ON ALTERNATIVE SUPERMATRIX REDUCTION

Steven Duplij *†‡
Physics Department, University of Kaiserslautern, Postfach 3049, D-67653 KAISERSLAUTERN, Germany

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Abstract

We consider a nonstandard odd reduction of supermatrices (as compared with the standard even one) which arises in connection with possible extension of manifold structure group reductions. The study was initiated by consideration of the generalized noninvertible superconformal-like transformations. The features of even- and odd-reduced supermatrices are investigated on a par. They can be unified into some kind of "sandwich" semigroups. Also we define a special module over even- and odd-reduced supermatrix sets, and the generalized Cayley-Hamilton theorem is proved for them. It is shown that the odd-reduced supermatrices represent semigroup bands and Rees matrix semigroups over a unit group.

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*Alexander von Humboldt Fellow
†On leave of absence from *Theory Division, Nuclear Physics Laboratory, Kharkov State University, KHARKOV 310077, Ukraine*
‡E-mail: duplij@physik.uni-kl.de

1 Introduction

According to the general theory of G-structures [7, 23, 30] various geometries are obtained by a reduction of a structure group of a manifold to some subgroup G of the tangent space endomorphisms. In the local approach using coordinate description this means that one should reduce a corresponding matrix in a given representation to some reduced form as a matter of fact. In the most cases this form is triangle, because of the simple observation from the ordinary matrix theory that the triangle matrices preserve the shape and form a subgroup. In supersymmetric theories, despite of appearance of odd subspaces and anticommuting variables, the choice of the reduction shape remained the same[24, 33, 44, 46], and a ground reason of this was the fully identity of the supermatrix multiplication with the ordinary one, and consequently the shape of the matrices from a subgroup was the same. However in fine search of nontrivial supersymmetric manifestations one can observe that the closure of multiplication can be also achieved for other shapes, but due to existence of zero divisors in the Grassmann algebra or in the ring over which a theory is defined. So the meaning of the reduction itself could be extended principally. Evidently, that some "good" properties of the transformations could be lost in this direction, but opening of new possibilities, beauty and interesting and unusual features which are distinctive for supersymmetric case only are the sufficient price for the surprises arisen and reason for them to investigate.

This paper was initiated by the study of superconformal symmetry semigroup extensions [14, 13]. Indeed superconformal transformations [3, 10, 9] appear as a result of the reduction of the structure group matrix to the triangle form [20, 19]. Also, the transition functions on semirigid surfaces [11, 22] (see [12]) occurred in the description of topological supergravity [21] have the same shape. In [14] we considered an alternative version of the reduction. The superconformal-like transformations obtained in this way have many unusual features, e. g. they are noninvertible and twist parity of the tangent space in the supersymmetric basis[1].

Here we study the alternative reduction of supermatrices from a more abstract viewpoint without connecting a special physical model.

[1]This situation is different from the case of Q-manifolds [1], where changing parity of the tangent space is done by hand from the first definitions.

2 Preliminaries

Let Λ be[2] a commutative Banach $\mathbb{Z}_2$-graded superalgebra over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$. The elements a from Λ_0 and Λ_1 are homogeneous and have the fixed even and odd parity defined as $|a| \stackrel{def}{=} \{i \in \{0,1\} = \mathbb{Z}_2 |\, a \in \Lambda_i\}$. The even homomorphism $\mathfrak{m}_b : \Lambda \to \mathbb{B}$, where $\mathbb{B}$ is a purely even algebra over $\mathbb{K}$, is called a body map, if for any other purely even algebra $\mathbb{A}$ and any homomorphism $\mathfrak{m}_a : \Lambda \to \mathbb{A}$ there is an even homomorphism $\mathfrak{m}_{ab} : \mathbb{B} \to \mathbb{A}$ such that $\mathfrak{m}_a = \mathfrak{m}_{ab} \circ \mathfrak{m}_b$. The kernel of $\mathfrak{m}_b$ is $\mathbb{S} \ \equiv \ker \mathfrak{m}_b \stackrel{def}{=} \{a \in \Lambda |\, \mathfrak{m}(a) = 0\}$ and is called the soul sector of Λ. If there are exists an embedding $\mathfrak{n} : \mathbb{B} \hookrightarrow \Lambda$ such that $\mathfrak{m} \circ \mathfrak{n} = id$, then Λ admits the body and soul decomposition $\Lambda = \mathbb{B} \oplus \mathbb{S}$, and a soul map can be defined as $\mathfrak{m}_s : \Lambda \to \mathbb{S}$. Usually the isomorphism $\mathbb{B} \ \cong \mathbb{K}$ is implied (which is not necessary in general and can lead to very nontrivial behavior of the body). This is the case when Λ is modeled with the Grassmann algebras $\wedge(N)$ having N generators [42, 41, 49] or $\wedge(\infty)$ [43, 5, 29], or with the free graded-commutative Banach algebras $\wedge_B E$ over Banach spaces [28, 36, 6]. The soul $\mathbb{S}$ is obviously a proper two-sided ideal of Λ which is generated by Λ_1. In case Λ is a Banach algebra (with a norm $||\cdot||$) soul elements are quasinilpotent [27], which means $\forall a \in \mathbb{S}$, $\lim\limits_{n\to\infty} ||a||^{1/n} = 0$. But in the infinite-dimensional case quasinilpotency of the soul elements does not necessarily lead to their nilpotency ($\forall a \in \mathbb{S}\, \exists n,\ a^n = 0$) [37]. These facts allow us to consider noninvertible morphisms on a par with invertible ones (in some sense), which gives, in proper conditions, many interesting and nontrivial results (see [14, 15, 16]).

The $(p|q)$-dimensional linear model superspace $\Lambda^{p|q}$ over Λ (in the sense of [42, 49, 29, 50]) is the even sector of the direct product $\Lambda^{p|q} = \Lambda_0^p \times \Lambda_1^q$. The even morphisms $\mathrm{Hom}_0\left(\Lambda^{p|q}, \Lambda^{m|n}\right)$ between superlinear spaces $\Lambda^{p|q} \to \Lambda^{m|n}$ are described by means of $(m+n) \times (p+q)$-supermatrices (for details see [4, 32]). In various physical applications supermatrices are reduced to some suitable form which is necessary for concrete consideration. For instance, in the theory of super Riemann surfaces [18, 45] the $(1+1)\times(1+1)$-supermatrices describing holomorphic morphisms of the tangent bundle have a triangle shape [19, 20].

[2] The standard notations can be found in [4, 32], and here we list some of them needed only.

Here we consider a special alternative reduction of supermatrices and study its features. We note that the supermatrix theory per se has many own problems [2, 17, 26] and unexpected conclusions (e.g. the lowering of the degree of characteristic polynomials comparing to the standard Cayley-Hamilton theorem[48, 47]).

For transparency and clarity we confine ourselves to $(1+1)\times(1+1)$-supermatrices[3] , and generalization to the $(m+n)\times(p+q)$ case is straightforward and can be mostly done by means of simple changing of notations.

3 Structure of $\mathrm{Mat}_{\Lambda}(1|1)$

In the standard basis in $\Lambda^{1|1}$ the elements from $\mathrm{Hom}_0\left(\Lambda^{1|1},\Lambda^{1|1}\right)$ are described by the $(1+1)\times(1+1)$-supermatrices [4]

$$M \equiv \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{Mat}_{\Lambda}(1|1) \tag{1}$$

where $a,b\in\Lambda_0$, $\alpha,\beta\in\Lambda_1$ (in the following we use Latin letters for elements from Λ_0 and Greek letters for ones from Λ_1). For sets of matrices we also use corresponding bold symbols, e. g. $\mathbf{M}\overset{def}{=}\{M\in\mathrm{Mat}_{\Lambda}(1|1)\}$. In this simple $(1|1)$ case the supertrace defined as $\mathrm{str}:\mathrm{Mat}_{\Lambda}(1|1)\to\Lambda_0$ and Berezinian defined as $\mathrm{Ber}:\mathrm{Mat}_{\Lambda}(1|1)\setminus\{M|\,\mathfrak{m}_b(b)=0\}\to\Lambda_0$ are

$$\mathrm{str}M = a-b, \tag{2}$$

$$\mathrm{Ber}M = \frac{a}{b}+\frac{\beta\alpha}{b^2}. \tag{3}$$

Now we define two types of possible reductions of M on a par and study some of their properties simultaneously.

Definition 1 *Even-reduced supermatrices are elements from* $\mathrm{Mat}_{\Lambda}(1|1)$ *having the form*

$$S \equiv \begin{pmatrix} a & \alpha \\ 0 & b \end{pmatrix} \in \mathrm{RMat}_{\Lambda}^{even}(1|1). \tag{4}$$

[3]which will allow us not to melt ideas by large formulas, and only this size will be used in the following consideration

Odd-reduced supermatrices are elements from $\mathrm{Mat}_\Lambda(1|1)$ *having the form*

$$T \equiv \begin{pmatrix} 0 & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{odd}(1|1)\,. \tag{5}$$

The name of the odd-reduced supermatrices follows naturally from $\mathrm{Ber}T = \beta\alpha/b^2 \Rightarrow (\mathrm{Ber}T)^2 = 0$ and

$$\mathrm{Ber}T^2 = \mathrm{Ber}\begin{pmatrix} \alpha\beta & \alpha b \\ \beta b & b^2 - \alpha\beta \end{pmatrix} = 0. \tag{6}$$

The explanation of the ground of the notations S and T comes from the fact that the even-reduced supermatrices give superconformal transformations which describe morphisms of the tangent bundle over the super Riemann surfaces [19], while the odd-reduced supermatrices give the superconformal-like transformations twisting the parity of the $(1|1)$ tangent superspace in the standard basis (see [14, 15]).

Assertion 2 $\mathbf{M}$ *is a direct sum of diagonal* $\mathbf{D}$ *and anti-diagonal (secondary diagonal)* $\mathbf{A}$ *supermatrices (the even and odd ones in the notations of [4])*

$$\mathbf{M} = \mathbf{D} \oplus \mathbf{A}, \tag{7}$$

where

$$D \equiv \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathbf{D} \equiv \mathrm{Mat}_\Lambda^{Diag}(1|1)\,, \tag{8}$$

$$A \equiv \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in \mathbf{A} \equiv \mathrm{Mat}_\Lambda^{Adiag}(1|1)\,, \tag{9}$$

and $\mathbf{D} \subset \mathbf{S}$ *and* $\mathbf{A} \subset \mathbf{T}$.

For the reduced supermatrices one finds

$$\mathbf{S} \cap \mathbf{T} = \begin{pmatrix} 0 & \alpha \\ 0 & b \end{pmatrix} \neq \emptyset. \tag{10}$$

Nevertheless, the following observation explains the fundamental role of $\mathbf{S}$ and $\mathbf{T}$.

Proposition 3 *The Berezians of even- and odd-reduced supermatrices are additive components of the full Berezinian*

$$\mathrm{Ber} M = \mathrm{Ber} S + \mathrm{Ber} T. \tag{11}$$

The first term in (11) covers all subgroups of even-reduced supermatrices from $\mathrm{Mat}_{\Lambda}(1|1)$, and only it was considered in the applications. But the second term is dual to the first in some sense and corresponds to all subsemigroups of odd-reduced supermatrices from $\mathrm{Mat}_{\Lambda}(1|1)$[4].

4 Invertibility and ideals of $\mathrm{Mat}_{\Lambda}(1|1)$

Denote the set of invertible elements of $\mathbf{M}$ by $\mathbf{M}^*$, and $\mathbf{I} = \mathbf{M} \setminus \mathbf{M}^*$. In [4] it was proved that $\mathbf{M}^* = \{M \in \mathbf{M} | \, \mathfrak{m}_b(a) \neq 0 \wedge \mathfrak{m}_b(b) \neq 0\}$. Then similarly $\mathbf{S}^* = \{S \in \mathbf{S} | \, \mathfrak{m}_b(a) \neq 0 \wedge \mathfrak{m}_b(b) \neq 0\}$ and $\mathbf{T}^* = \emptyset$, i. e. the odd-reduced matrices are noninvertible and $\mathbf{T} \subset \mathbf{I}$. Consider the invertibility structure of $\mathrm{Mat}_{\Lambda}(1|1)$ in more detail. Let us denote

$$\begin{array}{rcl} \mathbf{M}' & = & \{M \in \mathbf{M} | \, \mathfrak{m}_b(a) \neq 0\}, \\ \mathbf{M}'' & = & \{M \in \mathbf{M} | \, \mathfrak{m}_b(b) \neq 0\}, \\ \mathbf{I}' & = & \{M \in \mathbf{M} | \, \mathfrak{m}_b(a) = 0\}, \\ \mathbf{I}'' & = & \{M \in \mathbf{M} | \, \mathfrak{m}_b(b) = 0\}. \end{array} \tag{12}$$

Then $\mathbf{M} = \mathbf{M}' \cup \mathbf{I}' = \mathbf{M}'' \cup \mathbf{I}''$ and $\mathbf{M}' \cap \mathbf{I}' = \emptyset$, $\mathbf{M}'' \cap \mathbf{I}'' = \emptyset$, therefore $\mathbf{M}^* = \mathbf{M}' \cap \mathbf{M}''$ and $\mathbf{T} \subset \mathbf{M}''$. The Berezinian $\mathrm{Ber} M$ is well-defined for the matrices from $\mathbf{M}''$ only and is invertible when $M \in \mathbf{M}^*$, but for the matrices from $\mathbf{M}'$ the inverse $(\mathrm{Ber} M)^{-1}$ is well-defined and is invertible when $M \in \mathbf{M}^*$ too [4].

Under the ordinary matrix multiplication the set $\mathbf{M}$ is a semigroup of all $(1|1)$ supermatrices [35, 39, 40], and the set $\mathbf{M}^*$ is a subgroup of $\mathbf{M}$. In the standard basis $\mathbf{M}^*$ represents the general linear group $GL_{\Lambda}(1|1)$ [4]. According to the general definitions [8] a subset $\mathbf{I} \subset \mathbf{M}$ is called a right (left) ideal of the semigroup $\mathbf{M}$, if $\mathbf{I} \cdot \mathbf{M} \subset \mathbf{I}$ ($\mathbf{M} \cdot \mathbf{I} \subset \mathbf{I}$), where the point denotes the standard matrix set multiplication: $\mathbf{A} \cdot \mathbf{B} \overset{def}{=} \{\bigcup AB | \, A \in \mathbf{A}, \, B \in \mathbf{B}\}$.

[4] The relation (11) is a supersymmetric version of the obvious equality $\det M = \det D + \det A$, when D and A from (7) and (8) are ordinary matrices. The problem is that for A being a supermatrix $\mathrm{Ber}\, A$ is not defined at all.

An isolated ideal satisfies the relation [8]

$$M_1 M_2 \in \mathbf{I} \Rightarrow M_1 \in \mathbf{I} \vee M_2 \in \mathbf{I}, \tag{13}$$

and a filter $\mathbf{F}$ of the semigroup $\mathbf{M}$ is defined by

$$M_1 M_2 \in \mathbf{F} \Rightarrow M_1 \in \mathbf{F} \wedge M_2 \in \mathbf{F}. \tag{14}$$

Proposition 4 *1) The sets $\mathbf{I}$, $\mathbf{I}'$ and $\mathbf{I}'$ are isolated ideals of $\mathbf{M}$.*

2) The sets $\mathbf{M}^*$, $\mathbf{M}^{'}$ and $\mathbf{M}^{''}$ are filters of the semigroup $\mathbf{M}$.

3) The sets $\mathbf{M}'$ and $\mathbf{M}''$ are not subgroups[5], but subsemigroups of $\mathbf{M}$, which are $\mathbf{M}' = \mathbf{M}^* \cup \mathbf{J}'$ and $\mathbf{M}'' = \mathbf{M}^* \cup \mathbf{J}''$ with the isolated ideals $\mathbf{J}' = \mathbf{M}' \setminus \mathbf{M}^* = \mathbf{M}' \cap \mathbf{I}''$ and $\mathbf{J}'' = \mathbf{M}'' \setminus \mathbf{M}^* = \mathbf{M}'' \cap \mathbf{I}'$ respectively.

4) The ideal of the semigroup $\mathbf{M}$ is

$$\mathbf{I} = \mathbf{I}' \cup \mathbf{J}' = \mathbf{I}'' \cup \mathbf{J}''. \tag{15}$$

Proof. Let $M_3 = M_1 M_2$, then $a_3 = a_1 a_2 + \alpha_1 \beta_2$ and $b_3 = b_1 b_2 + \beta_1 \alpha_2$. Taking the body part we derive

$$\mathfrak{m}_b(a_3) = \mathfrak{m}_b(a_1)\, \mathfrak{m}_b(a_2), \tag{16}$$

$$\mathfrak{m}_b(b_3) = \mathfrak{m}_b(b_1)\, \mathfrak{m}_b(b_2). \tag{17}$$

1) The left-hand side of (16) and (17) vanishes iff the first or second multiplier of the right-hand side equals zero. Then use (13).

2) The left-hand side of (16) and (17) does not vanish iff the first and second multiplier of the right-hand side does not equal zero. Then use (14).

3) $\mathbf{J}'$ consists of the matrices with $\mathfrak{m}_b(a) \neq 0$, but $\mathfrak{m}_b(b) = 0$, and $\mathbf{J}''$ consists of the matrices with $\mathfrak{m}_b(a) = 0$, but $\mathfrak{m}_b(b) \neq 0$.

4) The ideal of $\mathbf{M}$ consists of the matrices with $\mathfrak{m}_b(a) = 0$ or $\mathfrak{m}_b(b) = 0$. Then use 3) and the definitions. □

Remark. Since the ideals $\mathbf{J}'$ and $\mathbf{J}''$ are isolated, i. e. $\mathbf{J}' \cap \mathbf{M}^* = \mathbf{J}' \cap \mathbf{M}^* = \emptyset$, they cannot be represented as sequences of elements from the group $\mathbf{M}^*$

[5] as it was incorrectly translated in [4], pp. 95, 103 (in the original Russian edition, Moscow, 1983, pp. 89, 93, the sets $\mathbf{M}'$ and $\mathbf{M}''$, denoted as $G'Mat(1,1|\Lambda)$ and $G''Mat(1,1|\Lambda)$ respectively, are called semigroups).

(viz. no one noninvertible element can be derived from a sequence of invertible ones, see, e. g. [27]), and so the statement "that any element of $G''Mat\,(p,q|\Lambda)$ (the semigroup $\mathbf{M}''$ here, and so the notation G''... misleads) is the limit of a sequence from $GMat\,(p,q|\Lambda)$ (the group $\mathbf{M}^{*}$ here)" holds only for invertible elements from $\mathbf{M}''$, i.e. belonging $\mathbf{M}^{*}$, and it means that elements from $\mathbf{M}^{*}$ can be obtained from a sequence of elements from $\mathbf{M}^{*}$, which is simply a group action.

Assertion 5 *For the odd-reduced matrices from (12) it follows* $\mathbf{T}\subset\mathbf{I}'$ *and* $\mathbf{A}\cap\mathbf{M}'=\mathbf{A}\cap\mathbf{M}''=\emptyset$*.*

5 Multiplication properties of odd-reduced supermatrices

In general the odd-reduced matrices do not form a semigroup, since

$$T_1T_2=\begin{pmatrix}\alpha_1\beta_2 & \alpha_1b_2\\ b_1\beta_2 & b_1b_2+\beta_1\alpha_2\end{pmatrix}\neq T. \tag{18}$$

But from (18) it follows that

$$\begin{array}{lcl}\mathbf{T}\cdot\mathbf{T}\cap\mathbf{T}\neq\emptyset & \Rightarrow & \alpha\beta=0,\\ \mathbf{T}\cdot\mathbf{T}\cap\mathbf{S}\neq\emptyset & \Rightarrow & \beta b=0,\end{array} \tag{19}$$

which can take place, because of the existence of zero divisors in Λ.

Proposition 6 *1) The subset* $\mathbf{T}^{SG}\subset\mathbf{T}$ *of the odd-reduced matrices satisfying* $\alpha\beta=0$ *form an odd-reduced subsemigroup of* $\mathbf{M}$*.*

2) In the odd-reduced semigroup $\mathbf{T}^{SG}$ *the subset of matrices with* $\beta=0$ *is a left ideal, and one with* $\alpha=0$ *is a right ideal, the matrices with* $b=0$ *form a two-sided ideal.*

5.1 Semigroup band representations

Let

$$Z_{\alpha}\,(t)=\begin{pmatrix}0 & \alpha t\\ \alpha & 1\end{pmatrix}\in\mathbf{Z}_{\alpha}\subset\mathbf{T}^{SG}, \tag{20}$$

i.e. $\mathbf{Z}_\alpha$ is a set of the odd-reduced matrices parameterized by the even parameter $t \in \Lambda_0$. Then $\mathbf{Z}_\alpha$ is a semigroup under the matrix multiplication (α numbers the semigroups) which is isomorphic to a one parameter semigroup with the multiplication

$$\{t_1\} *_\alpha \{t_2\} = \{t_1\}. \tag{21}$$

This semigroup is called a right zero semigroup $\mathcal{Z}_R = \{\bigcup \{t\}\,; *_\alpha\}$ and plays an important role (together with the left zero semigroup $\mathcal{Z}_L$ defined in a dual manner) in the general semigroup theory (e.g., see [8], Theorem 1.27, and [25]).

Let

$$B_\alpha(t,u) = \begin{pmatrix} 0 & \alpha t \\ \alpha u & 1 \end{pmatrix} \in \mathbf{B}_\alpha \subset \mathbf{T}^{SG}, \tag{22}$$

then $\mathbf{B}_\alpha$ is a matrix semigroup (numbered by α) which is isomorphic to a two Λ_0-parametric semigroup $\mathcal{B} = \{\bigcup \{t,u\}\,; *_\alpha\}$, where the multiplication is

$$\{t_1, u_1\} *_\alpha \{t_2, u_2\} = \{t_1, u_2\}. \tag{23}$$

Here every element is an idempotent (as in the previous case too), and so this is a rectangular band multiplication [25, 38].

Let $C_\alpha(t,u,v) = \begin{pmatrix} 0 & \alpha t \\ \alpha u & v \end{pmatrix} \in \mathbf{C}_\alpha \subset \mathbf{T}^{SG}$, then $\mathbf{C}_\alpha$ is a matrix semigroup isomorphic to a semigroup $\mathcal{B}_G = \{\bigcup \{t,u,v\}\,; *_\alpha\}$ where the multiplication is

$$\{t_1, u_1, v_1\} *_\alpha \{t_2, u_2, v_2\} = \{t_1 v_2, u_2 v_1, v_1 v_2\}. \tag{24}$$

The parameter v describes the difference of an element from an idempotent, since $\{t,u,v\}^2 - \{t,u,v\} = \{t(v-1), u(v-1), v(v-1)\}$.

Assertion 7 *The one and two parametric subsemigroups of the semigroup of odd-reduced supermatrices $\mathbf{T}^{SG}$ having vanishing Berezinian represent semigroup bands, viz. the left and right zero semigroups and rectangular bands.*

Theorem 8 *The continuous supermatrix representation of the Rees matrix semigroup over a unit group $G = e$ (see [8, 25]) is given by formulas (20) and (22).*

5.2 "Square root" of even-reduced supermatrices

Consider the second equation in (19).

Proposition 9 *The elements* $T^{\sqrt{S}}$*from the subset* $\mathbf{T}^{\sqrt{S}} \subset \mathbf{T}$ *of the odd-reduced matrices satisfying* $\beta b = 0$ *can be interpreted as "square roots" of the even-reduced matrices* S.

Example. 1) Let $T^{\sqrt{S}} = \begin{pmatrix} 0 & \alpha \\ \beta & \beta\gamma \end{pmatrix} \in \mathbf{T}^{\sqrt{S}}$, then $\left(T^{\sqrt{S}}\right)^2 = \alpha\beta \begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix} \in \mathbf{S} \setminus \mathbf{S}^*$.

2) If $\gamma = 0$ in 1), then we obtain $A^2 = \alpha\beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{D} \setminus \mathbf{D}^*$ (see (8), (9) and compare with $D^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$). This could be accepted as a definition of a square root of $\alpha\beta$ in some sense. Thus we have

$$\begin{aligned} \mathbf{D} \cdot \mathbf{D} &= \mathbf{D}, \\ \mathbf{A} \cdot \mathbf{A} &= \mathbf{D}, \end{aligned} \tag{25}$$

and the second relation could be formally considered as an "odd branch" of the root $\sqrt{D}$.

6 Unification of reduced supermatrices

Now we try to unify the even- and odd-reduced matrices (4) and (5) into a common abstract object. To begin with consider the multiplication table of all introduced sets including the even-reduced matrices products

$$\begin{aligned} \mathbf{S} \cdot \mathbf{S} &= \mathbf{S}, \\ \mathbf{D} \cdot \mathbf{D} &= \mathbf{D}, \\ \mathbf{D} \cdot \mathbf{S} &= \mathbf{S}, \\ \mathbf{S} \cdot \mathbf{D} &= \mathbf{S}, \end{aligned} \tag{26}$$

and ones for the odd-reduced matrices

$$\begin{array}{rcl} \mathbf{A}\cdot\mathbf{T} & = & \mathbf{S}, \\ \mathbf{A}\cdot\mathbf{S} & = & \mathbf{T}, \\ \mathbf{T}\cdot\mathbf{A} & = & \mathbf{S}^{st}, \\ \mathbf{S}\cdot\mathbf{A} & = & \mathbf{T}^{\Pi}, \\ \mathbf{S}\cdot\mathbf{T} & = & \mathbf{S}\cup\mathbf{T} \\ \mathbf{T}\cdot\mathbf{S} & = & \mathbf{T}. \end{array} \tag{27}$$

Here $st : \mathrm{Mat}_{\Lambda}(1|1) \to \mathrm{Mat}_{\Lambda}(1|1)$ is a supertranspose [4], i. e. $\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{st} = \begin{pmatrix} a & \beta \\ -\alpha & b \end{pmatrix}$. Also we use the Π-transpose [34] defined by Π: $\mathrm{Mat}_{\Lambda}(1|1) \to \mathrm{Mat}_{\Lambda}(1|1)$ and

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{\Pi} = \begin{pmatrix} b & \beta \\ \alpha & a \end{pmatrix}. \tag{28}$$

Note that the sets of matrices $\mathbf{S}$ and $\mathbf{T}$ are not closed under st and Π operations, but $\mathbf{S}^{st}\cap\mathbf{S} = \mathbf{D}$ and $\mathbf{T}^{\Pi}\cap\mathbf{T} = \mathbf{A}$.

First we observe from the first two relations of (27) that $\mathbf{A}$ plays a role of the left type-changing operator $\mathbf{A} : \mathbf{S} \to \mathbf{T}$ and $\mathbf{A} : \mathbf{T} \to \mathbf{S}$, while $\mathbf{D}$ does not change the type. Next from the first two relations of (26) it is obviously seen that the sets $\mathbf{S}$ and $\mathbf{D}$ are subsemigroups. Unfortunately, due to the next to last relation of (27) the set $\mathbf{T}$ has no clear abstract meaning. However, the last relation $\mathbf{T}\cdot\mathbf{S} = \mathbf{T}$ is important from another viewpoint: any odd-reduced morphism $\Lambda^{1|1} \to \Lambda^{1|1}$ corresponding to $\mathbf{T}$ can be represented as a product of odd- and even-reduced morphisms, such that

$$\begin{tikzcd} {} \arrow[r, "\mathbf{S}"] \arrow[dr, "\mathbf{T}"'] & {} \arrow[d, "\mathbf{T}"] \\ & {} \end{tikzcd} \tag{29}$$

is a commutative diagram. This decomposition is crucial in the application to the superconformal-like transformations construction (see [14]).

6.1 Reduced supermatrix set semigroup

To unify the introduced sets (26) and (27) we consider the triple products

$$\begin{array}{rcl} \mathbf{S}\cdot\mathbf{A}\cdot\mathbf{T} & = & \mathbf{S}, \\ \mathbf{T}\cdot\mathbf{A}\cdot\mathbf{T} & = & \mathbf{T}, \\ \mathbf{S}\cdot\mathbf{D}\cdot\mathbf{S} & = & \mathbf{S}, \\ \mathbf{T}\cdot\mathbf{D}\cdot\mathbf{S} & = & \mathbf{T}. \end{array} \tag{30}$$

Here we observe that the matrices $\mathbf{A}$ and $\mathbf{D}$ play the role of "sandwich" elements in a special $\mathbf{S}$ and $\mathbf{T}$ multiplication. Moreover, the sandwich elements are in one-to-one correspondence with the right sets on which they act, and so they are "sensible from the right". Therefore, it is quite natural to introduce the following

Definition 10 *A sandwich right sensible product of the reduced supermatrix sets* $\mathbf{R} = \mathbf{S}, \mathbf{T}$ *is*

$$\mathbf{R}_1 \odot \mathbf{R}_2 \stackrel{def}{=} \begin{cases} \mathbf{R}_1\cdot\mathbf{D}\cdot\mathbf{R}_2, & \mathbf{R}_2 = \mathbf{S}, \\ \mathbf{R}_1\cdot\mathbf{A}\cdot\mathbf{R}_2, & \mathbf{R}_2 = \mathbf{T}. \end{cases} \tag{31}$$

In terms of the sandwich product instead of (30) we obtain

$$\begin{array}{rcl} \mathbf{S}\odot\mathbf{T} & = & \mathbf{S}, \\ \mathbf{T}\odot\mathbf{T} & = & \mathbf{T}, \\ \mathbf{S}\odot\mathbf{S} & = & \mathbf{S}, \\ \mathbf{T}\odot\mathbf{S} & = & \mathbf{T}. \end{array} \tag{32}$$

Proposition 11 *The $\odot$-multiplication is associative.*

Proof. Consider the relations:

$$\begin{array}{rcccl} (\mathbf{T}\odot\mathbf{S})\odot\mathbf{T} & = & (\mathbf{T}\cdot\mathbf{D}\cdot\mathbf{S})\cdot\mathbf{A}\cdot\mathbf{T} & = & \mathbf{T}\cdot\mathbf{D}\cdot\mathbf{S}\cdot\mathbf{A}\cdot\mathbf{T}, \\ \mathbf{T}\odot(\mathbf{S}\odot\mathbf{T}) & = & \mathbf{T}\cdot\mathbf{D}\cdot(\mathbf{S}\cdot\mathbf{A}\cdot\mathbf{T}) & = & \mathbf{T}\cdot\mathbf{D}\cdot\mathbf{S}\cdot\mathbf{A}\cdot\mathbf{T}, \end{array} \tag{33}$$

where the last equalities follow from the associativity of the ordinary matrix multiplication. Therefore, $(\mathbf{T}\odot\mathbf{S})\odot\mathbf{T} = \mathbf{T}\odot(\mathbf{S}\odot\mathbf{T})$. Other associativity relations can be proved in a similar way[6]. □

[6] We stress here that the associativity does not follow from the associativity of the supermatrix multiplication only, but is a consequence of the special and nontrivial set multiplication table (32).

Definition 12 *The elements* $\mathbf{S}$ *and* $\mathbf{T}$ *form a semigroup under* $\odot$*-multiplication (31), which we call a reduced matrix set semigroup and denote* $\mathcal{RMS}_{set}$.

Comparing (21) and (32) we observe that the reduced matrix set semigroup can be viewed as a right zero semigroup having two elements.

Assertion 13 *The reduced matrix set semigroup is isomorphic to a special right zero semigroup, i. e.* $\mathcal{RMS}_{set} \cong \mathcal{Z}_R = \{\mathbf{R} = \mathbf{S}, \mathbf{T}; \odot\}$.

6.2 Scalars, anti-scalars and generalized modules

Now we introduce the analog of $\odot$-multiplication for the reduced matrices per se (not for sets). First we define the structure of generalized Λ-module in $\mathrm{Hom}_0\left(\Lambda^{1|1}, \Lambda^{1|1}\right)$ in some alternative way, the even part of which is described in [32] (in the ordinary matrix theory this is a trivial fact that the product of a matrix and a number is equal to a product of a matrix and a diagonal matrix having this number on the diagonal).

Definition 14 *In* $\mathrm{Mat}_\Lambda(1|1)$ *a scalar (matrix)* $E(x)$ *and anti-scalar (matrix)* $\mathcal{E}(\chi)$ *are defined by*

$$\begin{aligned} E(x) &\overset{def}{=} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \mathbf{D} = \mathrm{Mat}_\Lambda^{Diag}(1|1), \ x \in \Lambda_{0,} \\ \mathcal{E}(\chi) &\overset{def}{=} \begin{pmatrix} 0 & \chi \\ \chi & 0 \end{pmatrix} \in \mathbf{A} = \mathrm{Mat}_\Lambda^{Adiag}(1|1), \ \chi \in \Lambda_{1.} \end{aligned} \tag{34}$$

Assertion 15 *The Berezin's queer subalgebra* $Q_\Lambda(1) \equiv \begin{pmatrix} x & \chi \\ \chi & x \end{pmatrix} \subset \mathrm{Mat}_\Lambda(1|1)$ *[4] is a direct sum of the scalar and anti-scalar*

$$Q_\Lambda(1) = E(x) \oplus \mathcal{E}(\chi). \tag{35}$$

Assertion 16 *The anti-scalars anticommute* $\mathcal{E}(\chi_1)\mathcal{E}(\chi_2) + \mathcal{E}(\chi_2)\mathcal{E}(\chi_1) = 0$, *and so they are nilpotent.*

Proposition 17 *The structure of the generalized* $\Lambda_0 \oplus \Lambda_1$*-module in* $\mathrm{Hom}_0\left(\Lambda^{1|1}, \Lambda^{1|1}\right)$ *is defined by action of the scalars and anti-scalars (34).*

This means that everywhere we exchange the multiplication of supermatrices by even and odd elements from Λ with the multiplication by the scalar matrices and anti-scalar ones (34). The relations containing the scalars are well-known [32], but for the anti-scalars we obtain new dual ones. Consider their action on elements $\mathbf{M} \in \mathrm{Mat}_{\Lambda}(1|1)$ in more detail. First we need

Definition 18 *Left $\mathcal{P}$ and right $\mathcal{Q}$ anti-transpose are* $\mathrm{Hom}_0\left(\Lambda^{1|1}, \Lambda^{1|1}\right) \to \mathrm{Hom}_1\left(\Lambda^{1|1}, \Lambda^{1|1}\right)$ *mappings acting on $M \in \mathbf{M}$ as*

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{\mathcal{P}} = \begin{pmatrix} \beta & b \\ a & \alpha \end{pmatrix}, \tag{36}$$

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{\mathcal{Q}} = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}. \tag{37}$$

Corollary 19 *The anti-transpose is a square root of the parity changing operator (28) in the following sense*

$$\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \Pi. \tag{38}$$

Assertion 20 *The anti-transpose satisfy*

$$\begin{array}{rcl} \left(\mathcal{E}\left(\chi\right) M\right)^{\mathcal{P}} & = & \chi M \\ \left(\mathcal{E}\left(\chi\right) M\right)^{\mathcal{Q}} & = & \chi M^{\Pi} \\ \left(M\mathcal{E}\left(\chi\right)\right)^{\mathcal{P}} & = & M^{\Pi}\chi \\ \left(M\mathcal{E}\left(\chi\right)\right)^{\mathcal{Q}} & = & M\chi \end{array} \tag{39}$$

Thus the concrete realization of the right, left and two-sided generalized $\Lambda_0 \oplus \Lambda_1$-modules in $\mathrm{Hom}_0\left(\Lambda^{1|1}, \Lambda^{1|1}\right)$ is determined by the actions

$$\begin{array}{rcl} \mathcal{E}\left(\chi\right) M & = & \chi M^{\mathcal{P}}, \\ M\mathcal{E}\left(\chi\right) & = & M^{\mathcal{Q}}\chi, \\ \mathcal{E}\left(\chi_1\right) M\mathcal{E}\left(\chi_2\right) & = & \chi_1 M^{\Pi}\chi_2, \end{array} \tag{40}$$

together with the standard Λ-module structure [32]

$$\begin{array}{rcl} E\left(x\right) M & = & xM, \\ ME\left(x\right) & = & Mx, \\ E\left(x_1\right) ME\left(x_2\right) & = & x_1 M x_2. \end{array} \tag{41}$$

Corollary 21 *The generalized $\Lambda_0 \oplus \Lambda_1$-module relations are*

$$\begin{array}{rcl} (E(x) M) N & = & E(x)(MN) \\ (ME(x)) N & = & M(E(x) N) \\ M(NE(x)) & = & (MN) E(x) \\ (\mathcal{E}(\chi) M) N & = & \mathcal{E}(\chi)(MN) \\ (M\mathcal{E}(\chi)) N & = & M(\mathcal{E}(\chi) N) \\ M(N\mathcal{E}(\chi)) & = & (MN)\mathcal{E}(\chi) \end{array} \tag{42}$$

where $M, N \in \mathrm{Mat}_\Lambda(1|1)$.

Proposition 22 *The structure of the generalized $\Lambda_0 \oplus \Lambda_1$-module in $\mathrm{Hom}_1\left(\Lambda^{1|1}, \Lambda^{1|1}\right)$ is determined by the analogous actions of* odd scalar

$$E(\chi) \stackrel{def}{=} \left(\begin{array}{cc} \chi & 0 \\ 0 & -\chi \end{array}\right) \in \mathrm{Hom}_1\left(\Lambda^{1|1}, \Lambda^{1|1}\right) \tag{43}$$

and odd anti-scalar

$$\mathcal{E}(x) \stackrel{def}{=} \left(\begin{array}{cc} 0 & x \\ x & 0 \end{array}\right) \in \mathrm{Hom}_1\left(\Lambda^{1|1}, \Lambda^{1|1}\right) \tag{44}$$

respectively[7].

6.3 Reduced supermatrix sandwich semigroup

One way to unify the even- (4) and odd-reduced (5) supermatrices into an object analogous to a semigroup is consideration of the sandwich multiplication similar to (31), but on the level of matrices (not sets), by means of the scalars and anti-scalars as sandwich matrices. Indeed, the ordinary matrix product can be written as $M_1 M_2 = M_1 E(1) M_2$. But we cannot find an analog of this relation using anti-scalar, because among $\chi \in \Lambda_1$ there is no unity. Therefore, the only possibility to include $\mathcal{E}(\chi)$ into equal play is consideration of sandwich elements (34) having arbitrary (or fixed by other special conditions) both arguments x and χ. Thus we naturally come to

[7]¿From (43) and (44) it is clear why we use the name "anti-" and not "odd" (as in [4]) for the secondary diagonal matrices $\mathcal{E}(\chi)$.

Definition 23 *A sandwich right sensible* $\Lambda_0 \oplus \Lambda_1$*-product of the reduced supermatrices* $R = S$,T *is*

$$R_1 \star_X R_2 \stackrel{def}{=} \begin{cases} R_1 E(x) R_2, & R_2 = S, \\ R_1 \mathcal{E}(\chi) R_2, & R_2 = T, \end{cases} \tag{45}$$

where $X = \{x, \chi\} \in \Lambda_0 \oplus \Lambda_1$.

The $\star_X$-multiplication table coincides with (32). The associativity can be proved similar to (33). Therefore, we have

Proposition 24 *Under* $\Lambda_0 \oplus \Lambda_1$*-multiplication the reduced matrices form a semigroup which we call* a reduced matrix sandwich semigroup $\mathcal{RMSS}$.

Assertion 25 *The reduced matrix sandwich semigroup is isomorphic to a special right zero semigroup, i. e.* $\mathcal{RMSS} \cong \mathcal{Z}_R = \{R = \bigcup S \bigcup T; \star_X\}$.

6.4 Direct sum of reduced supermatrices

Another way to unify the reduced supermatrices is consideration of the connection between them and the generalized $\Lambda_0 \oplus \Lambda_1$-modules.

Definition 26 *The reduced supermatrix direct space* $\mathcal{RMDS}$ *is a direct sum of the even-reduced supermatrix space and the odd-reduced one.*

In terms of sets we have $\mathbf{R}_{\oplus} = \mathbf{S} \oplus \mathbf{T}$.

Assertion 27 *In* $\mathcal{RMDS}$ *the scalar is the Berezin's queer subalgebra* $Q_\Lambda(1)$ *(see (35)).*

Theorem 28 *In* $\mathcal{RMDS}$ *the scalars play the same role for the even-reduced supermatrices, as the anti-scalars for the odd-reduced ones.*

Corollary 29 *The eigenvalues of even- (4) and odd-reduced (5) supermatrices should be found from different equations, viz.*

$$\begin{aligned} SV &= E(x) V, \\ TV &= \mathcal{E}(\chi) V, \end{aligned} \tag{46}$$

where V *is a column vector, and they are*

$$\begin{aligned} x_1 = a, &\quad x_2 = b, \\ \chi_1 = \alpha, &\quad \chi_2 = \beta. \end{aligned} \tag{47}$$

(see (4) and (5)).

Definition 30 *The characteristic functions for even- and odd-reduced supermatrices are defined in $\mathcal{RMDS}$ by*

$$\begin{array}{rcl} H_S^{even}(x) & = & \operatorname{Ber}(E(x)-S), \\ H_T^{odd}(\chi) & = & \operatorname{Ber}(\mathcal{E}(\chi)-T). \end{array} \tag{48}$$

Remark. In the standard Λ-module over $\mathrm{Mat}_\Lambda(1|1)$ [4] one derives characteristic functions and eigenvalues for any matrix (and for odd-reduced too) from the first equations in (46) and (48) and obtains different results (see, e. g. [31, 47]).

Using (4), (5) we easily found

$$\begin{array}{rcl} H_S^{even}(x) & = & \dfrac{(x-a)(x-b)}{(x-b)^2}, \\ H_T^{odd}(\chi) & = & \dfrac{(\chi-\alpha)(\chi-\beta)}{b^2}. \end{array} \tag{49}$$

Here we observe the full symmetry between even- and odd-reduced supermatrices (for this purpose the cancellation in the first equation was avoided) and consistency with their $\Lambda_0 \oplus \Lambda_1$-eigenvalues (47).

The characteristic polynomial[8] of a supermatrix M is defined by $P_M(M) = 0$ and in complicated cases is constructed from the parts of the characteristic function $H_M(x)$ according to a special algorithm [31, 47]. Due to existence of zero divisors in Λ the degree of $P_M(x)$ can be less than $n = p+q$, $M \in \mathrm{Mat}_\Lambda(p|q)$. But this algorithm is not applicable for the odd-reduced and secondary diagonal supermatrices. As before, we introduce two dual characteristic polynomials and, using (49), obtain the Cayley-Hamilton theorem in $\mathcal{RMDS}$.

Theorem 31 (The generalized Cayley-Hamilton theorem) *The characteristic polynomials in the reduced supermatrix direct space are*

$$\begin{array}{rcl} P_S^{even}(x) & = & (x-a)(x-b), \\ P_T^{odd}(\chi) & = & (\chi-\alpha)(\chi-\beta). \end{array} \tag{50}$$

and $P_S^{even}(S) = 0$ for any S, but $P_T^{odd}(T) = 0$ for nilpotent b only.

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[8] For a nonsupersymmetric matrix M it evidently coincides with the characteristic function $P_M(x) = H_M(x) \equiv \det(Ix - M)$, where I is a unity matrix.

Proof. The even case is well-known, but for clarity we repeat it too, demonstrating the avoiding of multiplication of a matrix by a constant and using instead the scalars and anti-scalars (34), i. e. the introduced $\Lambda_0 \oplus \Lambda_1$-module structure. Thus, considering simultaneously the even and odd cases we obtain

$$P_S^{even}(S) = (S - E(a))(S - E(b)) = \begin{pmatrix} 0 & \alpha \\ 0 & b \end{pmatrix} \begin{pmatrix} a & \alpha \\ 0 & 0 \end{pmatrix} = 0, \tag{51}$$

$$\begin{gathered} P_T^{odd}(T) = (T - \mathcal{E}(\alpha))(T - \mathcal{E}(\beta)) = \\ \begin{pmatrix} 0 & 0 \\ \beta - \alpha & b \end{pmatrix} \begin{pmatrix} 0 & \alpha - \beta \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b^2 \end{pmatrix} = 0. \end{gathered} \tag{52}$$

□

7 Conclusions

We conclude that almost all above constructions are universal and ideas mostly do not depend on size of the supermatrices under consideration. In particular case of superconformal-like transformations it would be interesting to use the alternative reduction introduced here in building the objects analogous to super Riemann or semirigid surfaces, which can also lead to new topological-like models.

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