^\alpha \langle|\hat{\mu}|^2\rangle_T < \infty$.

Proof. Since $|\hat{\mu}(t)|^2 \leq 1$, we clearly have

$$\int_0^1 |\hat{\mu}(t)|^2 \, t^{\alpha-\epsilon-1} \, dt \; < \; \infty \, . \tag{5.10}$$

Since, by assumption, $\langle |\hat{\mu}|^2 \rangle_T < C T^{-\alpha}$, we also have

$$\begin{aligned} \int_1^\infty |\hat{\mu}(t)|^2 \, t^{\alpha-\epsilon-1} \, dt \; &= \; \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} |\hat{\mu}(t)|^2 \, t^{\alpha-\epsilon-1} \, dt \\ &\leq \; \sum_{n=0}^{\infty} (2^n)^{\alpha-\epsilon-1} \int_0^{2^{n+1}} |\hat{\mu}(t)|^2 \, dt \\ &\leq \; \sum_{n=0}^{\infty} (2^n)^{\alpha-\epsilon-1} C (2^{n+1})^{1-\alpha} \\ &= \; C 2^{1-\alpha} \sum_{n=0}^{\infty} 2^{-\epsilon n} \; < \; \infty \, . \quad \bigcirc \end{aligned} \tag{5.11}$$

By combining Lemmas 5.1 and 5.2 with Theorems 3.1 and 5.2, we obtain the following (weak) analog of Theorem 2.4:

Theorem 5.3. *If* $\alpha \in (0, 1)$*, then, for any* $0 < \epsilon < \min\{\alpha, 1-\alpha\}$*,*

$$\overline{\{\psi \mid \sup_T T^{\alpha+\epsilon} \langle |\hat{\mu}_\psi|^2 \rangle_T < \infty \}} \; \subseteq \; \overline{\{\psi \mid J_\alpha(\mu_\psi) < \infty \}} \; \subseteq \; \mathcal{H}_{\alpha c} \, ,$$

$$\mathcal{H}_{\alpha c} \; \subseteq \; \overline{\{\psi \mid \sup_T T^{\alpha} \langle |\hat{\mu}_\psi|^2 \rangle_T < \infty \}} \; \subseteq \; \overline{\{\psi \mid J_{\alpha-\epsilon}(\mu_\psi) < \infty \}} \, .$$

Remarks.

(i) Note that $J_\alpha(\mu_\psi) < \infty$ is the same as $\hat{\mu}_\psi \in L^2(R, t^{\alpha-1} \, dt)$.

(ii) Note that $\hat{\mu} \in L^2(R, dt)$ if and only if $\sup_T T \langle |\hat{\mu}|^2 \rangle_T < \infty$, and so Theorem 2.4 is equivalent to $\mathcal{H}_{ac} \; = \; \overline{\{\psi \mid \sup_T T \langle |\hat{\mu}_\psi|^2 \rangle_T < \infty \}}$.

Proof. $\overline{\{\psi \mid J_\alpha(\mu_\psi) < \infty \}} \subseteq \mathcal{H}_{\alpha c}$ follows from Lemma 5.1, along with the fact that $\mathcal{H}_{\alpha c}$ is closed. $\mathcal{H}_{\alpha c} \subseteq \overline{\{\psi \mid \sup_T T^{\alpha} \langle |\hat{\mu}_\psi|^2 \rangle_T < \infty \}}$ follows from Theorems 3.1 and 5.2. The other two inclusions follow from Lemma 5.2. $\bigcirc$

6. Lower bounds on $\langle\langle |X|^m \rangle\rangle_T$

Our main purpose in this section is to prove the following strengthened version of Theorem 2.6:

Theorem 6.1. *If H is self-adjoint on $\ell^2(Z^d)$ and $P_{\alpha c}\psi \neq 0$, where $P_{\alpha c}$ is the orthogonal projection on $\mathcal{H}_{\alpha c}$; then, for each $m > 0$, there exists a constant $C_{\psi,m}$, depending on ψ and m, such that for every $T > 0$*

$$\langle\langle |X|^m \rangle\rangle_T > C_{\psi,m} T^{m\alpha/d} .$$

Remarks.

(i) By Theorem 4.1, the condition $P_{\alpha c}\psi \neq 0$ is equivalent to the condition that μ_ψ is *not* supported on a set of zero h^α measure.

(ii) As the proof below will show, Theorem 6.1 follows from Theorem 3.2 in a similar way to that by which Corollary 2.3.1 follows from Theorem 2.3.

Proof. Let $\psi_{\alpha c} \equiv P_{\alpha c}\psi$, $\psi_{\alpha s} \equiv P_{\alpha s}\psi = (1-P_{\alpha c})\psi$. By Theorem 4.2, there exist mutually singular Borel measures $\mu_{\psi_{\alpha c},1}$ and $\mu_{\psi_{\alpha c},2}$, such that $d\mu_{\psi_{\alpha c}} = d\mu_{\psi_{\alpha c},1} + d\mu_{\psi_{\alpha c},2}$, where $\mu_{\psi_{\alpha c},1}$ is UαH and $\mu_{\psi_{\alpha c},2}(R) < \frac{1}{2}\|\psi_{\alpha c}\|^2$. Let $S_1 \subseteq R$ be a Borel set which supports $\mu_{\psi_{\alpha c},1}$, and has $\mu_{\psi_{\alpha c},2}(S_1) = 0$. Let P_{S_1} be the spectral projection on S_1, and denote $\psi_1 \equiv P_{S_1}\psi_{\alpha c} = P_{S_1}P_{\alpha c}\psi$, $\psi_2 \equiv (1-P_{S_1})\psi_{\alpha c}+\psi_{\alpha s} = (1-P_{S_1}P_{\alpha c})\psi$. Clearly, $\mu_{\psi_1} = \mu_{\psi_{\alpha c},1}$, such that μ_{ψ_1} is UαH, and we have

$$\|\psi_1\|^2 \;=\; \int d\mu_{\psi_1} \;=\; \int d\mu_{\psi_{\alpha c}} - \int d\mu_{\psi_{\alpha c},2} \;>\; \frac{1}{2}\|\psi_{\alpha c}\|^2 \,, \tag{6.1}$$

$$\psi \;=\; \psi_1 + \psi_2 \,, \qquad \|\psi_1\|^2 + \|\psi_2\|^2 \;=\; \|\psi\|^2 \;=\; 1 \,. \tag{6.2}$$

Let P_N be the projection on a sphere of radius N, defined by (2.5) (we consider N as a continuous variable, allowed to take any positive value). There exists a constant C_d, depending only on the space dimension d, such that:

$$\|P_N\|_1 \;=\; \mathrm{Tr}(P_N) \;<\; C_d N^d \,. \tag{6.3}$$

Thus, it follows from Theorem 3.2 that there exists a constant C_{ψ_1}, such that for any $T,\, N > 0$

$$\langle \|P_N\psi_1(t)\|^2 \rangle_T \;=\; \langle\langle \psi_1(t),\, P_N\psi_1(t) \rangle\rangle_T \;<\; C_{\psi_1}\|P_N\|_1 T^{-\alpha} \;<\; C_{\psi_1} C_d N^d T^{-\alpha} \,. \tag{6.4}$$

For each $T > 0$, we define

$$N_T \equiv \left(\frac{\|\psi_1\|^4 \, T^\alpha}{64 \, C_{\psi_1} C_d} \right)^{1/d} . \tag{6.5}$$

By (6.4) we have

$$\langle \|P_{N_T} \psi_1(t)\|^2 \rangle_T < \frac{\|\psi_1\|^4}{64} , \tag{6.6}$$

and thus

$$\begin{aligned} \langle \|P_{N_T} \psi(t)\|^2 \rangle_T &\le \left\langle \left(\|P_{N_T} \psi_1(t)\| + \|P_{N_T} \psi_2(t)\| \right)^2 \right\rangle_T \\ &\le \left\langle \left(\|P_{N_T} \psi_1(t)\| + \|\psi_2\| \right)^2 \right\rangle_T \\ &\le \left(\sqrt{\langle \|P_{N_T} \psi_1(t)\|^2 \rangle_T} + \|\psi_2\| \right)^2 \\ &< \left(\frac{\|\psi_1\|^2}{8} + \|\psi_2\| \right)^2 = \frac{\|\psi_1\|^4}{64} + \|\psi_2\|^2 + \frac{1}{4} \|\psi_2\| \|\psi_1\|^2 \\ &< \|\psi_2\|^2 + \frac{1}{2} \|\psi_1\|^2 = 1 - \frac{1}{2} \|\psi_1\|^2 . \end{aligned} \tag{6.7}$$

Since

$$\langle \|P_{N_T} \psi(t)\|^2 \rangle_T + \langle \|(1 - P_{N_T}) \psi(t)\|^2 \rangle_T = 1 , \tag{6.8}$$

we obtain

$$\langle \|(1 - P_{N_T}) \psi(t)\|^2 \rangle_T > \frac{1}{2} \|\psi_1\|^2 , \tag{6.9}$$

which implies

$$\langle \langle |X|^m \rangle \rangle_T > \frac{1}{2} \|\psi_1\|^2 N_T^m = \frac{\|\psi_1\|^2}{2} \left(\frac{\|\psi_1\|^4}{64 \, C_{\psi_1} C_d} \right)^{m/d} T^{m\alpha/d} . \quad \bigcirc \tag{6.10}$$

Theorem 6.1 also has a continuum analog, involving Schrödinger operators on $L^2(R^d)$, for which the moments of the position operator are defined by

$$(|X|^m \psi)(x) = |x|^m \psi(x) . \tag{6.11}$$

We need the following notions about classes of potentials on R^d:

Definition 6.1. *Let V be a real-valued measurable (Borel) function on R^d.*
(i) *V is said to lie in the class K_d if and only if:*

(a) $$\lim_{\epsilon \to 0} \left[\sup_x \int_{|x-y| \le \epsilon} |x-y|^{2-d} |V(y)| \, d^d y \right] = 0 , \qquad \textit{if } d \ge 3 .$$

(b) $$\lim_{\epsilon \to 0} \left[\sup_x \int_{|x-y| \le \epsilon} \ln(|x-y|^{-1}) |V(y)| \, d^2 y \right] = 0 , \qquad \textit{if } d = 2 .$$

(c) $$\sup_x \int_{|x-y| \le 1} |V(y)| \, dy < \infty , \qquad \textit{if } d = 1 .$$

(ii) *V is said to lie in the class K_d^{loc} if and only if $\chi_{B_r} V \in K_d$ for every $r < \infty$, where χ_{B_r} is the characteristic function of $B_r \equiv \{x \,|\, |x| \le r\}$.*

For every potential V, we define its positive and negative parts V_+, V_- by

$$V_+(x) \equiv \max\{V(x), 0\} , \qquad V_-(x) \equiv \max\{-V(x), 0\} . \tag{6.12}$$

Theorem 6.2. *Let $H = -\Delta + V$ on $L^2(R^d)$, where $V_+ \in K_d^{loc}$ and $V_- \in K_d$, and suppose that $P_{\alpha c}\psi \neq 0$, where $P_{\alpha c}$ is the orthogonal projection on $\mathcal{H}_{\alpha c}$; then, for each $m > 0$, there exists a constant $C_{\psi,m}$, depending on ψ and m, such that for every $T > 0$*

$$\langle\langle |X|^m \rangle\rangle_T > C_{\psi,m} T^{m\alpha/d} .$$

The following Lemma is a consequence of semi-group kernel inequalities, and is essentially due to Simon [31]:

Lemma 6.1. *Let $H = -\Delta + V$ on $L^2(R^d)$, where $V_+ \in K_d^{loc}$ and $V_- \in K_d$. Let S be a compact subset of R, and denote by P_S the spectral projection on S, and let Q_r be the operator of multiplication by χ_{B_r}, the characteristic function of $B_r \equiv \{x \,|\, |x| \le r\}$; then the operator $P_S Q_r P_S$ is trace class and its trace-norm obeys*

$$\|P_S Q_r P_S\|_1 \le C r^d ,$$

where C is independent of r.

Proof. It follows from Theorem B.9.1 of [31] (as a special case), that $P_S Q_r$ is Hilbert-Schmidt; and while it is not explicitly stated, it's easy to see from the proof in [31] that $\|P_S Q_r\|_2^2 \le C r^d$. Since $P_S Q_r P_S = P_S Q_r Q_r P_S = P_S Q_r (P_S Q_r)^*$, we have

$$\|P_S Q_r P_S\|_1 \;=\; \|P_S Q_r\|_2^2 \;\le\; C r^d \,. \quad \bigcirc \tag{6.13}$$

Proof of Theorem 6.2. This is a variant of the proof of Theorem 6.1: Clearly, we can repeat the initial steps of the proof of Theorem 6.1 and choose S_1 to be compact. (6.1) and (6.2) hold as before. Since P_{S_1} is a projection which commutes with e^{-iHt}, we have for every r

$$\|Q_r \psi_1(t)\|^2 \;=\; \langle \psi_1(t),\, Q_r \psi_1(t)\rangle \;=\; \langle \psi_1(t),\, P_{S_1} Q_r P_{S_1} \psi_1(t)\rangle \,. \tag{6.14}$$

Thus, replacing (6.3) by Lemma 6.1, we obtain an analog of (6.4) with Q_r replacing P_N, and r replacing N. The rest of the proof is essentially identical. $\bigcirc$

7. A pathological example

In this section, we study the Almost Mathieu operator $H_{\beta,\lambda,\theta}$ on $\ell^2(Z)$, defined by:

$$\begin{aligned} H_{\beta,\lambda,\theta} &= \Delta + V_{\beta,\lambda,\theta}\,, \qquad (\Delta\psi)(n) \;=\; \psi(n+1) + \psi(n-1)\,, \\ (V_{\beta,\lambda,\theta}\psi)(n) &= \lambda \cos(2\pi\beta n + \theta)\,\psi(n)\,. \end{aligned} \tag{7.1}$$

We call (following Avron-Simon [3]) an irrational number β a Liouville number, if there exists a sequence of rationals $\{p_n/q_n\}_{n=1}^{\infty}$ with $q_n \to \infty$ as $n \to \infty$, and a constant C, such that

$$\left|\beta - \frac{p_n}{q_n}\right| \;<\; C n^{-q_n} \tag{7.2}$$

for each n.

Theorem 7.1. *If β is a Liouville number, then for any $|\lambda| > 2$ and $\theta \in R$, $H_{\beta,\lambda,\theta}$ has purely zero-dimensional spectrum, namely, all its spectral measures are supported on a set of zero Hausdorff dimension.*

Remarks.

(i) A result of Avron-Simon [3], based on Gordon's theorem, says that under the conditions of Theorem 7.1 $H_{\beta,\lambda,\theta}$ has purely continuous spectrum. Thus, Theorem 7.1 implies that for Liouville β, $|\lambda| > 2$, and any real θ, $H_{\beta,\lambda,\theta}$ has purely zero-dimensional singular continuous spectrum.

(ii) Avron-Simon [3] have already shown that for Liouville β and $|\lambda| > 2$, the spectrum of $H_{\beta,\lambda,\theta}$ is purely singular (and thus singular-continuous) for a.e. θ.

(iii) The spectrum of $H_{\beta,\lambda,\theta}$, under the conditions of Theorem 7.1, is known to be a Cantor set [6] of positive Lebesgue measure [4,23,37].

Our proof of Theorem 7.1 is based on the following:

Proposition 7.1. (Avron, van Mouche, and Simon [4]) *Let $\sigma(\beta,\lambda,\theta)$ denote the spectrum of $H_{\beta,\lambda,\theta}$, then for any θ, $|\lambda| \geq 2$, and a pair of relatively prime integers p, q:*

$$|\sigma(p/q,\lambda,\theta)| \;\leq\; 4\pi\left(\frac{2}{|\lambda|}\right)^{\frac{q}{2}-1} .$$

Remark. Proposition 7.1 is not explicitly given in [4], but it follows immediately from their analysis. Essentially, it follows by duality from the main estimate in their "Proof of Theorem 2."

Consider now a periodic Jacobi matrix of period q, namely, $H = \Delta + V$, where $V(n+q) = V(n) \;\; \forall n$. Recall (or see, e.g., [38]) that the spectrum $\sigma(H)$ of H is the union of q bands (closed intervals), and it is precisely the set of energies for which the discriminant $D(E)$ obeys $|D(E)| \leq 2$. $D(E)$ is defined as the trace of the one-period transfer matrix, namely, $D(E) \equiv \mathrm{Tr}(\Phi_q(E))$, where $\Phi_q(E) \equiv T_q(E)T_{q-1}(E) \;\cdots\; T_1(E)$, and

$$T_n(E) \;\equiv\; \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} . \tag{7.3}$$

$D(E)$ is a polynomial of order q in E, with leading coefficient 1, and q real simple zeroes (one in each band). It is monotone on each band, and obeys $|D(E)| \geq 2$ in each of its $(q-1)$ local extremum points (which occur either inside gaps in $\sigma(H)$, or at points where two bands touch). We need the following technical lemma:

Lemma 7.1. *Let $H = \Delta + V$ be a periodic Jacobi matrix of period q. Consider a band $b = [E_1, E_2]$, of $\sigma(H)$, and let E_{g1} and E_{g2} be the (local) extremum points of the discriminant $D(E)$ just below and above b (one of the E_{gi}'s may be infinite if b is an extremal band). Let E_m be the zero of $D(E)$ inside b, then:*

(i) *For* $E \in (E_2, E_{g2})$, $\dfrac{|D(E)|}{2} > \dfrac{E - E_m}{e(E_2 - E_m)}$.

(ii) *For* $E \in (E_{g1}, E_1)$, $\dfrac{|D(E)|}{2} > \dfrac{E_m - E}{e(E_m - E_1)}$.

Proof. Assume that the band b is a nonextremal band of $\sigma(H)$, and that $D(E)$ is increasing on b, such that $D(E_1) = -2$ and $D(E_2) = 2$. (Otherwise, a simple variant of the proof below would yield the result.) We shall prove statement (i). (ii) is essentially the same. Define:

$$f(E) \equiv \frac{d}{dE}(\ln(D(E))) . \tag{7.4}$$

Since $D(E)$ can be expressed as:

$$D(E) = \prod_{j=1}^{q}(E - E_j) , \tag{7.5}$$

$f(E)$ can be written as:

$$f(E) = \sum_{j=1}^{q} \frac{1}{E - E_j} , \tag{7.6}$$

and we have:

$$f'(E) \equiv \frac{d}{dE} f(E) = -\sum_{j=1}^{q} \frac{1}{(E - E_j)^2} . \tag{7.7}$$

¿From (7.7) we see that:

$$f'(E) < \frac{-1}{(E - E_m)^2} , \tag{7.8}$$

and since E_{g2} is a zero of $f(E)$, we have for every $E \in (E_m, E_{g2})$:

$$f(E) = -\int_{E}^{E_{g2}} f'(E')\, dE' > \int_{E}^{E_{g2}} \frac{dE'}{(E' - E_m)^2} = \frac{1}{E - E_m} - \frac{1}{E_{g2} - E_m} . \tag{7.9}$$

Thus, we obtain for any $E \in (E_2, E_{g2})$:

$$\ln \frac{D(E)}{2} \;=\; \ln D(E) - \ln D(E_2) \;>\; \ln\left(\frac{E - E_m}{E_2 - E_m}\right) - 1 \;, \tag{7.10}$$

from which follows:

$$\frac{D(E)}{2} \;>\; \frac{E - E_m}{e(E_2 - E_m)} \;. \quad \bigcirc \tag{7.11}$$

Proof of Theorem 7.1. Let $\{p_n/q_n\}_{n=1}^{\infty}$ be a sequence of rationals obeying (7.2). Clearly, we can assume that p_n, q_n are relatively prime. For each n, $H_{p_n/q_n,\lambda,\theta}$ is a periodic Jacobi matrix of period q_n, and its spectrum $\sigma(p_n/q_n, \lambda, \theta)$ is a union of q_n bands:

$$\sigma(p_n/q_n, \lambda, \theta) \;=\; \bigcup_{m=1}^{q_n} b_m^n \;, \tag{7.12}$$

where $b_m^n = [E_{m,1}^n, E_{m,2}^n]$. Let $D_{p_n/q_n,\lambda,\theta}(E)$ be the discriminant of $H_{p_n/q_n,\lambda,\theta}$, and let $\{E_m^n\}_{m=1}^{q_n}$ be its zeroes (such that E_m^n is inside the band b_m^n). We define the intervals $\{B_m^n\}_{m=1}^{q_n}$ by

$$B_m^n \;\equiv\; [E_m^n - {q_n}^2(E_m^n - E_{m,1}^n)\,,\; E_m^n + {q_n}^2(E_{m,2}^n - E_m^n)] \;, \tag{7.13}$$

and let

$$S_n \;\equiv\; \bigcup_{m=1}^{q_n} B_m^n \;. \tag{7.14}$$

We define the set S by

$$S \;\equiv\; \liminf_{n\to\infty} S_n \;\equiv\; \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n \;. \tag{7.15}$$

The proof of Theorem 7.1 would now follow from showing:

(i) All the spectral measures of $H_{\beta,\lambda,\theta}$ are supported on S.

(ii) $\dim_{\mathrm{H}}(S) = 0$.

Proof of (i). By general principles (see, e.g., [5]), for a.e. E with respect to any spectral measure of $H_{\beta,\lambda,\theta}$ there exist polynomially bounded solutions of the Schrödinger equation $(H_{\beta,\lambda,\theta} - E)\psi = 0$ (generalized eigenfunctions), obeying, in particular,

$$|\psi(k)| < C(1 + |k|) \;, \tag{7.16}$$

for some constant C. We shall prove (i) by showing that if $E \notin S$, then (7.16) can't be obeyed.

Consider $E \notin S_n$. By Lemma 7.1 this implies

$$\frac{|D_{p_n/q_n,\lambda,\theta}(E)|}{2} \;>\; \frac{{q_n}^2}{e}\,. \tag{7.17}$$

Let $\Phi_{q_n}(E)$ be the one period transfer matrix for $H_{p_n/q_n,\lambda,\theta}$. Since $\mathrm{Tr}(\Phi_{q_n}(E)) = D_{p_n/q_n,\lambda,\theta}(E) > 2$, $\Phi_{q_n}(E)$ has two real eigenvalues of the form $e^{\pm\gamma q_n}$, or $-e^{\pm\gamma q_n}$, and $|D_{p_n/q_n,\lambda,\theta}(E)| = 2\cosh(\gamma q_n)$. We assume that the eigenvalues of $\Phi_{q_n}(E)$ are positive (otherwise the proof is essentially the same), and denote the corresponding normalized eigenvectors by $\vec{u}_\pm$. Let φ be a solution of $(H_{p_n/q_n,\lambda,\theta} - E)\varphi = 0$, obeying $|\varphi(0)|^2 + |\varphi(1)|^2 = 1$, and denote, for each k, $\vec{\varphi}_k \equiv (\varphi(k+1),\varphi(k))^T$. There exist constants a, b such that $\vec{\varphi}_0 = a\vec{u}_+ + b\vec{u}_-$, and thus

$$\vec{\varphi}_{q_n} \;=\; ae^{\gamma q_n}\vec{u}_+ + be^{-\gamma q_n}\vec{u}_-\,, \tag{7.18}$$

$$\vec{\varphi}_{-q_n} \;=\; ae^{-\gamma q_n}\vec{u}_+ + be^{\gamma q_n}\vec{u}_-\,. \tag{7.19}$$

Since the $\vec{u}_\pm$ are normalized, and since $\|\vec{\varphi}_0\| = 1$, we have $|a| + |b| \geq 1$. Suppose that $|a| \geq |b|$, then (7.18) implies $\|\vec{\varphi}_{q_n}\| \geq 2|a|\sinh(\gamma q_n) \geq \sinh(\gamma q_n)$. Similarly, if $|b| \geq |a|$, then (7.19) implies $\|\vec{\varphi}_{-q_n}\| \geq \sinh(\gamma q_n)$, such that in either case

$$\max\{\|\vec{\varphi}_{-q_n}\|, \|\vec{\varphi}_{q_n}\|\} \;\geq\; \sinh(\gamma q_n) \;=\; \sinh\left(\cosh^{-1}(D_{p_n/q_n,\lambda,\theta}(E)/2)\right)\,. \tag{7.20}$$

Consider now a solution ψ of $(H_{\beta,\lambda,\theta} - E)\psi = 0$, obeying $\vec{\psi}_0 = \vec{\varphi}_0$, and let

$$T_k^\beta(E) \;\equiv\; \begin{pmatrix} E - V_{\beta,\lambda,\theta}(k) & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{7.21}$$

By the same telescoping estimate used in proving Gordon's theorem [32], we have

$$\begin{aligned}
\|\vec{\psi}_{q_n} - \vec{\varphi}_{q_n}\| \;&\leq \|T^\beta_{q_n}\ \cdots\ T^\beta_1 - T^{p_n/q_n}_{q_n}\ \cdots\ T^{p_n/q_n}_1\| \\
&\leq q_n\left[\sup_k \max\{\|T^\beta_k\|, \|T^{p_n/q_n}_k\|\}\right]^{q_n-1} \max_{1\leq k\leq q_n} \|T^\beta_k - T^{p_n/q_n}_k\| \\
&\leq q_n(2+|\lambda|+|E|)^{q_n}|\lambda|2\pi q_n|\beta - p_n/q_n| \\
&\leq 2\pi|\lambda|C{q_n}^2(2+|\lambda|+|E|)^{q_n}n^{-q_n}\,,
\end{aligned} \tag{7.22}$$

and similarly

$$\|\vec{\psi}_{-q_n} - \vec{\varphi}_{-q_n}\| \;\leq\; 2\pi|\lambda|C{q_n}^2(2+|\lambda|+|E|)^{q_n}n^{-q_n} \;. \tag{7.23}$$

Thus, (7.20) implies

$$\max\{\|\vec{\psi}_{-q_n}\|, \|\vec{\psi}_{q_n}\|\} \;\geq\; \sinh\left(\cosh^{-1}(D_{p_n/q_n,\lambda,\theta}(E)/2)\right) - 2\pi|\lambda|C{q_n}^2(2+|\lambda|+|E|)^{q_n}n^{-q_n} \;, \tag{7.24}$$

for any solution ψ of $(H_{\beta,\lambda,\theta} - E)\psi = 0$ obeying $\|\vec{\psi}_0\| = 1$. The negative term in the r.h.s. of (7.24) clearly vanishes as $n \to \infty$, and the positive term can be written as $\tanh\left(\cosh^{-1}(D_{p_n/q_n,\lambda,\theta}(E)/2)\right)(D_{p_n/q_n,\lambda,\theta}(E)/2)$. Thus, (7.17) and (7.24) imply that for sufficiently large n

$$\max\{\|\vec{\psi}_{-q_n}\|, \|\vec{\psi}_{q_n}\|\} \;\geq\; \frac{{q_n}^2}{2e} \;. \tag{7.25}$$

If $E \notin S$, then there are infinitely many n's for which $E \notin S_n$ and (7.25) holds. Thus, for such E, there can be no solution of $(H_{\beta,\lambda,\theta} - E)\psi = 0$ which obeys (7.16), and this proves (i).

Proof of (ii). Since a countable union of sets with zero Hausdorff dimension has zero Hausdorff dimension, we see from (7.15) that it is sufficient to prove:

$$\dim_{\mathrm{H}}\left(\bigcap_{n=k}^{\infty} S_n\right) \;=\; 0 \;, \tag{7.26}$$

for each k. Since, for each m, $|B_m^n| = {q_n}^2|b_m^n|$, we have, by Proposition 7.1,

$$|B_m^n| \;\leq\; {q_n}^2|\sigma(p_n/q_n,\lambda,\theta)| \;\leq\; 4\pi{q_n}^2\left(\frac{2}{|\lambda|}\right)^{\frac{q_n}{2}-1} \;. \tag{7.27}$$

Thus, $|B_m^n| \to 0$, uniformly in m, as $n \to \infty$. Moreover, for any positive α we have

$$\sum_{m=1}^{q_n} |B_m^n|^\alpha \;\leq\; (4\pi)^\alpha {q_n}^{1+2\alpha}\left(\frac{2}{|\lambda|}\right)^{\alpha(\frac{q_n}{2}-1)} \;, \tag{7.28}$$

and the r.h.s. of (7.28) clearly vanishes as $n \to \infty$. Since each $S_n = \bigcup_{m=1}^{q_n} B_m^n$ with $n \geq k$ covers $\bigcap_{n=k}^{\infty} S_n$, this implies (7.26) and completes the proof. ◯

Theorem 7.1 indicates that spectral questions arising from the decomposition theory we have presented earlier are answerable. Our interest in it, in the context of the current paper, however, is mainly due to the following:

Theorem 7.2. *Let $\psi = \psi(0) = \delta_0$ and fix $\lambda, \theta \in R$; then there exists a Liouville number β and a sequence $\{T_n\}_{n=1}^{\infty}$, where $T_n \to \infty$ as $n \to \infty$, such that for $H = H_{\beta,\lambda,\theta}$*

$$\langle\langle |X|^2 \rangle\rangle_{T_n} \; > \; \frac{T_n^2}{\log T_n} \, ,$$

for every n.

Remarks.

(i) $\log T_n$ can be replaced by $F(T_n)$, where F is any positive valued monotonely increasing function, obeying $F(T) \to \infty$ as $T \to \infty$. The number β and the sequence $\{T_n\}_{n=1}^{\infty}$ will depend also on F in such case. Thus, Theorem 7.2 shows that the growth rate of $\langle\langle |X|^2 \rangle\rangle_T$ can be arbitrarily close to "ballistic" (at least for some time scales), even when H has purely zero-dimensional spectrum. This indicates that Theorem 6.1 is a strictly one-sided inequality, as far as singularity or continuity with respect to Hausdorff measures is concerned.

(ii) Del Rio et al. [9] use a strengthened variant of Theorem 7.2, which obtains the dynamical estimate uniformly in θ and in some range of suitable rank-one perturbations of $H_{\beta,\lambda,\theta}$, to show that the growth rate of $\langle\langle |X|^2 \rangle\rangle_T$ can be arbitrarily close to "ballistic," even in cases where H has pure-point spectrum with exponentially localized eigenvectors. This shows that Simon's result on the absence of ballistic motion [33] is optimal.

Lemma 7.2. *Let $H_1 = \Delta + V_1$ and $H_2 = \Delta + V_2$ on $\ell^2(Z)$, such that $|V_1(k)|, |V_2(k)| < C$ for all $k \in Z$ and some C. Let $T > 0$ and $\epsilon > 0$ be given, then there exist $L, \delta > 0$ such that if $|V_1(k) - V_2(k)| < \delta$ for all $|k| < L$, then*

$$\left| \langle\langle e^{-iH_1 t}\delta_0 \, , \, |X|^2 e^{-iH_1 t}\delta_0 \rangle\rangle_T \; - \; \langle\langle e^{-iH_2 t}\delta_0 \, , \, |X|^2 e^{-iH_2 t}\delta_0 \rangle\rangle_T \right| \; < \; \epsilon \, .$$

Proof. Clearly, it suffices to prove that

$$\left| \langle e^{-iH_1 t}\delta_0 \, , \, |X|^2 e^{-iH_1 t}\delta_0 \rangle \; - \; \langle e^{-iH_2 t}\delta_0 \, , \, |X|^2 e^{-iH_2 t}\delta_0 \rangle \right| \; < \; \epsilon \qquad (7.29)$$

for all $0 \le t \le T$. We have

$$
\begin{aligned}
&|\langle e^{-iH_1 t}\delta_0\,,\ |X|^2 e^{-iH_1 t}\delta_0\rangle - \langle e^{-iH_2 t}\delta_0\,,\ |X|^2 e^{-iH_2 t}\delta_0\rangle| \\
&\qquad = \left|\sum_{n\in Z} n^2 \left(|\langle \delta_n, e^{-iH_1 t}\delta_0\rangle|^2 - |\langle \delta_n, e^{-iH_2 t}\delta_0\rangle|^2\right)\right| \\
&\qquad \le 2\sum_{n\in Z} n^2 |\langle \delta_n, (e^{-iH_1 t} - e^{-iH_2 t})\delta_0\rangle| \\
&\qquad \le 2\sum_{n\in Z} n^2 \sum_{k=1}^{\infty} \frac{t^k}{k!} |\langle \delta_n, (H_1^k - H_2^k)\delta_0\rangle| \\
&\qquad \le 2\sum_{n\in Z} n^2 \sum_{k\ge|n|} \frac{T^k}{k!} |\langle \delta_n, (H_1^k - H_2^k)\delta_0\rangle|\,,
\end{aligned}
\tag{7.30}
$$

where the last inequality of (7.30) used the fact that the H_j's ($j = 1, 2$) are tridiagonal, such that $\langle \delta_n, H_j^k \delta_0\rangle = 0$ if $k < |n|$. For $k \le L$, $\langle \delta_n, H_j^k \delta_0\rangle$ depends on $V_j(m)$ only for $|m| < L$, and thus, for such k,

$$
|\langle \delta_n, (H_1^k - H_2^k)\delta_0\rangle| \ \le\ k\left(\sup_j \|H_j\|\right)^{k-1} \delta \ \le\ k(2+C)^{k-1}\delta\,. \tag{7.31}
$$

So, by separating the summation on n in the last expression of (7.30) into $|n| \le L$ and $|n| > L$, we see that the r.h.s. of (7.30) is bounded by

$$
2(2L+1)L^2\left[\delta T e^{T(2+C)} + 2\sum_{k>L} \frac{(T(2+C))^k}{k!}\right] \ +\ 4\sum_{|n|>L} n^2 \sum_{k\ge|n|} \frac{(T(2+C))^k}{k!}\,. \tag{7.32}
$$

The terms that are independent of δ in (7.32) are easily seen to vanish as $L \to \infty$; so choose L such that their sum is less than $\epsilon/2$, then choose δ such that the remaining term is also less than $\epsilon/2$. ◯

Proof of Theorem 7.2. We shall prove the existence of an appropriate Liouville number β by constructing its continued fraction expansion [19]:

$$
\beta \ =\ [m_1,\, m_2,\, m_3,\, \ldots\,] \ =\ \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 +\ \cdots}}}\,. \tag{7.33}
$$

We will show that an infinite sequence of quotients, $\{m_n\}_{n=1}^{\infty}$, can be chosen inductively, such that the resulting β, given by (7.33), is a Liouville number, and the dynamical estimate of Theorem 7.2 is obeyed for an appropriate sequence $\{T_n\}_{n=1}^{\infty}$.

Let β be given by (7.33) and denote

$$\beta_n \equiv [m_1, m_2, \ldots, m_n] . \tag{7.34}$$

The β_n's are rational, and for each n there are unique, relatively prime, p_n, q_n, such that $\beta_n = p_n/q_n$. It is well known [19] (and not hard to verify) that the q_n's obey

$$q_{n+1} = m_{n+1}q_n + q_{n-1} , \tag{7.35}$$

and that for each n:

$$|\beta - \beta_n| \leq \frac{1}{q_n q_{n+1}} . \tag{7.36}$$

Thus, if $m_1, m_2, \ldots, m_n$ (and thus β_n) are given, we can always make β arbitrarily close to β_n by choosing m_{n+1} to be sufficiently large.

Suppose now that $m_1, m_2, \ldots, m_n$ are given, and consider the time evolution of $\psi = \psi(0) = \delta_0$ with $H = H_{\beta_n,\lambda,\theta}$. Since $H_{\beta_n,\lambda,\theta}$ is a periodic Jacobi matrix, its spectrum is purely absolutely continuous. Thus, by Theorem 6.1, there exists a constant C_n such that $\langle\langle |X|^2\rangle\rangle_T > C_n T^2$. In particular, we can choose T_n large enough (and as large as we want) such that

$$\langle\langle |X|^2\rangle\rangle_{T_n} > \frac{2T_n^2}{\log T_n} . \tag{7.37}$$

By Lemma 7.2, there exists δ_n such that if $|\beta - \beta_n| < \delta_n$, then for $\psi = \psi(0) = \delta_0$ and $H = H_{\beta,\lambda,\theta}$ we have

$$\langle\langle |X|^2\rangle\rangle_{T_n} > \frac{T_n^2}{\log T_n} . \tag{7.38}$$

Choose $m_{n+1} \geq \max\{n^{q_n}, 1/\delta_n\}$, then clearly $|\beta - \beta_n| < \delta_n$, and thus (7.38) is obeyed. Moreover,

$$|\beta - \beta_n| < n^{-q_n} . \tag{7.39}$$

By continuing this construction indefinitely, (7.39) implies that we obtain a Liouville number, and the dynamical estimate (7.38) is obeyed for the sequence $\{T_n\}_{n=1}^{\infty}$. $\bigcirc$

8. Ergodic Schrödinger operators

In this section, we study ergodic Schrödinger operators on $\ell^2(Z^d)$. That is, we consider families of potentials of the form $\{V_\omega\}_{\omega\in\Omega}$, along with a probability measure dp on Ω, and assume that dp is stationary and ergodic with respect to all of the shift operators $\{T_j\}_{j\in Z^d}$, given by $(T_jV_\omega)(n) = V_\omega(n-j)$. Random, almost periodic, and periodic potentials can all be realized in the context of such families. We are interested in the spectral properties of the family $\{H_\omega\}_{\omega\in\Omega}$, where $H_\omega = \Delta + V_\omega$. An excellent introduction to the spectral theory of such families is given by Cycon et al. [8] (also see [5]). It is well known that, for such a family, there exist subsets of R: σ, σ_{ac}, σ_{sc}, and σ_{pp}, such that for a.e. ω (w.r.t. dp), σ is the spectrum of H_ω, and σ_{ac}, σ_{sc}, and σ_{pp} are, respectively, the absolutely continuous, singular continuous, and pure point spectra of H_ω. Our result in this section is a generalization of this to the various spectra introduced in Section 5, namely:

Theorem 8.1. *For any $\alpha\in[0,1]$, there exist subsets of R: $\sigma_{\alpha ds}$, $\sigma_{ed\alpha/\alpha s}$, $\sigma_{\alpha ac}$, $\sigma_{ed\alpha/s\alpha c}$, and $\sigma_{s\alpha dc}$, such that for a.e. ω they are, respectively, the α-dimension singular, α-singular of exact dimension α, absolutely continuous with respect to h^α, strongly α-continuous of exact dimension α, and strongly α-dimension continuous spectra of H_ω.*

Remark. Theorem 8.1 corresponds to the measure decomposition given by Corollary 4.1.3.

Proof. We shall rely on the proof of the corresponding theorem for the standard decomposition (σ_{ac}, σ_{sc}, σ_{pp}), as given by Cycon et al. [8]. It's easy to see, from the analysis given in [8], that it is sufficient to prove the weak measurability (in ω) of certain spectral projections. More precisely, it is sufficient to show that for each spectral measure, $\mu^\omega=\mu^\omega_\psi$, which is the spectral measure for H_ω and some fixed vector ψ, and for each Borel set A, $\mu^\omega_{\alpha ds}(A)$, $\mu^\omega_{ed\alpha/\alpha s}(A)$, $\mu^\omega_{\alpha ac}(A)$, $\mu^\omega_{ed\alpha/s\alpha c}(A)$, and $\mu^\omega_{s\alpha dc}(A)$ are measurable functions of ω. Since, for any finite Borel measure μ, $\mu_{ed\alpha/\alpha s}=\mu_{\alpha s}-\mu_{\alpha ds}$, $\mu_{\alpha ac}=\mu_{a\alpha s}-\mu_{\alpha s}$, $\mu_{ed\alpha/s\alpha c}=\mu_{a\alpha ds}-\mu_{a\alpha s}$, and $\mu_{s\alpha dc}=\mu-\mu_{a\alpha ds}$; it is clearly sufficient to show that $\mu^\omega(A)$, $\mu^\omega_{\alpha ds}(A)$, $\mu^\omega_{\alpha s}(A)$, $\mu^\omega_{a\alpha s}(A)$, and $\mu^\omega_{a\alpha ds}(A)$ are measurable. Cycon et al. [8] already show that $\mu^\omega(S)$ is measurable for any fixed Borel set S. The theorem would thus follow if we can show that $\mu^\omega_{\alpha ds}(A)$, $\mu^\omega_{\alpha s}(A)$, $\mu^\omega_{a\alpha s}(A)$, and $\mu^\omega_{a\alpha ds}(A)$ can be obtained by suitable countable operations from the restrictions of μ^ω to some fixed countable family of Borel sets. (That is, we need appropriate analogs of the lemma in the bottom of page 170 of [8].)

Let $\mathcal{J}$ be the family of finite unions of intervals, each of which has rational endpoints (note that $\mathcal{J}$ is countable). For each $\mathcal{J} \ni J = \bigcup_{\nu=1}^{N} I_\nu$ (where each I_ν is an interval with rational endpoints), we denote $\delta(J) \equiv \max_\nu |I_\nu|$, $M_\alpha(J) \equiv \sum_{\nu=1}^{N} |I_\nu|^\alpha$. The theorem is now implied by the following equalities, which hold for any Borel set $A \subseteq R$ and a finite Borel measure μ on R:

$$\mu_{\alpha s}(A) \;=\; \lim_{n\to\infty} \sup_{J\in\mathcal{J},\, M_\alpha(J)<1/n} \mu(A\cap J)\;, \tag{8.1}$$

$$\mu_{a\alpha s}(A) \;=\; \lim_{m\to\infty}\lim_{n\to\infty} \sup_{J\in\mathcal{J},\, M_\alpha(J)<m,\, \delta(J)<1/n} \mu(A\cap J)\;, \tag{8.2}$$

$$\mu_{\alpha ds}(A) \;=\; \lim_{m\to\infty}\lim_{n\to\infty} \sup_{J\in\mathcal{J},\, M_{(\alpha-1/m)}(J)<1/n} \mu(A\cap J)\;, \tag{8.3}$$

$$\mu_{a\alpha ds}(A) \;=\; \lim_{m\to\infty}\lim_{n\to\infty} \sup_{J\in\mathcal{J},\, M_{(\alpha+1/m)}(J)<1/n} \mu(A\cap J)\;. \tag{8.4}$$

(8.3) and (8.4) follow immediately from (8.1) and from the definition of $\mu_{\alpha ds}$ and $\mu_{a\alpha ds}$ (see Section 4), which is essentially equivalent to

$$\mu_{\alpha ds} \;=\; \lim_{m\to\infty} \mu_{(\alpha-1/m)s}\;, \qquad \mu_{a\alpha ds} \;=\; \lim_{m\to\infty} \mu_{(\alpha+1/m)s}\;. \tag{8.5}$$

The limit in (8.5) should be understood in the sense that $\mu_m \to \mu$ if $|\mu-\mu_m|(R)\to 0$, and so (8.1) clearly implies (8.3) and (8.4). It remains to prove (8.1) and (8.2).

Proof of (8.1). For each $\epsilon>0$, we have, by statement (i) of Theorem 4.2, $\mu_{\alpha c} = \mu_1^\epsilon + \mu_2^\epsilon$ with $\mu_2^\epsilon(R)<\epsilon$, and μ_1^ϵ UαH, such that $\mu_1^\epsilon(I) < C_\epsilon |I|^\alpha$ for any interval I with $|I|<1$. Thus, for each $J\in\mathcal{J}$ with $M_\alpha(J)<1/n$ we have $\mu(A\cap J) < \mu_{\alpha s}(A) + \epsilon + (C_\epsilon/n)$, from which we see that

$$\lim_{n\to\infty} \sup_{J\in\mathcal{J},\, M_\alpha(J)<1/n} \mu(A\cap J) \;\le\; \mu_{\alpha s}(A)\;. \tag{8.6}$$

Since $\mu_{\alpha s}$ is supported on some set S with $h^\alpha(S)=0$, there are, for each n, disjoint intervals $\{b_\nu\}_{\nu=1}^{\infty}$, such that $S\subseteq \cup_{\nu=1}^{\infty} b_\nu$ and $\sum_{\nu=1}^{\infty} |b_\nu|^\alpha < 1/2n$. Thus, for each $\epsilon>0$, there exists $J\in\mathcal{J}$ with $M_\alpha(J)<1/n$, and $\mu(A\cap J) > \mu_{\alpha s}(A)-\epsilon$. This implies the inverse inequality to (8.6), and thus completes the proof of (8.1).

Proof of (8.2). For each $\epsilon>0$, we have, by statement (ii) of Theorem 4.2, $\mu_{s\alpha c} = \mu_1^\epsilon + \mu_2^\epsilon$ with $\mu_2^\epsilon(R)<\epsilon$, and μ_1^ϵ USαH, such that for each $\eta>0$ there is $\delta>0$ so that $\mu_1^\epsilon(I) <$

$\eta|I|^\alpha$ if $|I| < \delta$. Thus, for sufficiently large n, we have for any $J \in \mathcal{J}$ with $M_\alpha(J) < m$ and $\delta(J) < 1/n$, $\mu(A \cap J) < \mu_{a\alpha s}(A) + \epsilon + \eta m$. From this we see that for each m

$$\lim_{n\to\infty} \sup_{J\in\mathcal{J},\, M_\alpha(J)<m,\, \delta(J)<1/n} \mu(A\cap J) \ \leq\ \mu_{a\alpha s}(A)\ , \tag{8.7}$$

and thus also

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{J\in\mathcal{J},\, M_\alpha(J)<m,\, \delta(J)<1/n} \mu(A\cap J) \ \leq\ \mu_{a\alpha s}(A)\ . \tag{8.8}$$

Since $\mu_{a\alpha s}$ is supported on some set S which has σ-finite h^α measure, there exists, for each $\epsilon > 0$, a set S_ϵ, such that $h^\alpha(S_\epsilon) < \infty$ and $\mu_{a\alpha s}(R \setminus S_\epsilon) < \epsilon$. S_ϵ has covers by disjoint intervals: $S_\epsilon \subseteq \cup_{\nu=1}^\infty b_\nu$, such that $\sup_\nu |b_\nu|$ is arbitrarily small, and $\sum_{\nu=1}^\infty |b_\nu|^\alpha$ is arbitrarily close to $h^\alpha(S_\epsilon)$. Thus, there exists sufficiently large m, such that for any n there is $J \in \mathcal{J}$ with $M_\alpha(J) < m$, $\delta(J) < 1/n$, and $\mu(A\cap J) > \mu_{a\alpha s}(A) - 2\epsilon$. This implies the inverse inequality to (8.8), and thus completes the proof of (8.2). ○

9. Discussion and open problems

The spectral decomposition theory arising from the measure decomposition theory of Rogers and Taylor extends the usual spectral decomposition theory in a natural way, and provides a rich collection of hierarchies, that can be used for spectral classification. Theorems 3.1 and 3.2 along with Theorem 5.2, Theorem 5.3, and Theorems 6.1 and 6.2 indicate that this classification contains some significant information about the dynamical behavior of the underlying quantum systems. Nevertheless, as Theorems 7.1 and 7.2, along with their variants by del Rio et al. [9,10], indicate, the long time behavior of dynamical quantities like $\langle |X|^2 \rangle$ is not completely determined by spectral properties such as singularity or continuity with respect to Hausdorff measures. Del Rio et al. [9,10], who study quantum dynamics in the Anderson localization regime, show that dynamical localization (namely, $\sup_t \langle |X|^2 \rangle < \infty$) is related to pure point spectrum which is accompanied by additional information on the spatial behavior of the eigenfunctions. It is an interesting open problem to obtain some relations of this kind for continuous spectrum. Another interesting open problem is: Are there any purely spectral (i.e., not involving information about spatial behavior of eigenfunctions) properties which imply meaningful (e.g., sub-ballistic by a power law) upper bounds on $\langle |X|^2 \rangle$? In particular, are there such bounds if the spectrum (as a set) is sufficiently "fractal"? For example, while $\dim_{\mathrm{H}}(\sigma(H))$ might not imply any bound on $\langle |X|^2 \rangle$, what happens if $\overline{\dim}_{\mathrm{B}}(\sigma(H))$, the upper box-counting dimension of the spectrum (see [12]), is small? Some recent results [18,39] suggest that even this need not bound $\langle |X|^2 \rangle$.

The above mentioned difficulties need not diminish the interest in classifying singular continuous spectra according to singularity or continuity with respect to Hausdorff measures. Such spectral classification allows one to distinguish between various operators with singular continuous spectrum, and we believe that this should help in improving our understanding of such operators. For example: Simon [35] has recently shown that barrier potentials in one-dimension with sufficiently sparse barriers lead to purely one-dimensional (but singular continuous) spectrum. Del Rio et al. [9,10] show that, while point spectrum can turn into one-dimensional spectrum under rank-one perturbations, the Anderson model (in any space dimension) has semi-stability, in the sense that the singular continuous spectrum arising from localized rank-one perturbations of typical realizations [11,14] is always purely zero-dimensional.

Another aspect of the results in [9] involves the genericity of singular continuous spectrum that has been recently discovered in a variety of contexts (largely by Simon and co-workers [11,21,34]; also see Gordon [14]). In the context of cyclic rank-one perturbations, where singular continuous spectrum has been shown to be generic (i.e., to occur for a dense G_δ of couplings) within every sub-interval of the spectrum in which there is no absolutely continuous spectrum, the results of [9] indicate that there is no specific type (in the sense of classification with respect to Hausdorff measures) of singular continuous spectrum which is generic

Acknowledgments

We would like to thank J. Avron, who should have been a rightful coauthor of this paper, and S. Fishman, for useful discussions that led to this work. We also benefited from discussions with R. del Rio, P. Hislop, S. Jitomirskaya, and B. Simon.

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