# SCALING OF FLUCTUATIONS & CRITICAL EXPONENTS

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### Abstract

We present a new technique to describe the abnormal behaviour of certain fluctuation observables in the critical regime of quantum statistical systems which undergo a phase transition. The idea is to rescale the local fluctuation operators by a relevant external parameter of the system, in addition to the usual scaling with the inverse square root of the volume. The scaling indices used in this scaling procedure are directly related to the critical exponents.

Furthermore, it is explained that this new method of scaling preserves the CCR structure of the algebra of macroscopic fluctuations.

Finally, scaling indices are computed for the relevant microscopic observables at all temperatures in a mean field approximation for a quantum anharmonic crystal. These indices yield the same critical exponents as predicted by mean field theory.

Key words : Abnormal fluctuation–observables, CCR–algebras, critical exponents.

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# 1 Introduction

The theory of normally scaled fluctuations and its connection with representations of CCR algebras is by now well understood [1, 2, 3, 4]. Also abnormally scaled fluctuations, appearing in systems with long range correlations, proved to be interesting since mathematical structures, other than CCR, arise when the fluctuations need anomalous scaling [5].

We present an alternative method to describe the abnormal behaviour of fluctuation observables of some equilibrium system at criticality. The idea is to scale the local *normal* fluctuations of an observable $A$ additionally with a relevant external parameter such as e.g. $(T - T_c)^{\delta_A}$ where $T$ is the temperature of the system. One can then determine the appropriate scaling index $\delta_A$ such that the limiting distribution

$$\lim_{T \to T_c +} \lim_{N \to \infty} \langle \exp[it(T - T_c)^{\delta_A} F_N(A)] \rangle_{T,\ldots,N},$$

where $F_N(A)$ is the normally scaled fluctuation of $A$, is a smooth, non–trivial function in $t$. The same procedure can be repeated for other external parameters. Note that the order of taking the limits is important and that there is no coupling at all of the external parameters to the infinite volume limit.

The scaling indices obtained this way, are directly related to the critical exponents as they are standardly defined in physics. Another important aspect of this method is that the limiting distributions determine operators on some Hilbert space which can be identified with the macroscopic fluctuation observables and which generate again a CCR algebra induced by a quasi–free state, as in the normal case.

In section 2 of this paper we briefly review the theory of normal, supernormal and subnormal fluctuations, as developed in [2, 6, 8, 7]. Section 3 introduces the anharmonic crystal, which is used in section 4 to explain the alternative way of scaling the fluctuations. Section 5 is meant as an illustration and contains a detailed analysis of the fluctuations of $Q$ and $P$ in the case of the quantum anharmonic crystal. An important feature of the illustration is the relation between the scaling indices and the critical exponents of the model.

## 2 Normal and Abnormal Fluctuations

Consider a quantum lattice system $(\mathcal{B}, \langle . \rangle)$, where $\mathcal{B}$ is a quasi-local $C^*$-algebra of microscopic observables, and $\langle . \rangle$ is a state on $\mathcal{B}$. The algebra $\mathcal{B}$ is defined as follows: let $\mathcal{A}$ be a $C^*$-algebra and denote by $\mathcal{A}_x$, $x \in \mathbb{Z}^d$, a copy of $\mathcal{A}$. Then define for all finite $\Lambda \subset \mathbb{Z}^d$, the minimal tensor product $\mathcal{A}_\Lambda = \otimes_{x\in\Lambda}\mathcal{A}_x$, which is also a $C^*$-algebra. Finally, $\mathcal{B} = \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}$. Typical examples are spin systems, where $\mathcal{A}$ is a matrix algebra, or the harmonic crystal, where $\mathcal{A}$ is generated by the momentum and position operators $P, Q$. The state $\langle . \rangle$ on $\mathcal{B}$ is assumed to be translation invariant and ergodic.

Define the normal local fluctuation observable $F_N(A)$ of an observable $A \in \mathcal{B}$ in the state $\langle . \rangle$ as

$$F_N(A) = |\Lambda_N|^{-1/2} \sum_{x\in\Lambda_N} \tau_x(A - \langle A \rangle),$$

where $\tau_x$ is the translation automorphism in $\mathcal{B}$ over the vector $x$ and $\Lambda_N = [-N, N]^d$, a d-dimensional cube in $\mathbb{Z}^d$. The main achievement of Goderis et al. was to give a mathematical meaning to the limits $N \to \infty$ of the $F_N(A)$. They proved the following. Let $\mathcal{B}_0$ be a subspace of $\mathcal{B}$. If $\forall A = A^* \in \mathcal{B}_0$ :

$$1) \quad 0 < \lim_{N\to\infty} \langle F_N(A)^2 \rangle < \infty \tag{1}$$

$$2) \quad \lim_{N\to\infty} \langle e^{itF_N(A)} \rangle = e^{-\frac{t^2}{2}\lim_{N\to\infty}\langle F_N(A)^2\rangle}, \tag{2}$$

then the system $(\mathcal{B}, \mathcal{B}_0, \langle . \rangle)$ is said to have normal fluctuations for $\mathcal{B}_0$. In this case, the limits (2) define a quasi–free state on and yield a regular representation of a CCR $C^*$-algebra [9]. This CCR algebra is called the algebra of normal fluctuations. The regularity of the representation implies the existence of Boson field operators satisfying the commutation relations

$$[F(A), F(B)] = \lim_{N\to\infty} \langle [F_N(A), F_N(B)] \rangle.$$

Goderis et al. also derived general cluster conditions under which the system $(\mathcal{B}, \mathcal{B}_0, \langle . \rangle)$ has normal fluctuations. For the precise statement and for further details, we refer to [2, 3].

For some systems however, this theory is not applicable. This is the case if the fluctuation variance (1) vanishes or diverges. Typical situations are thermodynamic systems at a phase transition and examples have already been studied : e.g. the anharmonic crystal or the Curie-Weiss model [10]. It is a well-known fact that at criticality susceptibilities may diverge. These susceptibilities correspond to the variance of some microscopic observable, e.g. the microscopic observable corresponding to the order parameter. The divergence of the fluctuation variance is caused by the long range correlations of this microscopic observable in the state.

In physics literature [11], the long range order is described by a critical exponent, indicating the decay, at large distances, of the correlation function:

$$\langle A\tau_x A\rangle - \langle A\rangle^2 \sim O\left(\frac{1}{|x|^{d-2+\eta}}\right) \quad \text{as} \quad x\to\infty.$$

To avoid the divergence of the variance, abnormal scaling was introduced: instead of scaling normally (i.e. $|\Lambda|^{-1/2}$), one defines a scaling index $\delta_A \in (0,1/2)$ such that the abnormally scaled fluctuations

$$F_N^{\delta_A}(A) \equiv \frac{1}{|\Lambda_N|^{1/2+\delta_A}} \sum_{x\in\Lambda_N} \tau_x(A - \langle A\rangle)$$

have a nontrivial distribution in the limit $N\to\infty$. Remark that the limiting distribution

$$\lim_{N\to\infty} \langle \exp(itF_N^{\delta_A}(A))\rangle = \phi_A(t)$$

is, in general, no longer Gaussian.

In situations where $\lim_{N\to\infty}\langle e^{itF_N(B)}\rangle = 1$, one uses the same technique of rescaling the local fluctuation abnormally, but with a scaling index $\delta_B < 0$ in this case.

Since the scaling index depends on the observable, the existence of a CCR algebra is no longer guaranteed; however, recent developments indicate that interesting structures appear also in this case [5].

# 3 The anharmonic crystal

The model describing anharmonic crystals, discussed in [6], will be used throughout as a guideline to present the new approach to study abnormal behaviour of fluctuations. Since a detailed analysis of this model can be found in [6, 12], it is permitted to be short in explaining the model and its features. The model is defined on a lattice, with a quantum mechanical particle associated to each lattice site. The observables of such a system can be described by an algebra $\mathcal{B} = \otimes_{i\in\mathbb{Z}^d}\mathcal{A}_i$, where each $\mathcal{A}_i$ is a copy of the $C^*$–algebra $\mathcal{A}$ associated to the Heisenberg group [13], with generators $\mathbb{1}, P, Q$ such that $[P, Q] = i\mathbb{1}$.

The local Hamiltonians of the model are specified by the operators :

$$H_N(h) = T_N + VW\left(\frac{1}{V}\sum_{l\in\Lambda_N} Q_l^2\right) + h\sum_{l\in\Lambda_N} Q_l, \tag{3}$$

where

$$T_N = \frac{1}{2m}\sum_{l\in\Lambda_N} P_l^2 + \frac{a}{2}\sum_{l\in\Lambda_N} Q_l^2 + \frac{1}{4}\sum_{l,l'\in\Lambda_N}\phi_{l,l'}(Q_l - Q_l')^2$$

and $\Lambda_N = \{l \in \mathbb{Z}^d : |l_\alpha| \leq N/2; \alpha = 1,..,d\}$, $V \equiv |\Lambda_N|$ is the cardinality of $\Lambda_N$.

The local Gibbs states of the model, describe the system in equilibrium at inverse temperature $\beta$ and are given by

$$\langle A\rangle_{\beta,h,N} = \frac{Tr(e^{-\beta H_N(h)}A)}{Tr(e^{-\beta H_\Lambda})}$$

where $A \in \mathcal{B}$.

This model is soluble in the sense that for all temperatures, the free energy density can be calculated, namely

$$\begin{aligned} f(T,h) \;&=\; \lim_{N\to\infty}\left\{\frac{1}{\beta|\Lambda_N|}\sum_{q\in\tilde{\Lambda}}\ln[2\sinh\beta\lambda\Omega_q(c_{h,N})]\right. \\ &\qquad \left. -\frac{1}{2}\frac{h^2}{\Delta(c_{h,N})} + W(c_{h,N}) - c_{h,N}W'(c_{h,N})\right\}, \end{aligned}$$

where $\tilde{\Lambda}_N$ denotes the reciprocal volume

$$\tilde{\Lambda}_N = \left\{q\,\middle|\, q_\alpha = \frac{2\pi}{N}n_\alpha; |n_\alpha| \leq \frac{N}{2}\right\}$$

and where $c_{h,N}$ is the solution of the self-consistency equation

$$\begin{aligned} c_{h,N} = \frac{1}{V}\langle \sum_{l\in\Lambda_N} Q_l^2\rangle_{\beta,h,N} = \\ \frac{h^2}{\Delta(c_{h,N})^2} + \frac{1}{V}\sum_{q\in\tilde{\Lambda}} \frac{\lambda}{2\Omega_q(c_{h,N})}\coth(\frac{\beta\lambda}{2}\Omega_q(c_{h,N})) \end{aligned} \tag{4}$$

and $\lambda = \hbar/m^{1/2}$,

$$\begin{aligned} \Omega_q(c)^2 &= \omega(q)^2 + \Delta(c) \\ \Delta(c) &= a + W'(c) \\ \omega(q)^2 &= \tilde{\phi}(0) - \tilde{\phi}(q). \end{aligned}$$

Taking first the limit $N \to \infty$ and subsequently the limit $h \to 0\pm$, shows that the model has a phase transition. The transition is described by the order parameter $\langle Q_0\rangle_\beta$ : there exists a critical line $(\beta_c, \lambda_c)$ (see Fig. 1) such that

$$\begin{cases} \text{if} \quad \beta \le \beta_c(\lambda) : \langle Q_0\rangle_\beta = 0 \\ \text{if} \quad \beta > \beta_c(\lambda) : \langle Q_0\rangle_{\beta,\pm} = \pm\sqrt{\rho(\beta,\lambda)}, \end{cases}$$

where

$$\rho(\beta,\lambda) = \lim_{h\to 0}\lim_{N\to\infty} \frac{h^2}{\Delta(c_{h,N})^2}$$

is the first term appearing in the self-consistency equation of the infinite volume state :

$$c_h = \frac{h^2}{\Delta(c_h)^2} + I_s(c_h, T, \lambda).$$

The integral

$$I_s(c_h, T, \Lambda) = \frac{\lambda}{(2\pi)^d}\int_{\mathcal{B}_d} d^d q \frac{\coth(\frac{\beta\lambda}{2}\Omega_q(c_h))}{2\Omega_q(c_h)}$$

where $\mathcal{B}_d = \{q \in \mathbb{R}^d : |q_\alpha| < \pi; \alpha = 1,..,d\}$ is the first Brillouin zone. Remark that the fact that $h \neq 0$ allows us to take this limit.

Fig. 1. Phase diagram of the model [6].

# 4 Scaling with external parameters

In [6, 8], it was observed that the fluctuations of the microscopic displacement observable and its canonical conjugate, the momentum observable show abnormal behaviour on the critical line. In the present section we will describe the new approach to study the abnormal behaviour of these observables.

The abnormal behaviour of the susceptibility $\chi_Q$ as a function of the external field, can be discussed by studying the following limiting distributions

$$\lim_{h\to 0\pm} \lim_{N_1\to\infty} \lim_{N_2\to\infty} \langle \exp[isF_{N_1}^{\delta_Q}(Q)] \rangle_{\beta,h,N_2} \tag{5}$$

where

$$F_{N_1}^{\delta_Q}(Q) = \frac{h^{\delta_Q}}{\sqrt{|\Lambda_{N_1}|}} \sum_{i\in\Lambda_{N_1}} \left( Q_i - \langle Q_i \rangle_{\beta,h,N_2} \right). \tag{6}$$

is the familiar normally scaled local fluctuation of the observable $Q$ in a volume $\Lambda_N$ multiplied with an extra scaling factor, which is some power of the external field strength. The problem is to determine the appropriate $\delta_Q$, such that the limiting distribution (5) is a smooth function of $s$.

Coupling the limits $N_1 = N_2 \to \infty$ is possible and does not change the results in a qualitative way.

The limit (5) is easily verified to be

$$\lim_{h\to 0\pm} \lim_{N_1\to\infty} \lim_{N_2\to\infty} \langle \exp[isF_{N_1}^{\delta_Q}(Q)]\rangle_{\beta,h,N_2}$$
$$= \lim_{h\to 0\pm} \exp -\frac{s^2}{2} h^{2\delta_Q} \chi_Q(h)$$

where

$$\chi_Q(h) = \lim_{N_1\to\infty} \lim_{N_2\to\infty} \langle \left(F_{N_1}^0(Q)\right)^2 \rangle_{\beta,h,N_2}$$

Rescaling the fluctuations in this way has two immediate consequences. First, the limiting characteristic functions of the fluctuations of *any* observable $A$, will be Gaussian, provided that an appropriate scaling index $\delta_A$ is used such that $\lim_{h\to 0+} \chi_A(h)$ is a non–zero, finite positive real number.

Secondly, this procedure will preserve the CCR structure of the fluctuations. This follows from the observation that

$$[F_{N_1}^{\delta_A}(A), F_{N_1}^{\delta_B}(B)] = \frac{h^{\delta_A+\delta_B}}{|\Lambda_{N_1}|} \sum_{x,y\in\Lambda_{N_1}} [A_x, B_y].$$

The infinite volume limit $N_1 \to \infty$ of this commutator is, by ergodicity of the states $\langle . \rangle_{\beta,h\neq 0}$, a c–number :

$$[F^{\delta_A}(A), F^{\delta_B}(B)] = h^{\delta_A+\delta_B} \langle \sum_{x\in\mathbb{Z}^d} [A, \tau_x B]\rangle_{\beta,h}. \tag{7}$$

This is the Law of Large Numbers, which lies at the basis of the CCR structure of normal fluctuations.

The limit $h \to 0$ of this commutator can only give a finite number, different from zero in the situation $\delta_A + \delta_B \leq 0$. For $\delta_A + \delta_B > 0$, it is trivial that the limit will be zero. The Cauchy–Schwarz inequality implies the finiteness of the limit

$$\lim_{N_1 \to \infty} \frac{h^{\delta_A + \delta_B}}{|\Lambda_{N_1}|} \sum_{x,y \in \Lambda_{N_1}} [A_x, B_y]$$

in a situation where $\delta_A + \delta_B < 0$ :

$$|\langle [F_{N_1}^{\delta_A}(A), F_{N_1}^{\delta_B}(B)] \rangle_{\beta,h}|^2 \leq 4 \langle \left( F_{N_1}^{\delta_A}(A) \right)^2 \rangle_{\beta,h} \langle \left( F_{N_1}^{\delta_B}(B) \right)^2 \rangle_{\beta,h}.$$

Since $\delta_A$ and $\delta_B$ are defined such that the variances on the r.h.s. are finite, the finiteness of the limit is established. The final conclusion is that under all circumstances the CCR structure of the limiting fluctuation observables is preserved.

Although the new technique of rescaling fluctuations was presented by means of an external field, it is clear that one can also work with other external variables, such as $|T - T_c|$ or $|\rho - \rho_c|$ in fluid systems where $\rho$ denotes the density. Doing so, one can obtain information on the critical exponents of the model one whishes to study, which is interesting from the physical point of view. Simultaneously, this technique is also interesting from the mathematical point of view, since the limiting fluctuation observables will always be characterisable as the Bose field operators of a macroscopic CCR algebra on a (possibly degenerated) symplectic space, with a macroscopic quasi–free state defined on it.

A last point is that one can use the same technique to study the asymptotic behaviour of the Fourier transform of the susceptibility $\hat{\chi}_A(q)$ . To that end, one defines the $q$-mode fluctuation :

$$F_{N_1,q}(A) = \frac{1}{|\Lambda_{N_1}|} \sum_{x \in \Lambda_{N_1}} e^{iqx}(A_x - \langle A_x \rangle_{\beta,h=0,N_2}).$$

and scales it with a factor $|q|^\varepsilon$, i.e.

$$F_{N_1,q}(A) = \frac{|q|^\varepsilon}{|\Lambda_{N_1}|} \sum_{x \in \Lambda_{N_1}} e^{iqx}(A_x - \langle A_x \rangle_{\beta,h=0,N_2}).$$

The scaling index $\varepsilon$ is directly related to the critical exponent which governs the correlation function [11]:

$$\eta = 2(1 - \epsilon)$$

# 5 Application : mean field critical exponents in the anharmonic crystal

The fluctuation–behaviour will be discussed of the single–site observables $Q$ and $P$, in the equilibrium states of the anharmonic crystal, using the additional scaling factors :

1. vanishing external field : $h \to 0$,
2. vanishing temperature difference $(T - T_c) \to 0+$
3. the momentum of the fluctuations $q \to 0$.

The results illustrate the ideas, introduced in the previous section. Moreover, $Q$ defines the order parameter of the model and consequently the scaling indices of $Q$ should correspond to the critical exponents of a mean field model. This is shown to be the case.

The resulting CCR algebras of macroscopic fluctuation observables will be commutative for all critical temperatures $T_c \neq 0$ with a quasi–free state defined on it. Only for $T_c = 0$, the algebra is non–abelian.

As explained above, it is sufficient to calculate the variances of the macroscopic fluctuations, in order to characterise the quasi–free state on the CCR algebra of macroscopic fluctuations. Therefore, we calculate :

$$\begin{aligned}
&\lim_{N_1\to\infty}\lim_{N_2\to\infty}\langle\left(F^{\delta}_{N_1}(Q)\right)^2\rangle_{\beta,h,N_2} \\
&= \frac{h^{2\delta}\lambda}{\sqrt{\Delta(c_h)}}\coth\left(\frac{\beta\lambda}{2}\sqrt{\Delta(c_h)}\right) \qquad (8)\\
&\lim_{N_1\to\infty}\lim_{N_2\to\infty}\langle\left(F^{\delta'}_{N_1}(P)\right)^2\rangle_{\beta,h,N_2} \\
&= \frac{h^{2\delta'}\lambda m\sqrt{\Delta(c_h)}}{2}\coth\left(\frac{\beta\lambda}{2}\sqrt{\Delta(c_h)}\right) \qquad (9)
\end{aligned}$$

$$\begin{aligned}
&\lim_{N_1\to\infty}\lim_{N_2\to\infty}\langle\left(F^{\alpha}_{N_1}(Q)\right)^2\rangle_{\beta,h=0,N_2} \\
&= \frac{|T-T_c|^{2\alpha}\lambda}{\sqrt{\Delta(c_h)}}\coth\left(\frac{\beta\lambda}{2}\sqrt{\Delta(c_h)}\right) \qquad (10)\\
&\lim_{N_1\to\infty}\lim_{N_2\to\infty}\langle\left(F^{\alpha'}_{N_1}(P)\right)^2\rangle_{\beta,h=0,N_2} \\
&= \frac{|T-T_c|^{2\alpha'}\lambda m\sqrt{\Delta(c_h)}}{2}\coth\left(\frac{\beta\lambda}{2}\sqrt{\Delta(c_h)}\right) \qquad (11)
\end{aligned}$$

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\langle F^{\varepsilon}_{q,N_1}(Q)^* F^{\varepsilon}_{q,N_1}(Q)\rangle_{\beta,h=0,N_2}$$

$$= \frac{|q|^{2\varepsilon}\lambda}{\Omega_q(\Delta(c_h))} \coth\left(\frac{\beta\lambda}{2}\Omega_q(\Delta(c_*))\right) \tag{12}$$

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\langle F^{\varepsilon'}_{q,N_1}(P)^* F^{\varepsilon'}_{q,N_1}(P)\rangle_{\beta,h=0,N_2}$$

$$= \frac{|q|^{2\varepsilon'}\lambda m \Omega_q(\Delta(c_h))}{2} \coth\left(\frac{\beta\lambda}{2}\Omega_q(\Delta(*))\right) \tag{13}$$

The results for these scaling indices are contained in the following theorems.

**Theorem 1**

*a)* *If* $T > T_c(\lambda_c)$ *then* $\delta' = \delta = 0.$

*b)* *If* $T < T_c(\lambda_c)$ *then* $\delta' = 0$ *and* $\delta = \frac{1}{2}.$

*c)* *If* $T = T_c(\lambda_c)$ *then* $\delta' = 0$ *and*

$$\delta = \begin{cases} \frac{2}{5} & \text{if} \quad d = 3 \\ \frac{1}{4} + 0 & \text{if} \quad d = 4 \\ \frac{1}{4} & \text{if} \quad d \geq 5 \end{cases} .$$

**Proof:**

The normality of the momentum fluctuations is immediate upon observing that for small $h$:

$$\lim_{N\to\infty}\langle \left(F^{\delta'}_N(P)\right)^2\rangle_{\beta,h,N} \sim \frac{h^{2\delta'} m}{2\beta}.$$

For the position fluctuation, we need to consider the three cases separately :

*a)* Since $\Delta(c_h) > 0$ if $h \to 0$, this is trivial.

*b)* In this case (see [6]) $\Delta(c_h) \sim h$, such that our statement follows.

*c)* Here the behaviour of $\Delta(c_h)$ as a function of $h$ has to be studied. This is done in exactly the same way as in [6]: We consider the behaviour of the self-consistency equation (4), as $h$ tends to zero. Therefore, rewrite (4) as

$$c_h - c_* = \frac{h^2}{\Delta(c_h)} + I_s(c_h, T_c, \lambda_c) - I_s(c_*, T_c, \lambda_c).$$

Using the fact that $\Delta(c_h) = (c_h - c_*)2W''(c_*) + o(c_h - c_*)$ and

$$\begin{aligned} I_s(c_h, T_c, \lambda_c) - I_s(c_*, T_c, \lambda_c) &= \frac{\Delta(c_h)}{\beta_c(2\pi)^d}\int_{|q|<\varepsilon} d^d q \frac{1}{s^2q^2(\Delta(c_h) + s^2q^2)} \\ &+ (c_h - c_*)\partial_c I_d^{(>\varepsilon)}(c_h'), \end{aligned}$$

where

$$I_d^{(>\varepsilon)} = \int_{|q|>\varepsilon} d^d q \frac{\lambda}{2\Omega_q(c_h)} \coth(\frac{\beta\lambda}{2}\omega(q)),$$

we get

$$\begin{aligned} \frac{\Delta(c_h)}{2W''(c_*)} &= \frac{h^2}{\Delta(c_h)} + \frac{\Delta(c_h)}{2W''(c_*)}\partial_c I_d^{(>\varepsilon)}(c_h') \\ &+ \frac{\Delta(c_h)}{\beta_c(2\pi)^d}\int_{|q|<\varepsilon} d^d q \frac{1}{s^2q^2(\Delta(c_h) + s^2q^2)}. \end{aligned}$$

The second term on the r.h.s. is always bounded. The third term, however, is dimension dependent:

$$I_3^{(<\varepsilon)}(c_*) - I_3^{(<\varepsilon)}(c_h) = \frac{4\pi\sqrt{\Delta(c_h)}}{\beta_c(2\pi)^3 s^3}\arctan(\frac{s\varepsilon}{\sqrt{\Delta(c_h)}})$$

$$I_4^{(<\varepsilon)}(c_*) - I_4^{(<\varepsilon)}(c_h) = \frac{K}{\beta_c s^4}\Delta(c_h)\ln(\frac{s^2\varepsilon^2 + \Delta(c_h)}{\Delta(c_h)}).$$

For $d \geq 5$ one uses:

$$I_d^{(<\varepsilon)}(c_*) - I_d^{(<\varepsilon)}(c_h) = (c_h - c_*)\partial_c I_d(c_h).$$

This leads, for $d = 3$, to the following self-consistency equation:

$$\frac{\Delta(c_h)}{2W''(c_*)} = \frac{h^2}{\Delta(c_h)^2} + \frac{\Delta(c_h)}{2W''(c_*)}\partial_c I_d^{(>\varepsilon)}(c_h') + \frac{4\pi\sqrt{\Delta(c_h)}}{\beta_c(2\pi)^3 s^3}\arctan(\frac{s\varepsilon}{\sqrt{\Delta(c_h)}}),$$

or $\Delta(c_h) \sim h^{4/5}$ as $h \to 0$. For $d = 4$ the behaviour of $\Delta(c_h)$ is given by the equation

$$h^2 = \Delta(c_h)^3 \ln(\frac{s^2\varepsilon^2}{\Delta(c_h)}),$$

which is somewhere between $h^{2/3} < \Delta(c_h) < h^{2/3}\ln(h)$.

For $d \geq 5$ we have, similarly, $\Delta(c_h) \sim h^{2/3}$. Now for small $h$, (8) can be written as

$$\lim_{N\to\infty} \langle F_\Lambda^\delta(Q)^2\rangle_{\beta_c,h,N} = \frac{2h^{2\delta}T_c}{\Delta(c_h)}.$$

¿From this, it is clear that this remains finite and nontrivial in the limit $h \to 0$ only if

$$\delta = \begin{cases} \frac{2}{5} & \text{if} \quad d = 3 \\ \frac{1}{4} + 0 & \text{if} \quad d = 4 \\ \frac{1}{4} & \text{if} \quad d \geq 5 \end{cases}.$$

■

**Theorem 2**

*If $T_c > 0$, $\lambda = \lambda_c$ and $T \to T_c+$ and if $\alpha$ and $\alpha'$ are defined by (10) and (11), then $\alpha = \frac{1}{2}$, $\alpha' = 0$. The value $\frac{1}{2}$ for the scaling index $\alpha$ agrees with the critical exponent $\gamma = 1$, predicted for mean field systems [11]:*

$$\chi_Q(T) \sim |T - T_c|^{-\gamma}.$$

**Proof:**

Again the behaviour of $\Delta(c_T)$ as $T \to T_c+$ is deduced from the behaviour of the gap equation, as $T \to T_c+$. The argument goes as follows: One can easily verify that $\partial_\beta I_d(c,T,\lambda)$ exists and is nonzero for $T \geq T_c$. Therefore, $c - c_* \sim (\beta - \beta_c)$, and hence $\Delta(c_T) \sim (T - T_c)$. Now, for small $\Delta(c)$ :

$$\lim_{N\to\infty} \langle F_N(Q)^2 \rangle_{T,h=0,N} \sim \frac{|T-T_c|^{2\alpha}}{\Delta(c)},$$

$$\lim_{N\to\infty} \langle F(P)^2 \rangle_{T,h=0,N} \sim |T-T_c|^{2\alpha'} \sqrt{\Delta(c)} \coth \frac{\beta\lambda_c}{2} \sqrt{\Delta(c)}.$$

This yields the results. ■

**Theorem 3**

*If $T = T_c > 0$ and $\lambda = \lambda_c$, then $\varepsilon = 1$, $\varepsilon' = 0$, where $\varepsilon$ and $\varepsilon'$ are defined by (12) and (13). The value $\varepsilon = 1$ implies that the critical exponent $\eta$, describing the asymptotic decay of the correlations for the orderparameter, equals* 0. *This agrees again with the standard knowledge [11].*

**Proof:**

For $T = T_c$, $\Delta(c_*) = 0$, and hence $\Omega_q(c_*) \sim |sq|$, where s is a constant. For small k, from (12), we have

$$\lim_{N\to\infty} \langle F^\varepsilon_{q,N}(Q)^* F^\varepsilon_{q,N}(Q) \rangle_{T_c,h=0,N} \sim \frac{|q|^{2\varepsilon}}{s^2|q|^2}.$$

This yields $\varepsilon = 1$. For the same reasons as in the previous theorems, $\varepsilon' = 0$.

■

So far, we may conclude that our new method of studying fluctuation behaviour has produced interesting results from the physical point of view : we find back the usual values for the critical exponents of a mean field system at the critical points $T_c > 0$.. Our results are also interesting from the mathematical point of view. It is important to realise that in fact we characterise for each point of the phase diagram a CCR algebra of macroscopic fluctuation observables and a quasi–free state on this algebra. The CCR algebra is abelian in the critical situations.

The non–abelian character of the macroscopic CCR algebra will be restored, when the critical behaviour of the fluctuations of $Q$ and $P$ is studied for the groundstates. This should be clear from the following expressions for the fluctuation variances :

$$\begin{aligned}
\lim_{N\to\infty}\langle\left(F_N^{\delta}(Q)\right)^2\rangle_{T=0,h,N} &= \frac{h^{2\delta}\lambda}{\Delta(c_h)},\\
\langle\left(F_N^{\delta'}(P)\right)^2\rangle_{T=0,h,N} &= h^{2\delta'}\Delta(c_h)\frac{\lambda m}{2},\\
\langle(F_N^{\alpha}(Q))^2\rangle_{T,h=0,N} &= \frac{|T|^{2\alpha}\lambda}{\Delta(c)},\\
\langle\left(F_N^{\alpha'}(P)\right)^2\rangle_{T,h=0,N} &= |T|^{2\alpha'}\Delta(c_h)\frac{\lambda m}{2},\\
\langle F_{q,N}^{\varepsilon}(Q)^* F_{q,N}^{\varepsilon}(Q)\rangle_{T=0,h=0,N} &= \frac{|q|^{2\varepsilon}\lambda}{\Omega_q(c)},\\
\langle F_{q,N}^{\varepsilon'}(P)^* F_{q,N}^{\varepsilon'}(P)\rangle_{T=0,h=0,N} &= |q|^{2\varepsilon'}\Omega_q(c)\frac{\lambda m}{2}.
\end{aligned}$$

The non-abelian CCR-algebra structure will always emerge, since the critical exponents for $Q$ and $P$ will always have an opposite sign. Again, this observation agrees with the one in [6]. The exact value of the critical exponent depends of course on the value of $\lambda$.

The scaling indices now no longer yield the same critical exponents. However, little is known about calculations of critical exponents in the ground state, therefore the results should be interpreted with caution.

**Theorem 4**

*For $T=0$ and $d\geq 2$ :*

*a) if $\lambda>\lambda_c$, then $\delta=\delta'=0$*

*b) if $0<\lambda\leq\lambda_c$, then $\delta=-\delta'=1/4$*

**Proof :**

If $T=0$, the self-consistency equation reads :

$$c_h=\frac{h^2}{\Delta(c_h)^2}+\frac{1}{(2\pi)^d}\int_{\mathcal{B}^d}\frac{\lambda}{2\Omega_q(c_h)}.$$

Making the same analysis as in theorem (1), leads to our result.

■

**Theorem 5**

*If $T_c=0$, $\lambda=\lambda_c$, then $\alpha=1/4$, $\alpha'=-1/4$.*

**Proof :**

The same reasoning as in theorem (2) is made, together with the fact that

$$\lim_{N\to\infty}\langle F_N(Q)^*F_N(Q)\rangle_N\sim\frac{|T-T_c|^{2\alpha}}{\sqrt{\Delta}(c)},$$
$$\lim_{N\to\infty}\langle F_N(P)^*F_N(P)\rangle\sim|T-T_c|^{2\alpha'}\sqrt{\Delta}(c),$$

which yields the result.

■

The critical exponent $\gamma$, would be 1/2 in this case.

Finally, we have :

**Theorem 6**

*If $T=0$, $\lambda=\lambda_c$, then $\varepsilon=1/2$ and $\varepsilon'=-1/2$.*

**proof :**

The proof follows the same lines as the proof of the previous theorems.

■

The critical exponent $\eta$, calculated with the result from this theorem, would equal 1.

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