MULTIPARAMETER PSEUDODIFFERENTIAL OPERATORS AND RELATED SPECTRAL ASYMPTOTICS

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Abstract

We introduce the notion of ellipticity with parameters and study elliptic boundary value problems with parameters in a bounded domain. Furthermore, the notion of the ellipticity with parameters for differential and pseudodifferential operator pencils is applied to the resolvent construction which leads to the definition of the corresponding complex powers and $\zeta-$function, i.e. to a new functional calculus of operators with promising applications to the multiparameter spectral theory. More precisely, we have proved the extension theorem for traces of the above defined kernels of the complex powers and the corresponding $\zeta-$function. The last appears to be a meromorphic function in C^m (m is the number of parameters) with polar sets of the first order and the corresponding holomorphic residue-forms. In particular, for $m = 2$ this investigation gives the opportunity to write the two - parameter spectral asymptotics in terms of these forms using the two - dimensional refinement of the Ikehara Tauberian theorem.

Contents

Some Notation

Here is the list of the conventional notation to be used below:

$\mathbb{N}$ - the set of all positive integers.
$\mathbb{Z}$ - the set of all integers.
$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
$\mathbb{R}$ - the set of all real numbers.
$\mathbb{C}$ - the set of all complex numbers.
$\mathbb{R}^n$ - the standard real n-dimensional euclidean space.
$\mathbb{C}^m$ - the standard complex m-dimensional euclidean space.
$\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n : \ x_n > \ (\text{resp.} <) \ 0\}$; $\overline{\mathbb{R}^n_\pm}$ - the closures of the half-spaces $\mathbb{R}^n_\pm$ in $\mathbb{R}^n$; $(\overline{\mathbb{R}^n_+})^m = \overline{\mathbb{R}^n_+} \times \overline{\mathbb{R}^n_+} \times \cdots \times \overline{\mathbb{R}^n_+}$ $-$ m times. $\mathbb{Z}^n_+ = (\mathbb{Z}_+)^n$.
$\mathbb{C}_\pm = \{z \in \mathbb{C} : \ \mathrm{Im} z > \ (\text{resp.} <) \ 0\}$.
$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \ . \ . \ . \ , \frac{\partial}{\partial x_n}\right)$, where $(x_1, ..., x_n) \in \mathbb{R}^n$.
$D = i^{-1}\frac{\partial}{\partial x}$, $D_j = i^{-1}\frac{\partial}{\partial x_j}$, where $i = \sqrt{-1} \in \mathbb{C}$.
$\partial^l_{x_j} = \frac{\partial^l}{\partial x^l_j}$, $l \in \mathbb{Z}_+$.
$D^\alpha = D_1^{\alpha_1} \ . \ . \ . \ D_n^{\alpha_n}$, where α is a multiindex, i.e. $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \in \mathbb{Z}_+$.
$\xi^\alpha = \xi_1^{\alpha_1} \ . \ . \ . \ \xi_n^{\alpha_n}$, where $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$.
$x \cdot \xi = x_1\xi_1 \ + \ \cdot \ \cdot \ \cdot \ + \ x_n\xi_n$, where $x = (x_1, ..., x_n) = (x', x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, ..., \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$.
$|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, where α is a multiindex, $\alpha = (\alpha', \alpha_n)$.
$\||q\||^l = \sum_{i=1}^m |q_i|^l$, if $q = (q_1, ..., q_m) \in \mathbb{C}^m$; $M_q = \min_{i=1,...,m} |q_i|$.
$\prod_{j=1}^k E_j = E_1 \times E_2 \times \cdots \times E_k$, where E_j are Hilbert spaces.
$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, $< x > = (1 + x_1^2 + \cdots + x_n^2)^{1/2}$ for $x = = (x_1, ..., x_n) \in \mathbb{R}^n$.
$\frac{l}{2\overline{b}} = \left\{\frac{l}{2b_1}, ..., \frac{l}{2b_m}\right\}$, where $l \in \mathbb{Z}_+$, $b \in \mathbb{N}^m$.
$\Delta = \Delta_x = \frac{\partial^2}{\partial x_1^2} \ + \ \cdot \ \cdot \ \cdot \ + \ \frac{\partial^2}{\partial x_n^2}$ - the Laplacian in $\mathbb{R}^n$.
δ_{jk} - the Kroneker symbol ($\delta_{jk} = 0$ as $j \neq k$ and 1 as $j = k$).
$[s]$ - the integer part of s .
$\triangleright$ means the end of the proof.
$:=$ means "equals by definition".
$\subset\subset$ means the compact inclusion.
$\mathrm{supp}\, u$ means the support of the function u .
$\partial\Omega$ means the boundary of an open set $\Omega \subset \mathbb{R}^n$, $\overline{\Omega}$ means its closure.
$C_0^\infty(\mathbb{R}^n)$ - the space of all infinitely smooth complex - valued functions with compact support.
$C^\infty(\overline{\Omega}, Y)$ - the space of smooth functions on Ω with the values in Y , which are continuous up to the boundary.

$S(\mathbb{R}^n)$ - the Schwartz space of C^∞ - functions on $\mathbb{R}^n$, with derivatives vanishing faster than any power of $|x|$ as $|x| \to \infty$.

The direct and the inverse Fourier transformations F , F^{-1} : $S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ and the Fourier transformation in the variables $x' \in \mathbb{R}^{n-1}$ are defined by the formulae

$$\widehat{u}(\xi) \equiv F(u)(\xi) \equiv F_{x\to\xi}(u) = \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi} u(x)\, dx \ ,$$

$$\widetilde{f}(x) \equiv F^{-1}(f)(x) \equiv F^{-1}_{\xi\to x}(f) = (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} f(\xi)\, d\xi \ ,$$

$$\widehat{u}(\xi') \equiv F'(u)(\xi') \equiv {F'}_{x'\to\xi'}(u) = \int_{\mathbb{R}^{n-1}_{x'}} e^{-ix'\cdot\xi'} u(x')\, dx' \ .$$

INTRODUCTION

Operators with parameters arise in different branches of partial differential equations, especially in equations with multidimensional time variables. In this paper we study boundary value problems for elliptic operators with parameters and then show, how to apply multiparameter theory to the definition of the corresponding complex powers of operator pencils, operator $\zeta-$function and, finally, to spectral asymptotics. We note that there exist quite a few works on multiparameter spectral asymptotics theory for partial differential operators (see in this case Guillemin V., Sternberg S. (1977), Kozlov S.M. (1984) and Boimatov K.Kh. (1991)).

We start with the operators with parameters, i.e. we solve the following elliptic boundary value problem with parameters in a bounded domain Ω :

$$Au \;\equiv\; A(x,D,q)\,u(x,q) \;=\; f(x,q)\ ,\quad x\in\Omega\subset\mathbb{R}^n\ ,\quad q\in\mathbb{C}^m\ , \qquad (0.0.1)$$

with boundary conditions on $\Gamma=\partial\Omega$:

$$B_\nu u|_\Gamma \;\equiv\; B_\nu(x,D,q)u(x,q)\big|_{x=x'} \;=\; g_\nu(x',q)\ ,\quad x'\in\Gamma\ ,\ \nu=1,...,k\ , \qquad (0.0.2)$$

where

$$A(x,D,q) \;=\; \sum_{|\alpha|+|\beta|\le 2k} a_{\alpha\beta}(x)q^\beta D_x^\alpha\ ,$$

$$B_\nu(x,D,q) \;=\; \sum_{|\alpha|+|\beta|\le k_\nu} b_{\alpha\beta}(x)q^\beta D_x^\alpha\ ,$$

i.e. $A(x,D,q)$ and $B_\nu(x,D,q)$ are differential operators with coefficients being C^∞ - functions in $x\in\overline{G}$ and polynomials in q_i for $i=1,...,m$, where

$$q_i\;\in\;Q_i\;=\;\{q_i\in\mathbb{C}^m\ :\ \arg q_i\;\in\;[\alpha_i,\beta_i]\}\ ,$$

thus $A(x,\xi,q)$ and $B_\nu(x,\xi,q)$ are polynomials in (ξ,q) of degrees $s=2k$ and k_ν $(\nu=1,...,k)$ respectively.

We denote by $A_0(x,\xi,q)$ and $B_{0,\nu}(x,\xi,q)$ the leading parts of the corresponding polynomials which are obtained by dropping the terms with degrees relatively to (ξ,q) being less then $2k$ and k_ν respectively.

We suppose the following conditions of the unique solvability of the problem (0.0.1) - (0.0.2) to be executed.

Condition 0.0.1 $\forall\,x\in\overline{\Omega}$, $\forall\,q\in Q\;=\;Q_1\times...\times Q_m$ *and* $\forall\,\xi\in\mathbb{R}^n$

$$A_0(x,\xi,q)\;\neq 0\quad for\quad |\xi|+M_q\neq 0\ , \qquad (0.0.3)$$

where $M_q\;=\;\min_{i=1,...,m}|q_i|$.

Condition 0.0.2 *Let us fix a boundary point* $x' \in \Gamma$ *and suppose that all operators in (0.0.1) - (0.0.2) are written in a local coordinate system* connected with this point *(i.e. the boundary is straightened and the* x_n-*axis is looking inside* Ω *together with the inner normal to* Γ *).*

We consider the problem on the half-axis $x_n > 0$

$$A_0 \big(0, \xi', -i \frac{d}{dx_n}, q\big)\, v(x_n) \;=\; 0\,, \quad x_n > 0\;, \tag{0.0.4}$$

$$B_{0,\nu} \big(0, \xi', -i \frac{d}{dx_n}, q\big)\, v(x_n)\big|_{x_n=0} = \; h_\nu\,, \quad \nu = 1, ..., k\;, \tag{0.0.5}$$

smoothly depending on parameters ξ' *and* q *, and suppose that* $\forall\, q \in Q$ *for* $|\xi'| + M_q \neq 0$ *this problem has a unique solution in the space* $\mathcal{M}$ *of all stable solutions to the equation (0.0.4) for all* $h_\nu \in \mathbb{C}$ *.*

Let us note that for $q = 0$ Condition 0.0.1 turns into the ordinary ellipticity condition and Condition 0.0.2 - into the classical Shapiro - Lopatinskij condition. Here are the main examples.

Example 0.0.3 *Let* $A(x, D)$ *be an elliptic operator* $(\mathrm{ord} A = 2k)$ *with a positive real leading part. Then the operator*

$$A(x, D) + q_1^k q_2^k$$

with Dirichlet boundary conditions

$$B_\nu \;=\; \Big(-i \frac{\partial}{\partial \vec{n}}\Big)^{\nu-1}\;, \quad \nu = 1, ..., k,$$

satisfies Conditions 0.0.1 and 0.0.2 in the cone $Q = Q_1 \times Q_2$ *, where* $Q_j = \big\{ |\arg q_j| \le \frac{\pi-\varepsilon}{2k} \big\}$ *,* $\varepsilon \in (0, \pi)$ *,* $j = 1, 2$ *.*

Example 0.0.4 *If* A_1 *and* A_2 $(\mathrm{ord} A_j = 2k_j\,,\ j = 1, 2)$ *are the two such operators as in Example 0.0.3, then the operator*

$$\big(A_1(x, D) + q_1^{2k_1}\big)\,\big(A_2(x, D) + q_2^{2k_1}\big)$$

with Dirichlet boundary conditions

$$B_\nu \;=\; \Big(-i \frac{\partial}{\partial \vec{n}}\Big)^{\nu-1}\;, \quad \nu = 1, ..., k_1 + k_2,$$

also satisfies Conditions 0.0.1 and 0.0.2, but in the cone $Q = Q_1 \times Q_2$ *, where* $Q_j = \big\{ |\arg q_j| \le \frac{\pi-\varepsilon}{2k_j} \big\}$ *,* $\varepsilon \in (0, \pi)$ *,* $j = 1, 2$ *.*

The main result of Chapter 1 is the unique solvability of the problem (0.0.1) - (0.0.2) in the scale of Sobolev spaces equipped with special norms depending on parameters q for sufficiently large $\sum_{j=1}^{m} |q_j|$ (and $M_q \geq q^{(0)}$ for some $q^{(0)} > 0$) under Conditions 0.0.1 and 0.0.2 (Proposition 1.3.6 and Theorem 1.4.2). The proof is obtained via a priori estimates which follow in a standard manner from the parametrix construction. The difference with the one - parameter case is in the fact, that though $\sum_{j=1}^{m} |q_j|$ is sufficiently large, some of q_j may be rather small (but not zero!).

By analogy with the one - parameter case (see, for example, Agranovich M.S. (1990)), the Calculus of elliptic differential operators with parameters (being developed in Chapter 1) leads to a number of interesting applications.

In this way in Chapter 2 we generalize (in the case of a closed manifold X) the results obtained in Chapter 1 for elliptic differential operators with parameters, to the case of pseudodifferential operator pencils of the form

$$A(x,D,q) \; = \sum_{\alpha+|\beta|=s} A_{\alpha,\beta}(x,D) q^{\beta} \; , \tag{0.0.6}$$

where $q^{\beta} = q_1^{\beta_1} \cdots q_m^{\beta_m}$, and $A_{\alpha,\beta}(x,D)$ are classical pseudodifferential operators of orders α . For operator pencils of the type (0.0.6) we obtain the parametrix construction with the modification of the notion of ellipticity with parameters from Chapter 1, and as application - a kernel $A_z(x,y)$ for the product of complex powers $A_1^{z_1} \cdots A_m^{z_m}$, where A_j are classical elliptic pseudodifferential operators of orders s_j . It is proved that this kernel appears to be holomorphic in the half - space

$$\{ z \in \mathbb{C}^m : \; \sum_{j=1}^{m} s_j \operatorname{Re} z_j \; < \; -n \}$$

and meromorphic in the hole $\mathbb{C}^m$ with first order polar sets

$$M_j \; := \; \{ z \in \mathbb{C}^m : \; \sum_{j=1}^{m} s_j z_j \; = \; j-n \}$$

and the corresponding holomorphic forms - residues $\mathbf{res}\, B_j(x)$ on them, being computed in terms of the symbol $b(x,\xi,q)$ of the parametrix B for $(A_1 - q_1^{s_1}) \cdots (A_m - q_m^{s_m})$.

Finally, we define the ζ $-$ function of the operator - vector $\{A_1, ..., A_m\}$ as the integral trace of the kernel A_z on the diagonal

$$\zeta_A(z) \; = \; \zeta_{A_1,...,A_m}(z_1, ..., z_m) \; := \; \int_X A_z(x,x)\, dx \; . \tag{0.0.7}$$

We see from (0.0.7) that $\zeta_A(z)$ is meromorphic in $\mathbb{C}^m$ with polar sets M_j and uniquely defined on them holomorphic forms - residues

$$\mathbf{res}_{M_j} \zeta_A \; = \; \int_X \mathbf{res}\, B_j(x)\, dx \; . \tag{0.0.8}$$

$$= \frac{(-1)^\nu}{s_\nu} \int_M \int_{|\xi|=1} b_{-n}^{(M_j),0}(x,\xi)\, dx đ\xi' dz[\nu]\big|_{M_j} \, .$$

We wish to discover the spectral sense of the designed function $\zeta_A(z)$.

Let all A_j $(\operatorname{ord} A_j = s_j)$ be positive self-adjoint elliptic differential operators with eigenvalues λ_{ji} and orthonormal system of eigenfunctions $\varphi_{ji}(x)$, $(j = 1, ..., m;\ i \in \mathbb{N})$. Then for $\operatorname{Re} z_j \ < \ -n/s_j$

$$\zeta_A(z) \ = \ \sum_{i_1,\ldots,i_m=1}^{\infty} \lambda_{1i_1}^{z_1} C_{i_1 i_2}^{\bar{1}2} \lambda_{2i_2}^{z_2} C_{i_2 i_3}^{\bar{2}3} \cdots C_{i_{m-1} i_m}^{\overline{m-1}m} \lambda_{m i_m}^{z_m} C_{i_m i_1}^{\bar{m}1} \ , \qquad (0.0.9)$$

where

$$C_{i_1 i_2}^{\bar{1}2} \ = \ \int_X \overline{\varphi_{1i_1}(x)}\, \varphi_{2i_2}(x)\, dx \quad \left(C_{i_2 i_1}^{\bar{2}1} \ = \ \overline{C_{i_1 i_2}^{\bar{1}2}}\right) \ ,$$

etc.

If all A_j are commuting operators then we have $C_{i_k i_{k+1}}^{\bar{k}k+1} \ = \ \delta_{i_k i_{k+1}}$ and

$$\zeta_A(z) \ = \ \sum_{i=1}^{\infty} \lambda_{1i}^{z_1} \cdots \lambda_{mi}^{z_m} \ .$$

Thus, we obtain as a special case the way to the joint spectral asymptotics theorem of commuting (pseudo)differential operators (see Guillemin V., Sternberg S. (1977)).

Now we restrict ourselves to the case $m = 2$, i.e. to

$$\zeta_A(z) \ = \ \sum_{i,j=1}^{\infty} \lambda_{1i}^{z_1} \lambda_{2j}^{z_2} |C_{\bar{i}j}|^2 \ ,$$

where $|C_{\bar{i}j}|^2 \ = \ \cos^2 \beta_{ij}$ and β_{ij} is the angle between the eigenvectors φ_{1i} and φ_{2j} . We introduce the distribution function

$$N(t_1,t_2) \ = \ \sum_{\lambda_{1i} \le t_1^{s_1},\, \lambda_{2j} \le t_2^{s_2}} |C_{\bar{i}j}|^2 \ , \qquad (0.0.10)$$

being the weighted number of eigenvalue pairs $\{\lambda_{1i}, \lambda_{2j}\}$ lying in the rectangle $(0, t_1^{s_1}] \times (0, t_2^{s_2}]$.

Clearly,

$$\zeta_A(z) \ = \ \int_0^\infty \int_0^\infty t_1^{s_1 z_1} t_2^{s_2 z_2}\, d_{t_1} d_{t_2} N(t_1,t_2) \ . \qquad (0.0.11)$$

Initially, the distribution function close to (0.0.10) was introduced in (Kozlov S.M. (1984)). Its asymptotics as $t_1, t_2 \to +\infty$ was established by the Tauberian method connected with Laplace transform (the study of the operator - function $\exp(-t_1 A_1 - t_2 A_2)$) and the refinement (due to S.M.Kozlov) of the multidimensional Tauberian theorems of (Vladimirov V.S., Drozhzhinov Yu.N., Zav'yalov B.I. (1986)).

The generalization of Kozlov's result to the case of arbitrary m is contained in the work (Boimatov K.Kh. (1991)).

For a fine survey of the one - parameter theory of spectral asymptotics see (Rosenblyum G.V., Solomyak M.Z., Shubin M.A. (1989)). A survey of multiparameter spectral theory is contained in (Volkmer H. (1988)). Multiparameter spectral asymptotics theory for ordinary differential operators was developed in the works (Binding P., Browne P.J. (1990), Binding P., Browne P.J., Seddighi K. (1992)).

In our case, we fix the direction $k = t_2/t_1$ in the $(t_1, t_2) -$ plane and find the asymptotics of the function $N(t, kt)$ as $t \to +\infty$ in terms of the above - constructed complex powers $A_1^{z_1} A_2^{z_2}$ and the *two - dimensional refinement of Ikehara Tauberian theorem* (see the book (Shubin M.A. (1978)) for the one - dimensional version).

Finally, we obtain the following asymptotic formula:

$$N(t, kt) \sim \frac{t^n}{n} \int_0^k dk' \, M^{-1}_{w \to k'} \int_X \int_{|\xi|=1} b_{-n}^{\left(-\frac{n-w+1}{s_1}, \frac{w-1}{s_2}\right),0}(x, \xi) \, dx đ\xi \,, \quad (0.0.12)$$

where M^{-1} is the inverse Mellin transform and $b_{-n}^{(z_1, z_2),0}$ is the homogeneous component of order $(-n)$ of the symbol of the operator $A_1^{z_1} A_2^{z_2}$.

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Chapter 1

Differential operators with parameters

1.1 Main functional spaces and interpolation inequalities

1.1.1

We denote by $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ (see, for example, Besov O.V., Il'in V.P., Nikolskij S.M. (1975), Kudryavtsev L.D., Nikolskij S.M. (1988), Egorov Yu.V., Shubin M.A. (1988)) the completion of the space $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_s = \|u\|_{s,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n_\xi} \left(1+|\xi|^{2s}\right)|Fu|^2\,d\xi\right)^{1/2}, \tag{1.1.1}$$

and also denote by $H^l(\mathbb{R}^n_+)$, $l \in \mathbb{Z}$ − the completion of the set of restrictions to $\mathbb{R}^n_+$ of the functions from $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{l,\mathbb{R}^n_+} = \left(\sum_{|\alpha|\le l}\int_{\mathbb{R}^n_+}|D^\alpha u|^2\,dx\right)^{1/2}, \tag{1.1.2}$$

or according to the equivalent norm

$$\left(\sum_{j+k\le l}\int_{\mathbb{R}^{n-1}_{\xi'}}|\xi'|^{2j}\int_0^\infty |D_n^k F'u|^2\,dx_n d\xi'\right)^{1/2}. \tag{1.1.3}$$

Note that, if $u \in H^l(\mathbb{R}^n)$ then $u\,|_{\mathbb{R}^n_+} \in H^l(\mathbb{R}^n_+)$ and

$$\|u\|_{l,\mathbb{R}^n_+} \;\le\; \|u\|_{l,\mathbb{R}^n} \;. \tag{1.1.4}$$

Inversely, there exists an extension operator (see Egorov Yu.V., Shubin M.A. (1988)) from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, such that

$$\|Lu(x)\|_{l,\mathbb{R}^n} \;\le\; C\,\|u(x)\|_{l,\mathbb{R}^n_+} \;, \tag{1.1.5}$$

Here and further we denote by C an arbitrary positive constant, which is independent of $u(x)$ and the parameters $q\in\mathbb{C}^m$ (if there are any) and may differ from inequality to inequality.

Also, we shall use the so-called trace inequality: let $u\in H^l(\mathbb{R}^n_+)$, $l>1/2$, then there exists $u(x',0)\in H^{l-1/2}(\mathbb{R}^{n-1}_{x'})$ such that

$$\|u(x',0)\|'_{l-1/2} \;\le\; C\,\|u(x)\|_{l,\mathbb{R}^n_+} \;, \tag{1.1.6}$$

where we denote by $\|\cdot\|'_s$ the norm in $H^s(\mathbb{R}^{n-1}_{x'})$.

1.1.2

The down-stated interpolation lemmas generalize the corresponding results of (Agranovich M.S., Dynin A.S. (1962), Agranovich M.S., Vishik M.I. (1964)) to the multiparameter case.

Lemma 1.1.1 *Let $q\in\mathbb{C}^m$, then for $0<k<l$ the following interpolation inequalities are valid:*

$$\sum_{i=1}^{m}|q_i|^{l-k}\|u\|_k \;\le\; C_{kl}\left[\|u\|_l \,+\, \sum_{i=1}^{m}|q_i|^{l}\|u\|_0\right] , \tag{1.1.7}$$

$$\sum_{i=1}^{m}|q_i|^{2(l-k)}\|u\|_k^2 \;\le\; C'_{kl}\left[\|u\|_l^2 \,+\, \sum_{i=1}^{m}|q_i|^{2l}\|u\|_0^2\right] , \tag{1.1.8}$$

where the norms are taken over $\mathbb{R}^n$ or $\mathbb{R}^n_+$, and the constants C_{kl} and C'_{kl} are independent of $u(x)$ and q .

Proof. It is sufficient to prove (1.1.8), whence follows (1.1.7). From the one-dimensional case we have

$$|q_i|^{2(l-k)}\left(1+|\xi|^{2k}\right) \;\le\; C\left[\left(1+|\xi|^{2l}\right) \,+\, |q_i|^{2l}\right]$$

for $i=1,...,m$, and thus

$$\sum_{i=1}^{m}|q_i|^{2(l-k)}\left(1+|\xi|^{2k}\right) \;\le\; C_{kl}\left[\left(1+|\xi|^{2l}\right) \,+\, \sum_{i=1}^{m}|q_i|^{2l}\right] ,$$

from where (1.1.8) follows after multiplication by $|Fu|^2$ and integration over ξ . With the help of (1.1.4), (1.1.5) we also obtain (1.1.7) and (1.1.8) in $\mathbb{R}^n_+$. $\triangleright$

Lemma 1.1.2 *Let* $l = 1/2$ *and* $u \in H^1(\mathbb{R}^n_+)$, *then*

$$\sum_{i=1}^{m} |q_i|^{1/2} \, \|u(x',0)\|_0' \ \le \ C\Big[\|u\|_1 \, + \, \sum_{i=1}^{m} |q_i| \, \|u\|_0\Big] \tag{1.1.9}$$

and

$$\sum_{i=1}^{m} |q_i| \, \|u(x',0)\|_0'^{\,2} \ \le \ C'\Big[\|u\|_1^2 \, + \, \sum_{i=1}^{m} |q_i|^2 \|u\|_0^2\Big] \ , \tag{1.1.10}$$

where the norms are taken over $\mathbb{R}^n$ *or* $\mathbb{R}^n_+$, *and the constants* C *and* C' *are independent of* $u(x)$ *and* q .

Proof. As in the previous lemma it is enough to prove (1.1.10) for smooth functions and in the case of $\mathbb{R}^n$ - norms in the right - hand side. Rewriting the left - hand side in the form

$$\sum_{i=1}^{m} |q_i| \int |v(\xi')|^2 \, d\xi' \ ,$$

where

$$v(\xi') \ = \ F'\left[u(x',0)\right] \ = \ \frac{1}{2\pi} \int e^{ix_n\xi_n} Fu(\xi) \, d\xi_n \ ,$$

we see that

$$|v(\xi')|^2 \ \le \ \frac{1}{2\pi} \left(\int |Fu(\xi)|^2 \, d\xi_n \right)^2$$

$$\le \ \frac{1}{2\pi} \int \Big(1 + \Big(\sum_{i=1}^{m} |q_i|\Big)^2 + |\xi_n|^2\Big) |Fu|^2 \, d\xi_n \cdot \int \frac{d\xi_n}{1 + \left(\sum_{i=1}^{m} |q_i|\right)^2 + |\xi_n|^2} \tag{1.1.11}$$

$$\le \ C \int \Big(1 + |\xi|^2 + \sum_{i=1}^{m} |q_i|^2\Big) |Fu|^2 \, d\xi_n \ \cdot \ \frac{\pi}{\sqrt{1 + \left(\sum_{i=1}^{m} |q_i|\right)^2}} \ ,$$

where we have used the Schwartz inequality twice and calculated the last integral in (1.1.11). Now, as trivially $\sum_{i=1}^{m} |q_i| \ < \ \sqrt{1 + \left(\sum_{i=1}^{m} |q_i|\right)^2}$, we get after integrating over ξ' that the left-hand side in (1.1.10) is less than

$$C \int \Big(< \xi >^2 + \sum_{i=1}^{m} |q_i| \Big) |Fu|^2 \, d\xi \ .$$

$\triangleright$

Also we use the parameter-dependent norms

$$|||u|||_l^2 \ := \ \|u\|_l^2 + \sum_{i=1}^m |q_i|^{2l} \, \|u\|_0^2 \ = \ \|u\|_l^2 \ + \ |||q|||^{2l} \|u\|_0^2 \ , \tag{1.1.12}$$

where the norms in the right - hand side are taken taken either over $\mathbb{R}^n$ or over $\mathbb{R}^n_+$. As follows from (1.1.8), the norms $\|\cdot\|_l$ and $|||\cdot|||_l$ are equivalent for the fixed $q \in \mathbb{C}^m$ and

$$|||u|||_l^2 \ \le \ \sum_{k=0}^l \sum_{i=1}^m |q_i|^{2(l-k)} \, \|u\|_k^2 \ \le \ C |||u|||_l^2 \ . \tag{1.1.13}$$

Now we can rewrite (1.1.4) - (1.1.6) in the new terms

$$|||u(x)|||_{l,\mathbb{R}^n_+} \ \le \ C |||u(x)|||_{l,\mathbb{R}^n} \ , \tag{1.1.14}$$

$$|||Lu(x)|||_{l,\mathbb{R}^n} \ \le \ C |||u(x)|||_{l,\mathbb{R}^n_+} \ , \tag{1.1.15}$$

$$|||u(x',0)|||'_{l-1/2} \ \le \ C |||u(x)|||_{l,\mathbb{R}^n_+} \ , \tag{1.1.16}$$

where $l \in \mathbb{N}$.

Remark 1.1.3 *The inequalities of this paragraph remain true if instead of* $\mathbb{R}^n_+$ *we consider a bounded domain* $G \subset \mathbb{R}^n$ *with sufficiently smooth boundary* Γ , *which is an* $(n-1)-$*dimensional surface admitting in every point local straightenings under local coordinate transformations.*

In this case the Sobolev norms are defined in a standard way as follows: for $l \in \mathbb{N}$ the space $H^l(G)$ is defined as a completion of the space $C^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{l,G} \ = \ \left(\sum_{|\alpha| \le l} \int_G |D^\alpha u|^2 \, dx \right)^{1/2} \ , \tag{1.1.17}$$

or, equivalently,

$$|||u|||_{l,G}^2 \ = \ \|u\|_{l,G}^2 \ + \ |||q|||^{2l} \, \|u\|_{0,G}^2 \ , \tag{1.1.18}$$

or with respect the equivalent norm which is defined by a partition of unity as follows: suppose $\{U_\alpha\}_{\alpha=1}^N$ is an open covering of G and $\sum_{\alpha=1}^N \varphi_\alpha(x) \equiv 1$ is the corresponding smooth partition of unity, then the norm (1.1.17) is equivalent to the norm $\left(\sum_{\alpha=1}^N \|\varphi_\alpha u\|_l^2 \right)^{1/2}$, where the norms are taken in local coordinates either over $\mathbb{R}^n$ or over $\mathbb{R}^n_+$, with the additional assumption in the last case that, if $\operatorname{supp} \varphi_\alpha \cap \Gamma \neq \emptyset$, then $\operatorname{supp} \varphi_\alpha \subset U_j$ for some U_j from the covering of the boundary Γ .

Let $\{\varphi_\beta\}$, $(\beta = 1, ..., r)$ be all functions from $\{\varphi_\alpha\}$ such that $\operatorname{supp} \varphi_\beta \cap \Gamma \neq \emptyset$. For $l \in \mathbb{N}$ the Sobolev space $H^l(\Gamma)$ is defined as a completion of the space $C^\infty(\overline{\Gamma})$ with respect to the norm

$$\|u\|'_{l,G} \;=\; \left(\sum_{\beta=1}^{r} \|\varphi_\beta u\|'^2_{l,\mathbb{R}^{n-1}}\right)^{1/2} . \tag{1.1.19}$$

This space is independent of local coordinates and the partition of unity for sufficiently small coverings. Of course, we also need the equivalent norm

$$|||u|||'^2_{l,\Gamma} \;=\; \|u\|'^2_{l,\Gamma} \;+\; |||q|||^{2l}\, \|u\|'^2_{0,\Gamma} \;. \tag{1.1.20}$$

Further, we define the spaces $H^s(I)$, where $I \subset \mathbb{R}$ is an open interval in the real line. It can be the line $\mathbb{R}$ itself or the half-line $\mathbb{R}_+$. We introduce the space $H^s(I)$ as a completion of the space of restrictions to I of smooth functions on $\mathbb{R}$, relatively to the norm

$$\|u\|_s \;=\; \left(\sum_{k=0}^{[s]} \int_I \left|u^{(k)}(t)\right|^2 dt \;+\; \int_I \int_I \frac{\left|u^{[s]}(t) \;-\; u^{[s]}(\tau)\right|^2}{|t \;-\; \tau|^{2s-2[s]+1}}\, dt\, d\tau\right)^{1/2} , \tag{1.1.21}$$

The equivalent norm can be introduced in terms of the operator L (see (1.1.5) and (Slobodetskij L.N. (1958)) for details). Let $I = (a,b)$, $b < +\infty$ and $J = (a, +\infty)$, then

$$\langle\, u \,\rangle \;:=\; \inf \|v\|_{s,J} \;, \tag{1.1.22}$$

where $v = Lu$ and "inf" is taken over all $v \in H^s(J)$ such that $v = u$ on I almost everywhere.

At last, if I and J are the two such intervals, then the norm in the corresponding anisotropic Sobolev space (see Slobodetskij L.N. (1958)) $H^{l,k}(I \times J)$ is

$$\|u\|^2_{l,k,I\times J} \;=\; \int_I \left\|u(t_1,t_2)\right\|^2_{k,J} dt_1 \;+\; \int_J \left\|u(t_1,t_2)\right\|^2_{l,I} dt_2 \;, \tag{1.1.23}$$

and, in general, by induction, if $I = I_1 \times \cdots \times I_m$ (in particular, we shall consider $I = (\mathbb{R}_+)^m$), then the norm in the space $H^{\vec{\alpha}}(I)$ is defined by

$$\|u\|^2_{\vec{\alpha},I} \;=\; \sum_{i=1}^{m} \int_{I[i]} \left\|u(t)\right\|^2_{\alpha_i, I_i} dt[i] \;, \tag{1.1.24}$$

where $I[i] \;=\; I_1 \times \cdots \times I_{i-1} \times I_{i+1} \times \cdots \times I_m$, $dt[i] \;=\; dt_1 ... dt_{i-1} dt_{i+1} ... dt_m$.

1.2 Some model examples of operators with constant coefficients

1.2.1

Let us consider in $\mathbb{R}^n$ the equation

$$A(D,q)u(x,q) \;=\; f(x,q) \qquad x \in \mathbb{R}^n \;,\; q \in \mathbb{C}^m \;, \tag{1.2.1}$$

where

$$A(D,q) \;=\; \sum_{|\alpha|+|\beta|=s} a_{\alpha\beta} q^\beta D_x^\alpha$$

is a differential operator with constant coefficients and $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \in \mathbb{Z}_+$, $\beta = (\beta_1, ..., \beta_m)$, $\beta_j \in \mathbb{Z}_+$ are multiindices.

We suppose that the symbol $A(\xi, q)$, which is obtained from $A(D,q)$ after substitution of D by ξ , is a homogeneous polynomial in (ξ, q) of degree s , and that each parameter q_i is varying in a closed angle

$$Q_i \;\subset\; \mathbb{C} \,:\; Q_i \;=\; \{q_i \in \mathbb{C} \,:\; \delta_i \leq \arg q_i \leq \gamma_i\} \tag{1.2.2}$$

with the vertex in the origin.

Proposition 1.2.1 *For* $l \geq s$ *operator* $A(D,q) \,:\, H^l(\mathbb{R}^n) \to H^{l-s}(\mathbb{R}^n)$ *is bounded and*

$$|||f|||_{l-s} \;\leq\; C|||u|||_l \;. \tag{1.2.3}$$

Proof. We easily obtain (1.2.3) applying (1.1.7) to the inequality

$$\left\|A(D,q)u\right\|_{l-s} \;\leq\; C \sum_{k=0}^{s} |||q|||^{l-k} \|u\|_k \;.$$

▷

Now we formulate the *ellipticity condition* for the operator $A(D,q)$.

Condition 1.2.2 $\forall\, q \in Q \;=\; Q_1 \times ... \times Q_m$ *and* $\forall\, \xi \in \mathbb{R}^n$

$$A(\xi, q) \;\neq 0 \quad for \quad |\xi| + M_q \neq 0 \;, \tag{1.2.4}$$

where $M_q \;=\; \min_{i=1,...,m} |q_i|$.

We note that, if $q = (q_1, ..., q_m) = 0$, then (1.2.4) turns into the standard ellipticity condition for the operator $A(D,0)$.

Also we denote by $Q_M \;=\; \{q \in Q \,:\, M_q > 0\}$.

Proposition 1.2.3 *Let* $l \geq s$ *. Then under Condition 1.2.2 ,* $\forall\, q \in Q_M$ *and* $f \in H^{l-s}(\mathbb{R}^n)$ *there exists the unique solution* $u = \mathcal{R}_0 f \in H^l(\mathbb{R}^n)$ *to the equation (1.2.1). Moreover for arbitrary* $q^{(0)} > 0$ *and arbitrary*

$$q \in Q_M^0 \;=\; \left\{ q \in Q_M : \, M_q \geq q^{(0)} \right\}$$

the following a priori estimate takes place

$$|||u|||_l \;=\; |||\mathcal{R}_0 f|||_l \;\leq\; C|||f|||_{l-s} \;. \tag{1.2.5}$$

Proof. Let $u \in H^l(\mathbb{R}^n)$, then $F\,[A(D,q)u] \;=\; A(\xi,q)Fu \;=\; Ff$ and it follows from (1.2.4) that $Fu \;=\; A^{-1}(\xi,q)Ff$. Then from the homogenity of A we obtain

$$|Fu| \;\leq\; C\,(|\xi| + |||q|||)^{-s}\,|Ff| \tag{1.2.6}$$

for $q \in Q_M$, and thus

$$\begin{aligned} &\left(1 + |\xi|^{2l} + |||q|||^{2l}\right) |Fu|^2 \\ &\leq\; C\,\frac{1 + |\xi|^{2l} + |||q|||^{2l}}{(|\xi| + |||q|||)^{2s}}\,|Ff|^2 \\ &\leq\; C\,\left(1 + |\xi|^{2l-2s} + |||q|||^{2l-2s}\right) |Ff|^2 \;, \end{aligned}$$

where $M_q \geq q^{(0)}$ and the constants C are independent of u, ξ, q . Integrating over ξ we obtain (1.2.5) and the uniqueness follows. The existence is the consequence of the formula

$$u \;=\; \mathcal{R}f \;=\; F^{-1}A^{-1}(\xi,q)Ff \;, \tag{1.2.7}$$

which is valid $\forall\, q \in Q_M$, and, as follows from (1.2.6), it defines $\forall\, f \in H^{l-s}(\mathbb{R}^n)$ the solution $u \in H^l(\mathbb{R}^n)$ to the equation (1.2.1). ▷

Remark 1.2.4 *Let* $n > 1$ *and suppose (1.2.4) is satisfied, then* $s = \mathrm{ord}A = 2k$ *and* $A(\xi + \lambda\xi_0, q)$ *as the polynomial in* λ *has for* $q \in Q_M$ *,* $|\xi| + M_q \neq 0$ *,* ξ_0 *orthogonal to* ξ *and* $\xi_0 \neq 0$ *, exactly* k *roots in the upper* $\lambda-$*half-plane and* k *roots in the lower half-plane, taking into account their multiplicities (for the proof see, for example, Agranovich M.S., Vishik M.I. (1964), Agranovich M.S. (1990)).*

Example 1.2.5 *The first main example of the above situation is the operator* $A(D) + q_1^k q_2^k$ *, where* $A(D)$ *is a homogeneous elliptic operator of order* $2k$ *with real coefficients and such that* $A(\xi) > 0$ *for* $\xi \neq 0$ *(for example,* $A = -\Delta$*). Then (1.2.4) is satisfied in the cone* $Q = Q_1 \times Q_2$ *, where*

$$Q_i \;=\; \left\{ q_i \in \mathbb{C} : \; |\arg q_i| \leq \frac{\pi - \varepsilon}{2k} \right\} \;,$$

with arbitrary $\varepsilon > 0$ *.*

Example 1.2.6 *The second example concerns the pair of operators* $A_1(D)$ *and* $A_2(D)$ *of the above type:*

$$\left(A_1(D) + q_1^{2k_1}\right)\left(A_2(D) + q_2^{2k_2}\right) ,$$

where $\operatorname{ord} A_1 = 2k_1$ *and* $\operatorname{ord} A_2 = 2k_2$ *. Obviously, (1.2.4) is satisfied in the cone* $Q_1 \times Q_2$ *, where*

$$Q_i = \left\{q_i \in \mathbb{C} : \ |\arg q_i| \le \frac{\pi - \varepsilon}{2k_i}\right\} .$$

1.2.2

Let us consider equation (1.2.1) in $\mathbb{R}^n_+$

$$A(D,q)u(x,q) = f(x,q) , \tag{1.2.8}$$

where $x = (x', x_n) \in \mathbb{R}^n_+$, $x_n \ge 0, \quad q \in Q \subset \mathbb{C}^m$, with boundary conditions

$$B_\nu(D,q)u(x,q)\big|_{x_n=0} = g_\nu(x') , \ \nu = 1, ..., k. \tag{1.2.9}$$

As in (1.2.1) $A(D,q)$ and $B_\nu(D,q)$ are differential operators with constant coefficients such that after substitution of D by ξ we obtain polynomials $A(\xi,q)$ and $B_\nu(\xi,q)$ which are homogeneous in $(\xi_1, ..., \xi_n, q_1, ..., q_m) \in \mathbb{R}^n \times Q$ of orders $2k$ and k_ν respectively. We denote by $\mathcal{A}$ a boundary value problem operator

$$\mathcal{A}u = \left\{A(D,q)u, \ B_1(D,q)u\big|_{x_n=0}, ..., \ B_k(D,q)u\big|_{x_n=0}\right\} \tag{1.2.10}$$

and by l_0 $-$ the $\max(2k, k_\nu + 1)$.

Proposition 1.2.7 *Let* $l \in \mathbb{Z}$, $l \ge l_0 = \max(2k, k_\nu + 1)$ *. Then the operator*

$$\mathcal{A} : \ H^l(\mathbb{R}^n_+) \longrightarrow H^{l-2k}(\mathbb{R}^n_+) \times \prod_{\nu=1}^{k} H^{l-k_\nu-1/2}(\mathbb{R}^{n-1}) \tag{1.2.11}$$

is bounded and

$$|||f|||_{l-2k} + \sum_{\nu=1}^{k} |||g_\nu|||'_{l-k_\nu-1/2} \le C|||u|||_l . \tag{1.2.12}$$

Proof. For the operator A the proof is the same as in Proposition 1.2.1. Using (1.1.6) we obtain for B_ν that

$$|||B_\nu(D,q)u|||'_{l-k_\nu-1/2} \le C|||B_\nu(D,q)u|||_{l-k_\nu} \le C|||u|||_l ,$$

where the last inequality is the analogue of (1.2.3) for B_ν and the proof follows. $\triangleright$

Let us formulate the analog of the Shapiro - Lopatinskij condition in our case, i.e. for the problem (1.2.8) - (1.2.9).

Suppose $f=0$ there. After the Fourier transformation F' in x' we obtain from (1.2.8) - (1.2.9) the boundary value problem for an ordinary differential equation on the half-axis:

$$A\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n) \;=\; 0\,, \quad x_n > 0\;, \tag{1.2.13}$$

$$B_\nu\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n)\big|_{x_n=0} = \; h_\nu\,, \quad \nu = 1, ..., k\;, \tag{1.2.14}$$

smoothly depending on parameters ξ' and q.

Denote by $\mathcal{M}$ $-$ the space of all stable solutions to the equation (1.2.13) for fixed ξ' and q. It follows from Remark 1.2.4 that $\dim \mathcal{M} = k$ for $n>1$. For $n=1$ we additionally assume that the equation $A(\lambda, q) = 0$ has relatively to λ equally k solutions in the upper and the lower half-planes for $q \in Q_M$.

Condition 1.2.8 $\forall\, q \in Q$ *for* $|\xi'| + M_q \neq 0$ *the problem (1.2.13) - (1.2.14) has the unique solution in the space* $\mathcal{M}$ *for all* $h_\nu \in \mathbb{C}$.

Remark 1.2.9 *We do not lose the generality assuming* $\big(F'_{x'\to\xi'} f\big)(\xi', x_n) \equiv 0$ *in (1.2.13). Indeed, if for fixed* ξ' *it equals to* $g(x_n)$, $g(x_n) \in \mathcal{M}$, *we note that the equation*

$$A\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n) \;=\; g(x_n)$$

always has a particular solution $v \in \mathcal{M}$. *Subtracting it we return to (1.2.13) with* $g(x_n) \equiv 0$.

Example 1.2.10 *Let* $A(D_1, ..., D_n, D_{n+1}, ..., D_{n+m})$ *and* $B_\nu(D_1, ..., D_n, D_{n+1}, ..., D_{n+m})$, $(n, m > 1)$ *be homogeneous differential operators in* $\mathbb{R}^{n+m}$ *of orders* $2k$ *and* k_ν *respectively, let* A *be elliptic and* A *and* B_ν *satisfy together the classical Shapiro - Lopatinskij condition (see, for example, Agranovich M.S., Vishik M.I. (1964), Egorov Yu.V., Shubin M.A. (1988)) for* $x_n = 0$. *Then the operators* $A(D,q)$ *and* $B_\nu(D,q)$ $(D = (D_1, ..., D_n)$, $q = (q_1, ..., q_m)$) *satisfy Conditions 1.2.2 and 1.2.8 for* $q \in \mathbb{R}^m \cap Q_M$ *or, equivalently, for each* q_i *varying in a real ray* $\{\arg q_i = 0 \text{ or } \pi\}$. *We are really interested not in rays, but in angles. The corresponding examples are given below (see Examples 1.3.4 and 1.3.5).*

In the sequel we need the algebraic form of Condition 1.2.8 . Let

$$\omega_1(x_n, \xi', q)\,, \;\ldots, \omega_k(x_n, \xi', q)$$

be an arbitrary basis in $\mathcal{M}$. Then an arbitrary solution of (1.2.13) - (1.2.14) can be written in the form

$$v(x_n) \;=\; \alpha_1\omega_1 \;+\;\cdots\;+\; \alpha_k\omega_k \;.$$

Substituting v into boundary conditions (1.2.14), we obtain the system of k linear algebraic equations with k unknowns $\alpha_1, ..., \alpha_k$. The condition of its unique solvability is

$$\det\left\{ B_j\big(\xi', -i\frac{d}{dx_n}, q\big)\,\omega_\nu(x_n,\xi',q)\Big|_{x_n=0}\right\} \;\neq\; 0$$

for $|\xi'| + M_q \neq 0$, $q\in Q$ and $j,\nu = 1,...,k$.

The crucial role in the following will be played by the canonical basis

$$\Omega_1(x_n,\xi',q)\,,\;\ldots,\Omega_k(x_n,\xi',q) \tag{1.2.15}$$

in $\mathcal{M}$ which is defined by the boundary conditions

$$B_j\big(\xi', -i\frac{d}{dx_n}, q\big)\,\Omega_l(x_n,\xi',q)\Big|_{x_n=0} \;=\; \delta_{jl} \;.$$

Obviously, a solution to the problem (1.2.13) - (1.2.14) with arbitrary h_ν is expressed in terms of the canonical basis as follows:

$$v(x_n) \;=\; h_1\Omega_1 \;+\;\cdots\;+\; h_k\Omega_k \;.$$

We also use the following integral representation for $\Omega_\nu(x_n,\xi',q)$ (see Agranovich M.S., Vishik M.I. (1964), Agranovich M.S. (1990) for the proof):

$$\Omega_\nu(x_n,\xi',q) \;=\; \int_{\gamma_+} e^{i\lambda x_n}\, A_+^{-1}(\xi',\lambda,q)\, N_\nu(\xi',\lambda,q)\, d\lambda \;, \tag{1.2.16}$$

for $|\xi'| + M_q \neq 0$, $q\in Q$ and $\nu = 1,...,k$, where γ_+ is a smooth curve in the upper half-plane $\operatorname{Im}\lambda > 0$, depending on (ξ',q) and surrounding all λ - zeroes of the polynomial

$$A_+(\xi',\lambda,q) \;=\; \prod_{j=1}^{k}\big(\lambda - \lambda_j^+(\xi',q)\big) \;, \tag{1.2.17}$$

where λ_j^+ - are all λ - zeroes of the polynomial $A(\xi',\lambda,q)$ with positive imaginary parts, $N_\nu(\xi',\lambda,q)$ are the C^∞-functions positively homogeneous of order $k - k_\nu - 1$ in all their variables and polynomial in λ .

Proposition 1.2.11 *Let $l \geq l_0$ and let Conditions 1.2.2 and 1.2.8 are fulfilled. Then, for arbitrary $q\in Q_M$, $f\in H^{l-2k}(\mathbb{R}^n_+)$ and $g_\nu \in H^{l-k_\nu-1/2}(\mathbb{R}^{n-1})$ the unique solution to the problem (1.2.8) - (1.2.9) exists.*

Moreover, for an arbitrary $q^{(0)} > 0$ *and* $\forall q \in Q_M^0$ *, we have the a priori estimate*

$$|||u|||_l \;\le\; C\{|||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||g_\nu|||'_{l-k_\nu-1/2}\} \qquad \textit{(1.2.18)}$$

Proof. We follow here the scheme of the proof of Theorem 3.1 from (Agranovich M.S., Vishik M.I. (1964)). As in Proposition 1.2.3 we start with the proof of an a priori estimate.

1). First, we reduce the problem to the case $f = 0$. Let L be the extension operator (see (1.1.5)). It follows from Proposition 1.2.3 that we can solve the equation $A(D,q)u = Lf$ in $\mathbb{R}^n$, and in the view of (1.2.7)

$$u_0 \;=\; F^{-1}A^{-1}(\xi,q)FLf \;, \qquad (1.2.19)$$

where $q \in Q_M$, is its particular solution and $u_0 \in H^l(\mathbb{R}^n)$.

Thus, $u_0|_{\mathbb{R}^n_+} \in H^l(\mathbb{R}^n_+)$, and from (1.2.5), (1.1.14) and (1.1.15) we obtain, that

$$|||u_0|||_{l,\mathbb{R}^n_+} \;\le\; C\,|||f|||_{l-2k,\mathbb{R}^n_+} \qquad (1.2.20)$$

for $q\,:\,M_q \ge q^{(0)}$. We now suppose that $u = u_0 + u_1$, where $u_1 \in H^l(\mathbb{R}^n_+)$ is a solution to the problem

$$\begin{cases} A(D,q)u_1 \;=\; 0 & \text{in } \mathbb{R}^n_+ \;, \\ B_\nu(D,q)u_1\big|_{x_n=0} \;=\; g_\nu - g_{0\nu} & \text{on } \mathbb{R}^{n-1}_{x'} \;, \end{cases}$$

where $g_{0\nu} \;=\; B_\nu(D,q)u_0\big|_{x_n=0}$. Suppose, we can estimate u_1 as follows:

$$|||u_1|||_l \;\le\; C\sum_{\nu=1}^{k} |||g_\nu - g_{0\nu}|||'_{l-k_\nu-1/2} \;. \qquad (1.2.21)$$

Then,

$$|||u|||_l \;\le\; |||u_0|||_l \;+\; |||u_1|||_l \;\le$$

$$\le\; \{|||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||g_\nu|||'_{l-k_\nu-1/2} \;+\; \sum_{\nu=1}^{k} |||g_{0\nu}|||'_{l-k_\nu-1/2}\}$$

Using (1.2.12) with the function u_0 and (1.2.20) we obtain

$$\sum_{\nu=1}^{k} |||g_{0\nu}|||'_{l-k_\nu-1/2} \;\le\; C\,|||u|||_l \;\le\; C\,|||f|||_{l-2k} \;.$$

Thus, the estimate (1.2.18) follows from (1.2.21), and we are able to suppose $f = 0$.

2). Applying the Fourier transform F' with respect to x' in (1.2.8), (1.2.9), we arrive to the problem on the half-axis:

$$A(\xi', D_n, q)\, F'u \;=\; 0 \;, \tag{1.2.22}$$

$$B_\nu(\xi', D_n, q)\, F'u\big|_{x_n=0} \;=\; F'g_\nu \;, \tag{1.2.23}$$

where $\nu = 1, ..., k$. Since $u \in H^l(\mathbb{R}^n_+)$, then $F'u$ is a stable solution of (1.2.22) for almost all (ξ', q) . So we can write the decomposition of $F'u$ in the canonical basis Ω_ν (see (1.2.15), (1.2.16)):

$$F'u \;=\; \sum_{\nu=1}^{k} F'g_\nu \times \Omega_\nu \;,$$

or, equivalently,

$$u \;=\; {F'}^{-1} \sum_{\nu=1}^{k} F'g_\nu \times \Omega_\nu \;. \tag{1.2.24}$$

In order to obtain the estimate of the H^l-norm of u , we have to estimate $D^j_{x_n}\Omega_\nu$. Differentiating (1.2.16) inside the integral, we get

$$|D^j_{x_n}\Omega_\nu|^2 \;\le\; \int_{\gamma_+} \left|\lambda^j e^{i\lambda x_n}\, A_+^{-1}(\xi',\lambda,q)\, N_\nu(\xi',\lambda,q)\right| |d\lambda| \;,$$

or, after substitution of λ by $\lambda_1 \left(|\xi'|^2 + |||q|||^2\right)^{1/2}$ and integration over λ_1 ,

$$|D^j_{x_n}\Omega_\nu|^2 \;\le$$

$$\le\; C\left[\left(|\xi'|^2 + |||q|||^2\right)^{j/2-k/2+(k-k_\nu-1)/2+1/2}\right]^2 \exp\Big\{-C_1 x_n \left(|\xi'|^2 + |||q|||^2\right)^{1/2}\Big\}$$

$$=\; C\left(|\xi'|^2 + |||q|||^2\right)^{j-k_\nu} \exp\Big\{-C_1 x_n \left(|\xi'|^2 + |||q|||^2\right)^{1/2}\Big\}$$

with C and C_1 independent of ξ', q, x_n and $M_q \ge q^{(0)}$ (we need it to estimate A_+^{-1}). Finally, integrating over x_n we get

$$\int_0^\infty |D^j_{x_n}\Omega_\nu|^2\, dx_n \;\le C\left(|\xi'|^2 + |||q|||^2\right)^{j-k_\nu-1/2}$$

and thus

$$\sum_{j+j_1\le l} |\xi'|^{2j_1} \int_0^\infty |D^j_{x_n}\Omega_\nu|^2\, dx_n \;+ |||q|||^{2l} \int_0^\infty |\Omega_\nu|^2\, dx_n$$

$$\le\; C\left(|\xi'|^2 + |||q|||^2\right)^{l-k_\nu-1/2} \;\le\; C\left(1 + |\xi'|^{2l-2k_\nu-1} \;+\; |||q|||^{2l-2k_\nu-1}\right) .$$

Multiplying the left- and the right-hand sides of the last inequality by $|F'g_\nu|^2$ and integrating over ξ' we obtain the desired estimate

$$\big\| |F'^{-1}\{F'g_\nu \cdot \Omega_\nu\}| \big\|^2_{l,\mathbb{R}^n_+} \;\le\; C \| |g_\nu| \|'^2_{l-k_\nu-1/2} \; . \qquad (1.2.25)$$

The existence follows from the formulae (1.2.19) and (1.2.24) applied to u_0 and u_1 We put

$$u \;=\; u_0 + u_1 \;=\; F^{-1}A^{-1}(\xi,q)FLf \,+$$

$$+\sum_{\nu=1}^{k} F'^{-1}\Big\{F'\big[g_\nu - \big(B_\nu(D,q)F^{-1}A^{-1}(\xi,q)FLf\big)\big|_{x_n=0}\big]\cdot\Omega_\nu\Big\}$$

$$=\; \mathcal{R}_0 f \;+\; \sum_{\nu=1}^{k}\mathcal{R}_\nu\big[g_\nu - \big(B_\nu(D,q)\mathcal{R}_0 f\big)\big|_{x_n=0}\big] \;:=\; \mathcal{R}(f,g_1,...,g_k)\;, \quad (1.2.26)$$

where

$$\mathcal{R}_0 f \;:=\; F^{-1}A^{-1}(\xi,q)FLf \qquad (1.2.27)$$

and

$$\mathcal{R}_\nu g_\nu \;:=\; F'^{-1}\{F'g_\nu\cdot\Omega_\nu\}\;. \qquad (1.2.28)$$

For $f \in H^{l-2k}(\mathbb{R}^n_+)$, $g_\nu \in H^{l-k_\nu-1/2}(\mathbb{R}^{n-1})$ and $q \in Q_M$ these formulae define the solution $u \in H^l(\mathbb{R}^n_+)$ to the problem (1.2.8) - (1.2.9). $\triangleright$

1.3 Elliptic boundary value problems with parameters in a bounded domain

Let G be a bounded domain in $\mathbb{R}^n_x$ with boundary $\Gamma \in C^\infty$, which is an $(n-1)$ - dimensional surface admitting a local straightening with the help of a smooth nondegenerate coordinate transformations.

Let us consider in G the equation

$$Au \;\equiv\; A(x,D,q)\,u(x,q) \;=\; f(x,q) \qquad (1.3.1)$$

with boundary conditions on Γ :

$$B_\nu u|_\Gamma \;\equiv\; B_\nu(x,D,q)u(x,q)\big|_{x=x'} \;=\; g_\nu(x',q)\;, \qquad (1.3.2)$$

where $x' \in \Gamma$ and $\nu = 1,...,k$. Here $A(x,D,q)$ and $B_\nu(x,D,q)$ are differential operators with coefficients, which are C^∞- functions in $x \in \overline{G}$ and polynomials in q_i for $i = 1,...,m$, where

$$q_i \in Q_i \;=\; \{q_i \,:\, \arg q_i \in [\alpha_i,\beta_i]\}\;.$$

Hence, $A(x,\xi,q)$ and $B_\nu(x,\xi,q)$ are polynomials in (ξ,q) of degrees $s=2k$ and k_ν $(\nu=1,...,k)$ respectively.

We denote by $A_0(x,\xi,q)$ and $B_{0,\nu}(x,\xi,q)$ the leading parts of the corresponding polynomials which are obtained by dropping the terms with degrees relatively to the variables (ξ,q) being less than $2k$ and k_ν respectively.

The operators $A_0(x,D,q)$ and $B_{0,\nu}(x,D,q)$ are called the *leading terms* of the corresponding operators . By analogy with the previous section, we introduce the operator of the boundary value problem

$$\mathcal{A}u \;=\big\{Au,\; B_1u|_\Gamma,...,\; B_ku|_\Gamma\big\}\;:\; H^l(G)\;\longrightarrow\; H^{l-2k}(G)\,\times\prod_{\nu=1}^{k}H^{l-k_\nu-1/2}(\Gamma)$$

and the number $l_0\;=\;\max(2k,k_\nu+1)$.

Proposition 1.3.1 *Let* $l\in\mathbb{Z}$, $l\geq l_0$. *Then the operator* $\mathcal{A}$ *is bounded and*

$$|||f|||_{l-2k}\;+\;\sum_{\nu=1}^{k}|||g_\nu|||'_{l-k_\nu-1/2}\;\leq\;C|||u|||_l\;, \tag{1.3.3}$$

where the norms are taken over G *and* Γ *(see (1.1.18) and (1.1.20)).*

bf Proof does not differ from the proof of Proposition 1.2.7. $\triangleright$

Let us now formulate conditions of the unique solvability of the problem (1.3.1) - (1.3.2) which are analogous to Conditions 1.2.2 and 1.2.8 from the previous section.

Condition 1.3.2 $\forall\,x\in\overline{G}$, $\forall\,q\in Q\;=\;Q_1\times...\times Q_m$ *and* $\forall\,\xi\in\mathbb{R}^n$

$$A_0(x,\xi,q)\;\neq 0\quad for\quad|\xi|+M_q\neq 0\;, \tag{1.3.4}$$

where $M_q\;=\;\min_{i=1,...,m}|q_i|$.

In the case $n=1$ we suppose in addition that the λ-roots of the polynomial $A_0(x,\lambda,q)$ are equally distributed between the upper and the lower half-planes.

Condition 1.3.3 *Let us fix a boundary point* $x'\in\Gamma$ *and suppose that the operators in (1.3.1) - (1.3.2) are written in a local coordinate system* connected with this point *(i.e. the boundary is straightened and the* x_n*-axis looks in one side with the inner normal to* Γ *).*

Consider the following problem on the half-axis $x_n>0$:

$$A_0\big(0,\xi',-i\frac{d}{dx_n},q\big)\,v(x_n)\;=\;0,\quad x_n>0\;, \tag{1.3.5}$$

$$B_{0,\nu}\big(0,\xi',-i\frac{d}{dx_n},q\big)\,v(x_n)\big|_{x_n=0}=\;h_\nu\;,\quad\nu=1,...,k\;, \tag{1.3.6}$$

smoothly depending on parameters ξ' *and* q .

We suppose that $\forall\, q \in Q$ *for* $|\xi'| + M_q \neq 0$ *this problem has the unique solution in the space* $\mathcal{M}$ *of all stable solutions of the equation (1.3.5) for all* $h_\nu \in \mathbb{C}$.

Note that for $q = 0$ Condition 1.3.2 turns into the ordinary ellipticity condition as well as Condition 1.3.3 turns into the classical Shapiro - Lopatinskij condition.

Example 1.3.4 *Let* $A(x, D)$ *be an elliptic operator* $(\mathrm{ord} A = 2k)$ *with a positive real leading part. Then the operator*

$$A(x, D) + q_1^k q_2^k$$

with Dirichlet boundary conditions

$$B_\nu \;=\; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1} \;, \quad \nu = 1, ..., k,$$

satisfies Conditions 1.3.2 and 1.3.3 in the cone $Q = Q_1 \times Q_2$, *where* $Q_j = \left\{|\arg q_j| \;\leq\; \frac{\pi-\varepsilon}{2k}\right\}$, $\varepsilon \in (0, \pi)$ *and* $j = 1, 2$.

Example 1.3.5 *If* A_1 *and* A_2 $(\mathrm{ord} A_j = 2k_j \,,\; j = 1, 2)$ *are the two such operators as in Example 1.3.4, then the operator*

$$\left(A_1(x, D) + q_1^{2k_1}\right)\left(A_2(x, D) + q_2^{2k_1}\right)$$

with Dirichlet boundary conditions

$$B_\nu \;=\; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1} \;, \quad \nu = 1, ..., k_1 + k_2,$$

also satisfies Conditions 1.3.2 and 1.3.3 in the cone $Q = Q_1 \times Q_2$, *where* $Q_j = \left\{|\arg q_j| \;\leq\; \frac{\pi-\varepsilon}{2k_j}\right\}$, $\varepsilon \in (0, \pi)$ *and* $j = 1, 2$.

Proposition 1.3.6 *Let* $l \in \mathbb{Z}$, $l \geq l_0$, $q^{(0)} > 0$ *and suppose Conditions 1.3.2 and 1.3.3 being fulfilled. Then there exists a constant* $q^{(1)} > 0$ *such that for all* $q \in Q_M^0 = \{q \in Q_M : \; M_q \geq q^{(0)}\}$ *with the property* $|||q||| = \sum_{i=1}^m |q_i| \geq q^{(1)}$, *the a priori estimate*

$$|||u|||_l \;\leq\; C\{|||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||g_\nu|||'_{l-k_\nu-1/2}\} \qquad (1.3.7)$$

is valid for the solutions of (1.3.1) - (1.3.2), where the norms are taken over G *and* Γ *(see (1.1.18) and (1.1.20)).*

Proof. We begin with the local situation and then glue the required estimate from local ones using a partition of unity.

A). Suppose $x_0 \in U \subset \operatorname{int} G$, $u \in H^l(G)$ and $\operatorname{supp} u \subset\subset U$, where U is an open set. Then (1.3.7) is equivalent to the inequality $|||u|||_l \le C|||A(x)u|||_{l-2k}$, where for brevity we write $A(x) = A(x, D, q)$, and analogously $A(x_0) = A(x_0, D, q)$.

It follows from Proposition 1.2.3 that

$$|||u|||_l \;\le\; C_1|||A_0(x_0)u|||_{l-2k} \;,$$

where the norms are taken over G . And thus

$$|||u|||_l \;\le\; C_1|||A(x)u|||_{l-2k} \;+\; \mathcal{R} \;,$$

where

$$\begin{aligned} \mathcal{R} \;&=\; C_1\big|\big|\big|\big(A(x) - A_0(x_0)\big)u\big|\big|\big|_{l-2k} \\ &\le\; C_1\big|\big|\big|\big(A(x) - A_0(x)\big)u\big|\big|\big|_{l-2k} \;+\; C_1\big|\big|\big|\big(A_0(x) - A_0(x_0)\big)u\big|\big|\big|_{l-2k} \;. \end{aligned}$$

We note that in $A(x) - A_0(x)$ the leading terms of the form $a(x)q^\alpha D^\beta$ $(|\alpha| + |\beta| = 2k)$ are absent. Hence, it follows from (1.1.13) that

$$\big|\big|\big|\big(A(x) - A_0(x)\big)u\big|\big|\big|_{l-2k} \;\le\; C_2|||u|||_{l-1} \;.$$

Conversely, $A_0(x) - A_0(x_0)$ consists only of leading terms, but for sufficiently small U the coefficients $a(x)$ may be made sufficiently small. Thus

$$|||a(x)q^\alpha D^\beta u|||_{l-2k} \;\le\; \varepsilon|||u|||_l \;+\; C_3|||u|||_{l-1} \;.$$

Now we have from (1.1.13)

$$|||q||| \cdot |||u|||_{l-1} \;=\; \sum_{i=1}^{m}|q_i|\; |||u|||_{l-1} \;\le\; C_4|||u|||_l \;, \tag{1.3.8}$$

hence

$$\mathcal{R} \;\le\; C_1\Big[\varepsilon + \frac{(C_2 + C_3)C_4}{\sum_{i=1}^m|q_i|}\Big]\;|||u|||_l \;,$$

where C_i are independent of u and q . Now, for sufficiently small ε and sufficiently large $|||q||| = \sum_{i=1}^m|q_i|$, $q \in Q_M^0$, we see that $\mathcal{R} \le 1/2|||u|||_l$ and thus, $|||u|||_l \le C|||A(x)u|||_{l-2k}$ for $C = 2C_1$.

B). We now suppose that $x_0 \in \Gamma$ and choose $U \ni x_0$ such that $U \subset U_\alpha$, where U_α is one of the boundary covering coordinate neighbourhoods. For the functions $u \in H^l(G)$ with $\operatorname{supp} u \subset U \cap \overline{G}$ we obtain from (1.2.18) (in local coordinates) that

$$|||u|||_l \;\le\; C\Big\{|||A_0(x_0)u|||_{l-2k} \;+\; \sum_{\nu=1}^{k}|||B_{\nu,0}(x_0)u|||'_{l-k_\nu-1/2}\Big\} \;.$$

In the same way as in part A) of the proof we obtain

$$|||u|||_l \;\leq\; C_1\{|||A(x)u|||_{l-2k} \;+\; \sum_{\nu=1}^{k}|||B_\nu(x)u|||'_{l-k_\nu-1/2}\} \;+\; \mathcal{R}\;,$$

where

$$\mathcal{R} \;=\; C_1\Big\{\Big|\Big|\Big|\big(A(x)-A_0(x_0)\big)u\Big|\Big|\Big|_{l-2k} \;+\; \sum_{\nu=1}^{k}\Big|\Big|\Big|\big(B_\nu(x)-B_{\nu,0}(x_0)\big)u\Big|\Big|\Big|'_{l-k_\nu-1/2}\Big\}\;.$$

The first norm in $\mathcal{R}$ is estimated as in part A) of the proof. For each term in the sum over ν we obtain applying the trace inequality (1.1.6) :

$$\Big|\Big|\Big|\big(B_\nu(x)-B_{\nu,0}(x_0)\big)u\Big|\Big|\Big|'_{l-k_\nu-1/2} \;\leq\; \Big|\Big|\Big|\big(B_\nu(x)-B_{\nu,0}(x_0)\big)u\Big|\Big|\Big|_{l-k_\nu}\;,$$

and now the right hand-side can be estimated as $\Big|\Big|\Big|\big(A(x)-A_0(x_0)\big)u\Big|\Big|\Big|_{l-2k}$ in part A) of the proof.

C). In order to end the proof we have to arrange a standard procedure with a partition of unity. Let $\{U_\alpha\}$ be a covering of $\bar{G}$ such that the results of items A) and B) are valid in each U_α , and let $\{U_\mu\}_{\mu=1}^{\mu_0}$ be its finite subcovering. Denote by C and $q^{(1)}$ the largest local ones. Now (1.3.7) is valid for all $q\,:\; |||q||| \;\geq\; q^{(1)}$, $q\,\in\,Q_M^0$ for all $u\,\in\,H^l(G)$ such that $\exists\mu\,\in\,\{1,...,\mu_0\}\,:$ $\operatorname{supp} u \subset U_\mu$.

We now take the corresponding to $\{U_\mu\}$ partition of unity $\{\varphi_\mu\}_{\mu=1}^{\mu_0}$. As for the function $\varphi_\mu u$ (1.3.7) is already proved, we have

$$\begin{aligned}|||u|||_l \;&\leq\; \sum_{\mu=1}^{\mu_0}\;|||\varphi_\mu u|||_l\\ &\leq\; C\sum_{\mu=1}^{\mu_0}\Big\{|||A(x)[\varphi_\mu u]|||_{l-2k} \;+\; \sum_{\nu=1}^{k}|||B_\nu(x)[\varphi_\mu u]|||'_{l-k_\nu-1/2}\Big\}\;,\end{aligned}$$

but $A(x)[\varphi_\mu u] \;=\; \varphi_\mu A(x)u \;+\; ...$, where in the terms denoted by "..." φ_μ is differentiated at least once and thus

$$\begin{aligned}&|||u|||_l\\ &\quad\leq\; C\sum_{\mu=1}^{\mu_0}\Big\{|||\varphi_\mu A(x)u|||_{l-2k} \;+\; \sum_{\nu=1}^{k}|||\varphi_\mu B_\nu(x)u|||'_{l-k_\nu-1/2}\Big\} \;+\\ &\qquad\qquad\qquad\qquad\qquad\qquad +\; C_1|||u|||_{l-1}\;.\end{aligned}$$

Once again we use (1.3.8) (together with the fact that the multiplication by $\varphi_\mu \in C^\infty$ is a bounded operator in H^s and $\|\varphi u\|_l^2 \;\leq\; \|u\|_l^2 \;+\; C_\varphi\|u\|_{l-1}^2$) and obtain

$$|||u|||_l$$

$$\leq \; C_3 \sum_{\mu=1}^{\mu_0} \Big\{ |||A(x)u|||_{l-2k} \; + \; \sum_{\nu=1}^{k} |||B_\nu(x)u|||'_{l-k_\nu-1/2} \Big\} \; +$$

$$+ \; \frac{C_2}{|||q|||} |||u|||_l \; .$$

Finally, we suppose $q^{(1)} \geq 2C_2$ and the proof follows. $\triangleright$

Naturally, the *uniqueness of the solution to the problem (1.3.1) - (1.3.2)* follows from (1.3.7) for sufficiently large $|||q|||\,,\; q \in Q_M^0$.

We now show that Conditions 1.3.2 and 1.3.3 are also necessary for the estimate (1.3.7) to take place for large $|||q|||\,,\; q \in Q_M^0$.

Proposition 1.3.7 *Let $l \in \mathbb{Z}$, $l \geq l_0$, $q^{(0)} > 0$ and suppose there exists a large constant $q^{(1)} > 0$ such that the estimate (1.3.7) is true for all solutions from $H^l(G)$ to the problem (1.3.1) - (1.3.2) when $|||q||| \geq q^{(1)}$ and $q \in Q_M^0$.*

Then Condition 1.3.2 is satisfied in every interior point $x_0 \in G$ and Condition 1.3.3 is satisfied in every boundary point $x_0 \in \Gamma$ where Condition 1.3.2 is already satisfied (if $n = 1$, we assume additionally the regular distribution of $\lambda-$roots of the equation $A_0(x, \lambda, q) = 0$).

Proof. A). First, we prove that $\forall x_0 \in G$ there exists $U_{x_0} \subset G$ and $q^{(1)}$: $\forall q \in Q_M^0$: $|||q||| \geq q^{(1)}$ and $\forall u \in H^l(G)$ with $\operatorname{supp} u \subset U_{x_0}$ the estimate

$$|||u|||_l \; \leq \; C_1 |||A_0(x_0)u|||_{l-2k} \; . \tag{1.3.9}$$

takes place. Indeed,

$$|||u|||_l \; \leq \; C|||A(x)u|||_{l-2k}$$

$$\leq \; C|||A_0(x_0)u|||_{l-2k} \; + \; \mathcal{R} \; ,$$

where

$$\mathcal{R} \; = \; C|||(A(x) - A_0(x_0))u|||_{l-2k} \; ,$$

but this was already estimated in the proof of part A) of Proposition 1.3.6, and thus, if U_{x_0} is sufficiently small and $|||q|||$ - sufficiently large , we obtain that $\mathcal{R} \leq 1/2|||u|||_l$, and the required estimate follows for $C_1 = 2C$.

B). By analogy, $\forall x_0 \in \Gamma$ there exists U_{x_0} and $q^{(2)}$ such that $\forall q \in Q_M^0$: $|||q||| \geq q^{(2)}$ and $\forall u \in H^l(G)$ with $\operatorname{supp} u \subset U_{x_0} \cap \overline{G}$ we have

$$|||u|||_l \; \leq \; C_2 \Big\{ |||A_0(x_0)u|||_{l-2k} \; + \; \sum_{\nu=1}^{k} |||B_{\nu,0}(x_0)u|||'_{l-k_\nu-1/2} \Big\} \; . \tag{1.3.10}$$

C). Suppose now Condition 1.3.2 is not satisfied in an interior point $x_0 \in G$, that is,

$$A_0(x_0, \xi_0, q_0) \; = \; 0 \tag{1.3.11}$$

for some (ξ_0, q_0) : $|\xi_0| + M_{q_0} \neq 0$, and let the estimate (1.3.9) be valid in the neighbourhood U_{x_0} and for $q \in Q^0_M$ we have $|||q||| \geq q^{(2)}$. Let $\varphi(x) \in C_0^\infty(U_{x_0})$ and $\varphi(x_0) \geq 0$. If $\xi_0 = 0$ then $M_{q_0} \neq 0$ and (1.3.11) means that in $A_0(x_0, D, q)$ the terms $C_\alpha q_1^{\alpha_1} ... q_m^{\alpha_m}$ with $\alpha_1 + \cdots + \alpha_m = 2k$ are absent. Putting $u = \varphi$ in (1.3.9) we get the estimate

$$\sum_{i=1}^m |q_i|^l \|\varphi\|_0 \ \leq\ C|||\varphi|||_l \ \leq\ C|||A_0(x_0)\varphi|||_{l-2k}$$

$$\leq\ C\Big[\|A_0(x_0)\varphi\|_{l-2k} \ +\ \sum_{i=1}^m |q_i|^{l-2k}\|A_0(x_0)\varphi\|_0\Big] \ \leq\ C\sum_{i=1}^m |q_i|^{l-1}$$

which is impossible for large $|||q|||$.

Let $\xi_0 \neq 0$, and introduce the function

$$u_\lambda(x) \ =\ e^{i\lambda x\cdot\xi_0}\varphi(x)$$

where $\lambda > 0$. Then we obtain from (1.3.9) with $q = \lambda q_0$ that

$$\|u_\lambda\|_l \ \leq\ |||u_\lambda|||_l \ \leq\ C\big|\big|\big|A(x_0, D, \lambda q_0)u_\lambda\big|\big|\big|_{l-2k}$$

$$\leq\ C\big\{\lambda^{l-2k}\|A(x_0, D, \lambda q_0)u_\lambda\|_0 \ +\ \|A(x_0, D, \lambda q_0)u_\lambda\|_{l-2k}\big\} \ , \qquad (1.3.12)$$

but $\|u_\lambda\|_l = O(\lambda^l)$ for large λ . As $A(x_0, D, \lambda q_0)e^{i\lambda x\cdot\xi_0} = 0$, we see that the right-hand side in (1.3.12) is $O(\lambda^{l-1})$, and again we approach to the contradiction.

D). Now let $x_0 \in \Gamma$ and suppose the coordinates in U_{x_0} are chosen to straighten the boundary. Suppose Condition 1.3.2 is satisfied in the point x_0 and Condition 1.3.3 is violated. In this case there exists a nontrivial stable solution $v(x_n)$ to the problem (1.3.5) - (1.3.6) with zero right-hand sides for certain (ξ_0, q_0) such that $|\xi'_0| + M_{q_0} \neq 0$. Then the function

$$w_\lambda(x) \ =\ e^{i\lambda x'\cdot\xi'_0}v(\lambda x_n) \ , \quad \lambda > 0 \ ,$$

is the solution to the problem

$$A_0(x_0, D, \lambda q_0)u \ =\ 0 \ , \qquad (1.3.13)$$

$$B_{\nu,0}(x_0, D, \lambda q_0)u\Big|_{x_n=0} \ =\ 0 \ , \quad \nu = 1, ..., k. \qquad (1.3.14)$$

In order to apply part B) of the proof, we introduce cut-off functions $\varphi(x')$, $\psi(x_n)$ such that $\varphi\psi \neq 0$ in some small neighbourhood U'_{x_0} , $\mathrm{supp}\,(\varphi\psi) = \overline{U'_{x_0}}$ and for some $\delta, h > 0$: $\psi(x_n) \equiv 1$ as $0 \leq x_n \leq \delta$ and $\psi(x_n) \equiv 0$ for $x_n \geq h$.

Let $u_\lambda(x) = w_\lambda(x)\varphi(x')\psi(x_n)$. Using the fact that w_λ is the solution of (1.3.13) and substituting u_λ in (1.3.10) with $q = \lambda q_0$, we compute the order of the right-hand side with respect to λ :

$$|||A(x_0, D, \lambda q_0)u_\lambda|||^2_{l-2k}$$

$$|||A(x_0, D, \lambda q_0)\varphi\psi w_\lambda \; - \; \varphi\psi A(x_0, D, \lambda q_0)w_\lambda|||^2_{l-2k}$$

$$|||D_1(\varphi\psi)\,\mathrm{Diff}^{2k-2}w_\lambda \; + \; \varphi\psi\,\mathrm{Diff}^{2k-1}w_\lambda|||^2_{l-2k}$$

$$\le \; C\lambda^{2l-2}\int_0^h |v(\lambda x_n)|^2\,dx_n \; + \; C\lambda^{2l-4} \; \le \; C\lambda^{2l-3} \; ,$$

where Diff^j is a homogeneous in D, λ differential operator of order j with constant coefficients.

By analogy,

$$|||B_{\nu,0}(x_0, D, \lambda q_0)u_\lambda|||'^2_{l-k_\nu-1/2} \; \le \; C\lambda^{2l-3} \; .$$

As for the left-hand side, there for $\xi_0 = 0$, $M_{q_0} \neq 0$ and large λ we have

$$|||u_\lambda|||^2_l \; \ge \; \sum_{i=1}^m |||q|||^{2l}\,\|u_\lambda\|^2_0$$

$$\ge \; C\lambda^{2l}\int_0^\delta |v(\lambda x_n)|^2\,dx_n \; \ge \; C\lambda^{2l-1} \; ,$$

and in the case $\xi_0 \neq 0$ and large λ we have

$$|||u_\lambda|||^2_l \; \ge \; \|u_\lambda\|^2_l$$

$$\ge \; C\lambda^{2l}\int_0^\delta |v(\lambda x_n)|^2\,dx_n \;\; + \; C\lambda^{2l-2} \; \ge \; C\lambda^{2l-1} \; .$$

Consequently, the left-hand side in (1.3.10) has the order $l-1/2$ in contradiction with the order $l-3/2$ of the right-hand side. ▷

1.4 The existence theorems

We start with the corresponding theorem for operators with coefficients, sufficiently close to the constant ones, and denote by $\mathcal{A}$ an operator in $\mathbb{R}^n_+$ of the form

$$\mathcal{A}u \; = \Big\{A(x, D, q)u, \; B_1(x, D, q)u\big|_{x_n=0}, ..., \; B_k(x, D, q)u\big|_{x_n=0}\Big\}.$$

Theorem 1.4.1 *A). Suppose* $l \in \mathbb{Z}$, $l \geq 2k$ *and let*

$$A(x,D,q) \;=\; \sum_{|\alpha|+|\beta|=2k} a_{\alpha,\beta}(x) q^\beta D^\alpha \ , \qquad a_{\alpha,\beta} \in C^\infty(\mathbb{R}^n) \ ,$$

be a homogeneous in (D,q) *differential operator in* $\mathbb{R}^n$ *. We assume that* $A(0,D,q)$ *satisfies Condition 1.3.2. There exists* $\varepsilon > 0$ *such that, if*

$$\big| a_{\alpha,\beta}(x) \;-\; a_{\alpha,\beta}(0) \big| \;<\; \varepsilon \ , \qquad \forall\, x \in \mathbb{R}^n \ , \tag{1.4.1}$$

then, for $q \in Q_M^0$ *with sufficiently large* $|||q|||$ *, the right inverse* $\mathcal{R}$ *to the operator* $A(x,D,q)$ *is defined and*

$$|||\mathcal{R} f|||_l \;\leq\; C |||f|||_{l-2k} \ . \tag{1.4.2}$$

B). Suppose $l \in \mathbb{Z}$, $l \geq \max(2k, k_j+1)$ *and let the operator* $A(x,D,q)$ *be as in part A) of the theorem with* $a_{\alpha,\beta} \in C^\infty(\mathbb{R}^n_+)$ *, and*

$$B_j(x,D,q) \;=\; \sum_{|\alpha|+|\beta|=k_j} b_{j,\alpha,\beta}(x) q^\beta D^\alpha \ ,$$

where $b_{j,\alpha,\beta} \in C^\infty(\mathbb{R}^n_+)$ *. Suppose that* $A(0,D,q)$ *satisfies Condition 1.3.2, and together with* $B_j(0,D,q)$ $(j = 1,...,k)$ *– Condition 1.3.3. Then there exists* $\varepsilon > 0$ *such that, if*

$$\big| a_{\alpha,\beta}(x) \;-\; a_{\alpha,\beta}(0) \big| \;<\; \varepsilon \quad (|\alpha| + |\beta| = 2k) \ , \qquad \forall\, x \in \mathbb{R}^n \ , \tag{1.4.3}$$

$$\big| b_{j,\alpha,\beta}(x) \;-\; b_{j,\alpha,\beta}(0) \big| \;<\; \varepsilon \quad (|\alpha| + |\beta| = k_j) \ , \qquad \forall\, x \in \mathbb{R}^n \ , \tag{1.4.4}$$

then, for $q \in Q_M^0$ *with sufficiently large* $|||q|||$ *, the right inverse* $\mathcal{R}$ *to the operator* $\mathcal{A}$ *is defined and*

$$|||\mathcal{R}(f, g_1, ..., g_k)|||_l \;\leq\; C\big\{ |||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||g_\nu|||'_{l-k_\nu-1/2} \big\} \ . \tag{1.4.5}$$

Proof. We show how item A) follows from Proposition 1.2.3, item B) follows from Proposition 1.2.7 in the same way.

Let $\mathcal{R}_0$ be the inverse to $A(0,D,q)$ being built in Proposition 1.2.3 $\forall\, q \in Q_M^0$. As follows from (1.2.5)

$$|||\mathcal{R}_0 f|||_l \;\leq\; C |||f|||_{l-2k} \ . \tag{1.4.6}$$

Consider an operator

$$A(x,D,q)\mathcal{R}_0 \;=\; A(0,D,q)\mathcal{R}_0 \;+\; T \;=\; I \;+\; T \ ,$$

where

$$T \;=\; \big[A(x,D,q) \;-\; A(0,D,q) \big] \mathcal{R}_0 \ .$$

By analogy with the proof of Proposition 1.3.6, part A), we see that

$$|||Tf|||_{l-2k} \;\le\; \eta(\varepsilon)|||\mathcal{R}_0 f|||_l \;+\; C|||\mathcal{R}_0 f|||_{l-1} \;,$$

where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, using the inequalities (1.3.8) and (1.4.6), we obtain

$$|||Tf|||_{l-2k} \;\le\; C_1\Big(\eta(\varepsilon) \;+\; \frac{C_3}{|||q|||}\Big)|||\,f|||_{l-2k} \;.$$

Thus, for sufficiently large $|||q|||$ and sufficiently small ε , we obtain that

$$|||Tf|||_{l-2k} \;\le\; 1/2|||\,f|||_{l-2k} \;,$$

and thus, operator T has a small norm. It implies , that the inverse $(I+T)^{-1}$ to the operator $(I+T)$ exists and is bounded in the norm $|||\cdot|||_{l-2k}$ and hence, in the norm $\|\cdot\|_{l-2k}$. Now we put

$$\mathcal{R} \;=\; \mathcal{R}_0(I+T)^{-1}$$

and observe that, the inequality (1.4.2) is valid for $\mathcal{R}$ and

$$A(x,D,q)\mathcal{R}f \;=\; f\,; \quad f \in H^{l-2k}(\mathbb{R}^n)\;.$$

$\triangleright$

Now we are ready to formulate the main theorem about elliptic operators with parameters.

Theorem 1.4.2 *Let* $l \in \mathbb{Z}$, $l \ge l_0 = \max(2k, k_j+1)$ *and suppose the boundary value problem (1.3.1) - (1.3.2) satisfy Conditions 1.3.2 and 1.3.3.*

Then for all $q \in Q^0_M$ *with sufficiently large* $|||q|||$ *a bounded inverse*

$$\mathcal{A}^{-1} \;:\; H^{l-2k}(G) \times \prod_{j=1}^{k} H^{l-k_j-1/2}(\Gamma) \;\longrightarrow\; H^l(G) \qquad \textit{(1.4.7)}$$

to the operator $\mathcal{A}$ *is defined.*

Proof. is rather standard. Using a sufficiently small partition of unity $\{\varphi_\nu\}_{\nu=1}^N$ such that $\varphi_\nu \in C_0^\infty(U_\nu)$ and $\sum_{\nu=1}^N \varphi_\nu(x) \equiv 1$ in a neighbourhood of G , we reduce the problem to the cases considered in Theorem 1.4.1.

Suppose that either $\operatorname{supp}\varphi_\nu \cap \Gamma = \emptyset$, or $\operatorname{supp}\varphi_\nu \subset U_j$, where $\bigcup_j U_j \supset \Gamma$. Moreover, for each U_ν we define an infinitely smooth function $\psi_\nu \in C_0^\infty(U_\nu)$ such that $\psi_\nu \equiv 1$ in a neighbourhood of $\operatorname{supp}\varphi_\nu(x)$,

$$D^\alpha\big[\varphi_\nu(x)\psi_\nu(x)\big] \;\equiv\; D^\alpha\varphi_\nu(x)$$

and ψ_ν has the same support properties as $\varphi_\nu(x)$. Obviously,

$$\mathcal{A}u \;=\; \sum_{\nu=1}^N \varphi_\nu \mathcal{A}(\psi_\nu u)\;. \qquad (1.4.8)$$

There are ingredients of two types in this sum: in those with $\operatorname{supp}\varphi_\nu, \psi_\nu \bigcap \Gamma \neq \emptyset$ we overcome to the local coordinates straightening the boundary in U_ν ; if $\operatorname{supp}\varphi_\nu,\, \psi_\nu \subset \operatorname{int} G$, we associate $\mathcal{A}$ with A for here $B_j(\psi_\nu u)\big|_\Gamma = 0$.

If $\operatorname{supp}\psi_\nu$ is sufficiently small the coefficients of the operator $\mathcal{A}$ on $\operatorname{supp}\psi_\nu$ are almost constant, and we construct an extension $\mathcal{A}_\nu$ of $\mathcal{A}$ from $\operatorname{supp}\psi_\nu$ to the whole $\mathbb{R}^n$ or $\mathbb{R}^n_+$ such that

$$\varphi_\nu \mathcal{A}(\psi_\nu u) \;=\; \varphi_\nu \mathcal{A}_\nu(\psi_\nu u) \;,$$

and its coefficients are almost constant too. We put

$$\mathcal{A}_\nu(x) \;=\; \alpha(x)\mathcal{A}(x) \;+\; [1-\alpha(x)]\mathcal{A}(x^{(\nu)}) \;,$$

where $\alpha(x) \in C_0^\infty$, $0 \le \alpha(x) \le 1$, $\alpha(x) \equiv 1$ on $\operatorname{supp}\psi_\nu$, $x^{(\nu)} \in \operatorname{supp}\psi_\nu$ and $x^{(\nu)} \in \Gamma$ if $\operatorname{supp}\psi_\nu \bigcap \Gamma \neq \emptyset$.

Operator

$$\mathcal{A}^0 u \;=\Big\{A_0 u,\; B_{1,0}u\big|_\Gamma, ...,\; B_{k,0}u\big|_\Gamma\Big\}$$

is called a leading part of operator $\mathcal{A}$. Here and further $\mathcal{A}^0_\nu, A_0, B_{j,0}$ are the leading parts of operators $\mathcal{A}_\nu, A, B_j$.

If $\operatorname{supp}\psi_\nu$ is sufficiently small, then it follows from Theorem 1.4.1 that the right inverse $\mathcal{R}^0_\nu$ to $\mathcal{A}^0_\nu$ is defined for $q \in Q^0_M$ with sufficiently large $|||q|||$. For these q we define a bounded operator

$$\mathcal{R}(f,g) \;=\; \sum_{\mu=1}^{N} \psi_\mu \mathcal{R}^0_\mu(\varphi_\mu f, \varphi_\mu g) \;:\; H^l(G,\Gamma) \longrightarrow H^l(G) \;, \tag{1.4.9}$$

where $g = (g_1, ..., g_k)$ and $H^l(G,\Gamma) \;=\; H^{l-2k}(G) \times \prod_{j=1}^k H^{l-k_j-1/2}(\Gamma)$ is equipped with a norm

$$|||(f,g)|||_l \;=\; |||f|||_{l-2k} \;+\; \sum_{j=1}^{k} |||g_j|||'_{l-k_j-1/2} \;. \tag{1.4.10}$$

As usual, we have to prove that $\mathcal{A}\mathcal{R} = I + T$, where $T = T(f,g)$ is an operator with the sufficiently small norm in $H^l(G,\Gamma)$. Thus we have

$$\mathcal{A}\mathcal{R}(f,g) \;=\; \sum_\mu \sum_\nu \varphi_\nu \mathcal{A}_\nu \psi_\nu \psi_\mu \mathcal{R}_\mu(\varphi_\mu f, \varphi_\mu g) \;,$$

where we consider only the terms with $\psi_\nu\psi_\mu \neq 0$. We put

$$u_\mu \;=\; \mathcal{R}^0_\mu(\varphi_\mu f, \varphi_\mu g)$$

and replace $\mathcal{A}_\nu$ by $\mathcal{A}_\mu$ in the last sum:

$$\mathcal{A}\mathcal{R}(f,g) \;=\; \sum_\mu \sum_\nu \varphi_\nu \mathcal{A}_\mu \psi_\nu \psi_\mu u_\mu$$

$$= \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \mathcal{A}_{\mu,0} u_\mu \; + \; T(f,g) \; ,$$

where
$$T(f,g) \;\; = \; \sum_\mu \sum_\nu \varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \; - \; \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu \; +$$

$$+ \; \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu [\mathcal{A}_\mu \; - \; \mathcal{A}_{\mu,0}] u_\mu \; . \tag{1.4.11}$$

As $\mathcal{A}_{\mu,0} \mathcal{R}^0_\mu = I$, it follows that,

$$\sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \mathcal{A}_{\mu,0} u_\mu \; = \; \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \varphi_\mu (f,g)$$

$$= \; \sum_\mu \sum_\nu \varphi_\nu \varphi_\mu (f,g) \; = \; (\sum_\nu \varphi_\nu)^2 (f,g) \; = \; (f,g) \; .$$

In order to estimate $T(f,g)$, we fix a pair of indices μ and ν and suppose $\mathcal{A}_\mu$ includes boundary operators (otherwize the work is easier).

For each addend in the first sum in (1.4.11) we have in terms of the norm (1.4.10)

$$\big\|\big|\varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \; - \; \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu \big|\big\|_l$$

$$\leq \; \Big\{ \big\|\big| [A \psi_\nu \psi_\mu \; - \; \psi_\nu \psi_\mu A] u_\mu \big|\big\|_{l-2k} \; +$$

$$+ \; \sum_{j=1}^{k} \big\|\big| [B_j \psi_\nu \psi_\mu \; - \; \psi_\nu \psi_\mu B_j] u_\mu \big|\big\|_{l-k_j} \Big\}$$

$$\leq \; C \| |u_\mu| \|_{l-1} \; , \tag{1.4.12}$$

where as in the proof of Theorem 1.4.1 we use the fact that operators in square brackets do not contain the leading terms.

Now, it follows from (1.1.13) and (1.4.5) for $\mathcal{R}_\mu$ that

$$\big\|\big|\varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \; - \; \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu \big|\big\|_l$$

$$\leq \; \frac{C}{\| |q| \|} \, \big\|\big| (\varphi_\mu f, \varphi_\mu g) \big|\big\|_l \; . \tag{1.4.13}$$

As $\|\varphi u\|^2_l \; \leq \; \|u\|^2_l \; + \; C_\varphi \|u\|^2_{l-1}$, we can get reed of φ_μ in the right - hand side of (1.4.13) and return to the initial coordinates .

The addends in the second sum in (1.4.11) we estimate in the same way, for the operator $\mathcal{A}_\mu - \mathcal{A}^0_\mu$ does not contain leading terms too. We arrive to inequality

$$\big\|\big| T(f,g) \; \big|\big\|_l \; \leq \; \frac{C}{\| |q| \|} \, \big\|\big| (f,g) \big|\big\|_l \; .$$

Consequently, for $q \in Q^0_M$ with sufficiently large $|||q|||$ we obtain that $\|T\| \leq 1/2$ and $\mathcal{A}\mathcal{R} = I + T$ $\big($ I is the identity map in $H^l(G,\Gamma)$ $\big)$ possess the bounded inverse $(\mathcal{A}\mathcal{R})^{-1}$. Finally, we define

$$\mathcal{A}^{-1} \;=\; \mathcal{R}(\mathcal{A}\mathcal{R})^{-1}$$

being the bounded operator from $H^l(G,\Gamma)$ to $H^l(G)$ and the right inverse to $\mathcal{A}$. Thus, it follows, that $\mathcal{A}$ maps $H^l(G)$ onto $H^l(G,\Gamma)$, and is one-to-one for $q \in Q^0_M$ with sufficiently large $|||q|||$, and for these q , $\mathcal{A}^{-1}$ also is a left inverse to $\mathcal{A}$. $\triangleright$

We conclude this section with a few general remarks about possible generalizations and further results.

Remark 1.4.3 *A). If the problem (1.3.1) - (1.3.2) satisfies Conditions 1.3.2 and 1.3.2 on the rays* $\{\arg q_i = c_i\ \}$ *then, for all* $q \in Q^0_M$ *with sufficiently large* $|||q|||$ *, this is a Fredholm problem with zero index (this result follows directly from the work (Agranovich M.S., Dynin A.S. (1962))).*

B). All propositions and theorems form Sections 2-4 of this chapter can be reformulated for elliptic boundary value problems for systems with parameters (only the formulae for canonical basis need to be changed).

C). The above results can be reformulated for operators acting on manifolds and in sections of bundles over them.

D). It would be interesting to consider pseudodifferential boundary problems depending on a number of parameters. The one - parameter case was considered in the book (Grubb, G. (1986)), also see (Grubb, G. (1989)).

Remark 1.4.4 *We note that the parameter set* $\{q \in \mathbb{C}^n\ :\ q \in Q^0_M,\ |||q||| \geq q^{(1)}\}$ *is a relatively complete Reinhart domain. It can be made logarithmically convex if* $q^{(1)}$ *is slightly increased (see Shabat B.V. (1985)).*

Chapter 2

Multiparameter pseudodifferential operators and related spectral asymptotics

2.1 Pseudodifferential operators with parameters

In this Section we consider classical (or polyhomogeneous) pseudodifferential operator pencils polynomially depending on parameters and suppose that these parameters $q = (q_1, ..., q_m)$ vary in the direct product Q of closed cones $Q_j = \{q_j \in \mathbb{C} : \delta_j \leq \arg q_j \leq \gamma_j\} : Q = Q_1 \times \cdots \times Q_m$. More precisely, we deal with operators of the form

$$A(x, D, q) = \sum_{l+|\beta|=s} A_{l,\beta}(x, D) q^\beta , \tag{2.1.1}$$

where $q^\beta = q_1^{\beta_1} \cdots q_m^{\beta_m}$ and $|\beta| = \sum_{j=1}^m \beta_j$. Here $A_{l,\beta}(x, D)$ are classical pseudodifferential operators of order l with leading symbols $a_{0;l,\beta}(x, \xi)$ (see Agranovich M.S. (1990), Shubin M.A. (1978) for the definition). If $|\beta| = s$ the operators $A_{l,\beta}(x, D)$ are the operators of multiplication by smooth functions $a_{0;0,\beta}(x) = a_{0\beta}(x) \in C^\infty(M)$. The number s we call the order of operator (2.1.1). Throughout this Section we suppose that M is a smooth compact manifold and T^*M is its cotangent bundle.

Definition 2.1.1 *A pseudodifferential operator (ΨDO) of the form (2.1.1) is called elliptic with parameters in Q if*

$$a_0(x,\xi,q) \;=\; \sum_{l+|\beta|=s} a_{0;l,\beta}(x,\xi)q^{\beta} \;\neq\; 0 \tag{2.1.2}$$

for all $(x,\xi)\in T^*M$ and $q\in Q$ such that $|\xi|+M_q\neq 0$, where $M_q = \min_{i=1,\dots,m}|q_i|$.

The function $a_0(x,\xi,q)$ is positively homogeneous in (ξ,q) of order s. We note that for $q=(q_1,...,q_m)=0$, this definition implies the ordinary ellipticity of the operator $A_{s,0}(x,D)$.

We also consider more general classes of classical ΨDOs depending on a set of parameters $q\in Q$. The corresponding symbols admit asymptotic decompositions of the form

$$b(x,\xi,q) \;=\; \sum_{j=0}^{\infty} b_{s-j}(x,\xi,q) \tag{2.1.3}$$

for $|\xi|+M_q\geq 1$, where $b_{s-j}(x,\xi,q)$ is positively homogeneous in (ξ,q) of order $s-j$, i.e.

$$b_{s-j}(x,t\xi,tq) \;=\; t^{s-j}b_{s-j}(x,\xi,q) \tag{2.1.4}$$

for $t>0$. We denote the set of such symbols by $\mathbf{CS}^s(M\times\mathbb{R}^n,Q)$ and the corresponding set of pseudodifferential operators by $\mathbf{CL}^s(M,Q)$. The definition of ellipticity with parameters for the class $\mathbf{CL}^s(M,Q)$ is analogous to Definition 2.1.1. If the relations (2.1.3) and (2.1.4) are valid for $|\xi|\geq 1$ and $t\geq 1$ we obtain the respective classes $\mathbf{CSP}^s(M\times\mathbb{R}^n,Q)$ and $\mathbf{CLP}^s(M,Q)$.

Theorem 2.1.2 *Let $t\geq s$ and suppose Condition (2.1.2) is satisfied. Then for all $q\in Q_M=\{q\in Q:\, M_q>0\}$ with sufficiently large $|||q|||$ the a priori estimate*

$$|||u|||_t \;\leq\; C|||f|||_{t-s} \tag{2.1.5}$$

holds for the solutions of the equation $A(x,D_x,q)u(x)=f(x)$ (see (1.1.12) for the definition of $|||\cdot|||$ norm). Also, there exists an inverse operator $\mathcal{R}_A = A^{-1}(x,D_x,q)$ which is rationaly dependent on q classical ΨDO of order $-s$. As usual, we call $\mathcal{R}_A$ the resolvent of A. It belongs to the classes $\mathbf{CL}^{-s}(M,Q)$ if $A(x,D_x,q)$ is a differential operator and to the classes $\mathbf{CLP}^{-s}(M,Q)$ if $A(x,D_x,q)$ is a ΨDO.

Proof. A). We start with the case where $A(x,D_x,q)$ is a differential operator in $\mathbb{R}^n$ with infinitely smooth coefficients which are independent of x for sufficiently large $|x|$. Then each of operators $A_{l,\beta}(x,D)$ has the complete symbol $a_{l,\beta}(x,\xi) = \sum_{j=0}^{l} a_{j;l,\beta}(x,\xi)$, where $\operatorname{ord} a_{0;l,\beta}=l$. We put $a_j(x,\xi,q) = \sum_{l+|\beta|=s-j} a_{j;l,\beta}(x,\xi)q^{\beta}$ and construct rational in $q\in Q_M$

functions $b_k(x,\xi,q)$ for $k=0,1,...$ being positively homogeneous in (ξ,q) of order $(-s-k)$ and such that

$$\Big[\sum_{l+|\beta|=s}\Big(\sum_{j=0}^{l} a_{j;l,\beta}(x,\xi)\Big)q^\beta\Big]\cdot\sum_{k=0}^{\infty} b_k(x,\xi,q) \;=\; 1\,. \tag{2.1.6}$$

Equating the symbols of the same homogenity order in the left- and in the right-hand sides of (2.1.6) and applying the composition rules for the symbols of ΨDOs we obtain the chain of equalities from which $b_0, b_1, ...$ can be computed successively:

$$a_0(x,\xi,q)b_0(x,\xi,q) \;=\; \sum_{l+|\beta|=s} a_{0;l,\beta}(x,\xi)q^\beta\cdot b_0(x,\xi,q) \;=\; 1\,,$$

$$a_0 b_k \;+\; \sum_{j+r+|\gamma|=k}\frac{1}{\gamma!}\partial_\xi^\gamma a_j\cdot D_x^\gamma b_r \;=\; 0\,, \quad (r<k).$$

Thus, in the view of (2.1.2), we obtain for $(\xi,q)\neq 0$ and $q\in Q$ that

$$b_0(x,\xi,q) \;=\; a_0^{-1}(x,\xi,q)\,,$$

$$b_1(x,\xi,q) \;=\; \frac{-a_1(x,\xi,q)}{a_0^2(x,\xi,q)} \;+\; \frac{-i\sum_{\nu=1}^n \partial_{\xi_\nu} a_0(x,\xi,q)\partial_{x_\nu} a_0(x,\xi,q)}{a_0^3(x,\xi,q)}\,, \tag{2.1.7}$$

etc. Finally, for all $k\geq 0$

$$b_k(x,\xi,q) \;=\; \sum_{r\geq\frac{k+s}{s}}\frac{b_{k,r}(x,\xi,q)}{a_0^r(x,\xi,q)}\,, \tag{2.1.8}$$

where the sum in the right-hand side is finite and $b_{k,r}(x,\xi,q)$ are some polynomials in $a_0,...,a_s$ and their derivatives and, hence, are the polynomials in q and ξ with smooth coefficients which are independent of x for large $|x|$. The functions $b_{k,r}$ are positively homogeneous in (ξ,q) of orders $s(r-1)-k\geq 0$ from where the inequality $r\geq\frac{k+s}{s}$ in (2.1.8) follows. Analogously, the derivatives $\partial_x^\alpha\partial_\xi^\gamma b_k$ are positively homogeneous in (ξ,q) and rational in q functions.

We now introduce the operators with the symbols b_k

$$B_k(x,D,q)f(x) \;=\; \int e^{i\langle x,\xi\rangle} b_k(x,\xi,q)(Ff)(\xi)đ\xi\,, \tag{2.1.9}$$

for $q\in Q$, $q\neq 0$, and define a parametrix of order N for A by the formula

$$B^{(N)}(q) \;=\; B_0(q)+\cdots+B_N(q)\,. \tag{2.1.10}$$

By now $B^{(N)}(q)$ is not defined for sufficiently small $|q_i|$. We extend this operator in the way to define it in the whole cone Q and even in somewhat larger set in the neighbourhood of 0 . Let $\theta(\tau) \in C^\infty(\mathbb{R}_+)$ be a cut-off function being equal to 0 for $\tau \le 1/2$ and 1 for $\tau > 1$. We put

$$b_{\theta k}(x,\xi,q) \;=\; \prod_{j=1}^{m} \theta\big(|\xi|^2 + |q_j|^2\big)\, b_k(x,\xi,q) \tag{2.1.11}$$

and define $Q_{(\rho)}$, where $\rho = (\rho_1,...,\rho_m)$, $\rho_j > 0$, as the sum of Q and polydisk $B^m_\rho = \{q \in \mathbb{C}^m : |q_j| \le \rho_j\} : Q_{(\rho)} = Q \cup B^m_\rho$. The functions (2.1.11) are defined for $q \in Q_{(\rho)}$, where ρ_j are sufficiently small in the way that $a_0(x,\xi,q) \ne 0$ for $|q_j| \le \rho_j$, and $|\xi|^2 + |q_j|^2 \ge 1/2$ for all $j \in \{1,...,m\}$. Finally, these functions coincide with $b_k(x,\xi,q)$ if $\min_{j=1,\dots,m} |q_j| \ge 1$. We now put

$$B_{\theta k}(q) f(x) \;=\; \int e^{i\langle x,\xi\rangle} b_{\theta k}(x,\xi,q)(Ff)(\xi) đ\xi \tag{2.1.12}$$

instead of (2.1.9) and

$$B^{(N)}_\theta(q) \;=\; B_{\theta 0}(q) + \cdots + B_{\theta N}(q)\,. \tag{2.1.13}$$

instead of (2.1.10). Obviously, $B^{(N)}_\theta(q) \in \mathbf{CL}^{-s}(M,Q)$.

B). Let us consider now the case, where $A_{l,\beta}(x,D)$ are classical pseudodifferential operators with the symbols

$$a_{l,\beta}(x,\xi) \;=\; a_{0;l,\beta}(x,\xi) + a_{1;l,\beta}(x,\xi) + \cdots .$$

Defining b_k through the formulae (2.1.7) and (2.1.8), we see that $b_{kr}(x,\xi,q)$ for $k \ge 1$ do not vanish if $r \ge 2$ and for $s(r-1) - k < 0$ these functions have singularity as $\xi = 0$.

That is why we review all the construction from the beginning starting not from the homogeneous functions $a_{j;l,\beta}(x,\xi)$, but from their smooth for $\xi = 0$ modifications

$$a_{\theta,j;l,\beta}(x,\xi) \;=\; \theta(|\xi|) a_{j;l,\beta}(x,\xi)\ , \qquad j = 0,1,2,\dots\ . \tag{2.1.14}$$

Additionally, we put

$$a_{\theta,0}(x,\xi,q) \;=\; \sum_{l+|\beta|=s} a_{\theta,0;l,\beta}(x,\xi) q^\beta + 1 - \theta(|\xi|)\,.$$

Thus, instead of the asymptotic sum $\sum_{j=0}^\infty a_j(x,\xi,q)$ we obtain the sum $\sum_{j=0}^\infty a_{\theta,j}(x,\xi,q)$, where for $\xi = 0$ the functions $a_{\theta,j}(x,0,q)$, are defined through the equalities $a_{\theta,0}(x,0,q) = 1$, and $a_{\theta,j}(x,0,q) = 0$ if $j > 0$. Thus we obtain that

$$a_{\theta,j;l,\beta}(x,\xi) \;\in \mathbf{S}^{s-j-l}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\ , \qquad j = 0,1,2,...,$$

where $\mathbf{S}^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ is the space of symbols from the standard Hörmander class (see, for example, Shubin M.A. (1978)), and

$$a(x,\xi,q) \;\sim\; a_{\theta,0}(x,\xi,q) + a_{\theta,1}(x,\xi,q) + \cdots ,$$

where $a_{\theta,j}(x,\xi,q)$ is positively homogeneous in (ξ,q) for $|\xi| \geq 1$ of order $s-j$, i.e.

$$a_{\theta,j}(x,t\xi,tq) \;=\; t^{s-j} a_{\theta,j}(x,\xi,q) ,$$

for $t \geq 1$. Finally, $a_{\theta,0}(x,\xi,q) \neq 0$ for all $(x,\xi,q) \in \mathbb{R}^n \times \mathbb{R}^n \times Q_{(\rho)}$, for sufficiently small $\rho_j > 0$.

Now, by analogy with the formulae (2.1.7), (2.1.8) and using the composition formula

$$\sum_{j=0}^{\infty} a_{\theta,j} \circ \sum_{k=0}^{\infty} b_{\theta,k} \sim 1 ,$$

we construct rational in q functions $b_{\theta,k}(x,\xi,q)$ which are positively homogeneous in (ξ,q) for $|\xi| \geq 1$ of order $-s-k$, i.e.

$$b_{\theta,k}(x,t\xi,tq) \;=\; t^{-s-k} b_{\theta,k}(x,\xi,q) , \tag{2.1.15}$$

for $t \geq 1$. For $b_{\theta,k}$ we obtain the formulae analogous to (2.1.7) and (2.1.8) substituting a_k by $a_{\theta,k}$. Instead of (2.1.8) we get the following formula:

$$b_{\theta,k}(x,\xi,q) \;=\; \sum_{p\geq 2} \frac{b_{\theta,k,p}(x,\xi,q)}{a^p_{\theta,0}(x,\xi,q)} , \tag{2.1.16}$$

where $k \geq 1$, the sum in the right-hand side is finite and $b_{\theta,k,p}(x,\xi,q)$ are some polynomials in $a_{\theta,0}$, $a_{\theta,1}, \ldots$ and their derivatives, and $b_{\theta,k}$ are defined differently in the cases of differential and pseudodifferential operators . We note that

$$b_{\theta,k,p} \;\in \mathbf{S}^{s(p-1)-k}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \tag{2.1.17}$$

and

$$b_{\theta,k,p}(x,t\xi,tq) \;=\; t^{s(p-1)-k} b_{\theta,k,p}(x,\xi,q)$$

for $|\xi| \geq 1$ and $t \geq 1$.

The function $b_{\theta,k,2} = -a_{\theta,k}$ differs from zero for sufficiently large k and thus, in the sum (2.1.16) we have $p \geq 2$ in contrast to (2.1.8).

Now we are able to define the corresponding to $b_{\theta,k}$ operators $B_{\theta,k}(q)$ and $B^{(N)}_\theta(q)$, $q \in Q_{(\rho)}$, by the formulae (2.1.11) and (2.1.13).

The symbol $b^{(N)}_\theta = b_{\theta,0} + \cdots + b_{\theta,N}$ of the operator $B^{(N)}_\theta(q)$ satisfies the inequalities

$$\left|\partial^\alpha_x \partial^\beta_\xi b^{(N)}_\theta(x,\xi,q)\right| \;\leq\; C_{\alpha,\beta,N} \left(1+|\xi|\right)^{-\beta} \left(1+|\xi|+|q_1|+\cdots+|q_m|\right)^{-s} \tag{2.1.18}$$

obviously following from (2.1.16), (2.1.17) for all multiindices α, β and with the constants $C_{\alpha,\beta,N}$ being independent of (x, ξ, q). The estimates (2.1.18) give us the opportunity to review the proof of Proposition 1.2.3 and obtain the estimate

$$|||B_\theta^{(N)}(q)f|||_t \ \leq \ C|||f|||_{t-s} \tag{2.1.19}$$

for all $t \in \mathbb{R}$.

C). We finally consider the ΨDO A of the form (2.1.1) on a closed manifold M. We define the parametrix of order N to A by the formula

$$B_{(N)}(q) \ = \ B_\theta^{(N)}(q) \ = \ \sum_{i=1}^{K} \psi_i B_{\theta i}^{(N)}(q)(\phi_i \cdot) \ , \tag{2.1.20}$$

where $\{\phi_i\}_{i=1}^K$ is the partition of unity on M subordinate to a covering $\{\mathcal{O}_i\}_{i=1}^K$ of the manifold M, $\{\psi_i\}_{i=1}^K$ is the system of functions from $C^\infty(M)$ and such that $\operatorname{supp}\psi_i \subset \mathcal{O}_i$, $\psi_i \equiv 1$ in some neighbourhood of $\operatorname{supp}\phi_i$ for each $i \in \{1, ..., K\}$, $B_{\theta i}^{(N)}$ is the parametrix of order N for the operator A_i of the form (2.1.1) defined in the neighbourhood Ω_i with the properties $\Omega_i \supset \operatorname{supp}\psi_i$, and $\Omega_i \subset \mathcal{O}_i$. Operator A can be written in the form $A \sim \sum_i \phi_i A_i(\psi_i \cdot)$ up to infinitely smoothing operators and the estimate (2.1.19) remains valid for operators $B_{(N)}(q)$ on manifold M.

Further, in the same way as in Section 4 of Chapter 1 (see inequalities (1.4.11) - (1.4.13) in the proof of Theorem 1.4.2) we prove that

$$A \cdot B_{(N)}(q) \ = \ I + T_{(N)}(q) \ ,$$

where

$$|||T_{(N)}(q)f|||_{l+N+1} \ \leq \ |||f|||_l$$

for sufficiently large $|||q|||$, and, hence, it has a small norm as operator acting from H^s into H^{s+l}. Then the operator $I + T_{(N)}(q)$ is invertible, and there exists a right inverse operator to A being also the left inverse:

$$A^{-1}(q) \ = \ \mathcal{R}_A(q) \ = \ B_{(N)}(q)[I + T_{(N)}(q)]^{-1} \ . \tag{2.1.21}$$

▷

Corollary 2.1.3 *The following estimates follow easily from (2.1.5):*

$$|||\mathcal{R}_A f|||_l \ \leq \ \frac{C}{1 + |||q|||^s}|||f|||_l \tag{2.1.22}$$

for all l *and*

$$\|\mathcal{R}_A f\|_0 \ \leq \ \frac{C}{1 + |||q|||^s}\|f\|_0 \tag{2.1.23}$$

Corollary 2.1.4 *We have* $\mathcal{R}_A(q) - B_{(N)}(q) \in \mathbf{CL}^{-s-N-1}(M, Q)$ *in the case where* A *is a differential operator and* $\mathcal{R}_A(q) - B_{(N)}(q) \in \mathbf{CLP}^{-s-N-1}(M, Q)$ *in the case where* A *is a pseudodifferential operator. We refer to the book (Shubin M.A. (1978), Proposition 11.2) for the proof.*

2.2 Complex powers of differential operator pencils

2.2.1 Definition

In this Section we consider a special case of differential operator pencils of the form (2.1.1) where $A(x,D,q)$ is decomposed into multipliers:

$$A(x,D,q) \;=\; \left(A_1 - q_1^{s_1}\right)\cdots\left(A_m - q_m^{s_m}\right). \tag{2.2.1}$$

Here $s_1+\cdots+s_m = |s| = \operatorname{ord} A$, $A_j = A_j(x,D)$ are parameter-elliptic in the angles $Q_j = \{q_j \in \mathbb{C} : \frac{\pi-\varepsilon}{s_j} \le \arg q_j \le \frac{\pi+\varepsilon}{s_j}\}$ differential operators of orders $\operatorname{ord} A_j = s_j$ on a closed manifold (or in a domain in $\mathbb{R}^n$) M with leading symbols $a_j^0(x,\xi)$.

In other words we suppose that

A). $a_j^0(x,\xi) - \lambda_j \neq 0$ for $\xi \neq 0$ and $\lambda_j = q_j^{s_j} \in \Lambda_j = \{\lambda_j \in \mathbb{C} : \pi - \varepsilon \le \arg\lambda_j \le \pi+\varepsilon\}$.

B). The spectrum of each A_j does not intersect with Λ_j (this condition is included only for convenience and is not essential, see in this case Section 10 in the book (Shubin M.A. (1978))).

It follows from B) that there exists $\rho_j > 0$ such that the spectrum of each A_j does not intersect with the whole circle $B^1_{2\rho_j} = \{\lambda_j \in \mathbb{C} : |\lambda_j| < 2\rho_j\}$.

Now let $\Gamma_j \subset \Lambda_j \cup B^1_{\rho_j}$ be a curve in the λ_j-complex plane of the form $\Gamma_j^1 \cup \Gamma_j^2 \cup \Gamma_j^3$, where

$$\Gamma_j^1 \;=\; \{\lambda_j \in \mathbb{C} : \lambda_j = r_j e^{\imath\pi},\;\; +\infty > r_j > \rho_j\},$$

$$\Gamma_j^2 \;=\; \{\lambda_j \in \mathbb{C} : \lambda_j = \rho_j e^{\imath\varphi_j},\;\; +\pi > \varphi_j > -\pi\},$$

$$\Gamma_j^3 \;=\; \{\lambda_j \in \mathbb{C} : \lambda_j = r_j e^{-\imath\pi},\;\; \rho_j < r_j < +\infty\}.$$

We put $\Gamma = \Gamma^1 \times\cdots\times \Gamma^m$ and consider the integral

$$A_z \;=\; \left(\frac{\imath}{2\pi}\right)^m \int_\Gamma \lambda_1^{z_1}\cdots\lambda_m^{z_m}\,\mathcal{R}_A(\lambda)\,d\lambda_1...d\lambda_m, \tag{2.2.2}$$

where $\mathcal{R}_A(\lambda)$ is the parametrix for A being built during the proof of Theorem 2.1.2. We note that in (2.2.2) $z = (z_1,...,z_m) \in \mathbb{C}^m$, and $\lambda_j^{z_j}$ is defined in the standard way as follows: $\lambda_j^{z_j}$ is holomorphic in $\lambda_j \in \mathbb{C}\backslash(-\infty,0]$, $\lambda_j^{z_j} = e^{z_j \ln\lambda_j}$ for $\lambda_j > 0$ (we take the branch of logarithm such that $\ln\lambda_j \in \mathbb{R}$ for $\lambda_j > 0$). Thus, on Γ_j we have $\lambda_j^{z_j} = e^{z_j\ln\lambda_j} = e^{z_j \ln|\lambda_j| + \imath z_j \arg\lambda_j} = |\lambda_j|^{z_j}\, e^{\imath z_j \arg\lambda_j}$, where $-\pi < \arg\lambda_j < \pi$.

In order to investigate the convergence of (2.2.2) we need the following lemma from analysis (see Zorich V.A.(1984), Ch. XI, Sect.6).

Lemma 2.2.1 *Let* $|s| = s_1 + ... + s_m$, $s_j \in \mathbb{Z}_+$, *then the following improper integrals are convergent:*

$$\int_{|x_1|+...+|x_m|>1} \frac{|x_1|^{s_1 t_1} \cdots |x_m|^{s_m t_m}}{1 + |x_1|^{|s|} + ... + |x_m|^{|s|}}\, dx_1^{s_1} ... dx_m^{s_m} \ < \ \infty \qquad (2.2.3)$$

if $\sum_{j=1}^m s_j t_j \ < \ 0$ *and*

$$\int_{|x_1|+...+|x_m|>1} \frac{|x_1|^{s_1 t_1} \cdots |x_m|^{s_m t_m}}{\big(1 + |x_1| + ... + |x_m|\big)^{|s|+p}}\, dx_1^{s_1} ... dx_m^{s_m} \ < \ \infty \qquad (2.2.4)$$

if $\sum_{j=1}^m s_j t_j \ < \ p$

Now, in the view of the estimates (2.1.23) and (2.2.3) we see that the integral (2.2.2) is convergent in the operator norm in $L^2(M)$ if $\sum_{j=1}^m s_j \mathbf{Re}\, z_j \ < \ 0$. In particular, it is convergent if $\mathbf{Re}\, z_j \ < \ 0$ for all $j = 1, ..., m$. Indeed, in the view of inequality (2.1.22) we obtain that the integral (2.2.2) also converges in the operator norm of $H^l(M)$ for $l \in \mathbb{R}$, and thus A_z maps $H^l(M)$ into itself if $\sum_{j=1}^m s_j \mathbf{Re}\, z_j \ < \ 0$. That is why A_z maps $C^\infty(M)$ into $C^\infty(M)$ and $D'(M)$ into $D'(M)$ (see (Shubin M.A. (1978)).

Proposition 2.2.2 A_z *is the holomorphic operator-valued function in the domain* $\Pi_0 = \big\{ z \in \mathbb{C}^m : \ \sum_{j=1}^m s_j \mathbf{Re}\, z_j \ < \ 0 \big\}$ *with the values in the algebra of bounded operators in* $H^l(M)$ *for all* $l \in \mathbb{R}$.

Proof. Only holomorphicity in z_j needs to be proved. Differentiating (2.2.2) in z_j we obtain the integral

$$\left(\frac{\imath}{2\pi}\right)^m \int_\Gamma \lambda_1^{z_1} \cdots \lambda_m^{z_m} \ln \lambda_j \, \mathcal{R}_A(\lambda)\, d\lambda_1 ... d\lambda_m \,, \qquad (2.2.5)$$

which is uniformly convergent in the domain $\big\{ z \in \mathbb{C}^m : \ \sum_{j=1}^m s_j \mathbf{Re}\, z_j \ \le \ -\varepsilon \ < \ 0 \big\}$. Thus A_z is holomorphic and $\partial A_z / \partial z_j$ equals to the integral (2.2.5). $\triangleright$

2.2.2 Symbols of Complex Powers

We now construct the homogeneous components $b^{(z),0}_{s\cdot z - j}(x, \xi)$ $(s \cdot z = s_1 z_1 + \cdots + s_m z_m)$ of the symbol of operator A_z for $\sum_{j=1}^m s_j \mathbf{Re}\, z_j \ < \ 0$ according to the formula

$$b^{(z),0}_{s\cdot z-j}(x,\xi) \ = \ \left(\frac{\imath}{2\pi}\right)^m \int_\Gamma \lambda_1^{z_1} \cdots \lambda_m^{z_m} b^0_{-|s|-j}(x,\xi,\lambda)\, d\lambda_1 ... d\lambda_m \,, \qquad (2.2.6)$$

where $j = 0, 1, 2, ...$ and $b^0_{-|s|-j}(x, \xi, \lambda) \ = \ b_j(x, \xi, \lambda_1^{1/s_1}, ..., \lambda_m^{1/s_m})$ from (2.1.7) and (2.1.8). We can overwrite (2.1.7) and (2.1.8) as follows

$$b^0_{-|s|-j}(x,\xi,\lambda) \ = \ \sum_{p \ge \frac{j+|s|}{|s|}} \frac{b^0_{j,p}(x,\xi,\lambda)}{(a^0_1 - \lambda_1) \cdots (a^0_m - \lambda_m)} \,. \qquad (2.2.7)$$

In particular,

$$b^0_{-|s|}(x,\xi,\lambda) \;=\; \frac{1}{(a^0_1-\lambda_1)\cdots(a^0_m-\lambda_m)}\,.$$

It follows from the parameter-ellipticity of A_j $(j=1,...,m)$ that there exist ρ_j such that $a^0_j(x,\xi)-\lambda_j\neq 0$ if $|\lambda_j|<2\rho_j$, $\xi\neq 0$ and $\lambda_j\in\mathbb{C}$.

Then $b^0_{-|s|-k}(x,\xi,\lambda)$ are holomorphic in the polydisk $\left\{\lambda\in\mathbb{C}^m : |\lambda_j|< 2\rho_j\right\}$ and satisfy the following estimate (cf. (2.1.18))

$$\left|\partial^\alpha_x\partial^\beta_\xi b^0_{-|s|-k}(x,\xi,\lambda)\right| \tag{2.2.8}$$

$$\leq\; C_{\alpha,\beta,k}\left(1+|\xi|\right)^{-\beta}\left(1+|\xi|+|\lambda_1|^{1/s_1}+\cdots+|\lambda_m|^{1/s_m}\right)^{-|s|-k}$$

for sufficiently large $|\xi|$. Hence, the integral in (2.2.6) is independent of ρ_j's for sufficiently small ρ_j's. We note that $b^{(z),0}_{s\cdot z-k}(x,\xi)$ is positively homogeneous in ξ of order $s\cdot z-k$. Indeed, for $t>0$ we have

$$\begin{aligned}
b^{(z),0}_{s\cdot z-k}(x,t\xi) \;&=\; \left(\frac{\imath}{2\pi}\right)^m\int_\Gamma \lambda_1^{z_1}\cdots\lambda_m^{z_m}\,b^0_{-|s|-k}(x,t\xi,\lambda)\,d\lambda_1...d\lambda_m\\
&=\;\left(\frac{\imath}{2\pi}\right)^m\int_{\Gamma^t}(t^{s_1}\mu_1)^{z_1}\cdots(t^{s_m}\mu_m)^{z_m}\\
&\quad b^0_{-|s|-k}(x,t\xi,t^{s_1}\mu_1,...,t^{s_m}\mu_m)\,\cdot t^{|s|}\,d\mu_1...d\mu_m\\
&=\;\left(\frac{\imath}{2\pi}\right)^m t^{s\cdot z-k}\int_\Gamma \mu^z\,b^0_{-|s|-k}(x,\xi,\mu)\,d\mu\\
&=\; t^{s\cdot z-k}b^{(z),0}_{s\cdot z-k}(x,\xi)\,,
\end{aligned}$$

where $\Gamma^t=\Gamma^t_1\times\cdots\times\Gamma^t_m$ and the curve Γ^t_j has the same shape as Γ_j but with the circle part of the radius $t^{-s_j}\rho_j$ instead of ρ_j .

We have the following analog of Proposition 2.2.2 for symbols $b^{(z),0}_{s\cdot z-k}$.

Proposition 2.2.3 *For all multiindices α β and $\xi\neq 0$ the function $\partial^\alpha_x\partial^\beta_\xi$ $b^{(z),0}_{s\cdot z-j}(x,\xi)$ is holomorphic in z in the domain $\Pi_j=\left\{z\in\mathbb{C}^m:\sum_{i=1}^m s_i\mathbf{Re}\,z_i- j<0\right\}$.*

Proof of this statement is based on the estimation of the integrand in (2.2.6) with the help of (2.2.8) and the subsequent reduction to (2.2.4), see Lemma 2.2.1.

Before we overcome to the kernels of operators A_z we have to smooth out the symbols, introduce the holomorphic families of ΨDOs and prove the structure theorem.

The smoothed symbols are defined as in previous Section, see (2.1.14):

$$b^{(z)}_{s\cdot z-j}(x,\xi)\;=\;\theta(|\xi|)b^{(z),0}_{s\cdot z-j}(x,\xi)\,.$$

Let $B^{(z);i}_{s\cdot z-j}$ be an operator with the symbol $b^{(z)}_{s\cdot z-j}(x,\xi)$ in the coordinate neighbourhood $\mathcal{O}_i$, $i\in\{1,...,K\}$. Then, as in (2.1.20) we put

$$B^{(z)}_{s\cdot z-j} \;=\; \sum_{i=1}^{K} \psi_i B^{(z);i}_{s\cdot z-j}\phi_i \tag{2.2.9}$$

and

$$B^{(z)}_{(N)} \;=\; \sum_{j=1}^{N} B^{(z)}_{s\cdot z-j}\,. \tag{2.2.10}$$

We now turn to holomorphic families of ΨDOs $\mathcal{O}(G,\mathbf{L}^t_{1,0}(M))$, where $G\subset\mathbb{C}^m$ and by $\mathbf{L}^t_{1,0}(M)$ we denote the standard ΨDOs from the Hörmander class (see , for example, Shubin M.A. (1978)). For $M\subset\mathbb{R}^n$ the corresponding symbol class is denoted by $\mathbf{S}^t_{1,0}(M\times\mathbb{R}^n)$.

Definition 2.2.4 *Let Ω be a domain in $\mathbb{R}^n$ and G – in $\mathbb{C}^m$, $a(x,\xi,z)\in C^\infty(\Omega\times\mathbb{R}^n\times G)$ and $a(x,\xi,z)$ is holomorphic in z. We say that $a(x,\xi,z)\in\mathcal{O}(G,\mathbf{S}^t_{1,0}(M\times\mathbb{R}^n))$ if for all multiindices α, β, γ and arbitrary compact sets $K_1\subset\Omega$, $K_2\subset G$ there exists a constant $C=C(\alpha,\beta,\gamma,K_1,K_2)$ such that*

$$\left|\partial^\alpha_x\partial^\beta_\xi\partial^\gamma_z a(x,\xi,z)\right| \;\le\; C\langle\xi\rangle^{t-|\alpha|}\,, \tag{2.2.11}$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $(x,\xi,z)\in K_1\times\mathbb{R}^n\times K_2$.

The calculus of the corresponding holomorphic families of ΨDOs can be built in the standard way, see Shubin M.A. (1978). The structure theorem now looks as follows.

Theorem 2.2.5 *1). For an arbitrary $z\in\Pi_0=\left\{z\in\mathbb{C}^m:\ \sum_{j=1}^m s_j\mathbf{Re}\,z_j<0\right\}$ we have $A_z\in\mathbf{CLP}^{s\cdot z}(M)$.*

2). For an arbitrary $N\ge 0$ and for all $t\le N+1$

$$\mathcal{R}^{(z)}_{(N)} \;=\; A_z-B^{(z)}_{(N)} \;\in\; \mathcal{O}\big(\Pi_t,\mathbf{L}^{t-N-1}_{1,0}(M)\big)\,, \tag{2.2.12}$$

where $\Pi_t=\left\{z\in\mathbb{C}^m:\ \sum_{j=1}^m s_j\mathbf{Re}\,z_j<t\right\}$.

Proof reproduces the proof of Theorem 11.2 from the book (Shubin M.A. (1978)).

1). We prove first that assertion 1) of the theorem follows from assertion 2). Indeed, let

$$B^{(z)} \;\sim\; \sum_{j=0}^{\infty} B^{(z)}_{s\cdot z-j}\,.$$

this formula means that in every coordinate neighbourhood $\mathcal{O}_i$ we have to construct the asymptotic sum

$$b^{(z);i}(x,\xi) \;\sim \sum_{j=0}^{\infty} b^{(z);i}_{s\cdot z-j}(x,\xi)$$

then take the operator $B^{(z);i}$ in $\mathcal{O}_i$ with the symbol $b^{(z);i}(x,\xi)$ and finally put $B^{(z)} = \sum_{i=1}^{K} \psi_i B^{(z);i}\phi_i$, where ψ_i and ϕ_i are the same as in (2.1.20). Clearly, $B^{(z)} \in \mathbf{CLP}^{s\cdot z}(M)$. Taking the smaller domain Π_0 in (2.2.12) and passing to the limit as $N \to \infty$ we obtain that $A_z - B^{(z)} \in \mathcal{O}\big(\Pi_0, \mathbf{L}^{-\infty}(M)\big)$. It means that $A_z \in \mathbf{CLP}^{s\cdot z}(M)$ in Π_0.

2). Suppose $z \in \Pi_t$ and consider the operator

$$'B^{(z)}_{(N)} \;=\; \Big(\frac{\imath}{2\pi}\Big)^m \int_\Gamma \lambda_1^{z_1}\cdots\lambda_m^{z_m}\, B_{(N)}(\lambda)\, d\lambda_1 ... d\lambda_m\,, \tag{2.2.13}$$

where $B_{(N)}(\lambda) \;=\; B^{(N)}_\theta(q)\Big|_{q_j=\lambda_j^{1/s_j}}$ is defined by the formula (2.1.13).

Obviously, $'B^{(z)}_{(N)} - B^{(z)}_{(N)} \in \mathcal{O}\big(\Pi_t, \mathbf{L}^{-\infty}(M)\big)$ as in every coordinate neighbourhood we have

$$'b^{(z)}_{s\cdot z-j}(x,\xi) \;=\; b^{(z),0}_{s\cdot z-j}(x,\xi) \;=\; b^{(z)}_{s\cdot z-j}(x,\xi)$$

for $|\xi| \ge 1$. Thus, it is sufficient to prove (2.2.12) for the operator $'\mathcal{R}^{(z)}_{(N)} = A_z - {}'B^{(z)}_{(N)}$.

Let $'r^{(z)}_{(N)}(x,\xi)$ be a symbol of operator $'\mathcal{R}^{(z)}_{(N)}$ in some local coordinate system and $r_{(N)}(x,\xi,\lambda)$ be a symbol of operator $\mathcal{R}_{(N)}(\lambda) = \mathcal{R}_A(\lambda) - B_{(N)}(\lambda)$ (see Corollary 2.1.4 with q substituted by λ). Then

$$'r^{(z)}_{(N)}(x,\xi) \;=\; \Big(\frac{\imath}{2\pi}\Big)^m \int_\Gamma \lambda_1^{z_1}\cdots\lambda_m^{z_m}\, r_{(N)}(x,\xi,\lambda)\, d\lambda_1 ... d\lambda_m \tag{2.2.14}$$

and

$$\partial_x^\alpha \partial_\xi^\beta \partial_z^\gamma\, 'r^{(z)}_{(N)}(x,\xi) \tag{2.2.15}$$

$$= \Big(\frac{\imath}{2\pi}\Big)^m \int_\Gamma (\ln\lambda_1)^{\gamma_1}\cdots(\ln\lambda_m)^{\gamma_m}\, \lambda_1^{z_1}\cdots\lambda_m^{z_m}\, \partial_x^\alpha\partial_\xi^\beta r_{(N)}(x,\xi,\lambda)\, d\lambda_1 ... d\lambda_m$$

Now, in the view of Corollary 2.1.4 and (2.2.8) we obtain the estimate

$$\big|\partial_x^\alpha\partial_\xi^\beta r_{(N)}(x,\xi,\lambda)\big| \;\le\; C_{\alpha,\beta,K}\,\big(1+|\xi|\big)^{-\beta}\,\big(1+|\xi|+|||q|||\big)^{-|s|-N-1}\,, \tag{2.2.16}$$

where K is some compact in our coordinate neighbourhood. As obviously $\big(1+|\xi|+|||q|||\big)^{-|s|-N-1} \le \big(1+|\xi|\big)^{t-N-1}\cdot\big(1+|||q|||\big)^{-|s|-t}$ for all $t < N+1$, we have

$$\big|\partial_x^\alpha\partial_\xi^\beta r_{(N)}(x,\xi,\lambda)\big| \;\le\; C_{\alpha,\beta,K}\,\big(1+|\xi|\big)^{t-N-1-\beta}\,\big(1+|\xi|+|||q|||\big)^{-|s|-t}. \tag{2.2.17}$$

Substituting (2.2.17) into (2.2.15) and applying Proposition 2.2.3 and Lemma 2.2.1 we obtain the inclusion (2.2.12). ▷

2.2.3 The analytic continuation of kernels

We now consider the kernels of the complex powers A_z. As follows from Theorem 2.2.5, item 1), for all $z \in \mathbb{C}^m$ such that $\sum_{j=1}^m s_j \mathbf{Re}\, z_j < -n$, $n = \dim M$, we can represent the kernel of operator A_z in the form $A_z(x,y)dy$. This is the density depending on the parameter $x \in M$ which can be written in the above form in local coordinates defined $\mathcal{O}_y \subset M$, where dy is the Lebesgue measure defined by the local coordinates in $y \in \mathcal{O}_y$ and $A_z(x,y)$ is a continuous function on $M \times \mathcal{O}_y$. For simplicity we call the kernel the function $A_z(x,y)$ itself. Our aim is to construct an analytic continuation of the kernel $A_z(x,y)$ to the whole $\mathbb{C}^m$.

As follows from the general theory of ΨDOs, the kernel $A_z(x,y)$ for close $x, y \in \mathcal{O}_y \subset M$ can be represented in the form

$$A_z(x,y) \;=\; \int e^{\imath (x-y)\cdot \xi} b(x,\xi,z)\, đ\xi\,,$$

where $b(x,\xi,z) = \sum_{j=0}^\infty b^{(z)}_{s\cdot z - j}(x,\xi)$ is the complete symbol of the corresponding proper ΨDO. Once again, we note that the kernel $A_z(x,y)$ is continuous and holomorphic in z if $\sum_{j=1}^m s_j \mathbf{Re}\, z_j < -n$. For $x = y$ we obtain $A_z(x,x) = \int b(x,\xi,z)\, đ\xi$.

We now formulate the main theorem about the analytic continuation of kernels $A_z(x,y)$.

Theorem 2.2.6 *Let $\mathcal{O}$ be a fixed coordinate neighbourhood on M and let $A_z(x,y)$ be the kernel of operator A_z being defined for $\sum_{j=1}^m s_j \mathbf{Re}\, z_j < -n$ and $x, y \in \mathcal{O}$. Then*

1). For $x \neq y$ the kernel $A_z(x,y)$ can be continued to an entire function in z vanishing for $z \in (\mathbb{Z}_+)^m$.

2). For $x = y$ the trace $A_z(x,x)$ can be continued to a meromorphic function in the whole $\mathbb{C}^m \ni z$ with the only polar sets

$$M_j \;=\; \left\{ z \in \mathbb{C}^m :\; \sum_{i=1}^m s_i z_i \;=\; j - n \right\}$$

of the first order and the corresponding to $A_z dz$ holomorphic residue-forms

$$\gamma_j(x) \;=\; \frac{(-1)^\nu}{s_\nu} \int_{|\xi|=1} b^{(M_j),0}_{-n}(x,\xi)\, đ\xi' dz[\nu]\big|_{M_j} \tag{2.2.18}$$

on them, where $b^{(M_j),0}_{-n}(x,\xi) = b^{(z),0}_{-n}(x,\xi)\big|_{M_j} = b^{(z),0}_{s\cdot z - j}(x,\xi)\big|_{M_j}$, $\nu \in \{1,...,m\}$, $dz[\nu] = dz_1 \wedge ... \wedge dz_{\nu-1} \wedge dz_{\nu+1} \wedge ... \wedge dz_m$ and $đ\xi' = (2\pi)^{-n} d\xi'$.

3). In particular, if $z = (l_1,...,l_m) \in (\mathbb{Z}_+)^m$ and $z \in M_j$, we have $\gamma_j(x) = 0$ and the value $\kappa_z(x)$ of the analytic extension of the trace $A_z(x,x)$ can be computed directly in the following special cases:

A) $\kappa_z(x) = 0$ *if and there exists at least one coordinate* $z_\mu \in \mathbb{Z}_+$ *and the whole* $z \notin M_j$;

B) $\kappa_z(x) = 0$ *if* $z \in M_j$, $\mathbf{Re}\, z_\mu > -1$ *for all* $\mu \in \{1,...,m\}$ *and there exist at least two coordinates* z_μ *and* z_ν $(\mu \neq \nu)$ *such that* $z_\mu, z_\nu \in \mathbb{Z}_+$;

C) if $z \in M_j$, $\mathbf{Re}\, z_\mu > -1$ *for all* $\mu \in \{1,...,m\}$ *and there exists a unique coordinate* z_μ *of* z *such that* $z_\mu \in \mathbb{Z}_+$ *, then*

$$\kappa_z(x) \;=\; \frac{(-1)^{m-1}\prod_{\alpha\neq\mu}\sin\pi z_\alpha}{(-1)^{z_\mu} s_\mu \pi^{m-1}} \cdot \qquad \text{(2.2.19)}$$

$$\cdot \int_{|\xi|=1}\int_0^\infty \cdots \int_0^\infty r_1^{z_1}\cdots r_m^{z_m}\, b^0_{-|s|-s\cdot z-n}(x,\xi,-r)\, dr_1\cdots dr_m đ\xi\,.$$

Remark 2.2.7 *A). Assertion 1) of the Theorem takes place uniformly in* $x \in K_1$ *and* $y \in K_2$ *, where* K_1 *and* K_2 *are the nonintersecting compacts in* $\mathcal{O}$ *, i.e.* $A_z(x,y)$ *continues to an entire function with the values in* $C(K_1,K_2)$ *.*

B). Analogously, assertions 2) and 3) are uniform in $x \in K$ *, where* K *is a compact in* $\mathcal{O}$ *, i.e.* $A_z(x,x)$ *continues to a meromorphic function with the values in* $C(K)$ *with the polar sets* M_j *,* $j = 0,1,...$ *of the first order, residue-forms* $\gamma_j(x)\big|_K$ *on them and the values* $\kappa_z(x)\big|_K$ *in the points described in assertion 3).*

3). We note here that only Item 2). of the Theorem is of use in Spectral Theory (see the next Section).

Proof. 1). The structure Theorem 2.2.5 gives us the opportunity to reduce the initial problem to the problem of holomorphicity, poles and kernels of operators $B^{(z)}_{(N)}$. Indeed, if $\mathcal{R}^{(z)}_{(N)}(x,y)$ is the kernel of operator $\mathcal{R}^{(z)}_{(N)} \;=\; A_z - B^{(z)}_{(N)}$ and $r^{(z)}_{(N)}(x,\xi)$ is its symbol in some coordinate neighbourhood $\mathcal{O}$, then for $x,y \in \mathcal{O}$ we have

$$\mathcal{R}^{(z)}_{(N)}(x,y) \;=\; \int r^{(z)}_{(N)}(x,\xi)e^{\imath(x-y)\cdot\xi}\, đ\xi\,,$$

where the integral is convergent if $\sum_{j=1}^m s_j \mathbf{Re}\, z_j \;<\; N-n$ and defines for these z a holomorphic function in z with the values in $C(K_1,K_2)$, where K_1 and K_2 are arbitrary compacts in $\mathcal{O}$.

2). The kernel of each $B^{(z)}_{(N)}$ is in its turn the sum of the kernels of the form

$$B^{(z)}_{s\cdot z-j}(x,y) \;=\; \int b^{(z)}_{s\cdot z-j}(x,\xi)e^{\imath(x-y)\cdot\xi}\, đ\xi\,.$$

The standard integration by parts (see Shubin M.A. (1978), Sect.12) shows that off the diagonal $x=y$ the kernel $B^{(z)}_{s\cdot z-j}(x,y)$ is the entire function in z with the values in $C(K_1,K_2)$, where K_1 and K_2 are arbitrary nonintersecting compacts in $\mathcal{O}$, i.e. the assertion 1) of the Theorem is proved modulo the part containing the zeroes in integer points.

3). We have to investigate the traces

$$B^{(z)}_{s\cdot z-j}(x,x) \;=\; \int b^{(z)}_{s\cdot z-j}(x,\xi)\, đ\xi \;=\; \int \theta(|\xi|) b^{(z);0}_{s\cdot z-j}(x,\xi)\, đ\xi\,, \qquad (2.2.20)$$

where $\theta \in C^{\infty}(\mathbb{R}_+)$ is a cut-off function being equal to 0 for $|\xi| \le 1/2$ and 1 for $|\xi| > 1$ (cf.(2.1.14)).

Representing (2.2.20) in polar coordinates we obtain

$$B^{(z)}_{s\cdot z-j}(x,x) \;=\; \left(\int_0^{\infty} \theta(r) r^{s\cdot z-j+n-1}\, dr\right)\left(\int_{|\xi|=1} b^{(z);0}_{s\cdot z-j}(x,\xi)\, đ\xi'\right), \quad (2.2.21)$$

where $đ\xi'$ is the element of the surface area of the unit sphere in $\mathbb{R}^n$. The second integral in (2.2.21) is the entire function in z with the values in $C(K)$ for an arbitrary compact $K \in \mathcal{O}$.

The first multiplier is treated as follows:

$$\int_0^{\infty} \theta(r) r^{s\cdot z-j+n-1}\, dr \;=\; \int_0^{1} \theta(r) r^{s\cdot z-j+n-1}\, dr \;+\; \int_1^{\infty} r^{s\cdot z-j+n-1}\, dr\,.$$

Here the first term is the entire function in z and the second one can be computed explicitly:

$$\int_1^{\infty} r^{s\cdot z-j+n-1}\, dr \;=\; -\frac{1}{s\cdot z - j + n}\,. \qquad (2.2.22)$$

Thus, the sets $M_j = \left\{z \in \mathbb{C}^m : \sum_{i=1}^m s_i z_i = j - n\right\}$ for $j = 0, 1, ...$ are the polar sets of the first order for the functions $B^{(z)}_{s\cdot z-j}(x,x)$. In this case the corresponding to $A_z(x,y)dz$ holomorphic residue-forms $\gamma_j(x)$ are uniquely defined on M_j (see Aizenberg L.A., Yuzhakov A.P. (1979)):

$$\gamma_j(x) \;=\; -\frac{(-1)^{\nu-1}}{s_\nu} \int_{|\xi|=1} b^{(M_j),0}_{-n}(x,\xi)\, đ\xi' dz[\nu]\big|_{M_j}$$

where $b^{(M_j),0}_{-n}(x,\xi) = b^{(z),0}_{-n}(x,\xi)\big|_{M_j} = b^{(z),0}_{s\cdot z-j}(x,\xi)\big|_{M_j}$, $\nu \in \{1, ..., m\}$ and $dz[\nu] = dz_1 \wedge ... \wedge dz_{\nu-1} \wedge dz_{\nu+1} \wedge ... \wedge dz_m$.

The reader being interested only in Spectral Theory can omit the rest part of this proof.

4). Here we start computing the values of $A_z(x,y)$ for $z \in (\mathbb{Z}_+)^m$, i.e. $z = (l_1, ..., l_m)$, $l_i \in \mathbb{Z}_+$.

As for the right inverses we have

$$\left[(A_1 - \lambda_1)(A_2 - \lambda_2)\cdots(A_m - \lambda_m)\right]^{-1}$$

$$= \ (A_m - \lambda_m)^{-1}(A_{m-1} - \lambda_{m-1})^{-1} \cdots (A_1 - \lambda_1)^{-1} \, ,$$

then writing down (2.2.2) in these terms as an iterated integral and using the one-parameter theory of complex powers (see Shubin M.A. (1978)), we obtain that for $z = l = (l_1, ..., l_m) \in (\mathbb{Z}_+)^m$ the kernel $A_z(x, y)$ is indeed the kernel of differential operator $A_m^{l_m} \cdots A_1^{l_1}$ and thus is equal to zero for $x \neq y$.

In particular, in this case $b^{(l),0}_{s\cdot l-j}(x,\xi)$ are the homogeneous components of the symbol of differential operator $A_m^{l_m} \cdots A_1^{l_1}$ and are equal to zero if $s \cdot l - j < 0$ and, in particular, if $s \cdot l - j = -n$. Thus, $\gamma_j(x) = 0$ if $z \in (\mathbb{Z}_+)^m$ and $z \in M_j$.

5). In order to prove the rest assertions of item 3) of the Theorem, i.e. to compute the values $\kappa_z(x)$ we use the kernels $'B^{(z)}_{s\cdot z-k}(x, y)$ of the components of operators $'B^{(z)}_{(N)}$ defined by (2.2.13). We remind that in the given local coordinate neighbourhood $\mathcal{O}$

$$'B^{(z)}_{s\cdot z-k}(x,y) \ = \ \left(\frac{\imath}{2\pi}\right)^m \int e^{\imath(x-y)\cdot\xi} \int_\Gamma \lambda^z b_{-|s|-k}(x,\xi,\lambda)\, d\lambda d\!\!{}^{-}\!\xi \, , \qquad (2.2.23)$$

where $b_{-|s|-k}(x,\xi,\lambda) \ = \ \theta(\xi,\lambda) b^0_{-|s|-k}(x,\xi,\lambda)$, $\theta(\xi,\lambda) \ = \ \prod_{j=1}^m \theta\big(|\xi|^2 + \lambda_j^{2/s_j}\big)$ and $b^0_{-|s|-k}$ are defined by formula (2.2.7).

The presence of nonholomorphic in λ function $\theta(\xi,\lambda)$ dictates the choice of radii ρ_j of the polydisk part of Γ. We denote them by $\rho_j^{s_j}$ where $\rho_j > 0$. As the function $b^0_{-|s|-k}(x,\xi,\lambda)$ is homogeneous in $(\xi, \lambda_1^{1/s_1}, ..., \lambda_m^{1/s_m})$, there exist constants $L_j > 0$, $j = 1, ..., m$, such that $b^0_{-|s|-k}(x,\xi,\lambda)$ is holomorphic in λ if $|\lambda_j| < L_j^{s_j} |\xi|^{s_j}$ for all $j \in \{1, ..., m\}$. We now take

$$\rho_j \ < \ \frac{L_j}{2\sqrt{L_j^2 + 1}} \, . \qquad (2.2.24)$$

Then if $|\xi|^2 + \lambda_j^{2/s_j} \ \geq 1/4$ and $\lambda_j \in \Gamma_j$ we have that either $|\lambda_j| \ > \ \rho_j^{s_j}$ (i.e. we are on the straight part of Γ_j) or $|\lambda_j| \ = \ \rho_j^{s_j}$ and

$$L_j^{s_j} |\xi|^{s_j} \ \geq \ L_j^{s_j} \big(1/4 \ - \ \lambda_j^{2/s_j}\big)^{s_j/2}$$

$$= \ L_j^{s_j} \big(1/4 \ - \ \rho_j^2\big)^{s_j/2} \ > \ \rho_j^{s_j} \ = \ |\lambda_j| \, .$$

Now, as $\theta\big(|\xi|^2 + \lambda_j^{2/s_j}\big) \ = \ 0$ if $|\xi|^2 + \lambda_j^{2/s_j} \ \leq \ 1/2$, we can assume the function $b^0_{-|s|-k}(x,\xi,\lambda)$ to be holomorphic inside the polydisk part of Γ. Thus we obtain that in the same way as in part 1) of this proof the kernels $'B^{(z)}_{s\cdot z-j}(x,y)$ are defined and holomorphic in z for $\sum_{i=1}^m s_i \mathbf{Re}\, z_i \ < \ j$. Also, they continue to entire functions in z for $x \neq y$. Indeed, integrating by parts in (2.2.23) we obtain that

$$'B^{(z)}_{s\cdot z-j}(x,y) \;=\; \left(\frac{\imath}{2\pi}\right)^m \int e^{\imath(x-y)\cdot\xi}|x-y|^{-2M} \int_\Gamma \lambda^z \Delta^M_\xi\big(b_{-|s|-j}(x,\xi,\lambda)\big)\,d\lambda d\!\!\bar{}\,\xi\,, \tag{2.2.25}$$

where $M \in \mathbb{N}$ is an arbitrary number, and the holomorphicity in the domain $\sum_{i=1}^m s_i \mathbf{Re}\, z_i < j + 2M$ follows.

6). We now show that $'B^{(z)}_{s\cdot z-j}(x,y) = 0$ for $x \neq y$ and $z \in (\mathbb{Z}_+)^m$. Indeed, if $z \in (\mathbb{Z}_+)^m$ then $\lambda^z = \lambda_1^{z_1}\cdots\lambda_m^{z_m}$ is the single-valued function and the integrals over the straight parts of Γ_j in (2.2.25) vanish. On the polydisk part we observe that

$$\Delta^M_\xi\big(b_{-|s|-j}(x,\xi,\lambda)\big) \;=\; \sum_{|\alpha|+|\beta|=2M} C_{\alpha,\beta}\big[\partial^\alpha_\xi b^0_{-|s|-j}(x,\xi,\lambda)\big]\partial^\beta_\xi\big[\theta(\xi,\lambda)\big]\,,$$

where $\partial^\alpha_\xi b^0_{-|s|-j}(x,\xi,\lambda)$ is holomorphic in λ inside the polydisk part of Γ and $\partial^\beta_\xi\theta(\xi,\lambda)$ is constant in λ_j when $|\lambda_j| = const$.

The same fact is true for

$$'\mathcal{R}^{(z)}_{(N)}(x,y) \;=\; A_z(x,y) - \sum_{j=0}^N {'B}^{(z)}_{s\cdot z-j}(x,y)\,.$$

Indeed, we have

$$'\mathcal{R}^{(l)}_{(N)}(x,y) \;=\; A_l(x,y) \;=\; {'B}^{(l)}_{s\cdot l-j}(x,y) \;=\; 0$$

for $x \neq y$.

7). Let us continue analytically the trace of the kernel $'B^{(z)}_{s\cdot z-j}(x,y)$ to the whole $\mathbb{C}^m$.

We have

$$'B^{(z)}_{s\cdot z-j}(x,x) \;=\; \left(\frac{\imath}{2\pi}\right)^m \int d\!\!\bar{}\,\xi \int_\Gamma \theta(\xi,\lambda)\lambda^z b^0_{-|s|-j}(x,\xi,\lambda)\,d\lambda\,. \tag{2.2.26}$$

Let Γ^1_i be a part of Γ_i lying in the domain $|\xi|^2 + |\lambda_i|^{2/s_i} > 1$ and Γ^2_i be the leftover one. Clearly,

$$\int_{\Gamma_i} \theta_i\lambda_i^{z_i} b^0_{-|s|-j}\,d\lambda_i \;=\; \int_{\Gamma^1_i} \lambda_i^{z_i} b^0_{-|s|-j}\,d\lambda_i \;+\; \int_{\Gamma^2_i} \theta_i\lambda_i^{z_i} b^0_{-|s|-j}\,d\lambda_i \tag{2.2.27}$$

and the last integral is the entire function in z_i. It vanishes if $z_i \in \mathbb{Z}_+$, see item 6) of the proof. That is why it is sufficient to consider the integrals with at least one integration over Γ^1_i.

7.A). We first consider the part of (2.2.26) where all the integrals are over Γ_i^1, i.e.

$$I(z) \;=\; \left(\frac{\imath}{2\pi}\right)^m \int đ\xi \int_{\Gamma_1^1\times\cdots\times\Gamma_m^1} \lambda^z b^0_{-|s|-j}(x,\xi,\lambda)\,d\lambda\,. \tag{2.2.28}$$

Suppose $\rho=\rho_1=...=\rho_m$, then for $|\xi|\geq\sqrt{1-\rho^2}$ we have $\Gamma_i^1=\Gamma_i$ and for $|\xi|<\sqrt{1-\rho^2}$ the curve Γ_i consists of the two rays. We put $\Gamma_{i,\xi}^1=\Gamma_i^1$ for $|\xi|<\sqrt{1-\rho^2}$ and $\Gamma_{i,\xi}^1=|\xi|^{s_i}(1-\rho^2)^{-s_i/2}\Gamma_i$ for $|\xi|\geq\sqrt{1-\rho^2}$. In the view of the Cauchy theorem we can substitute the integration over $\Gamma_1^1\times\cdots\times\Gamma_m^1$ by the integration over $\Gamma_{1,\xi}^1\times\cdots\times\Gamma_{m,\xi}^1$. Changing the variables by putting $\lambda_j=e^{\pm\imath\pi}t_j^{s_j}$ on the straight part of $\Gamma_{j,\xi}^1$ and $\lambda_j=\rho^{s_j}(1-\rho^2)^{-s_j/2}e^{\imath\varphi_j}$, $-\pi\leq\varphi_j<\pi$, on the circle part of $\Gamma_{j,\xi}^1$, we overcome to the integrals

$$I_1(\xi,z) \;=\; \left(\frac{\imath}{2\pi}\right)^m \int_{\Gamma_{1,\xi}^1\times\cdots\times\Gamma_{m,\xi}^1} \lambda^z b^0_{-|s|-j}(x,\xi,\lambda)\,d\lambda$$

for $|\xi|<\sqrt{1-\rho^2}$. As in this case the integration is over the two rays for each $j\in\{1,...,m\}$, we obtain that

$$I_1(\xi,z) \;=\; s_1\cdots s_m\,\frac{\prod_{j=1}^m \sin\pi z_j}{\pi^m}\int_{\sqrt{1-|\xi|^2}}^{+\infty}\cdots\int_{\sqrt{1-|\xi|^2}}^{+\infty} t_1^{s_1z_1+s_1-1}\cdots$$

$$\cdots t_m^{s_mz_m+s_m-1}b^0_{-|s|-j}(x,\xi,-t_1^{s_1},...,-t_m^{s_m})\,dt_1...dt_m\,.$$

In the case where $|\xi|\geq\sqrt{1-\rho^2}$ we have the sum of superpositions of the integrals over $t_1,...,t_m$, where each one can be of the form

$$I_2(\xi,z,\cdot) \;=\; s_j\,\frac{\sin\pi z_j}{\pi}\int_{\alpha|\xi|}^{+\infty} t_j^{s_jz_j+s_j-1}b^0_{-|s|-j}(x,\xi,\cdot,-t_j^{s_j},\cdot)\,dt_j\,,$$

or of the form

$$I_3(\xi,z,\cdot) \;=\; \frac{1}{2\pi}\int_{-\pi}^{+\pi}(\alpha|\xi|)^{s_jz_j+s_j}e^{\imath\varphi_j(z_j+1)}b^0_{-|s|-j}\big(x,\xi,\cdot,(\alpha|\xi|)^{s_j}e^{\imath\varphi_j},\cdot\big)\,d\varphi_j\,,$$

where $\alpha=\rho/\sqrt{1-\rho^2}$ and by "$\cdot$" we denote the leftover variables which are obtained from λ_k and can be either $-t_k^{s_k}$ or $(\alpha|\xi|)^{s_k}e^{\imath\varphi_k}$, $k\neq j$.

It is sufficient to compute the integrals in the two-parameter case ($m=2$, $\lambda=(\lambda_1,\lambda_2)$), the general case follows then easily by induction. Thus, for $m=2$ we have

$$I(z) \;=\; \int_{|\xi|<\sqrt{1-\rho^2}} I_1(\xi,z)\,đ\xi$$

$$+ \int_{|\xi|\geq\sqrt{1-\rho^2}} \left[I_{2,2}(\xi,z) + I_{2,3}(\xi,z) + I_{3,2}(\xi,z) + I_{3,3}(\xi,z)\right] đ\xi \,,$$

where

$$I_{2,2}(\xi,z) \;=\; s_1 s_2 \frac{\sin \pi z_1 \, \sin \pi z_2}{\pi^2} \int_{\alpha|\xi|}^{+\infty} \int_{\alpha|\xi|}^{+\infty} t_1^{s_1 z_1 + s_1 - 1} \cdot$$

$$\cdot t_2^{s_2 z_2 + s_2 - 1} b^0_{-|s|-j}(x, \xi, -t_1^{s_1}, -t_2^{s_2}) \, dt_1 dt_2 \,,$$

$$I_{2,3}(\xi,z) \;=\; s_1 \frac{\sin \pi z_1}{\pi} \int_{\alpha|\xi|}^{+\infty} t_1^{s_1 z_1 + s_1 - 1} \cdot$$

$$\cdot \frac{1}{2\pi} \int_{-\pi}^{+\pi} (\alpha|\xi|)^{s_2 z_2 + s_2} e^{\imath \varphi_2 (z_2+1)} b^0_{-|s|-j}(x, \xi, -t_1^{s_1}, \alpha^{s_2} |\xi|^{s_2} e^{\imath\varphi_2}) \, dt_1 d\varphi_2 \,,$$

the integral $I_{3,2}(\xi,z)$ is of the same form, but the indices 2 and 3 change their places,

$$I_{3,3}(\xi,z) \;=\; \frac{1}{4\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} (\alpha|\xi|)^{s\cdot z + |s|} e^{\imath\varphi_1(z_1+1)} e^{\imath\varphi_2(z_2+1)} \cdot$$

$$\cdot b^0_{-|s|-j}(x, \xi, \alpha^{s_1}|\xi|^{s_1} e^{\imath\varphi_1}, \alpha^{s_2}|\xi|^{s_2} e^{\imath\varphi_2}) \, d\varphi_1 d\varphi_2 \,,$$

(here $s = (s_1, s_2)$ and $m = 2$).

7.A.1). In this item we consider the sum

$$\int_{|\xi|<\sqrt{1-\rho^2}} I_1(\xi,z) \, đ\xi \;+\; \int_{|\xi|\geq\sqrt{1-\rho^2}} I_{2,2}(\xi,z) \, đ\xi \,,$$

indeed being the integral of a homogeneous function in (ξ, t_1, t_2) of order $s \cdot z + |s| - 2 - |s| - j$ over a domain of conic shape. Introducing the polar coordinates in the $(\xi,t)-$space, we see that this integral is absolutely convergent for $s \cdot \mathbf{Re}\, z - 2 - j < -n - 2$ and in the view of (2.2.22) it can be written in the form

$$- s_1 s_2 \frac{\sin \pi z_1 \, \sin \pi z_2}{\pi^2 (s \cdot z - j + n)} \int_{\mathbf{S}_\rho^{n+1}} t_1^{s_1 z_1 + s_1 - 1} \cdot$$

$$\cdot t_2^{s_2 z_2 + s_2 - 1} b^0_{-|s|-j}(x, \xi, -t_1^{s_1}, -t_2^{s_2}) \, d(\xi,t)' \,,$$

where $\mathbf{S}_\rho^{n+1} = \left\{(\xi,t) \in \mathbb{R}^{n+2} : \, |\xi|^2 + t_1^2 + t_2^2 = 1, \; t_1 > \rho, \; t_2 > \rho\right\}$ and $d(\xi,t)'$ is the surface element of the unit sphere $\mathbf{S}^{n+1}$ in the $(\xi,t)-$space. The last itegral is well defined (for $t_i > \rho > 0$) and is the entire function in $z = (z_1, z_2)$. Thus, the whole expression can have only one polar set $s \cdot z - j + n = 0$. Surely, it vanishes if both z_1 and z_2 are integers, no matter if $s \cdot z - j + n$ is zero or not.

7.A.2). Let us consider the term with $I_{3,3}(\xi,z)$, i.e. the integral

$$\int_{|\xi|\geq\sqrt{1-\rho^2}} I_{3,3}(\xi,z) \, đ\xi \,.$$

Here we overcome to the polar coordinates in the $\xi-$space and in the view of (2.2.22) we obtain

$$\int_{|\xi|\geq\sqrt{1-\rho^2}} I_{3,3}\,đ\xi \;=\; \frac{C}{s\cdot z-j+n}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} e^{\imath\varphi_1(z_1+1)}e^{\imath\varphi_2(z_2+1)}\,\cdot$$

$$\cdot\int_{|\xi|=1} b^0_{-|s|-j}(x,\xi,\alpha^{s_1}e^{\imath\varphi_1},\alpha^{s_2}e^{\imath\varphi_2})\,d\xi' d\varphi_1 d\varphi_2\,, \tag{2.2.29}$$

where $C = const$ and thus, this term is meromorphic too with the same polar set. If $z_\beta = l_\beta \in \mathbb{Z}_+$, $\beta = 1,2$, we obtain zero for each integration over φ_β. Indeed,

$$\int_{-\pi}^{+\pi} e^{\imath\varphi_\beta(z_\beta+1)}b^0_{-|s|-j}(x,\xi,\alpha^{s_\beta}e^{\imath\varphi_\beta,\cdot})\,d\varphi_\beta \;=$$

$$=\; -\imath\int_{|w|=1} w^l b^0_{-|s|-j}(x,\xi,\alpha^{s_\beta}w,\cdot))\,dw \;=\; 0$$

as $b^0_{-|s|-j}(x,\xi,\lambda)$ is holomorphic in λ_β for $|\lambda_\beta| < \alpha^{s_\beta}$. Thus the integral (2.2.29) vanishes if $z\in(\mathbb{Z}_+)^2$ and even in the case where $s\cdot z-j+n=0$.

7.A.3). We now consider the integrals containing $I_{3,2}(\xi,z)$ and $I_{2,3}(\xi,z)$, i.e.

$$\int_{|\xi|\geq\sqrt{1-\rho^2}} I_{2,3}(\xi,z)\,đ\xi \;=$$

$$\frac{s_1\sin\pi z_1}{\pi}\int_{|\xi|\geq\sqrt{1-\rho^2}}\int_{\alpha|\xi|}^{+\infty} t_1^{s_1z_1+s_1-1}\,\frac{1}{2\pi}\int_{-\pi}^{+\pi}(\alpha|\xi|)^{s_2z_2+s_2}e^{\imath\varphi_2(z_2+1)}\cdot$$

$$\cdot b^0_{-|s|-j}(x,\xi,-t_1^{s_1},\alpha^{s_2}|\xi|^{s_2}e^{\imath\varphi_2})\,dt_1 d\varphi_2 đ\xi\,.$$

Here we have to overcome to polar coordinates in the $\xi-$space and also to change the variable t_1 into $|\xi|\tau_1$. Thus we get

$$\int_{|\xi|\geq\sqrt{1-\rho^2}} I_{2,3}(\xi,z)\,đ\xi \;=\; \frac{s_1\sin\pi z_1}{2\pi^2}$$

$$\cdot\int_{\sqrt{1-\rho^2}}^{+\infty} R^{s\cdot z+|s|-|s|-j+n-1}dR\int_{|\xi|=1}\int_{\alpha}^{+\infty}\tau_1^{s_1z_1+s_1-1}\int_{-\pi}^{+\pi}\alpha^{s_2z_2+s_2}e^{\imath\varphi_2(z_2+1)}$$

$$\cdot b^0_{-|s|-j}(x,\xi',-\tau_1^{s_1},\alpha^{s_2}e^{\imath\varphi_2})\,d\tau_1 d\varphi_2 đ\xi'\,,$$

from where we see that the integral in R converges if $s\cdot\mathbf{Re}\,z-j+n<0$, and the integral in τ_1 in this case converges automatically as the inequality $s_1\mathbf{Re}\,z_1+s_1-|s|-j = s_1\cdot\mathbf{Re}\,z_1-s_2-j < 0$ follows from the previous one. The integration over τ_1 gives the polar set $M_j{}' = \{z\in\mathbb{C}^2 : z_2 = -1-n/s_2\,,\ s_1z_1+s_2(-1-n/s_2) = j-n\}$ which is contained in M_j, thus remaining the single polar set of this integral. Thus the whole integral vanishes if $z \in (\mathbb{Z}_+)^2$ and even in the case where $s \cdot z - j + n = 0$, i.e. $z \in M_j$, as $z \notin (\mathbb{Z}_+)^2$ on $M_j{}'$.

7.B). Finally, we have to consider the remainder terms containing the integrations over Γ_i^2. As above, it is sufficient to consider the superposition of the two integrals $(m = 2)$:

$$I_{1,2} \;=\; \int d\!\!\!{}^-\xi \int_{|\xi|^2+|\lambda_1|^{2/s_1}>1} \int_{|\xi|^2+|\lambda_2|^{2/s_2}<1} b^0_{-|s|-j}(x,\xi,\lambda)\cdot$$

$$\cdot\theta(|\xi|^2 + |\lambda_2|^{2/s_2})\, d\lambda_1 d\lambda_2$$

Here we have $|\xi|^2+|\lambda_2|^{2/s_2} \le 1$ on Γ_2^2 and thus $|\xi|^2 \le 1-\rho^2$ and we can remove Γ_1^1 by $\Gamma_1^{1,\xi}$ and consider only the integration "I_1" over $t_1 \in \big(\sqrt{1-|\xi|^2}, +\infty\big)$. As for $\Gamma_2^{2,\xi}$, we obtain only the integration over $t_2 \in \big(0, \sqrt{1-|\xi|^2}\big)$. Thus we have

$$I_{1,2} \;=\; s_1 s_2 \,\frac{\sin\pi z_1 \,\sin\pi z_2}{\pi^2} \int_{|\xi|\le\sqrt{1-\rho^2}} d\!\!\!{}^-\xi \int_{\sqrt{1-|\xi|^2}}^{+\infty} t_1^{s_1 z_1 + s_1 - 1}\cdot$$

$$\cdot\int_0^{\sqrt{1-|\xi|^2}} t_2^{s_2 z_2 + s_2 - 1} b^0_{-|s|-j}(x,\xi,-t_1^{s_1},-t_2^{s_2})\theta(|\xi|^2 + t_2^2)\, dt_1 dt_2 \,,$$

This integral also has the only polar set $\big\{z \in \mathbb{C}^2 \,:\, z_2 = -1 - n/s_2 \,,\; s_1 \mathbf{Re}\, z_1 + s_2(-1 - n/s_2) \;=\; j - n\big\}$ which is contained in M_j. Thus the whole integral vanishes if $z \in (\mathbb{Z}_+)^2$ and even if $z \in M_j$.

This ends the proof of the meromorphic continuation of traces $'B^{(z)}_{s\cdot z-j}(x,x)$ (see (2.2.26)).

8). Moreover, these traces are equal to zero if $z \in (\mathbb{Z}_+)^m$ $(m > 1)$ and even if at least one coordinate $z_\mu \in \mathbb{Z}_+$, except for the case $z \in M_j$. We wish to compute them if $z \in M_j$, $\mathbf{Re}\, z_\mu > -1$ and at least one coordinate z_j is a positive integer. As the integral in (2.2.26) converges for $s\cdot\mathbf{Re}\, z = j - n < j$, we can divide it into the sum of the two integrals: the one is over the ball $|\xi| \le 1$ and the other is over its exterior part $|\xi| > 1$. The standard procedure shows (see (2.2.27) and the expressions for I_1, I_2, I_3) that the integral over $|\xi| \le 1$ vanishes if for some $i \in \{1, ..., m\}$ there exists z_i such that $z_i \in \mathbb{Z}_+$. Suppose now $|\xi| \ge 1$, then we see that $\theta(|\xi|^2 + \lambda_i^{2/s_i}) \;=\; 1$ and thus, it is sufficient to consider the following integral

$$I \;=\; \Big(\frac{\imath}{2\pi}\Big)^m \int_{|\xi|\ge 1} d\!\!\!{}^-\xi \int_\Gamma \lambda^z b^0_{-|s|-j}(x,\xi,\lambda)\, d\lambda \tag{2.2.30}$$

instead of (2.2.26). Once again, passing to polar coordinates in $\xi-$space we get

$$I \;=\; -\,\frac{1}{s\cdot z - j + n} \int_{|\xi|=1} \Big(\frac{i}{2\pi}\Big)^m \int_\Gamma \lambda^z b^0_{-|s|-j}(x,\xi,\lambda)\, d\lambda d\!\!\!{}^-\xi' \,.$$

Suppose all $\mathbf{Re}\, z_\mu > -1$. Then we are able to reduce to a point all circular parts of the curves Γ_i and

$$I \;=\; \frac{(-1)^{m-1}\prod_{\mu=1}^{m}\sin\pi z_\mu}{\pi^m(s\cdot z-j+n)}\int_{|\xi|=1}\int_0^\infty r^z b^0_{-|s|-j}(x,\xi,-r)\,dr d\!\!{}^-\xi' .$$

Thus we see that under the condition $\mathbf{Re}\, z_\mu > -1$ the integral I vanishes if at most two of the coordinates z_μ are integers, no matter $z \in M_j$ or not, if there exists only one $z_\mu \in \mathbb{Z}_+$ and the whole $z \notin M_j$, then I vanishes too. At last, if there exists only one $\mu \in \{1,...,m\}$ such that $z_\mu \in \mathbb{Z}_+$ and the whole $z \in M_j$, then the integral I can be calculated through the right hand side of the formula (2.2.19).

9). To end the proof of Theorem 2.2.6 we need to check that the obtained zeroes of $'B^{(z)}_{s\cdot z-j}(x,x)$ indeed coincide with the ones of $A_z(x,x)$. Indeed, the difference

$$'\mathcal{R}^{(z)}_{(N)}(x,y) \;=\; A_z(x,y) - {}'B^{(z)}_{(N)}(x,y)$$

continues to a holomorphic function in z for $s\cdot\mathbf{Re}\, z < N-n$ with the values in $C(K\times K)$, where K is an arbitrary compact in M ,. Thus in the half-space $s\cdot\mathbf{Re}\, z < N-n$ the functions $A_z(x,y)$ and $'B^{(z)}_{(N)}(x,y)$ have the same polar sets with the same residue forms. Moreover, for $z = l \in (\mathbb{Z}_+)^m$ and $x \neq y$ we have $A_l(x,y) = {}'B^{(l)}_{(N)}(x,y)$ if $s\cdot l < N-n$. In particular, it means that $A_l(x,x) = {}'B^{(l)}_{(N)}(x,x) = 0$. Further, if $z \in M_j$, $\mathbf{Re}\, z_\mu > -1$ and at least one of the coordinates z_μ is a positive integer, then, taking into consideration that the polar sets and the residue-forms of $A_z(x,x)$ and $'B^{(z)}_{(N)}(x,x)$ coincide, we obtain that $\kappa_z(x) = I$. Both are equal to zero if there is more than one positive integer z_μ and are equal to the right hand side of the formula (2.2.19) if there is exactly one positive integer z_μ and the others have $\mathbf{Re}\, z_\nu > -1$. This ends the proof of Theorem 2.2.6. ▷

2.3 ζ−function and spectral asymptotics in the plane

Let $A_z(x,y)dy$ be a kernel of operator A_z (see (2.2.2)) on a closed manifold M ($\dim M = n$) being built in the previous section. Putting $x = y$ in it we obtain the properly defined density $A_z(x,x)dx$ on M, which can be integrated over M.

Definition 2.3.1 *The function*

$$\zeta_A(z) \;=\; \zeta_{A_1,\dots,A_m}(z_1,...,z_m) \;=\; \int_M A_z(x,x)\,dx \qquad (2.3.1)$$

is called the ζ−function of the operator pencil $(A_1-\lambda_1)\cdots(A_m-\lambda_m)$.

The following Theorem is an immediate consequence of Theorem 2.2.6.

Theorem 2.3.2 *The function* $\zeta_A(z)$ *being defined through the formula (2.3.1) for* $\sum_{j=1}^m s_j \mathbf{Re}\, z_j < -n$ *can be continued to a meromorphic function in* $\mathbb{C}^m$ *with polar sets of the first order on the hyperplanes*

$$M_j \;=\; \{z \in \mathbb{C}^m \;:\; \sum_{i=1}^m s_i z_i \;=\; j-n\}$$

and the uniquely defined on them holomorphic residue-forms

$$\mathbf{res}_{M_j}\, \zeta_A \;=\; \int_M \gamma_j(x)\, dx \qquad (2.3.2)$$

$$=\; \frac{(-1)^\nu}{s_\nu} \int_M \int_{|\xi|=1} b_{-n}^{(M_j),0}(x,\xi)\, dx đ\xi' dz[\nu]\big|_{M_j} \,.$$

Moreover, $\zeta_A(z) = 0$ *if there exist at least two positive integer coordinates in* z *and is finite if there is only one positive integer coordinate in* z *and the others have* $\mathbf{Re}\, z_\nu > -1$ *(indeed it is also zero if there is one positive integer coordinate in* z *but the whole* $z \notin M_j$ *).*

We now wish to clarify the spectral role of the designed $\zeta-$function.

Proposition 2.3.3 *Suppose for all* $j \in \{1,...,m\}$ *we have* $\mathbf{Re}\, z_j < -n/s_j$ *and let* A_j *be self-adjoint positive differential operators* $(\mathrm{ord}\, A_j = s_j)$ *on* M. *Let* λ_{ji} $(i = 1,2,...)$ *be the eigenvalues of the operator* A_j *and* $\varphi_{ji}(x)$ *be the corresponding orthonormal system of eigenfunctions. Then*

$$\zeta_A(z) \;=\; \sum_{i_1,\dots,i_m=1}^{\infty} \lambda_{1i_1}^{z_1} C_{i_1 i_2}^{\overline{1},2}\, \lambda_{2i_2}^{z_2} C_{i_2 i_3}^{\overline{2},3} \cdots C_{i_{m-1} i_m}^{\overline{m-1},m}\, \lambda_{mi_m}^{z_m}\, C_{i_m i_1}^{\overline{m},1} \,, \qquad (2.3.3)$$

where

$$C_{i_1 i_2}^{\overline{1},2} \;=\; \int_M \overline{\varphi_{1i_1}(x)} \varphi_{2i_2}(x)\, dx \,, \qquad C_{i_2 i_2}^{\overline{2},1} \;=\; \int_M \overline{\varphi_{2i_2}(x)} \varphi_{1i_1}(x)\, dx \;=\; \overline{C_{i_1 i_2}^{\overline{1},2}} \,,$$

etc.

If A_j *are commuting operators then we have* $C_{i_k i_{k+1}}^{\overline{k},k+1} \;=\; \delta_{i_k i_{k+1}}$, *and*

$$\zeta_A(z) \;=\; \sum_{i=1}^{\infty} \lambda_{1i_1}^{z_1} \lambda_{2i_2}^{z_2} \cdots \lambda_{mi_m}^{z_m} \,. \qquad (2.3.4)$$

As a special case we obtain here the way to the Theorem due to Guillemin V. and Sternberg S. (1977) about the joint spectrum asymptotics. The series in (2.3.3) and (2.3.4) are absolutely convergent for the z's above and are uniformly convergent for $\mathbf{Re}\, z_j < -n/s_j - \varepsilon$.

Proof. Suppose for all $j \in \{1,...,m\}$ we have $\mathbf{Re}\, z_j < -n/s_j$, then overcoming in (2.2.2) to the iterated integral we see that

$$A_z(x,y) \;=\; \int_M \cdots \int_M A_{m,z_m}(x,y_m)A_{m-1,z_{m-1}}(y_m,y_{m-1}) \cdots$$

$$\cdot A_{1,z_1}(y_2,y_1)\, dy_2 \cdots dy_m\,, \tag{2.3.5}$$

where each of the functions A_{j,z_j} is continuous for $\mathbf{Re}\, z_j < -n/s_j$ and can be decomposed into the Fourier series through the orthogonal system of functions

$$\{\varphi_{ji}(y_{j+1})\overline{\varphi_{jk}(y_j)}\}_{i,k=1}^{\infty}\,,$$

where we put $x = y_{m+1}$, $y = y_1$ and

$$A_{j,z_j}(y_{j+1},y_j) \;=\; \sum_{i=1}^{\infty} \lambda_{ji}^{z_j} \varphi_{ji}(y_{j+1})\overline{\varphi_{ji}(y_j)}\,, \tag{2.3.6}$$

where each series is convergent in $L^2(M\times M)$ and also absolutely and uniformly (see Shubin M.A. (1978)). Multiplying the series (2.3.6) for all $j \in \{1,...,m\}$ and integrating over $y_2,...,y_m$ we finally obtain (2.3.3). $\triangleright$

In the rest of the Section we bound ourselves with the case $m = 2$, i.e.

$$\zeta_A(z) \;=\; \sum_{i_1,i_2=1}^{\infty} \lambda_{1i_1}^{z_1}\lambda_{2i_2}^{z_2}\left|C_{\bar{i}_1 i_2}\right|^2, \tag{2.3.7}$$

where $C_{\bar{i}_1 i_2} \;=\; C_{i_1 i_2}^{\bar{1},2}$, $C_{i_1,\bar{i}_2} \;=\; C_{i_2 i_1}^{\bar{2},1}$, $C_{i_2 i_1}^{\bar{2},1} \;=\; \overline{C_{i_1 i_2}^{\bar{1},2}}$ and $\left|C_{\bar{i}_1 i_2}\right|^2 = C_{\bar{i}_1 i_2}C_{i_1\bar{i}_2}$. In the special case where A_1 and A_2 are commuting operators we have $C_{\bar{i}_1 i_2} \;=\; \delta_{i_1 i_2}$ and

$$\zeta_A(z) \;=\; \sum_{i=1}^{\infty} \lambda_{1i}^{z_1}\lambda_{2i}^{z_2}\,. \tag{2.3.8}$$

Let us introduce the function

$$N(t) \;=\; N(t_1,t_2) \;=\; \sum_{\lambda^{1/s}\preceq t} |C_{\bar{i}j}|^2\,, \tag{2.3.9}$$

where $\lambda^{1/s} \preceq t$ means that both $\lambda_{1i}^{1/s_1} \le t_1$, $\lambda_{2j}^{1/s_2} \le t_2$ and the sum is over all eigenvalues of operators A_1 and A_2 with respect to their multiplicities. In the case of commuting operators we observe that

$$N(t_1,t_2) \;=\; \sum_{\lambda^{1/s}\preceq t} 1\,, \tag{2.3.10}$$

i.e. coincides with the number of eigenvalues $\{\lambda_{1i}, \lambda_{2j}\}$ of the joint spectrum lying in the rectangle $(0, t_1^{s_1}] \times (0, t_2^{s_2}]$. Clearly, the function $N(t_1, t_2)$ vanishes if $t_i < \lambda_{1i}$ and is connected with $\zeta_A(z)$ through the relation

$$\zeta_A(z) \;=\; \int_0^\infty \int_0^\infty t_1^{s_1 z_1} t_2^{s_2 z_2}\, d_{t_1} d_{t_2} N(t_1, t_2)\,. \tag{2.3.11}$$

Let us assume that $N(t)$ admits the following asymptotic expansion as $t_1, t_2 \to +\infty$ in the way that $t_1 = t$, $t_2 = kt$, $k = const$ (i.e. we wish to estimate the asymptotics of the weighted number of eigenvalue pairs lying in the sequence of similar rectangles, see (2.3.9)):

$$N(t_1, t_2) \;=\; c_1(k) t^{\alpha_1} + c_2(k) t^{\alpha_2} + \cdots + c_p(k) t^{\alpha_p} + O(t^{\alpha_{p+1}})\,, \tag{2.3.12}$$

where $\mathbf{Re}\,\alpha_1 > \mathbf{Re}\,\alpha_2 > \ldots > \mathbf{Re}\,\alpha_{p+1}$. Then

$$\zeta_A(z) \;=\; \sum_{l=1}^{p} \int_1^\infty \left[\int_0^\infty c_l{}'(k) k^{s_2 z_2}\, dk\right] t^{s_1 z_1 + s_2 z_2 + 1} d(t^{\alpha_l}) \;+\; f_p(z)\,,$$

where $f_p(z)$ is holomorphic in z for $\sum s_j \mathbf{Re}\, z_j < -\mathbf{Re}\,\alpha_{p+1}$. As obviously

$$\int_1^\infty t^{s_1 z_1 + s_2 z_2 + 1} d(t^{\alpha_l}) \;=\; \frac{-\alpha_l}{s_1 z_1 + s_2 z_2 + \alpha_l}\,,$$

then

$$\zeta_A(z) \;=\; -\sum_{l=1}^{p} \frac{\alpha_l}{s_1 z_1 + s_2 z_2 + \alpha_l} \int_0^\infty c_l{}'(k) k^{s_2 z_2}\, dk \;+\; f_p(z)\,. \tag{2.3.13}$$

Thus, the function $\zeta_A(z)$ has for $\sum s_j \mathbf{Re}\, z_j < -\mathbf{Re}\,\alpha_{p+1}$ the polar sets $M_l = \{z \in \mathbb{C}^2 : s_1 z_1 + s_2 z_2 = -\alpha_l\}$ of the first order with the uniquely defined on them residue-forms

$$\mathbf{res}_{M_l} \zeta_A \;=\; \frac{(-1)^\nu \alpha_l}{s_\nu} \int_0^\infty c_l{}'(k) k^{s_2 z_2}\, dk dz[\nu]\big|_{M_l}\,, \tag{2.3.14}$$

where $\nu = 1$ or 2.

Thus we see that, the notion of polar sets and residue-forms on them gives us an opportunity to write the formal asymptotics for $N(t)$ through the inverse Mellin transformation of the function $c_l{}'(k)$.

Suppose, the first term in the asymptotics (2.3.12) is known then, comparing (2.3.2) with (2.3.14) we obtain from the most "left" polar set $s_1 z_1 + s_2 z_2 = -n$ that $\alpha_1 = n$ and, in the view of the uniqueness of the residue-form on M_1, that

$$n \int_0^\infty c_l{}'(k) k^{s_2 z_2}\, dk \;=\; \int_M \int_{|\xi|=1} b_{-n}^{(-n/s_1 - s_2 z_2/s_1, z_2), 0}(x, \xi)\, dx d\!\!{}^{-}\xi'\,, \tag{2.3.15}$$

or

$$\mathbb{M}_{k\to z_2}[{c_1}'](s_2z_2+1) \;=\; \frac{1}{n}\int_M\int_{|\xi|=1} b_{-n}^{(-n/s_1-s_2z_2/s_1,z_2),0}(x,\xi)\,dx đ\xi' \,,$$

where $\mathbb{M}$ is the Mellin transformation.

We now refer to the Kozlov's lemma (see Proposition 1 in Kozlov S.M. (1984)) from where it follows that there exists $L>0$ such that $N(t,kt) < C(1+t+k)^L$. As the function $N(t,kt)$ is piecewise continuous in k and of bounded variation on each finite interval, there exists its inverse Mellin transformation and

$${c_1}'(k) \;=\; \frac{1}{n}\mathbb{M}^{-1}_{w\to k}\int_M\int_{|\xi|=1} b_{-n}^{((-n-w+1)/s_1,(w-1)/s_2),0}(x,\xi)\,dx đ\xi' \,,$$

where $w=s_2z_2+1$. As, obviously, $c_1(0)=0$ we have

$$c_1(k) \;=\; \frac{1}{n}\int_0^k dk'\,\mathbb{M}^{-1}_{w\to k'}\int_M\int_{|\xi|=1} b_{-n}^{((-n-w+1)/s_1,(w-1)/s_2),0}(x,\xi)\,dx đ\xi' \,.$$

As the integrand here is entire analytic function in w then, in the view of the properties of Mellin transformation, the function ${c_1}'(k)$ is analytic in $k\in(0,\infty)$ and rapidly decreasing for $k\to 0$. Thus, $c_1(k)$ is also analytic in $k\in(0,\infty)$, that is a priori supposed in (2.3.12) and the desired asymptotics has the form

$$N(t,kt) \;\sim\; \frac{t^n}{n}\int_0^k dk'\,\mathbb{M}^{-1}_{w\to k'}\int_M\int_{|\xi|=1} b_{-n}^{((-n-w+1)/s_1,(w-1)/s_2),0}(x,\xi)\,dx đ\xi' \,. \tag{2.3.16}$$

This formula follows immediately from the following two-dimensional version of the Ikehara Tauberian Theorem (see the book (Shubin M.A. (1978) for the respective one-dimensional version).

Theorem 2.3.4 *Suppose $N(t)$ is a nondecreasing function in each variable in $\mathbb{R}_+\times\mathbb{R}_+$ vanishing for $t_i\le 1$, $i=1,2$, and such that:*

A) there exists $L>0$ such that $N(t_1,t_2) < C(1+t_1+t_2)^L$,

B) the integral

$$\zeta(z) \;=\; \int_1^\infty\int_1^\infty t_1^{s_1z_1}t_2^{s_2z_2}\,d_{t_1}d_{t_2}N(t_1,t_2)$$

is convergent for $s_1\mathbf{Re}\,z_1+s_2\mathbf{Re}\,z_2 < -k_0$, $k_0>0$, and there exists a holomorphic function $A(z)$ such that the function $\zeta(z)+\frac{A(z)}{s_1z_1+s_2z_2+k_0}$ can be continuously extended to the closed half-space $s_1\mathbf{Re}\,z_1+s_2\mathbf{Re}\,z_2\le -k_0$.

Then for $t_1, t_2 \to \infty$, $t_2 = kt_1 = kt$ *we have the asymptotics*

$$N(t,kt) \;\sim\; \frac{t^{k_0}}{k_0} \int_0^k dk' \,\mathbb{M}^{-1}_{w\to k'} \, A\big((-k_0 - w + 1)/s_1, (w-1)/s_2\big)\,, \qquad \textit{(2.3.17)}$$

where $w = s_2 z_2 + 1$, $A\big((-k_0 - w + 1)/s_1, (w-1)/s_2\big) \;=\; A\big((-k_0 - s_2 z_2)/s_1, z_2\big)$ $= \; A(z_1, z_2)\big|_{\{s_1 z_1 + s_2 z_2 = -k_0\}} := \tilde{A}(w)$.

Proof. 1). It is convenient to consider instead of $\zeta(z)$ the function $f(z) = \zeta(-z)$. Then the integral

$$f(z) \;=\; \int_1^\infty \int_1^\infty t_1^{-s_1 z_1} t_2^{-s_2 z_2} \, d_{t_1} d_{t_2} N(t_1, t_2)$$

is convergent for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 \;>\; k_0$ and the function $f(z) - \frac{A(-z_1,-z_2)}{s_1 z_1 + s_2 z_2 - k_0}$ is continuous for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 \;\geq\; k_0$.

2). We now reduce the problem to the case $k_0 = 1$. Let $f_1(z) \;=\; f(k_0 z_1, k_0 z_2)$, then

$$\begin{aligned} f_1(z) \;&=\; \int_1^\infty \int_1^\infty t_1^{-k_0 s_1 z_1} t_2^{-k_0 s_2 z_2} \, d_{t_1} d_{t_2} N(t_1, t_2) \\ &=\; \int_1^\infty \int_1^\infty \tau_1^{-s_1 z_1} \tau_2^{-s_2 z_2} \, d_{\tau_1} d_{\tau_2} N_1(\tau_1, \tau_2)\,, \end{aligned}$$

where $N_1(\tau_1, \tau_2) \;=\; N(\tau_1^{1/k_0}, \tau_2^{1/k_0})$. As we have

$$f(k_0 z) \;-\; \frac{A(-k_0 z)}{s_1 z_1 + s_2 z_2 - k_0} \;\;=\;\; f_1(z) \;-\; \frac{A(-k_0 z)/k_0}{s_1 z_1 + s_2 z_2 - 1}\,,$$

then the asymptotics $N(t,kt) \;\sim\; c_1(k) t^{k_0}$ is equivalent to the asymptotics $N_1(\tau, k\tau) \;\sim\; c_1(k^{1/k_0})\, \tau/k_0$, or

$$N_1(\tau, k\tau) \;\sim\; \tau/k_0 \int_0^{k^{1/k_0}} dk' \,\mathbb{M}^{-1}_{w\to k'} \, A\big((-k_0 - w + 1)/s_1, (w-1)/s_2)\big)\,,$$

where $w = -s_2 z_2 + 1$. Thus the Theorem is reduced to the following statement:

Let $N(t)$ be a nondecreasing function and the integral

$$f(t) \;=\; \int_1^\infty \int_1^\infty t_1^{-s_1 z_1} t_2^{-s_2 z_2} \, dN(t)$$

converges for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 \;>\; 1$ in the way that the function $f(z) - \frac{A(-k_0 z)/k_0}{s_1 z_1 + s_2 z_2 - 1}$ is continuous for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 \;\geq\; 1$. Then for $t_1, t_2 \to \infty$, $t_2 = kt_1 = kt$ we have

$$N(t,kt) \;\sim\; t/k_0 \int_0^{k^{1/k_0}} dk' \,\mathbb{M}^{-1}_{w\to k'} \, A\big((-k_0 - w + 1)/s_1, (w-1)/s_2\big)\,,$$

where $w = -s_2 z_2 + 1$.

We note that, from the continuity of the difference $f(z) - \frac{A(-k_0 z)/k_0}{s_1 z_1 + s_2 z_2 - 1}$ for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 \geq 1$, and from the fact that $f(z) \geq 0$ for real (z_1, z_2) such that $s_1 z_1 + s_2 z_2 \geq 1$, it follows that $A(-k_0 z) > 0$ for real (z_1, z_2) such that $s_1 z_1 + s_2 z_2 \geq 1$, or $A(z) > 0$ for real (z_1, z_2) such that $s_1 z_1 + s_2 z_2 \leq -k_0$.

3). We now put $t = e^x$ and $\phi(k, x) = N(e^x, k e^x)$, where $\phi(k, x)$ is a nondecreasing function in x and k vanishing for negative x or k. We have

$$f(z) = \int_0^\infty e^{-(s_1 z_1 + s_2 z_2) x}\, d_x \psi(x, w), \tag{2.3.18}$$

where $\psi(x, w) = \mathbb{M}_{w \to k'} [d_k \phi]$ is the Mellin transformation of the measure $d_k \phi(k, x) = d_k N(e^x, k e^x)$. In the view of the Kozlov's lemma (see Proposition 1 in Kozlov S.M. (1984)) indeed exists $L > 0$ such that $\phi(k, x) < C(1 + t + k)^L$, i.e. $\psi(x, w)$ is correctly defined and holomorphic in $\mathbf{Re}\, w < -L$, or, equivalently, for $\mathbf{Re}\, z_2 > (L+1)/s_2$. In this case the inverse Mellin transform from w to k also exists.

The integral in (2.3.18) converges for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 > 1$ and the function $f(z) - \frac{A(-k_0 z)/k_0}{s_1 z_1 + s_2 z_2 - 1}$ is continuous for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 \geq 1$. We have to prove that

$$\lim_{x \to +\infty} e^{-x} \phi(k, x) = \frac{1}{k_0} \int_0^{k^{1/k_0}} dk'\, \mathbb{M}^{-1}_{w \to k'}\, \tilde{A}(w), \tag{2.3.19}$$

where $w = -s_2 z_2 + 1$ and for the convenience we put $\tilde{A}(w) = A\big((-k_0 - w + 1)/s_1, (w-1)/s_2\big)$.

As the left-hand side in (2.3.19) is independent of k_0 we can put $k_0 = 1$ and the rest of the proof would not change.

4). We now put $H(k, x) = e^{-x} \phi(k, x)$, and observe that from the monotonicity of the function $\phi(k, x)$ follows $H(k, y) \geq H(k, x) e^{x-y}$ for $y \geq x$ and fixed k. Integrating (2.3.18) by parts for $s_1 \mathbf{Re}\, z_1 + s_2 \mathbf{Re}\, z_2 > 1$ we get that

$$\begin{aligned} f(z) &= (s_1 z_1 + s_2 z_2) \int_0^\infty e^{-(s_1 z_1 + s_2 z_2) x} \psi(x, w)\, dx \\ &= (s_1 z_1 + s_2 z_2) \int_0^\infty e^{(-s_1 z_1 - s_2 z_2 + 1) x} \psi_H(x, w)\, dx, \end{aligned}$$

where $\psi_H(x, w) = \mathbb{M}_{k \to w = -s_2 z_2 + 1} \big[d_k H(k, x)\big]$, We note that

$$\int_0^\infty e^{(-s_1 z_1 - s_2 z_2 + 1) x}\, dx = \frac{1}{s_1 z_1 + s_2 z_2 - 1},$$

and thus

$$\frac{f(z)}{s_1 z_1 + s_2 z_2} - \frac{\tilde{A}(w)}{s_1 z_1 + s_2 z_2 - 1} = \int_0^\infty e^{(-s_1 z_1 - s_2 z_2 + 1) x} \big[\psi_H(x, w) - \tilde{A}(w)\big]\, dx$$

$$= \frac{1}{s_1 z_1 + s_2 z_2} \Big(f(z) - \frac{\tilde{A}(w)}{s_1 z_1 + s_2 z_2 - 1} - \tilde{A}(w) \Big) := \mathbb{U}(z,w)\,.$$

Putting $h_\varepsilon(t,w) = \mathbb{U}(z,w)\big|_{s_1 z_1 + s_2 z_2 = 1+\varepsilon+\imath t}$, where $\varepsilon > 0$, $t \in \mathbb{R}$, we obtain that

$$h_\varepsilon(t,w) = \int_0^\infty e^{-\varepsilon x - \imath t x} \big[\psi_H(x,w) - \tilde{A}(w)\big]\, dx\,. \tag{2.3.20}$$

In the new terms Theorem 2.3.4 looks as follows:

Theorem 2.3.5 *Let $H(k,x)$ be a real function vanishing for $x < 0$ and satisfying for all $x, y \in \mathbb{R}$ and $k \in \mathbb{R}_+$ to the condition*

$$H(k,y) \geq H(k,x) e^{x-y} \quad \text{if} \quad y \geq x\,. \tag{2.3.21}$$

Let the integral in (2.3.20) be absolutely convergent for all $\varepsilon > 0$ and the limit $\lim_{\varepsilon\to 0} h_\varepsilon(t,w) = h(t,w)$ exists and is uniform if $|t| \leq 2\lambda$ and $\mathbf{Re}\, w < -L - \delta$ for all finite λ and δ.

Then for each $k \in \mathbb{R}_+$ there exists a limit

$$\lim_{x\to+\infty} H(k,x) = \int_0^k dk' \, \mathbb{M}^{-1}_{w\to k'} \tilde{A}(w)\,.$$

Proof. We introduce a cut-off function $\tilde{\rho}(t)$, $t \in \mathbb{R}$, being the Fourier transformation of a nonnegative function $\rho(v) \in L^1(\mathbb{R})$, i.e.

$$\tilde{\rho}(t) = \int e^{\imath t v} \rho(v)\, dv\,.$$

We can choose $\rho(v)$ in the way that $\tilde{\rho} \in C_0^\infty(\mathbb{R})$ and $\tilde{\rho}(0) = 1$, i.e.

$$\int_{-\infty}^\infty \rho(v)\, dv = 1\,.$$

In order to prove the Theorem we need the following Main Lemma.

Lemma 2.3.6 *For each fixed $\lambda > 0$ and $\mathbf{Re}\, w < -L - \lambda$*

$$\lim_{y\to+\infty} \int_{-\infty}^\infty H(k, y - v/\lambda) \rho(v)\, dv = \tilde{\tilde{A}}(k)\,, \tag{2.3.22}$$

where $\tilde{\tilde{A}}(k) = \int_0^k dk' \, \mathbb{M}^{-1}_{w\to k'} \tilde{A}(w)$.

Proof. 1). Let $\tilde{\rho}_\lambda(t) = \tilde{\rho}(t/\lambda)$ and $\rho_\lambda(v) = \lambda \rho(\lambda v)$, then

$$\int_{-\infty}^\infty H(k, y - v/\lambda) \rho(v)\, dv = \int_{-\infty}^\infty H(k, y - v) \rho_\lambda(v)\, dv$$

and in the view of the fact that ρ_λ has the same properties as ρ we can see that it is sufficient to prove the Lemma in the case $\lambda = 1$.

2). We introduce a function $\varphi_\varepsilon(t,x) = \tilde{\rho}(t)h_\varepsilon(t,w)$ having a compact support in t and calculate its inverse Fourier transformation $F^{-1}_{t\to y}[\varphi_\varepsilon]$. As we have

$$h_\varepsilon(t,w) \;=\; F\big[\theta(v)\big(\psi_H(v,w) - \tilde{A}(w)\big)e^{-\varepsilon v}\big]\,,$$

then

$$F^{-1}[\varphi_\varepsilon](y,w) \;=\; \int_{-\infty}^{\infty} e^{\imath ty}\varphi_\varepsilon(t)\,đt \tag{2.3.23}$$

$$F^{-1}\big[h_\varepsilon(t,w)\big] \;*\; F^{-1}\big[\tilde{\rho}(t)\big](y) \;=\; \int_0^{\infty} e^{-\varepsilon x}\big(\psi_H(v,w) - \tilde{A}(w)\big)\rho(y-x)\,dx\,.$$

The last integral in (2.3.23) converges everywhere and in usual sense and we can overwrite (2.3.23) as follows

$$\int_{-\infty}^{\infty} e^{\imath ty}\varphi_\varepsilon(t,w)\,đt \;+\; \tilde{A}(w)\int_0^{\infty} e^{-\varepsilon x}\rho(y-x)\,dx \tag{2.3.24}$$

$$=\; \int_0^{\infty} \psi_H(x,w)e^{-\varepsilon x}\rho(y-x)\,dx\,.$$

and suppose $\varepsilon \to 0$. As we have $\mathrm{supp}_t\,\varphi_\varepsilon(t,w) \subset \mathrm{supp}\,\tilde{\rho}(t)$ and $\varphi_\varepsilon(t,w) \to \varphi(t,w) = \tilde{\rho}h(t,w)$ uniformly in $t \in \mathrm{supp}\,\tilde{\rho}(t)$ and $w \in \mathbb{P}_{M,\delta} = \{w \,:\, -M < \mathbf{Re}\,w < -L-\delta\}$, for all $M > L+\delta$, we see that the first integral in the left-hand side of (2.3.24) has the limit as $\varepsilon \to 0$ for all $y \in \mathbb{R}$, $w \in \mathbb{P}_{M,\delta}$. The same is true for the second integral due to Lebesgue Theorem. That is why the integral in the right-hand side has the limit as $\varepsilon \to 0$ for all respective y and w. We overwrite it in the form

$$\mathbb{M}_{k\to w}\,d_k\left[\int_0^{\infty} H(k,x)e^{-\varepsilon x}\rho(y-x)\,dx\right], \tag{2.3.25}$$

where we have changed the order of integration for the fixed ε. There is a function of a bounded variation uniformly in ε in the square brackets of (2.3.25), thus due to the first Helly's Theorem (see i.e. Kolmogorov A.N., Fomin S.V. (1976)), we can overcome to the limit under the sign of integral over k. Then, in the view of the Tauberian condition (2.3.21), the function $H(k,x)e^{-\varepsilon x}\rho(y-x)$ monotonely converges to the limit $H(k,x)\rho(y-x)$ as $\varepsilon \to 0$ and in the view of the B.Levy's Theorem (also see Kolmogorov A.N., Fomin S.V. (1976)) we obtain

$$\int_{-\infty}^{\infty} e^{\imath ty}\varphi_\varepsilon(t,w)\,đt \;+\; \tilde{A}(w)\int_0^{\infty} e^{-\varepsilon x}\rho(y-x)\,dx \tag{2.3.26}$$

$$=\; \mathbb{M}_{k\to w}\,d_k\left[\int_0^{\infty} H(k,x)\rho(y-x)\,dx\right].$$

We now suppose $y \to +\infty$ and obtain due to Riemann Lemma

$$\lim_{y\to+\infty} \int_{-\infty}^{\infty} e^{\imath ty} \varphi_\varepsilon(t,w)\, đt \;=\; 0\,. \tag{2.3.27}$$

Also, we have

$$\lim_{y\to+\infty} \int_0^{\infty} \rho(y-x)\, dx \;=\; 1\,. \tag{2.3.28}$$

It follows from (2.3.27) and (2.3.28) that

$$\lim_{y\to+\infty} \mathbb{M}_{k\to w}\, d_k \left[\int_0^{\infty} H(k,x)\rho(y-x)\, dx \right] \;=\; \tilde{A}(w)\,. \tag{2.3.29}$$

Once again, applying the first Helly's Theorem, we obtain

$$\mathbb{M}_{k\to w}\, d_k \left[\lim_{y\to+\infty} \int_0^{\infty} H(k,x)\rho(y-x)\, dx \right] \;=\; \tilde{A}(w)\,. \tag{2.3.30}$$

Applying $\mathbb{M}^{-1}$ to (2.3.29) we obtain

$$\lim_{y\to+\infty} \int_0^{\infty} H(k,x)\rho(y-x)\, dx \;=\; \int_0^{k} dk'\, \mathbb{M}^{-1}_{w\to k'} \left[\tilde{A}(w)\right], \tag{2.3.31}$$

and the Lemma is proved. ▷

To end the proof of the Theorem it is sufficient to prove that

$$\lim_{x\to+\infty} H(k,x) \;=\; \int_0^{k} dk'\, \mathbb{M}^{-1}_{w\to k'} \left[\tilde{A}(w)\right] \tag{2.3.32}$$

for the fixed $k \in (0,+\infty)$. The proof of the last statement does not differ from the appropriate proof in Section 14.4 from the book (Shubin M.A. (1978)). ▷

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