Semigroups and Scattering Theory for a Scalar Field on Curved Space-Time

Edward P. Furlani
Millard Fillmore College, SUNY at Buffalo,
Buffalo, New York 14214

Abstract

A classical theory is developed for the time development and scattering of a minimally coupled scalar field on closed spacetimes that evolve from initial, to final static states. The time development is obtained by reformulating the field equation as an abstract Cauchy problem on a Hilbert space. Constraints are imposed on the metric that enable the use of semigroup theory, and the field solution is obtained from Cauchy data via application of a two parameter semigroup of evolution operators. The scattering theory is also formulated on a Hilbert space, and the wave operators and scattering operator are constructed from the evolution operators. It is shown that semigroup theory most readily applies to spacetimes that undergo contraction.

1 Introduction

The time development and scattering of the scalar field on curved space-time are of interest both at a classical level, and as a foundation from which to build a semiclassical quantum theory in which the field itself is quantized in the presence of a classical "background" metric [1] [2]. In the later case, it is useful to reformulate the classical problem in a Hilbert space setting to facilitate the quantization. For static spacetimes, the adaptation to Hilbert space is straightforward [1] [3] [4]. However, this is progressively more difficult for spacetimes with more complex "time dependent" metrics. In this article we consider the time development and scattering of a minimally coupled scalar field on closed spacetimes that evolve from initial, to final static states. To obtain the time development, the field problem is reformulated as an abstract Cauchy problem on a Hilbert space. Once in this form, conditions are imposed on the metric that enable the use of semigroup theory. We prove the existence of a two parameter semigroup of evolution operators, and obtain the field solution as a mapping of Cauchy data. For the scattering theory, we construct the wave operators, and then the scattering operator, all of which are represented in terms of the evolution operators. Lastly, examples are given of metrics to which the theory applies. We show that Semigroup theory most readily applies to closed spacetimes that undergo "contraction" from initial, to final static states.

2 Preliminary Concepts

Let $(\mathcal{M}, g)$ be a globally hyperbolic oriented time-oriented smooth Lorentzian manifold consisting a four-dimensional manifold $\mathcal{M}$ and a smooth metric g with signature $(-1, 1, 1, 1)$ which is of the form

$$g_{\mu\nu}(t,x) = \begin{pmatrix} -1 & 0 \\ 0 & \gamma(t,x) \end{pmatrix}. \tag{1}$$

Thus, $\mathcal{M} \approx R \times \Omega$ and there is a global time coordinate, and a family of Cauchy Surfaces $(\Omega, \gamma(t,x))$ which we restrict to be compact and without boundary. The function $\gamma(t,x) \in C^\infty(R \times \Omega)$ is a smooth Riemannian metric on Ω for each $t \in R$. We further assume that $\gamma(t,x)$ has the following time development,

$$\gamma(t,x) = \begin{cases} \gamma_T(x) & (T \leq t) \\ \tilde{\gamma}(t,x) & (0 < t < T) \\ \gamma_0(x) & (t \leq 0) \end{cases} . \tag{2}$$

Thus, $(\mathcal{M}, g)$ represents a closed universe that evolves from an initial static state $(\Omega, \gamma_0(x))$, to a final static state $(\Omega, \gamma_T(x))$. From this point on, the x dependence of $\gamma(t,x)$ is suppressed when appropriate, and the standard convention is adopted in which Greek subscripts apply to $(\mathcal{M}, g)$ taking values from 0 to 4, and Latin subscripts apply to $(\Omega, \gamma(t))$ and range from 1 to 3.

Since the two manifolds $(\mathcal{M}, g)$ and $(\Omega, \gamma(t))$ have different dimensions and their metrics have different signatures care is needed when discussing operations that are common to both. To avoid confusion, different notation is used to distinguish these operations. Let d^4 be the exterior derivative on $(\mathcal{M}, g)$, and let δ^4 be the codifferential, $\delta^4 = * \, d^4 *$, where $*$ is the star operator that sends p-forms to $(n-p)$-forms and satisfies $(*)^2 = (-1)^{p+1}$. The D'Alembertian is given by $\Box = -(\,\delta^4 d^4 + d^4 \delta^4\,)$.

The exterior derivative on $(\Omega, \gamma(t))$, denoted d, is independent of the metric and is therefore common to all Cauchy surfaces. However, the codifferential $\delta(t) = (-1)^p *_t \, d *_t$ depends on the star operator $*_t$ which, in turn, depends on the metric and is therefore indexed by t ($*_t^2 = 1$). The Laplace-Beltrami operator $\triangle(t) = \delta\,(t)\, d + d\, \delta(t)$ is also indexed. A detailed description of these operators can be found in references 5 and 6.

Let $\mathcal{H}_t(\Omega)$ denote the Hilbert space of complex-valued functions on Ω that are square integrable with respect to the measure $\mu_\gamma(t)$ induced by $\gamma(t)$. Thus, $\mathcal{H}_t(\Omega)$ is endowed with a norm $\|\cdot\|_t = \langle \cdot, \cdot \rangle_t^{\frac{1}{2}}$, defined by the inner product

$$\langle f, g \rangle_t \;=\; \int_\Omega \overline{f}\, g\, \sqrt{\gamma(t)}\; dx^1 \wedge dx^2 \wedge dx^3.$$

We take the operator closures of d, $\delta(t)$ and $\triangle(t) = \delta(t)\, d + d\, \delta(t)$ defined on $C^\infty(\Omega)$ and have these as unbounded Hilbert space operators. The Laplace-Beltrami operator has the following representation in local coordinates,

$$\triangle(t)\, f = \frac{-1}{\sqrt{\gamma(t)}}\, \partial_i \left(\gamma^{ij}(t)\, \sqrt{\gamma(t)}\, \partial_j\, f \right), \tag{3}$$

where $f \in C^\infty(\Omega)$ and $\delta(t)\, f \equiv 0$. It is well known that $\triangle(t)$ extends to a positive self-adjoint operator on $\mathcal{H}_t(\Omega)$, and that this space decomposes as follows [7],

$$\mathcal{H}_t(\Omega) \;=\; Ran(\triangle(t)) \,\oplus\, Ker(\triangle(t)). \tag{4}$$

The subspace $Ker(\triangle(t))$ consists of harmonic functions which are constants for compact manifolds such as $(\Omega, \gamma(t))$.

We introduce another operator $\triangle'(t) = \partial_t \triangle(t)$ on smooth functions,

$$\triangle'(t) \equiv \partial_t \left(\frac{-1}{\sqrt{\gamma(t)}}\, \partial_i \left(\gamma^{ij}(t)\, \sqrt{\gamma(t)}\, \partial_j\, f \right) \right) \,. \tag{5}$$

Since $C^\infty(\Omega) \subset D(\triangle'^*(t))$ (adjoint) is dense in $\mathcal{H}_t(\Omega)$, $\triangle'(t)$ is closable, and we use the same notation to denote its closure.

Finally, note that the spaces $\{\mathcal{H}_t(\Omega)\}_{t\in[0,T]}$ are setwise equivalent. To see this, choose $t,\ t' \in [0,T]$ and let $\mu_\gamma(t),\ \mu_\gamma(t')$ denote the measures induced by $\gamma(t)$ and $\gamma(t')$, respectively. Consider,

$$\int_\Omega |f|^2 \sqrt{\gamma(t)}\ dx^1 \wedge dx^2 \wedge dx^3 = \int_\Omega |f|^2 \frac{\sqrt{\gamma(t)}}{\sqrt{\gamma(t')}} \sqrt{\gamma(t')}\ dx^1 \wedge dx^2 \wedge dx^3,$$

where $\frac{\sqrt{\gamma(t)}}{\sqrt{\gamma(t')}}$, which is smooth, bounded, and strictly positive, is the Radon Nikodym derivative of $\mu_\gamma(t)$ with respect to $\mu_\gamma(t')$. Thus the measures $\{\mu_\gamma(t)\}_{t\in[0,T]}$ are mutually absolutely continuous, and since $\frac{\sqrt{\gamma(t)}}{\sqrt{\gamma(t')}}$ is bounded, $f \in \mathcal{H}_t(\Omega) \Leftrightarrow f \in \mathcal{H}_{t'}(\Omega)$. Having established these preliminary results we turn to the field problem.

3 The Cauchy Problem

In this section, we obtain an abstract solution to the minimally coupled Klein Gordon equation

$$\Box\phi + m^2\phi = 0, \tag{6}$$

where $m \in (0,\infty)$. For a metric of the form (1), this equation reduces to

$$\partial_t^2\phi \ + K(t)\,\phi = \ 0, \tag{7}$$

where

$$K(t) = \triangle(t) + m^2. \tag{8}$$

Note that $K(t)$ is strictly positive, self-adjoint and injective on $\mathcal{H}_t$ with $D(K(t)) = D(\triangle(t))$. Therefore, $K^{-1}(t)$ is positive, self-adjoint and bounded, and $K^{\pm\frac{1}{n}}(t)$ are positive and self-adjoint for $n = 2, 4, \ldots$.

The second-order equation (7) can be written as a first-order system,

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi(t)\\ \pi(t)\end{pmatrix} \ = \ -iH(t)\begin{pmatrix}\phi(t)\\ \pi(t)\end{pmatrix}, \tag{9}$$

where $\pi(t) = \partial_t \phi(t)$, and

$$H(t) = \; i \begin{pmatrix} 0 & I \\ -K(t) & 0 \end{pmatrix} . \tag{10}$$

It is well known that $H(t)$ with $D(H(t)) = D(K(t)) \oplus D(K^{\frac{1}{2}}(t))$ is self-adjoint on the Hilbert space

$$\mathcal{H}_{K(t)}(\Omega) \;\equiv\; D(K^{\frac{1}{2}}(t)) \oplus \mathcal{H}_t(\Omega), \tag{11}$$

with inner product

$$\langle\, F \,, G \,\rangle_{K(t)} \;\equiv\; \langle\; K^{\frac{1}{2}}(t)\, f_1 \,,\, K^{\frac{1}{2}}(t)\, g_1 \;\rangle_t \;+\; \langle\; f_2 \,,\, g_2 \;\rangle_t, \tag{12}$$

where $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}_{K(t)}$ [9] [10]. Since $H(t)$ is self-adjoint it gives rise to a group of unitary operators $W_t(s) \equiv \exp(-i\, H(t)\, s\,)$ on $\mathcal{H}_{K(t)}$. These operators can also be viewed as a contraction semigroup with generator $-i\, H(t)$ for $s \geq 0$. Therefore, from the Hille-Yosida theorem we have

$$\|\; R(\lambda : -i\, H(t))\, F \,\|_{_{K(t)}} \;\leq \frac{1}{\lambda} \|\;\; F \,\|_{_{K(t)}} \;\;, \tag{13}$$

where $R(\lambda \,:\, -i\, H(t)) \,=\, [\lambda \;I - \;(-iH(t))]^{-1}$ (resolvent) and $\lambda \in (0, \infty) \subset \rho(-i\, H(t))$ (resolvent set) [11].

It is important to note that the above results apply for $H(t)$ on $\mathcal{H}_{K(t)}$ for each $t \in [0, T]$. However, the goal is to reformulate (9) as a "time-dependent" operator equation on a single Hilbert space (independent of t), thereby transforming the classical dynamics to an abstract Cauchy problem. Once in this form, we apply standard results from the theory of semigroups to obtain the solution. Before we proceed, a brief review of these results is in order [11] [12].

Definition 1 *Let X be a Banach space. A family $U(t)$ of linear operators from X to X is called a C_0 semigroup if for each $F \in X$,*

$$\|U(t)\, F\,\|_X < \infty\,,$$

$$U(t+s)\, F = U(t)U(s)\, F\,,$$

$$U(0)\, F = F\,,$$

$$t \rightarrow U(t)\, F is\ continuous\ for t \geq 0\,.$$

Definition 2 *Let X be a Banach space. A family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of C_0 semigroups on X is called stable if there are constants $B \geq 1$ and β (stability constants) such that*

$$(\beta, \infty) \subset \rho(-i\,H(t))\,, \tag{14}$$

for $t \in [0,T]$ and

$$\left\| \, \Pi_{j=1}^{k} R(\lambda : A(t_j)) \right\|_X \leq \frac{B}{\lambda^k}\,, \tag{15}$$

for every finite sequence $0 \leq t_1 \leq t_2, \ldots, t_k \leq T$, $k = 1, 2, \ldots$. In (15) the product of resolvent operators is "time-ordered" with the factors that contain larger t_j to the left of ones with smaller t_j [12].

The following theorem establishes the criteria that are sufficient for the existence of an abstract solution to evolution equations of the form

$$\partial_t\, F(t) = A(t)\, F(t). \tag{16}$$

Theorem 1 *Let $\{A(t)\}_{t\in[0,T]}$ be a stable family of infinitesimal generators of C_0 semigroups on a Banach space X with stability constants β and B. If $D(A(t)) \equiv D(A)$ is independent of t and for each $F \in D(A)$, $A(t)\,F$ is continuously differentiable in X then there exists a unique two-parameter semigroup of evolution operators $U(t,s)$, $0 \leq s \leq t \leq T$ satisfying*

$$\|U(t,s)\|_X \leq B \exp(\beta\,(t-s)\,),$$

for $0 \leq s \leq t \leq T$, and

$$U(t,s)\, D(A) \subset D(A)\,,$$

$$\partial_t\, F(t) = A(t)\, F(t)\,,$$

where $F(t) = U(t,s)\ F(s)$, and $F(s)$ is the Cauchy data at $t = s$ [12].

We return now to the first-order equation (9). To apply Theorem 1 we first identify the operator $-i\,H(t)$ with $A(t)$ and the "out" Hilbert space $\mathcal{H}_{K(T)}$ with X. The next step is to establish conditions on the metrics $\{\gamma(t)\}_{t\in[0,T]}$ that satisfy the criteria of Theorem 1.

Metric Conditions: Let $f \in \mathcal{H}_t(\Omega)$ and $g \in C^\infty(\Omega)$. We assume that the family of metrics $\{\gamma(t)\}_{t\in[0,T]}$ satisfies the following conditions:

$$\| f \|_T^2 \leq \| f \|_{t'}^2 \leq \| f \|_t^2 \leq M_T \| f \|_T^2 \ , \tag{17}$$

$$\left\| \triangle^{\frac{1}{2}}(T)\, g \right\|_T^2 \leq \left\| \triangle^{\frac{1}{2}}(t')\, g \right\|_{t'}^2 \leq \left\| \triangle^{\frac{1}{2}}(t)\, g \right\|_t^2 \leq \widetilde{M}_{\triangle^{\frac{1}{2}}} \left\| \triangle^{\frac{1}{2}}(T)\, g \right\|_T^2 \ , \tag{18}$$

for $0 \leq t \leq t' \leq T$, with constants M_T, $\widetilde{M}_{\triangle^{\frac{1}{2}}} \geq 1$, and

$$\| \triangle(t)\, g \|_t^2 \leq M_\triangle(t,t') \ \| \triangle(t')\ g \|_{t'}^2 + M_{\triangle^{\frac{1}{2}}}(t,t') \left\| \triangle^{\frac{1}{2}}(t')\, g \right\|_{t'}^2 \ , \tag{19}$$

$$\| \triangle'(t)\, g \|_t^2 \leq M'_\triangle \ \| \triangle(\tau)\, g \|_\tau^2 + M'_{\triangle^{\frac{1}{2}}} \left\| \triangle^{\frac{1}{2}}(\tau)\, g \right\|_\tau^2 , \tag{20}$$

$\forall\ t,\ t' \in [0,T]$ and some $\tau \in [0,T]$, where $M_\triangle(t,t')$, $M'_\triangle > 0$, and $M_{\triangle^{\frac{1}{2}}}(t,t')$, $M'_{\triangle^{\frac{1}{2}}} \geq 0$.

Remark 1 *The metric conditions are related to the criteria of Theorem 1 as follows: Conditions (17) - (19) imply that the family of operators $\{-i\, H(t)\}_{t\in[0,T]}$ is stable, and the domain $D(-i\, H(t))$ is independent of t. Condition (20) implies that $H(t)F$ is continuously differentiable on $\mathcal{H}_{K(T)}$.*

Remark 2 *The first condition (17) implies that the "volume" $V(t) \equiv \|1\|_t^2$ of the Cauchy surface $(\Omega, \gamma(t))$ is either constant, or contracts as a function of time.*

Conditions (17) - (19) imply that the domains $D(K(t))$ and $D(K^{\frac{1}{2}}(t))$ are independent of time, i.e., $D(K(t)) = D(\triangle(t)) \equiv D(\triangle)$ and $D(K^{\frac{1}{2}}(t)) = D(\triangle^{\frac{1}{2}}(t)) \equiv D(\triangle^{\frac{1}{2}})$ (Appendix). This, in turn, implies that the spaces the spaces $\left\{\mathcal{H}_{K(t)}\right\}_{t\in[0,T]}$ are setwise equivalent, and

Proposition 1 *The domain $D(H(t)) \equiv D(H) = D(\triangle) \oplus D(\triangle^{\frac{1}{2}})$ is independent of t.*

Proof: Recall that $D(H(t)) = D(K(t)) \oplus D(K^{\frac{1}{2}}(t))$ and apply Propositions 5 and 6 of the Appendix. $\Box$

Another consequence of the metric conditions is

Proposition 2 *Let $F \in \mathcal{H}_{K(T)}$, if a family of metrics $\{\gamma(t)\}_{t\in[0,t]}$ satisfies the metric conditions (17), and (18) then*

$$\| F \|^2_{\mathcal{H}_{K(T)}} \leq \| F \|^2_{\mathcal{H}_{K(t')}} \leq \| F \|^2_{\mathcal{H}_{K(t)}} \leq \widetilde{M} \| F \|^2_{\mathcal{H}_{K(T)}} , \tag{21}$$

for $0 \leq t \leq t' \leq T$, where $\widetilde{M} \geq 1$.

Proof: First, note that (18) holds for all $f \in D(\triangle^{\frac{1}{2}}(t)) = D(\triangle^{\frac{1}{2}})$ independent of t. Let $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{H}_{K(T)} = D(K^{\frac{1}{2}}(t)) \oplus \mathcal{H}_t$, and consider,

$$\begin{aligned} \| F\|^2_{\mathcal{H}_{K(t)}} &= \left\| K^{\frac{1}{2}}(t)\, f_1 \right\|^2_t + \| f_2\|^2_t , \\ &= \left\| \triangle^{\frac{1}{2}}(t')\ f_1 \right\|^2_t + m^2 \| f_1 \|^2_t + \| f_2 \|^2_t . \end{aligned} \tag{22}$$

Equation (21) follows immediately from (17), (18), and (22) with $\widetilde{M} = \max\{M, \widetilde{M}_{\triangle^{\frac{1}{2}}}\}$. $\Box$

Note that (21) implies that

$$\frac{1}{\widetilde{M}} \| F \|^2_{\mathcal{H}_{K(t)}} \leq \| F \|^2_{\mathcal{H}_{K(t')}} \leq \widetilde{M} \| F \|^2_{\mathcal{H}_{K(t)}} , \tag{23}$$

$\forall\, t, t' \in [0,T]$, and therefore the spaces $\left\{\mathcal{H}_{K(t)}\right\}_{t\in[0,T]}$ have equivalent norms.

The following propositions are needed for the application of Theorem 1.

Proposition 3 *The operators $\{-iH(t)\}_{t\in[0,T]}$ represent a stable family of infinitesimal generators of C_0 semigroups on $\mathcal{H}_{K(T)}$.*

Proof: We verify the criteria of Definition 2. First, recall that $-i\,H(t)$ is the infinitesimal generator of a C_0 contraction semigroup $W_t(s) \equiv \exp(-i\,H(t)\,s)$ on $\mathcal{H}_{K(t)}$. This and (21) imply that $W_t(s)$ is a C_0 semigroup on $\mathcal{H}_{K(T)}$. Hence $\{-iH(t)\}_{t\in[0,T]}$ is a family of infinitesimal generators of C_0 semigroups on $\mathcal{H}_{K(T)}$. It remains to show that this family is stable.

Let $\rho_t(-i\,H(t))$ and $\rho_T(-i\,H(t))$ denote the resolvent sets of $-i\,H(t)$ as an operator on $\mathcal{H}_{K(t)}$ and $\mathcal{H}_{K(T)}$, respectively. From the Hille-Yosida theorem we have $(0,\infty) \subset \rho_t(-i\,H(t))\ \forall\ t \in [0,T]$. This implies that $[\lambda - (-i\,H(t))]$ is bijective on $\mathcal{H}_{K(t)}$ for $\lambda \in (0,\infty)$. Since $\mathcal{H}_{K(t)}$ and $\mathcal{H}_{K(T)}$ are setwise equivalent we have $[\lambda - (-i\,H(t))]$ bijective on $\mathcal{H}_{K(T)}$ for $\lambda \in (0,\infty)$ and thus

$$(0,\infty) \subset \rho_T(-i\,H(t)), \tag{24}$$

$\forall\ t \in [0,T]$. Also, from (13) and (17) it follows that,

$$\begin{aligned} \|\ R(\lambda : -i\,H(t))\ F\ \|_{_{K(T)}} &\leq \|\ R(\lambda : -i\,H(t))\ F\ \|_{_{K(t)}}\ , \\ &\leq \frac{1}{\lambda}\ \|\ F\ \|_{_{K(t)}}\ , \end{aligned} \tag{25}$$

$\forall\ t \in [0,T]$. Let $0 \leq t_1 \leq t_2, \ldots, t_k \leq T$ be a time ordered partition of $[0,T]$. From (25) and repeated application of (13) and (17) we have,

$$\begin{aligned} \left\|\ \Pi_{j=1}^{k}\, R(\lambda : -i\,H(t_j))\ F\ \right\|_{_{K(T)}} &\leq \left\|\ \Pi_{j=1}^{k}\, R(\lambda : -i\,H(t_j))\ F\ \right\|_{_{K(t_k)}}\ , \\ &\leq \frac{1}{\lambda}\ \left\|\ \Pi_{j=1}^{k-1}\, R(\lambda : -i\,H(t_j))\ F\ \right\|_{_{K(t_k)}}\ , \\ &\leq \frac{1}{\lambda}\ \left\|\ \Pi_{j=1}^{k-1}\, R(\lambda : -i\,H(t_j))\ F\ \right\|_{_{K(t_{k-1})}}\ , \\ &\leq \frac{1}{\lambda^2}\ \left\|\ \Pi_{j=1}^{k-2}\, R(\lambda : -i\,H(t_j))\ F\ \right\|_{_{K(t_{k-1})}}\ , \\ &\ \ \vdots \\ &\leq \frac{1}{\lambda^k}\ \|\ F\ \|_{_{K(t_1)}}\ , \\ &\leq \frac{\sqrt{\widetilde{M}}}{\lambda^k}\ \ \|\ F\ \|_{_{K(T)}}\ . \end{aligned}$$

Thus $\{-iH(t)\}_{t\in[0,T]}$ is a stable family of operators with stability constants $\beta = 0$ and $B = \sqrt{\widetilde{M}}$. $\square$

Proposition 4 *For each $F \in D(H(t)) = D(H)$, $-iH(t)\,F$ is continuously differentiable.*

Proof: Let $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in D(H) = D(\triangle) \oplus D(\triangle^{\frac{1}{2}})$. We want to show that there is a vector $F'(t) \in \mathcal{H}_{K(T)}$ that is continuous with respect to t, and satisfies

$$\lim_{h \to 0} \left\| \frac{-iH(t+h)\,F - (-iH(t)\,F)}{h} - F'(t) \right\|_{\mathcal{H}_{K(T)}} = 0.$$

Since,

$$-iH(t+h)\,F - (-iH(t)\,F) = -i \begin{pmatrix} 0 \\ \triangle(t+h)\,f_1 - \triangle(t)\,f_1 \end{pmatrix},$$

the proof reduces to showing that $\triangle(t)\,f$ is continuously differentiable. Let $f \in D(\triangle(t))$ and let $\{f_n\}$ be a Cauchy sequence of smooth functions such that

$$\lim_{n \to \infty} \|f_n - f\|^2_{G(\triangle(t))} = 0. \tag{26}$$

Consider the sequence $\{\triangle'(t) f_n\}$. ¿From (17) and (20) we have

$$\begin{aligned} \|\triangle'(t)(f_n - f_m)\|^2_T &\leq \|\triangle'(t)(f_n - f_m)\|^2_t\,, \\ &\leq M'_\triangle\, \|\triangle(\tau)(f_n - f_m)\|^2_\tau + M'_{\triangle^{\frac{1}{2}}} \left\|\triangle^{\frac{1}{2}}(\tau)(f_n - f_m)\right\|^2_\tau, \\ &\leq \|\triangle(\tau)(f_n - f_m)\|^2_\tau \left(M'_\triangle + M'_{\triangle^{\frac{1}{2}}}\, \|(f_n - f_m)\|^2_\tau\right). \end{aligned} \tag{27}$$

However, (26), which is true for $\tau \in [0, T]$, implies that the right hand side of (27) tends to zero (Proposition 5). Therefore, given any $\epsilon > 0$, there exists $N(\epsilon)$ independent of t such that

$$\|\triangle'(t)(f_n - f_m)\|_T < \epsilon,$$

as long as $n, m \geq N(\epsilon)$, which shows that $\{\triangle'(t) f_n\}$ is uniformly convergent with respect to $\|\cdot\|_T$ for $t \in [0, T]$. Now, since $\left\{f'_n(t)\right\} \equiv \{\triangle'(t) f_n\}$ is uniformly convergent on $[0, T]$, and $\{f_n(t)\} \equiv \{\triangle(t) f_n\}$ converges to $f(t) \equiv \triangle(t) f_n$ for each $t \in [0, T]$, it follows that $f(t)$ is differentiable with $f'_n(t) \to f'(t)$ [14] [15]. Moreover, since each element of $\left\{f'_n(t)\right\}$ is continuous, $f'(t)$ is also continuous, and therefore $\triangle(t) f$ is continuously differentiable. $\square$

We are finally ready to prove the existence of the evolution operators.

Theorem 2 *Let* $\{\gamma(t)\}_{t\in[0,T]}$ *be a family of metrics satisfying the metric conditions (17)–(20) then there exists a unique two-parameter semigroup of evolution operators* $\mathcal{U}(t,s)$, $0 \le s \le t \le T$ *satisfying*

$$\|\mathcal{U}(t,s)\|_{\mathcal{H}_{K(T)}} \le \widetilde{M} \qquad for \qquad 0 \le s \le t \le T, \tag{28}$$

$$\mathcal{U}(t,s)\, D(H) \subset D(H), \tag{29}$$

$$\partial_t\, F(t) = -iH(t)\, F(t), \tag{30}$$

where

$$H(t) = \ i \begin{pmatrix} 0 & I \\ -K(t) & 0 \end{pmatrix},$$

with $D(H(t)) = D(H)$ *independent of time, and* $F(t) = \mathcal{U}(t,s)\ F(s)$, *where* $F(s)$ *is the Cauchy data at* $t = s$.

Proof: We verify that the criteria of Theorem 1 are satisfied, i.e., that $\{-iH(t)\,\}_{t\in[0,T]}$ is a stable family of infinitesimal generators of C_0 semigroups on $\mathcal{H}_{K(T)}$, $D(-iH(t)) \equiv D(H)$ independent of t, and for each $F \in D(H)$, $-iH(t)\,F$ is continuously differentiable in $\mathcal{H}_{K(T)}$. These were proven in Propositions 1, 3 and 4, respectively. $\Box$

Theorem 2 gives the evolution operator for the period $0 \le s \le t \le T$. The evolution operators for $t \le s \le 0$ and $T \le s \le t$ are the "free" operators

$$\mathcal{U}_0(t,s) \ = \exp(-iH(0)\,(t-s)), \tag{31}$$

and

$$\mathcal{U}_T(t,s) \ = \exp(-iH(T)\,(t-s)), \tag{32}$$

respectively. Thus, there are three distinct evolution operators corresponding to three different epochs, i.e.,

$$U(t,s) \ = \begin{cases} \mathcal{U}_T(t,s) & (T \le s \le t) \\ \mathcal{U}(t,s) & (0 \le s \le \ t \le T) \\ \mathcal{U}_0(t,s) & (t \le s \le 0) \end{cases} . \tag{33}$$

To obtain the time evolution between arbitrary Cauchy surfaces, one needs to take the appropriate combinations of these operators.

Finally, for the scattering problem we need the inverse of $\mathcal{U}_T(t,s)$ and $\mathcal{U}(t,s)$. Since $H(T)$ is self-adjoint on $\mathcal{H}_{K(T)}$, $\mathcal{U}_T(t,s)$ is unitary with inverse

$\mathcal{U}_T^{-1}(t,s) = \mathcal{U}_T(s,t)$. The operator $\mathcal{U}(t,s)$ also has an inverse. To see this, it suffices to show that

$$\mathcal{U}(t,s)F^1(s) = \mathcal{U}(t,s)F^2(s) \Longrightarrow F^1(s) = F^2(s). \tag{34}$$

for real-valued $F^1(s)$ and $F^2(s)$. To this end, consider the symplectic form,

$$\Lambda_t(F,G) \equiv \left\langle \overline{f_1}, g_2 \right\rangle_t - \left\langle \overline{f_2}, g_1 \right\rangle_t. \tag{35}$$

It is well known that real-valued solutions of hyperbolic equations on globally hyperbolic manifolds preserve this form, i.e., $\Lambda_t(F(t),G(t)) = \Lambda_s(F(s),G(s))$ where $F(t) = \mathcal{U}(t,s)F(s)$, $G(t) = \mathcal{U}(t,s)G(s)$, and $F(s)$, $G(s)$ are real-valued [16] [17] [18] [19]. To prove (34) let $F(s) = F^1(s) - F^2(s)$, $F(t) = \mathcal{U}(t,s)F(s)$ and $G(s)$ be any smooth real-valued vector. Assume $F(t) = 0$ and consider,

$$\Lambda_t(F(t),G(t)) = 0 \Longrightarrow \Lambda_s(F(s),G(s)) = 0.$$

Since $G(s)$ is arbitrary we have $F(s) = 0$. Thus, $\mathcal{U}(t,s)$ is injective with inverse $\mathcal{U}^{-1}(t,s)$. Having obtained the classical dynamics, we proceed to the scattering theory.

4 Scattering Theory

The goal of scattering theory is to compare the asymptotic behavior of a field solution as $t \to -\infty$ to its asymptotic behavior as $t \to +\infty$, that is, to construct the scattering operator. Note that this scattering problem is different from what one usually encounters in two respects. First, a typical scattering problem entails an asymptotic comparison of a free and interacting dynamics which are described by evolution operators, say $\mathcal{U}_0(t,0)$ and $\mathcal{U}(t,0)$, respectively. However, in our case the 'interacting" evolution operator $\mathcal{U}(t,0)$ is aysmptotically compared to two "different" free evolution operators $\mathcal{U}_0(t,0)$ and $\mathcal{U}_T(t,0)$ as $t \to -\infty$ and $t \to \infty$, respectively. Second, the "generator" of the "free" dynamics usually has a continuous spectrum which reflects the fact there are no bound states. However, in our case the generators of the "free" dynamics have discrete spectra (the Laplace-Beltrami operator has a discrete spectrum on compact manifolds). Nevertheless, scattering occurs in the sense that the time evolution of the metric perturbs the time development of the field. Also, note that this problem is formally similar to an external field problem in which the field is "turned on" for a finite period of time [13].

To develop the scattering theory, we show that for any choice of data $\Phi \in \mathcal{H}_{K(T)}$ for the interacting dynamics $\Phi(t) = \mathcal{U}(t,0)\,\Phi$ there are unique "free" fields $\Phi_{in}(t)$, and $\Phi_{out}(t)$ that describe the asymptotic behavior of $\Phi(t)$ in the distant past and future, respectively, i.e.,

$$\lim_{t\to-\infty} \|\Phi_{in}(t) - \Phi(t)\|_{\mathcal{H}_{K(T)}} = 0, \tag{36}$$

and

$$\lim_{t\to\infty} \|\Phi_{out}(t) - \Phi(t)\|_{\mathcal{H}_{K(T)}} = 0. \tag{37}$$

where

$$\Phi_{in}(t) = \mathcal{U}_0(t,0)\,\Phi_{in}, \tag{38}$$

and

$$\Phi_{out}(t) = \mathcal{U}_T(t,0)\,\Phi_{out}. \tag{39}$$

With this in mind, we define the "in" and "out" spaces

$$\mathcal{H}_{in} = \left\{\Phi \in \mathcal{H}_{K(T)} : \exists\ \Phi_{in} \in \mathcal{H}_{K(T)} \text{ with } \lim_{t\to-\infty} \Phi_{in}(t) - \Phi(t) = 0\right\},$$

and

$$\mathcal{H}_{out} = \left\{\Phi \in \mathcal{H}_{K(T)} : \exists\ \Phi_{out} \in \mathcal{H}_{K(T)} \text{ with } \lim_{t\to+\infty} \Phi_{out}(t) - \Phi(t) = 0\right\}.$$

Note that throughout this section we choose the reference data to be at time zero; however, this is arbitrary and the same analysis applies to reference data on any other Cauchy surface.

The analysis begins with the construction of the wave operators W_{in} and W_{out} that map the data Φ of the perturbed solution to the data of the "in" and "out" field, respectively, i.e., $\Phi_{in} = W_{in}\,\Phi$ and $\Phi_{out} = W_{out}\,\Phi$. We treat (36) first. This case is trivial because $\mathcal{U}(t,0) = \mathcal{U}_0(t,0)$ for $t \le 0$. Thus, $\Phi_{in} = \Phi$ and

$$W_{in} = I. \tag{40}$$

Note, $\mathcal{H}_{in} = Ran(W_{in}^{-1}) = D(W_{in}) = \mathcal{H}_{K(T)}$.

Next, we treat (37). This case is also easy since $\mathcal{U}(t,s) = \mathcal{U}_T(t,s)$ for $T \leq s \leq t$. One can readily verify that

$$W_{out} = \mathcal{U}_T^{-1}(T,0)\,\mathcal{U}(T,0). \tag{41}$$

Since $\mathcal{U}_T^{-1}(T,0)$ is bounded, W_{out} makes sense. It is easy to check that $W_{out}^{-1} = \mathcal{U}^{-1}(T,0)\,\mathcal{U}_T(T,0)$ is also bounded. Also, $\mathcal{H}_{out} = Ran(W_{out}^{-1}) = D(W_{out}) = \mathcal{H}_{K(T)}$, and therefore

$$\mathcal{H}_{in} = \mathcal{H}_{out} = \mathcal{H}_{K(T)}.$$

Finally, we define the classical scattering operator $S_{cl} \equiv W_{out} W_{in}^{-1}$,

$$S_{cl} = \mathcal{U}_T^{-1}(T,0)\,\mathcal{U}(T,0). \tag{42}$$

Note that,

$$S_{cl}\Phi_{in} = \Phi_{out}, \tag{43}$$

and therefore S_{cl} correlates the data for the past and future asymptotics of the perturbed dynamics. The construction of this operator completes the scattering theory.

5 Examples

To apply the theory developed above, one needs to determine whether a given metric satisfies conditions (17) - (20). To demonstrate the applicability of the theory, we give two examples of metrics that satisfy these conditions.

For the first example, we assume that the family of metrics is of the form $\{\gamma(t,x)\}_{t\in[0,T]} = \{\alpha(t)\,\gamma(x)\}_{t\in[0,T]}$. Thus the line element for this spacetime is of the form,

$$ds = -dt^2 + \alpha(t)\,\gamma_{ij}(x)\,dx^i dx^j. \tag{44}$$

Robertson-Walker spacetimes (e.g. de Sitter space) have this form of conformally static metric, however for those metrics $\alpha(t)$ is not subject to the constraints imposed here [20]. Specifically, we assume that $\alpha(t) > 0$ is a smooth function with

$$\alpha(t,x) = \begin{cases} 1 & t \leq 0 \\ \alpha(T) & t \geq T \end{cases}.$$

The metric conditions (17) - (20) reduce to the following constraints on $\alpha(t)$,

$$\alpha^{\frac{3}{2}}(T) \leq \alpha^{\frac{3}{2}}(t') \leq \alpha^{\frac{3}{2}}(t) \leq M_T\, \alpha^{\frac{3}{2}}(T), \tag{45}$$

$$\alpha^{\frac{5}{2}}(T) \leq \alpha^{\frac{5}{2}}(t') \leq \alpha^{\frac{5}{2}}(t) \leq \widetilde{M}_{\triangle^{\frac{1}{2}}}\, \alpha^{\frac{5}{2}}(T), \tag{46}$$

for $0 \leq t \leq t' \leq T$, and

$$\alpha^{\frac{7}{2}}(t) \leq M_{\triangle}(t,t')\, \alpha^{\frac{7}{2}}(t') + M_{\triangle^{\frac{1}{2}}}(t,t')\, \alpha^{\frac{5}{2}}(t'), \tag{47}$$

$$(\alpha'(t))^2 \alpha^{\frac{3}{2}}(t) \leq M'_{\triangle} \alpha^{\frac{7}{2}}(\tau) + M'_{\triangle^{\frac{1}{2}}} \alpha^{\frac{5}{2}}(\tau), \tag{48}$$

$\forall\ t,\, t' \in [0,T]$ and some $\tau \in [0,T]$. Note that the volume of the Cauchy surface is either constant or decreases with time i.e., $V(t') \leq V(t)$ for $0 \leq t \leq t' \leq T$.

For the second example, we assume that the family of metrics is of the form $\{\gamma(t,x)\}_{t\in[0,T]} = \{\alpha(t,x)\,\gamma(0,x)\}_{t\in[0,T]}$ where $\alpha(t,x) > 0$ is a smooth function with

$$\alpha(t,x) = \begin{cases} 1 & t \leq 0 \\ \alpha(T,x) & t \geq T \end{cases}.$$

We further assume that there is a finite atlas of charts $\{\Theta_k, \chi_k\}_{k\in[1,N]}$ covering Ω such that the metrics $\{\gamma(t,x)\}_{t\in[0,T]}$ have a diagonal form in each chart, i.e.,

$$\gamma(t,x) = \begin{pmatrix} \gamma^{11}(t,x) & 0 & 0 \\ 0 & \gamma^{22}(t,x) & 0 \\ 0 & 0 & \gamma^{33}(t,x) \end{pmatrix}, \tag{49}$$

for $t \in [0,T]$ where

$$\gamma^{ii}(t,x) = \alpha(t,x)\,\gamma^{ii}(0,x). \tag{50}$$

Examples of manifolds with this property include the sphere and torus. We show that the metrics $\{\gamma(t,x)\}_{t\in[0,T]}$ satisfy conditions (17) - (20) if $\alpha(t,x)$ satisfies the following conditions: First, we assume that there exists $\epsilon > 0$ such that

$$\epsilon < \frac{\alpha(t',x)}{\alpha(t,x)} < 1, \tag{51}$$

$\forall\ x \in \Omega$, with $0 \leq t < t' \leq T$. This condition implies that

$$\frac{\alpha(t,x)}{\alpha(t',x)} < \frac{1}{\epsilon}, \tag{52}$$

and

$$\frac{\sqrt{\gamma(t,x)}}{\sqrt{\gamma(t',x)}} = \left(\frac{\alpha(t,x)}{\alpha(t',x)}\right)^{\frac{3}{2}} < \epsilon^{-\frac{3}{2}}, \tag{53}$$

$\forall\, x \in \Omega,\ t,\ t' \in [0,T]$. We further assume that for each chart (Θ_k, χ_k),

$$\max_i \sup_{x\in\Theta_k} \left| \partial_i \left(\frac{\alpha(t,x)}{\alpha(t',x)} \right) \right| < Q, \tag{54}$$

$$\sup_{x\in\Theta_k} \left| \partial_t \ln\left(\alpha(t,x)\right)\right| < P, \tag{55}$$

and

$$\max_i \sup_{x\in\Theta_k} \left| \partial_i \left(\partial_t \ln\left(\alpha(t,x)\right)\right)\right| < P', \tag{56}$$

$\forall\, t,\ t' \in [0,T]$, where $Q,\ P,\ P' > 0$.

Given the constraints above, it is easy to show that the metric condition (17) is satisfied with $M_T = \epsilon^{-\frac{3}{2}}$. To verify the remaining conditions we make use of the relation

$$\begin{aligned} \left\| \triangle^{\frac{1}{2}}(t)\, g \right\|_t^2 &= \langle g, \triangle(t)\, g\rangle_t\,, \\ &= \langle dg, d\, g\rangle_t\,, \\ &\equiv \sum_{k=1}^{N} \int_{\chi_k(\Theta_k)} \eta_k \sum_{i=1}^{3} \gamma^{ii}(t)\ \left|\partial_i g\right|^2 \sqrt{\gamma(t)}\ d^3x, \end{aligned} \tag{57}$$

where $\{\eta_k\}_{k=[1,N]}$ is a partition of unity subordinate to the atlas $\{\Theta_k, \chi_k\}_{k\in[1,N]}$. To verify condition (18) we write $\left\|\triangle^{\frac{1}{2}}(t)\, g\right\|_t^2$ in terms of (57), apply (52) and (53) and obtain $\widetilde{M}_{\triangle^{\frac{1}{2}}} = \epsilon^{-\frac{5}{2}}$.

The remaining conditions require some analysis. For condition (19) choose any chart (Θ_k, χ_k) and consider the following representation of $\triangle(t)\, f$ in local coordinates,

$$\begin{aligned} \triangle(t)\, f &= \sum_{i=1}^{3} \frac{-1}{\sqrt{\gamma(t)}}\, \partial_i \left(\gamma^{ii}(t) \sqrt{\gamma(t)}\, \partial_i f \right), \\ &= \sum_{i=1}^{3} \frac{\sqrt{\gamma(t')}}{\sqrt{\gamma(t)}} \frac{-1}{\sqrt{\gamma(t')}}\, \partial_i \left(\frac{\gamma^{ii}(t)\sqrt{\gamma(t)}}{\gamma^{ii}(t')\sqrt{\gamma(t')}} \gamma^{ii}(t') \sqrt{\gamma(t')}\, \partial_i f \right), \end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{5}{2}\left(\frac{\alpha(t,x)}{\alpha(t',x)}\right)^{\frac{3}{2}} \sum_{i=1}^{3} \partial_i \left(\frac{\alpha(t,x)}{\alpha(t',x)}\right) \frac{\sqrt{\gamma(t')}}{\sqrt{\gamma(t)}}\, \gamma^{ii}(t')\, \partial_i f \\
& \qquad + \frac{\alpha(t,x)}{\alpha(t',x)} \triangle(t')\, f \; . \qquad (58)
\end{aligned}
$$

By applying (52), (53) and (54) to (58), and then summing the contributions from all the charts we obtain

$$
\|\triangle(t)\, f\,\|_t^2 \le M_\triangle \,\|\triangle(t')\, f\,\|_{t'}^2 + M_{\triangle^{\frac{1}{2}}} \left\|\triangle^{\frac{1}{2}}(t')\, f\,\right\|_{t'}^2 , \qquad (59)
$$

where

$$
M_\triangle = \left(\frac{1}{\epsilon}\right)^{\frac{7}{2}},
$$

and

$$
M_{\triangle^{\frac{1}{2}}} = \frac{25}{4}\left(\frac{1}{\epsilon}\right)^{\frac{9}{2}} Q^2\, \gamma_{\max},
$$

where

$$
\gamma_{\max} = \max_i \sup_{x\in\Omega} \gamma^{ii}(t,x)). \qquad (60)
$$

We apply a similar analysis for condition (20). Choose any chart (Θ_k, χ_k) and consider the following representation of $\triangle'(t)\, f$ in local coordinates,

$$
\begin{aligned}
\triangle'(t)\, f \quad &= \quad \sum_{i=1}^{3} \partial_t \left(\frac{-1}{\sqrt{\gamma(t)}}\, \partial_i \left(\gamma^{ii}(t)\,\sqrt{\gamma(t)}\,\partial_i\, f\,\right)\right), \\
&= \quad \partial_t(\ln(\alpha(t,x)))\, \triangle(t)\, f \\
& \qquad -\frac{5}{2}\sum^{3} \partial_i[\partial_t(\ln(\alpha(t,x)))]\, \gamma^{ii}(t)\; \partial_i\, f\, . \qquad (61)
\end{aligned}
$$

By applying (55) and (56) to (61), integrating over all charts, and then substituting the results into (59) with $t' = 0$ we obtain

$$
\|\triangle'(t)\, f\,\|_t^2 \le M_\triangle' \,\|\triangle(0)\, f\,\|_0^2 + M'_{\triangle^{\frac{1}{2}}} \left\|\triangle^{\frac{1}{2}}(0)\, f\,\right\|_0^2 ,
$$

where

$$
M_\triangle' = P^2\, M_\triangle,
$$

and

$$M'_{\triangle\frac{1}{2}} = P^2\, M_{\triangle\frac{1}{2}} + \left(P'\right)^2 \frac{25}{4}\,\gamma_{\max}.$$

Thus, the metric conditions are satisfied. Finally, note that the constraint (51) implies that the volume of the Cauchy surfaces decreases monotonically with time i.e., $V(t') < V(t)$ for $0 < t < t' < T$.

6 Discussion

Recall that one of the motivations for this work was to develop a foundation for a semiclassical quantum theory. Now that the classical problem is solved, we give a brief description of a method that could presumably be used for quantization. This method is essentially due to Isham and Dimock [2] [16].

The goal of the quantum problem is to construct a field operator that satisfies the field equation in a distributional sense, is causal, and satisfies the canonical commutation relations (CCRs). The first step is the construction of operator-valued "data" on an arbitrary Cauchy surface. This Cauchy data provides a representation of the CCRs on the Fock space over the square-integrable functions on this surface. The time development mimics the classical problem with the classical evolution operator transferred to the test functions inside the arguments of the field operators. Thus, the interacting field solves an operator-valued Cauchy problem in addition to the other quantum requirements.

The quantum scattering theory also mimics the classical theory. Given the Cauchy data for the interacting field one defines "data" for the "in" and "out" fields by transferring the respective classical wave operators from the interacting "data" to their test functions. The time development of the "in" and "out" data is realized in a distributional sense using the respective "free" evolution operators. Thus, representations of the "in", "out", and interacting fields are obtained, all on the same Fock space. To describe scattering, the "out" field is represented in terms of the "in" field by transferring the classical scattering operator from the "in" data to their test functions. To complete the theory one shows that this mapping between the "in" and "out" data is unitarilly implementable. This is equivalent to proving the existence of the quantum scattering operator. For this final step, we note the work of Wald and Fulling et al [21] [22]. Wald has proven the existence of the S-matrix on spacetimes that are flat off of compact sets, and Fulling, Narcowich and Wald have adapted this work to closed spacetimes that are initially, and finally static. Since these are the spacetimes considered here, we are optimistic that this work can be applied to our problem. Finally, although the method outlined above appears to be viable, numerous technical details remain to be addressed.

Acknowledgments

The author would like to thank Professor J. Dimock for numerous helpfuld-iscussions

Appendix

This appendix contains two propositions that establish the time invariance of the domain of the Laplace-Beltrami operator and its square root.

Proposition 5 *If a family of metrics $\{\gamma(t)\}_{t\in[0,T]}$ satisfies the metric conditions (17), (18) and (19) then the domain $D(K(t)) = D(\triangle(t)) \equiv D(\triangle)$ is independent of t.*

Proof: Choose any $t \in [0,T]$ and $f \in D(K(t))$, and let $\{f_n\}$ be a Cauchy sequence of smooth functions that converge to f in the graph norm ($C^\infty(\Omega)$ is a core for $K(t)$), i.e.,

$$\lim_{n\to\infty} \| f_n - f\|^2_{G(K(t))} = 0, \tag{62}$$

where

$$\begin{aligned} \| f \|^2_{G(K(t))} &= \| f \|^2_t + \left\| \left(\triangle(t) + m^2\right) f \right\|^2_t \\ &= \| \triangle(t)\ f \|^2_t + 2m^2 \left\| \triangle^{\frac{1}{2}}(t)\ f \right\|^2_t + (1+m^4)\ \| f \|^2_t . \end{aligned}$$

Note that the second term on the right hand side makes sense since $f \in D(\triangle(t)) \subset D(\triangle^{\frac{1}{2}}(t))$.

Now, since $\mathcal{H}_t(\Omega)$ and $\mathcal{H}_{t'}(\Omega)$ are setwise equivalent, $\{f_n\}$, $f \in \mathcal{H}_{t'}(\Omega)$. Let $t' \in [0,T]$, we show that $f \in D(K(t'))$ by showing that $\{f_n\}$ is Cauchy with respect to the graph norm $\|\cdot\|_{G(K(t'))}$. Note, $(f_n - f_m) \in C^\infty(\Omega) \subset D(K(t')) \subset D(K^{\frac{1}{2}}(t'))$, and consider,

$$\begin{aligned} \|(f_n - f_m)\|^{\ 2}_{G(K(t'))} &= \| \triangle(t')\ (f_n - f_m)\|^2_{t'} + 2m^2 \left\| \triangle^{\frac{1}{2}}(t')\ (f_n - f_m) \right\|^2_{t'} \\ &\quad + (1+m^4)\ \|(f_n - f_m)\|^2_{t'} . \end{aligned}$$

¿From (17) and (18) we have

$$\|(f_n - f_m)\|^2_{t'} \le M_T \|\ (f_n - f_m)\ \|^2_t\ , \tag{63}$$

and

$$\left\|\triangle^{\frac{1}{2}}(t')\,(f_n - f_m)\right\|^2_{t'} \le \widetilde{M}_{\triangle^{\frac{1}{2}}} \left\| \triangle^{\frac{1}{2}}(t)\,(f_n - f_m) \right\|^2_t , \tag{64}$$

and from (19),

$$
\begin{aligned}
\| \triangle(t')\,(f_n - f_m)\|_{t'}^2 \;\; &\leq \;\; M_{\triangle}(t',t)\; \| \triangle(t)\,(f_n - f_m)\, \|_t^2 \\
&\quad + M_{\triangle^{\frac{1}{2}}}(t',t)\; \left\| \triangle^{\frac{1}{2}}(t)\,(f_n - f_m)\, \right\|_t^2 . \quad (65)
\end{aligned}
$$

Since $\{f_n\}$ is Cauchy with respect to $\| \cdot \|_{G(K(t))}$, the right hand sides of (63), (64) and (65) approach zero as $n,\, m \to \infty$. Thus,

$$
\lim_{n,m\to\infty} \| \, f_n - f_m \, \|_{G(K(t'))} = 0,
$$

and since $K(t')$ is closed, $\lim_{n\to\infty} f_n \;\to\; f \;\in\; D(K(t'))$ which shows that $D(K(t)) \subset D(K(t'))$. However, since t and t' are arbitrary it follows that $D(K(t)) = D(K(t')) \equiv D(\triangle)$ independent of t. $\square$

Proposition 6 *If a family of metrics $\{\gamma(t)\}_{t\in[0,T]}$ satisfies the metric conditions (17), and (18) then the domain $D(K^{\frac{1}{2}}(t)) \equiv D(\triangle^{\frac{1}{2}})$ is independent of t.*

Proof: This proof is very similar to that of Proposition (5). Let $t \in [0,T]$, $f \in D(K^{\frac{1}{2}}(t))$, and $\{f_n\}$ be a Cauchy sequence such that

$$
\lim_{n\to\infty} \| \, f_n \, - f \, \|_{G(K^{\frac{1}{2}}(t))}^2 = 0,
$$

where

$$
\begin{aligned}
\| \, f \, \|_{G(K^{\frac{1}{2}}(t))}^2 \;\; &= \;\; \| \, f \, \|_t^2 + \left\| \, (\triangle(t) + m^2)^{\frac{1}{2}} \; f \, \right\|_t^2 , \\
&= \;\; \left\| \, \triangle^{\frac{1}{2}}(t) \; f \, \right\|_t^2 + (1+m^2) \;\; \| \, f \, \|_t^2 , \\
&= \;\; \| \;\; f \, \|_{G(\triangle^{\frac{1}{2}}(t))}^2 + m^2 \;\; \| \, f \, \|_t^2 . \qquad (66)
\end{aligned}
$$

It follows from (66) that $D(K^{\frac{1}{2}}(t)) = D(\triangle^{\frac{1}{2}}(t))$. Now, we know that $\{f_n\}$, $f \;\in\; \mathcal{H}_{t'}(\Omega)$. It remains to show that $f \;\in\; D(K^{\frac{1}{2}}(t'))$ for any $t' \;\in\; [0,T]$. Consider,

$$
\|(f_n - f_m)\|_{G(K^{\frac{1}{2}}(t'))}^{\,2} \; = \left\| \triangle^{\frac{1}{2}}(t') \;\, (f_n - f_m) \right\|_{t'}^2 + (1+m^2) \;\, \| \, (f_n - f_m) \|_{t'}^2 .
$$

From, (63), (64) and (66) we have

$$\lim_{n,m\to\infty} \|(f_n - f_m)\|^{\,2}_{G(K^{\frac{1}{2}}(t'))} = 0,$$

which shows that $\{f_n\}$ is Cauchy with respect to $\|\cdot\|_{G(K^{\frac{1}{2}}(t'))}$,and since $K^{\frac{1}{2}}(t')$ is closed, $\lim_{n\to\infty} f_n \to f \in D(K^{\frac{1}{2}}(t'))$ which shows that $D(K^{\frac{1}{2}}(t)) \subset D(K^{\frac{1}{2}}(t'))$. However, since t and t' are arbitrary it follows that $D(K^{\frac{1}{2}}(t)) = D(K^{\frac{1}{2}}(t')) \equiv D(\triangle^{\frac{1}{2}})$ is independent of t. $\square$

References

[1] Fulling, S.: Aspects of quantum field theory on curved spacetimes. Cambridge: Cambridge University Press 1989

[2] Isham, C.: Quantum field theory in curved space-time: a general mathematical framework. In: Bluler, K., Petry, H., Reetz, A. (eds.) Differential geometric methods in mathematical physics II, p. 459. Berlin. Heidelberg, New York: Springer. 1978.

[3] Kay, B.: Linear spin-zero fields in external gravitational and scalar fields I. Commun. Math. Phys. **62**, 55 (1978)

[4] Furlani, E. P.: Quantization of the electromagnetic field on static spacetimes. J. Math. Phys. **36**, 1063 (1995)

[5] Misner, C. W., Thorne, K. S., Wheeler, J. A.: Gravitation. New York.: W. H. Freeman 1973

[6] Misner, C. W., Wheeler J. A.: Classical physics as geometry: gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space. Ann. Phys. (U.S.A.) **2**, 525-603 (1957)

[7] Cycon, H. L., Froese, R. G., Kirsch W., Simon, B.: Schrodinger operators: with application to quantum mechanics and global geometry. New York.: Springer-Verlag 1987

[8] Okubo, T. :Differential geometry. New York: Marcel Dekker 1987

[9] Reed, M., Simon, B.: Methods of Mathematical Physics I. New York: Academic Press 1980

[10] Reed, M., Simon, B.: Methods of Mathematical Physics II. New York: Academic Press 1975

[11] Goldstein, J. A.: Semigroups of linear operators and applications. New York: Oxford University Press 1985

[12] Pazy, A.: Semigroups of linear operators and applications to partial differential equations. New York: Springer-Verlag 1983

[13] Reed, M., Simon, B.: Methods of Mathematical Physics III. New York: Academic Press. 1979

[14] Nachbin, L.: Introduction to functional analysis: banach spaces and differential calculus. Monographs and textbooks in pure and applied math. Vol 60. New York: Marcel Dekker 1981

[15] Klambauer, G.: Mathematical analysis. Monographs and textbooks in pure and applied mathematics. Vol 31. New York: Marcel Dekker 1975

[16] Dimock, J.: Algebras of local observables on a manifold. Commum. Math. Phys. **77**, 219-228 (1980)

[17] Dimock, J.: Quantized electromagnetic field on a manifold. Rev. Math. Phys. **4**, 223-233 (1992)

[18] Bruhat, Y. C., Morette, C. D.: Hyperbolic partial differential equations on a manifold. In: DeWitt, C. M., Wheeler, J. A. (eds.) Battelle rencontres. Lectures in mathematics and physics . New York: Benjamin 1967

[19] Leray, J.: Hyperbolic differential equations. Lecture notes. Institute for advanced study. Princeton 1953 (unpublished)

[20] Hawking, S., Ellis, G.: The large scale structure of space-time. Cambridge: Cambridge University Press.1973

[21] Wald, R.: Existence of the S-matrix in quantum field theory in curved spacetime. Ann. Phys. (N.Y.) **118**, 490-510 (1979)

[22] Fulling, S., Narcowich, F., Wald. R.: Signularity structure of the two-point function in quantum field theory in curved spacetime II. Ann. Phys. (N.Y.) **136**, 243-272 (1981)