New Methods and Structures in the Theory of the Multi-Mode Dicke Laser Model

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Abstract

We base a treatment of the dissipative, multi-mode version of Dicke laser model on the theory of completely positive dynamical semigroups and quantum Markov processes. This leads to new results at both the mathematical and physical levels. On the physical side, it provides a generalisation of the Hepp-Lieb model that admits both chaotic and polychromatic laser radiation. On the mathematical side, it extends the theory of dynamical semigroups to a regime where the generators are perturbed by unbounded derivations.

PACS numbers: 02.20.Mp, 03.65.Db, 05.45.+b, 42.55.Ah

I. Introduction

In a seminal work on constructive non-equilibrium statistical mechanics of open systems, Hepp and Lieb[1] (HL) proved that the Dicke laser model, equipped with optical pumps and reservoirs, undergoes a phase transition from normal to coherent radiation when the pumping strength reaches a critical value. This result substantiated earlier, more heuristic, ideas of Graham and Haken[2,3], concerning phase transitions far from equilibrium. In a more general setting, subsequent works on the mathematical structure of open quantum systems represented their dynamics by completely positive (CP) one-parameter contractive Markovian semigroups[4−7], which, under suitable conditions, possess minimal dilations to quantum Markov processes[8−10]. Evidently, models of the HL type, that are governed by singular couplings, are *prima facie* candidates for such processes.

In the present article, we provide a new, quantum stochastic treatment of the *multi-mode* dynamical Dicke laser, or maser, model, with the aim of extending both the physical picture of HL and the theory of perturbations of CP semigroups. We note that, since this is a non-equilibrium model, driven by pumps and sinks, it is radically different from both the multi-mode equilibrium model of ref. 11 and the internally driven dynamical ones of refs. 12 and 13. Our main results may be summarised as follows. On the physical side, we show that the model undergoes transitions, far from equilibrium, to phases of chaotic and polychromatic, as well as monochromatic, laser radiation. Thus, it provides the framework for a theory comprising a rich variety of radiative structures. On the mathematical side, we extend the perturbation theory of one-parameter CP semigroups[14,15] to a regime where the generators are perturbed by *unbounded* *-derivations of the algebra of observables.

Our model, $\Sigma^{(N,n)}$, consists of N two-level atoms coupled to n radiation modes, each atom and mode interacting also with reservoirs that serve to provide optical pumping and damping. The physics represented by this model is thus essentially the same as that of HL, though our mathematical formulation is quite different from theirs. For, whereas the HL model was a *conservative* system, comprising the atoms, radiation and reservoirs, with their various interactions, the present model is an *open dissipative* one, in which the action of the reservoirs is incorporated into the structure of the dynamical semigroup, governing the evolution of the matter and radiation only. This formulation enables us to employ the technical machinery of the theories of CP semigroups and quantum stochastic processes.

We shall present the main structure of the theory in Sections II-IV, leaving the technical constructions and proofs of Propositions to Sections V and VI. Thus, in Section II, we shall formulate the model $\Sigma^{(N,n)}$ as a $W^\star-$dynamical system, whose evolution is governed by a CP semigroup of contractions of its algebra of observables. This semigroup is constructed from those of the component parts of the system, and the microdynamics it engenders is specified by Props. 2.3 and 2.4. It will be seen that it provides a concrete example of a non-quasi-free semigroup with unbounded generator.

In Section III, we specify the macroscopic description of the model. Our main results here are given by Props. 3.4 and 3.5, which serve to reduce the macro-dynamics of the model to a classical deterministic form, in the limit where N tends to infinity, with n fixed and finite. In fact, our resultant phenomenological equations constitute a generalisation of those of

HL.

Section IV encapsulates the physical content of this article at the macroscopic level. Here, we show that the phenomenological equations yield dynamical phase transitions, not only from incoherent to coherent monochromatic radiation, but also to more complex phases of optical chaos and multi-mode activation, which have been discussed in the optics literature[3,16–18].

Section V is devoted to the explicit construction of the dynamical semigroup of the model $\Sigma^{(N,n)}$, via the dilation of this model to a quantum Markov process, and thence to the proofs of the main Proposition of Section II. We remark here that, by Stinespring's theorem[19], the advent of dilations is essential to CP maps.

Section VI is devoted to the proofs of the main Propositions of Section III. Here, the extraction of a classical, deterministic, macroscopic law from the underlying quantum dynamics of the system is achieved by a method, originally devised in the context of certain hydrodynamical limits[20,21].

We conclude, in Section VII with a brief discussion of the results obtained here and some outstanding problems.

II. The Model

Our model is one of matter interacting with radiation, and may be described as follows. The matter consists of N identical two-level atoms and and radiation of n modes. The atoms are coupled to the radiation by dipolar interactions. Furthermore, each element of the model, whether atom or mode, is coupled to its own reservoir, an atomic reservoir consisting of a pump and a sink, and a mode reservoir of a sink only (cf. ref. 1). For physical purposes, we note that the radiation modes could be photons or phonons, and that, in the latter case, they could be of the accoustic or optical kind.

We formulate the model as a quantum dynamical system $\Sigma = (\mathcal{A}, T, \phi)$, where $\mathcal{A}$ is a $W^\star-$algebra of observables, $\{T(t)|t{\in}\mathbf{R}_+\}$ is a one-parameter semigroup of normal CP contractions of $\mathcal{A}$, and ϕ is a normal state on this algebra. We recall here that the generator of a *strongly continuous* CP semigroup of contractions of a $C^\star-$algebra takes the Lindblad form[7]

$$L = i[H, .]_- + \sum(V_j^\star(.)V_j - \frac{1}{2}[V_j^\star V_j, .]_+) \tag{2.1}$$

where H, which is self-adjoint, the $V_j's$ and $\sum V_j^\star V_j$ are elements of the algebra. In the present context, however, we shall generally be dealing with *weakly continuous*, CP semigroups of contractions of $W^\star-$algebras, with unbounded generators.

We shall now build the model Σ from its elements.

A. The Single Atom. We take the single two-level atom to be a quantum dynamical system $\Sigma_{at} = (\mathcal{A}_{at}, T_{at}, \phi_{at})$, with the following specifications.

$\mathcal{A}_{at}$, the $W^\star-$algebra of observables of the atom, consists of the 2-by-2 matrices with complex entries, and is therefore the linear span of the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ and the identity I. Its structure is thus given by the relations

$$\sigma_x^2=\sigma_y^2=\sigma_z^2=I;\ \sigma_x\sigma_y=-\sigma_y\sigma_x=i\sigma_z,\ etc. \tag{2.2}$$

We define the spin raising and lowering operators

$$\sigma_\pm=\frac{1}{2}(\sigma_x\pm i\sigma_y) \tag{2.3}$$

$\{T_{at}(t)|t\in\mathbf{R}_+\}$ is a strongly continuous one-parameter semigroup of normal CP contractions of $\mathcal{A}_{at}$, representing the dynamics of the atom. We assume that its generator, L_{at}, is of the form (2.1), where $H=\epsilon\sigma_z/2$, the $V's$ are of the form $b_\pm^{1/2}\sigma_\pm$ and $b_z^{1/2}\sigma_z$, where ϵ and $b_\pm$ are positive constants and b_z is a non-negative one. Thus, L_{at} is given by the formula

$$L_{at}\sigma_\pm=-(\gamma_\perp\mp i\epsilon)\sigma_\pm;\ L_{at}\sigma_z=-\gamma_\parallel(\sigma_z-\eta I);\ L_{at}I=0 \tag{2.4}$$

where

$$\gamma_\perp=\frac{1}{2}(b_++b_-)+2b_z;\ \gamma_\parallel=(b_++b_-);\ \eta=\frac{b_+-b_-}{b_++b_-}$$

One sees from these equations that b_+ represents the strength of the interaction of the atom with a pump, while $b_-,\ b_z$ represent its interactions with sinks. Further, the formulae for $\gamma_\perp,\ \gamma_\parallel$ and η imply that the values of these constants are constrained by the conditions

$$\gamma_\parallel\le 2\gamma_\perp;\ -1<\eta<1 \tag{2.5}$$

Finally, we take ϕ_{at} to be the unique $T_{at}-$invariant state on $\mathcal{A}_{at}$, and this is given by

$$\phi_{at}(\sigma_z)=\eta;\ \phi_{at}(\sigma_\pm)=0 \tag{2.6}$$

We note that, by (2.5), this state is faithful. Further, by (2.6), the condition that it carries an inverted population is that $\eta>0$.

Note. This atomic model is not the same as that of HL, which was built from a pair of Fermi oscillators in such a way that the damping constants $\gamma_\parallel$ and $\gamma_\perp$ were identical. Here, however, they are generally different, as is standard in quantum optics.

B. The Matter. We now assume that the matter consists of N non-interacting copies of Σ_{at}, located on the sites $r=1,..,N$ of a one-dimensional lattice. Thus, to each site r, we assign a copy, $\Sigma_r=(\mathcal{A}_r,T_r,\phi_r)$, of Σ_{at}, and then represent the matter as the $W^\star-$dynamical system $\Sigma_{mat}=(\mathcal{A}_{mat},T_{mat},\phi_{mat})$, where the elements of this triple are the tensor products of the $\mathcal{A}_r's,\ T_r's$ and $\phi_r's$, respectively. $\mathcal{A}_{mat}$ is therefore faithfully represented as the linear transformations of the Hilbert space $\mathcal{H}_{mat}=\mathbf{C}^{2N}$.

We identify the spin component $\sigma_{u,r}$ $(u=x,y,z,\pm)$, of the atom at r with the element of $\mathcal{A}_{mat}$, given by the tensor product of N elements of $\mathcal{A}_{at}$, of which the r'th is σ_u and the others are I. It follows from these specifications and (2.4) that the generator, L_{mat}, of the semigroup T_{mat} is given by

$$L_{mat}\sigma_{\pm,r}=-(\gamma_\perp\mp i\epsilon)\sigma_{\pm,r};\ L_{mat}\sigma_{z,r}=-\gamma_\parallel(\sigma_{z,r}-\eta I);\ L_{mat}I=0 \tag{2.7}$$

C. The Radiation. We assume that the radiation model, Σ_{rad}, corresponds to n modes, with frequencies $\omega_0, . ., \omega_{n-1}$, each mode being coupled to its own sink. We formulate Σ_{rad} as a $W^\star -$ dynamical system $(\mathcal{A}_{rad}, T_{rad}, \phi_{rad})$, as follows.

First, we construct $\mathcal{A}_{rad}$ as an algebra generated by creation and destruction operators for the radiation modes in the following standard way. We define the Hilbert space $\mathcal{H}_{rad}$ and the closed, densely defined operators $\{a_l^\star, a_l | l = 0, .. , n-1\}$ in this space by the following conditions.

(1) there is a unit vector, Φ_{rad} in $\mathcal{H}_{rad}$, such that $a_l \Phi_{rad} = 0$ for $l = 0, .. ., n-1$;

(2) $\mathcal{H}_{rad}$ is generated by the application to Φ_{rad} of the polynomials in the $a^\star$'s; and

(3) the a's and $a^\star$'s satisfy the canonical commutation relations

$$[a_l, a_m^\star]_- = \delta_{lm} I;\ \ [a_l, a_m]_- = 0 \tag{2.8}$$

We then define $\mathcal{A}_{rad}$, the algebra of observables of Σ_{rad}, to be $\mathcal{L}(\mathcal{H}_{rad})$, the set of bounded operators in $\mathcal{H}_{rad}$, and we take ϕ_{rad} to be the vacuum state $(\Phi_{rad}, .\Phi_{rad})$.

We define the Weyl map $z = (z_0, .. ., z_{n-1}) \rightarrow W(z)$ of $\mathbf{C}^n$ into $\mathcal{A}_{rad}$ by the standard prescription

$$W(z) = \exp i(z.a + (z.a)^\star),\ \ with\ z.a = \sum_{l=0}^{n-1} z_l a_l \tag{2.9}$$

Thus, by (2.8), W satisfies the Weyl algebraic relation

$$W(z)W(z') = W(z+z')\exp(iIm(z,z')_n) \tag{2.10}$$

where $(.,.)_n$ is the $\mathbf{C}^n$ inner product. The algebra of polynomials in $\{W(z) | z \in \mathbf{C}^n\}$ is therefore just their linear span, and is ultraweakly dense in $\mathcal{A}_{rad}$.

We formulate the dynamical semigroup T_{rad} by Vanheuverszwijn's prescription[22,23] for normal quasi-free CP semigroups of contractions on the CCR algebra. In this scheme, the action of $T_{rad}(t)$ on $W(z)$ is given by

$$T_{rad}(t)[W(z)] = W(\xi(t)z)\exp(-\theta(t)) \tag{2.11}$$

where $\xi(t) : \mathbf{C}^n \rightarrow \mathbf{C}^n$ and $\theta : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ are defined in terms of the frequencies ω_l and damping constants κ_l of the modes by the formulae

$$(\xi(t)z)_l = z_l \exp(-(i\omega_l + \kappa_l)t)\ for\ l = 0, .. ., n-1 \tag{2.12}$$

and

$$\theta(t) = \frac{1}{2}(\|z\|_n^2 - \|\xi(t)z\|_n^2) \tag{2.13}$$

where $\|.\|_n$ is the $\mathbf{C}^n$ norm. The generator of the semigroup T_{rad} is

$$L_{rad} = \sum_{l=0}^{n-1} (i[\omega_l a_l^\star a_l, .]_- + 2\kappa_l a_l^\star (.) a_l - \kappa_l [a_l^\star a_l, .]_+) \tag{2.14}$$

We note that this is an unbounded version of a Lindblad generator (cf. (2.1)).

D. The Interacting System. This is the system, Σ, formed by coupling Σ_{mat} to Σ_{rad}, by interactions specified below. We formulate Σ as a $W^{\star}-$dynamical system $(\mathcal{A}, T, \phi)$, as follows.

The algebra of observables $\mathcal{A}$ is the tensor product, $\mathcal{A}_{mat}\otimes\mathcal{A}_{rad}$, of those of the matter and radiation. Thus, $\mathcal{A}$ is an algebra of operators in the Hilbert space $\mathcal{H} = \mathcal{H}_{mat}\otimes\mathcal{H}_{rad}$. We shall identify $\mathcal{A}_{mat}\otimes I_{rad}$ and $I_{mat}\otimes\mathcal{A}_{rad}$ with $\mathcal{A}_{mat}$ and $\mathcal{A}_{rad}$, respectively, thus rendering them intercommuting subalgebras of $\mathcal{A}$. Correspondingly, if $A_{mat}\in\mathcal{A}_{mat}$ and $B_{rad}\in\mathcal{A}_{rad}$, we denote the tensor product $A_{mat}\otimes B_{rad}$ by $A_{mat}B_{rad}$.

We take the state ϕ to be the tensor product $\phi_{mat}\otimes\phi_{rad}$. Thus,

$$\phi(A_{mat}A_{rad}) = \phi_{mat}(A_{mat})\phi_{rad}(A_{rad}) \ \forall A_{mat}\in\mathcal{A}_{mat}, A_{rad}\in\mathcal{A}_{rad} \tag{2.15}$$

We denote by $\mathcal{N}(\mathcal{A})$ the set of normal states on $\mathcal{A}$. Since $\mathcal{A} = \mathcal{L}(\mathcal{H})$, each normal state ψ corresponds to a unique density matrix ρ_{ψ} in $\mathcal{H}$, with $\psi\equiv Tr(\rho_{\psi}.)$. Thus, denoting by $\mathcal{H}_{HS}$ the Hilbert-Schmidt space $\{A\in\mathcal{L}(\mathcal{H})|Tr(A^{\star}A) < \infty\}$, with inner product $(A, B)_{HS} = Tr(A^{\star}B)$ and corresponding norm $\|.\|_{HS}$, we may represent ψ by the vector $\rho_{\psi}^{1/2}$ in this space, according to the formula

$$\psi(A) = (\rho_{\psi}^{1/2}, A\rho_{\psi}^{1/2})_{HS} \ \forall A\in\mathcal{A} \tag{2.16}$$

Hence, ψ has a canonical extension[24] to the unbounded affiliates, Q, of $\mathcal{A}$, for which $Q\rho_{\psi}^{1/2}\in\mathcal{H}_{HS}$, i.e.,

$$\psi(Q) = (\rho_{\psi}^{1/2}, Q\rho_{\psi}^{1/2})_{HS} \tag*{(2.16)$'$}$$

Turning now to the dynamics, we assume that the coupling between the matter and radiation is dipolar and corresponds to an interaction Hamiltonian of the form

$$H_{int} = iN^{-1/2}\sum_{r=1}^{N}\sum_{l=0}^{n-1}\lambda_l(\sigma_{-,r}a_l^{\star}\exp(-2\pi ik_lr) - h.c.), \tag{2.17}$$

where k_l is the wave-number of the k'th mode and the $\lambda's$ are real-valued, $N-$independent coupling constants. We shall provide further specifications of k_l in §3.

We note here that the unboundedness of H_{int} presents serious problems when one seeks to construct the dynamical semigroup, T, for Σ. One might envisage, of course, that this interaction leads simply to a contribution $i[H_{int}, .]$ to the generator of T, so that this takes the form

$$L = L_{mat} + L_{rad} + L_{int} \tag{2.18}$$

where

$$L_{int} = i[H_{int}, .] \tag{2.19}$$

and L_{mat}, L_{rad} are identified with $L_{mat}\otimes I, I\otimes L_{rad}$, respectively. However, in view of the fact that both L_{rad} and L_{int} are unbounded, it is not clear from existing results on one-parameter semigroups (e.g. refs. 14, 15, 25, 26) whether the L of this formula would generate one. In fact, we shall show in §5 that it is indeed the generator of a CP semigroup, T, by a construction based on the theory of quantum Markov processes[9].

We shall denote by $T^\star(\mathbf{R}_+)$ the one-parameter semi-group of transformations of $\mathcal{N}(\mathcal{A})$ dual to T, and by ψ_t the image of a normal state ψ under $T^\star(t)$. Thus,

$$\psi_t(A)\equiv(T^\star(t)\psi)(A) = \psi(T(t)A)\ \forall A\in\mathcal{A},\ t\in\mathbf{R}_+ \tag{2.20}$$

In order to formulate the evolution of Σ in states sufficiently regular to yield well-defined photon statistics, we introduce the following definitions.

Definition 2.1. (1) We define $\mathcal{M}$ to be the set of all monomials $a^\sharp_{l_1}\ldots a^\sharp_{l_m}$, where m is an arbitrary positive integer and $a^\sharp_l$ is either a_l or $a^\star_l$.

(2) We define $\mathcal{D}^{(0)}$ to be the set of vectors of $\mathcal{H}$, that lie in the domain of the operators $\mathcal{M}$, and, for $\Psi\in\mathcal{D}^{(0)}$ and fixed m, we define $M_m(\Psi)$ to be the maximum value of $\{\|a^\sharp_{l_1}\ldots a^\sharp_{l_m}\Psi\| | l_1 . . l_m\in[0, n-1]\}$.

(3) We define $\mathcal{D}$ to be subset of $\mathcal{D}^{(0)}$, whose elements, Ψ, satisfy the condition that

$$\sum\nolimits_{m=1}^{\infty} M_m(\Psi)v^m/m! < \infty\ \forall v\in\mathbf{R}_+ \tag{2.21}$$

Thus, $\mathcal{D}$ is dense in $\mathcal{H}$.

Definition 2.2. We define $\mathcal{F}(\mathcal{A})$ to be the *−algebra of polynomials in the elements of $\mathcal{A}_{mat}$, the Weyl operators $W_{rad}(f)$ and the creation and destruction operators $a, a^\star$.

The following Proposition will be proved in §5, following the construction there of the semigroup T.

Proposition 2.3. *Let $\Psi\in\mathcal{D}$ and let $\psi = (\Psi, .\Psi)$ be the corresponding pure state on $\mathcal{A}$. Then the evolute, ψ_t, of ψ has a canonical extension to $\mathcal{F}(\mathcal{A})$ and satisfies the equation of motion*

$$\frac{d}{dt}\psi_t(Q) = \psi_t(LQ)\ \forall Q\in\mathcal{F}(\mathcal{A}) \tag{2.22}$$

Extension to Initial Mixed States. This last result is easily generalised to initial mixed states by the following procedure. We define $\mathcal{D}_{HS}$ to be the subset of $\mathcal{H}_{HS}$, whose elements, Θ, satisfy the analogue of (2.21), obtained by replacing Ψ, $\|.\|$ by Θ, $\|.\|_{HS}$, respectively, in Def. 2.1. In view of this definition of $\mathcal{D}_{HS}$ and the representation (2.16) of mixed states, it is a straightforward matter to establish the following counterpart of Prop. 2.3 for such states.

Proposition 2.4. *Let ψ be a normal state on $\mathcal{A}$ corresponding to a density matrix ρ, which satisfies the regularity condition that $\rho_\psi^{1/2}\in\mathcal{D}_{HS}$. Then the evolute, ψ_t, of ψ has a canonical extension to $\mathcal{F}(\mathcal{A})$, which satisfies the equation of motion (2.22).*

III. The Macroscopic Dynamics

We formulate the macroscopic description of the model in terms of the global intensive observables

$$s_l^{(N)} = N^{-1}\sum_{r=1}^{N} \sigma_{-,r}\exp(-2\pi i k_l r);\ l = 0,..,n-1 \tag{3.1}$$

and

$$p_l^{(N)} = N^{-1}\sum_{r=1}^{N} \sigma_{z,r}\exp(-2\pi i k_l r);\ l = 0,..,n-1 \tag{3.2}$$

together with the operators

$$\alpha_l^{(N)} = N^{-1/2}a_l;\ l = 0,..,n-1 \tag{3.3}$$

corresponding to a scaling of the number operators $a_l^\star a_l$ in units of N (cf. HL). We denote the set $\{s_l^{(N)}, s_l^{(N)\star}, p_l^{(N)}, p_l^{(N)\star}, \alpha_l^{(N)}, \alpha_l^{(N)\star} | l = 0, 1, .., n-1\}$ by $\mathbf{M}^{(N)}$. To simplify the model, we choose

$$k_l = \frac{l}{n} \tag{3.4}$$

so that $\mathbf{M}^{(N)}$ is a Lie algebra w.r.t. commutation. Specifically, by (3.1)-(3.4), its non-zero Lie brackets are the following ones, and their adjoints.

$$[s_l^{(N)}, s_m^{(N)\star}] = -N^{-1}p_{[l-m]}^{(N)};\ [s_l^{(N)}, p_m^{(N)}] = 2N^{-1}s_{[l-m]}^{(N)};$$

$$[s_l^{(N)\star}, p_m^{(N)}] = -2N^{-1}s_{[l+m]}^{(N)\star};\ [a_l^{(N)}, a_m^{(N)\star}] = N^{-1}I\delta_{lm} \tag{3.5}$$

where $[l \pm m] = l \pm m\ (mod\ n)$. Thus, the observables $\mathbf{M}^{(N)}$ become classical in the limit $N \to \infty$. Further, by (2.2), (2.3), (3.1) and (3.2),

$$\|s_l^{(N)}\| = 1;\ \|p_l^{(N)}\| = 1\ for\ l = 0, .., n-1 \tag{3.6}$$

and

$$p_0^{(N)\star} = p_0^{(N)};\ and\ p_l^{(N)\star} = p_{n-l}^{(N)}\ for\ l = 1, .., n-1 \tag{3.7}$$

By (2.17) and (3.1)-(3.3), the interaction Hamiltonian H_{int} is a function of the macro-observables only, i.e.,

$$H_{int}^{(N)} = iN\sum_{l=0}^{n-1} \lambda_l(\alpha_l^{(N)\star}s_l^{(N)} - \alpha_l^{(N)}s_l^{(N)\star}) \tag{3.8}$$

Our objective will be to extract the dynamics of $\mathbf{M}^{(N)}$ from the microscopic equation of motion (2.22), in a limit where $N \to \infty$ and n remains fixed and finite. Since N is not fixed here, we shall indicate the dependence of $\Sigma, \mathcal{A}, T, \phi, \psi, L, H_{int}$ and $\mathcal{D}$ on this number by the superscript (N).

For finite N, the macroscopic dynamics is governed by the action of $L^{(N)}$ on $\mathbf{M}^{(N)}$. This is given by the following equations, which ensue from (2.17)-(2.19) and (3.1)-(3.4).

$$L^{(N)}\alpha_l^{(N)} = A_l^{(N)};\ L^{(N)}s_l^{(N)} = S_l^{(N)};\ L^{(N)}p_l^{(N)} = P_l^{(N)} \tag{3.9}$$

where

$$A_l^{(N)} = -(i\omega_l + \kappa_l)\alpha_l^{(N)} + \lambda_l s_l^{(N)} \tag{3.10a}$$

$$S_l^{(N)} = -(i\epsilon + \gamma_\perp)s_l^{(N)} + \sum\nolimits_{m=0}^{n-1}\lambda_m p_{[l-m]}^{(N)}\alpha_m^{(N)} \tag{3.10b}$$

and

$$P_l^{(N)} = -\gamma_\parallel(p_l^{(N)} - \eta_l^{(N)}I) - 2\sum\nolimits_{m=0}^{n-1}\lambda_m(\alpha_m^{(N)\star}s_{[l+m]}^{(N)} + \alpha_m^{(N)}s_{[m-l]}^{(N)\star}) \tag{3.10c}$$

where

$$\eta_l^{(N)} = \eta\delta_{l,0} + O(N^{-1}) \tag{3.10d}$$

We shall assume that the initial state, $\psi^{(N)}$, of $\Sigma^{(N)}$ corresponds to a vector $\Psi^{(N)}$ in $\mathcal{D}^{(N)}$, and that the number of photons it carries does not increase faster than N, i.e. that, for some finite constant B,

$$\psi^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)}) < B\ for\ l = 0,..,n-1\ and\ \forall N\in\mathbf{N} \tag{3.11}$$

The following definitions provide a framework for the anticipated classical limit of the dynamics of the macro-observables.

Definition 3.1. (1) We define the space $X = \mathbf{C}^n\times\mathbf{C}^n\times\tilde{\mathbf{C}}^n$, the third factor being the subset of $\mathbf{C}^n$, whose elements $(c_0,..,c_{n-1})$ satisfy the conditions that $c_0 = \overline{c}_0$ and $\overline{c}_l = c_{n-l}$ for $l = 1,..,n-1$. Thus, by (3.1)-(3.3) and (3.7), X is the 'phase space' for the expectation values of the macroscopic observables, the means of $\alpha^{(N)}$, $s^{(N)}$ and $p^{(N)}$ being harboured by the factors $\mathbf{C}^n$, $\mathbf{C}^n$ and $\tilde{\mathbf{C}}^n$, respectively.

(2) We denote the points (α, s, p) (or (z, w, v)) of X by x (or y), and correspondingly, we define the measure dx (resp. dy) to be the product of those of the mutually independent real and imaginary components of α, s, p (resp. z, w, v). With these identifications, we equip X with the bilinear form

$$x.y = \sum\nolimits_{l=0}^{n-1}[(\alpha_l z_l + s_l w_l + c.c.) + p_l v_l] \tag{3.12}$$

and define the inverse Fourier transformation $\hat{f}\to f$ of $L^2(X, dx)$ by the formula

$$f(x) = \int_X dy\hat{f}(y)\exp(ix.y) \tag{3.13}$$

(3) We employ the following notation for the derivatives of the differentiable functions on X w.r.t. their (complex) arguments.

$$df(\alpha, s, p) = \frac{\partial f}{\partial \alpha}.d\alpha + +\frac{\partial f}{\partial \overline{\alpha}}.d\overline{\alpha} + \frac{\partial f}{\partial s}.ds + \frac{\partial f}{\partial \overline{s}}.d\overline{s} + \frac{\partial f}{\partial p}.dp$$

(4) We define $\mathcal{C}_0(X)$ to be the $C^\star-$algebra of bounded continuous functions with compact support on X, with supremum norm $\|.\|_0$, and $\mathcal{C}_0^{(1)}(X)$ to be the Banach space of its continuously differentiable elements, with norm $\|.\|_1$ given by the formula

$$\|f\|_1 = \|f\|_0 + \sum_{l=0}^{n-1} (\|\frac{\partial f}{\partial \alpha_l}\|_0 + \|\frac{\partial f}{\partial \overline{\alpha}_l}\|_0 + \|\frac{\partial f}{\partial s_l}\|_0 + \|\frac{\partial f}{\partial \overline{s}_l}\|_0 + \|\frac{\partial f}{\partial p_l}\|_0)$$

(5) We define the generalised Weyl map $U^{(N)} : X \to \mathcal{L}(\mathcal{H})$ by the formulae

$$U^{(N)}(z, w, v) = \exp i((z.\alpha^{(N)} + w.s^{(N)} + h.c) + v.p^{(N)}) \ \forall (z, w, v) \in X \tag{3.14}$$

where, for $z = (z_0, . ., z_{n-1})$, $w = (w_0, . ., w_{n-1})$ and $v = (v_0, . ., v_{n-1})$,

$$z.\alpha^{(N)} = \sum_{l=0}^{n-1} z_l \alpha_l^{(N)}; \ w.s^{(N)} = \sum_{l=0}^{n-1} w_l s_l^{(N)}; \ v.p^{(N)} = \sum_{l=0}^{n-1} v_l p_l^{(N)}$$

(6) We define the characteristic function $\mu_t^{(N)} : X \to \mathbf{C}$, for $t \in \mathbf{R}_+$, by the formula

$$\mu_t^{(N)} = \psi_t^{(N)} \circ U^{(N)} \tag{3.15}$$

Proposition 3.2. *(1) Under the above assumptions, $\mu_t^{(N)}$ tends pointwise, as $N \to \infty$, to the characteristic function, μ_t, of a classical probability measure, m_t, on X, i.e.*

$$\lim_{N \to \infty} \mu_t^{(N)}(y) = \int_X \exp i(x.y) dm_t(x) \tag{3.16}$$

convergence being uniform on the $X-$compacts.

(2) The evolution of this measure is governed by the following equation of motion.

$$\frac{d}{dt} \int_X dm_t f = \int_X dm_t \mathcal{L} f \ \forall f \in \mathcal{C}_0^{(1)}(X), t \in \mathbf{R}_+ \tag{3.17}$$

where

$$\mathcal{L} = A.\frac{\partial}{\partial \alpha} + S.\frac{\partial}{\partial s} + \overline{A}.\frac{\partial}{\partial \overline{\alpha}} + \overline{S}.\frac{\partial}{\partial \overline{s}} + P.\frac{\partial}{\partial p} \tag{3.18}$$

and $A=(A_0,\ldots,A_{n-1})$, $S=(S_0,\ldots,S_{n-1})$ *and* $P=(P_0,\ldots,P_{n-1})$ *are the functions from* X *into* $\mathbf{C}^n$, $\mathbf{C}^n$ *and* $\tilde{\mathbf{C}}^n$, *respectively, given by*

$$A_l(\alpha,s,p)=-(i\omega_l+\kappa_l)\alpha_l+\lambda_l s_l \tag{3.19a}$$

$$S_l(\alpha,s,p)=-(i\epsilon+\gamma_\perp)s_l+\sum_{m=0}^{n-1}\lambda_m p_{[l-m]}\alpha_m \tag{3.19b}$$

and

$$P_l(\alpha,s,p)=-\gamma_\parallel(p_l-\eta\delta_{l,0})-2\sum_{m=0}^{n-1}\lambda_m(\overline{\alpha}_m s_{[l+m]}+\alpha_m\overline{s}_{[m-l]}) \tag{3.19c}$$

with $[l\pm m]=l\pm m$ *(mod* n*).*

Definition 3.3. We define $\mathcal{K}$ to be the classical flow in X given by the following equations of motion for the evolute $x_t=(\alpha_t,s_t,p_t)$ of $x=(\alpha,s,p)$.

$$\frac{d\alpha_t}{dt}=A(\alpha_t,s_t,p_t);\ \frac{ds_t}{dt}=S(\alpha_t,s_t,p_t);\ \frac{dp_t}{dt}=P(\alpha_t,s_t,p_t) \tag{3.20}$$

Comment. These constitute a slight generalisation of the HL phenomenological equations, and correspond to an obvious classical limit obtained by taking the expectation values of equns. (3.9) for the state $\psi_t^{(N)}$.

Proposition 3.4. *(1) Equations (3.20) have a unique global solution. Thus, if* $x=(\alpha,s,p)$ *is the initial value of* $x_t=(\alpha_t,s_t,p_t)$, *these equations define a one-parameter semigroup* $\{\tau(t)|t\in\mathbf{R}_+\}$ *of transformations of* X, *such that* $\tau(t)x=x_t$.

(2) Correspondingly, the equation of motion (3.17) for m_t *has the unique solution*

$$\int_X dm_t f=\int_X dm_0 f\circ\tau(t)\ \forall f\in\mathcal{C}_0(X),\ t\in\mathbf{R}_+ \tag{3.21}$$

We shall provide the proofs of Props. (3.2) and (3.4) in §6. Here, we note the following immediate implication of them.

Proposition 3.5. *If* m_0 *is the Dirac measure* δ_{x_0}, *with support at* x_0, *then* $m_t=\delta_{\tau(t)x_0}$. *Thus, if the macro-observables are initially dispersion-free, then they subsequently remain so and evolve according to the classical deterministic law* $(\alpha,s,p)\rightarrow(\alpha_t,s_t,p_t)$, *given by the equations of motion (3.20).*

Comment. The assumption that m_0 has point support signifies that the initial state of the system corresponds to a pure phase, in the thermodynamic limit.

IV. The Dynamical Phase Transitions

We shall now investigate the dynamics of the classical system $\mathcal{K}$, since, by Props. 3.4 and 3.5, it is this that governs the evolution of the macroscopic observables, and thus the large scale optical patterns, of Σ.

A. Transition to Laser Light. By equations (3.19) and (3.20), $\mathcal{K}$ has a stationary state, given by the phase point, $x^{(0)}$, at which $p_0 = \eta$ and all other components of α, s and p vanish. In order to investigate its stability, we introduce a 'small' perturbation $x'_t = (\alpha'_t, s'_t, p'_t)$ of $x^{(0)}$, and infer from (3.19) and (3.20) that its *linearised* equations of motion are

$$\frac{d\alpha'_{t,l}}{dt} = -(i\omega_l + \kappa_l)\alpha'_{t,l} + \lambda_l s'_{t,l}; \ \frac{ds'_{t,l}}{dt} = -(i\epsilon + \gamma_\perp)s'_{t,l} + \lambda_l \eta \alpha'_{t,l}; \ \frac{dp'_{t,l}}{dt} = -\gamma_\parallel p'_{t,l} \tag{4.1}$$

¿From the last equation, we see that $p'_{t,l}$ decays to zero, and so the stability of p_t is guaranteed. To test for stability of α_t, s_t, it suffices to look at the solutions of the first two equations of (4.1) for which $\alpha'_{t,l}$ and $s'_{t,l}$ are constant multiples of $\exp(-\zeta_l t)$, with ζ_l a complex constant. Thus, by (4.1), ζ_l is determined by the roots of the quadratic equation

$$(\zeta_l - \kappa_l - i\omega_l)(\zeta_l - \gamma_\perp - i\epsilon) - \lambda_l^2\eta = 0 \tag{4.2}$$

The real parts of the roots of this equation both positive, i.e. $\alpha'_{t,l}$ and $s'_{t,l}$ decay to zero, provided that η is less than the critical value $\eta_l^{(c)}$, given by

$$\eta_l^{(c)} = \frac{\kappa_l\gamma_\perp}{\lambda_l^2}[1 + \frac{(\epsilon - \omega_l)^2}{(\kappa_l + \gamma_\perp)^2}] \tag{4.3}$$

Hence, the fixed point $x^{(0)}$ is stable provided that η is less than $min_l\{\eta_l^{(c)}\}$.

We now assume that this minimum, which we denote by $\eta^{(1)}$, is attained at just one value, k, of l. Then, if η is increased beyond $\eta^{(1)}$, the real part of a root of (4.2) becomes positive, for $l = k$, and so the components α_k, s_k of the phase point $x^{(0)}$ become unstable. On the other hand, for $\eta > \eta^{(1)}$, the equations of motion (3.20) have a periodic solution

$$\alpha_{t,l} = \alpha_k\delta_{l,k}\exp(-i\nu t); \ s_{t,l} = s_k\delta_{l,k}\exp(-i\nu t); \ p_{t,0} = \eta^{(1)}\delta_{l,0} \tag{4.4}$$

where

$$\nu = \frac{\gamma_\perp\omega_k + \epsilon\kappa_k}{\gamma_\perp + \kappa_k} \tag{4.5}$$

$$|\alpha_k| = \frac{1}{2}[\frac{\gamma_\perp(\eta - \eta^{(1)})}{\kappa_k}]^{\frac{1}{2}} \tag{4.6}$$

and

$$s_k = -\frac{\kappa_k(\gamma_\perp + \kappa_k) + i(\omega_k - \epsilon))\alpha_k}{\lambda_k(\gamma_\perp + \kappa_k)} \tag{4.7}$$

Further, by the Hopf bifurcation theory[27,28], this periodic orbit is stable for $\eta - \eta^{(1)}$ sufficiently small. Evidently, it corresponds to coherent laser light in the mode $l = k$. Thus, the following result, obtained by HL for the single mode case, prevails here too.

Proposition 4.1. *Under the specified conditions, including the initial pure phase assumption of Prop. 3.5, there is a Hopf bifurcation in $\mathcal{K}$, corresponding to a phase transition in*

Σ from a non-radiant state to a coherently radiating one, as η increases past the critical value $\eta^{(1)}$.

B. Further Transitions: Polychromatic and Chaotic Laser Light. In the single mode case, it was proved by HL that, in the particular case that $\gamma_{\perp} = \gamma_{\parallel} = \kappa$, the periodic orbit of $\mathcal{K}$, and hence the coherent monochromatic radiation of Σ, is stable for all $\eta > \eta^{(1)}$.

In general, however, it is clear from the theory of classical dynamical systems[29,30] that the equations of motion (3.19) and (3.20) for $\mathcal{K}$ provide the framework for further bifurcations, of the kinds arising in hydrodynamics, as η is increased. Specifically, we have the following possibilities.

(I) There could be a bifurcation of the simply periodic orbit of $\mathcal{K}$ into a strange attractor[29] at some value $\eta^{(2)}(> \eta^{(1)})$ of η. In this case, there would be a transition from monochromatic to chaotic radiation when η increased beyond the value $\eta^{(2)}$ (cf. refs. 16-18).

(II) There could be a succession of bifurcations, corresponding to the activation of different modes, according to the Landau mechanism for the onset of turbulence[31]. This would then correspond to polychromatic radiation, and would simulate optical chaos when the number of activated modes became very large.

In fact, we shall now adapt an argument of Haken[16] to the present model to show that the scenario (I) is achieved even when there is only one mode and when the resonance condition $\epsilon = \omega$ is fulfilled. Thus, assuming these conditions, we note first note that, by (4.3),

$$\eta^{(1)} = \frac{\kappa\gamma_{\perp}}{\lambda^2} \tag{4.8}$$

Thus, introducing a transformation of variables, $(\alpha_t, s_t, p_t) \rightarrow (e_t, c_t, f_t)$, by the prescription

$$e_t = 2\left(\frac{\kappa}{\gamma_{\parallel}(\eta - \eta^{(1)})}\right)^{1/2}\alpha_t \exp(i\epsilon t);\ c_t = 2\lambda(\kappa\gamma_{\parallel}(\eta - \eta^{(1)}))^{-1/2}s_t \exp(i\epsilon t);\ f_t = \frac{p_t}{\eta^{(1)}} \tag{4.9}$$

where the subscript $l(= 0)$ has been omitted, we see from (4.8) and (4.9) that the equations of motion(3.19) and (3.20) take the following form.

$$\frac{de_t}{dt} = -\kappa(e_t - c_t);\ \frac{dc_t}{dt} = -\gamma_{\perp}(c_t - e_t f_t);\ \frac{df_t}{dt} = -\gamma_{\parallel}(f_t - g + (g-1)Re(e_t\bar{c}_t)) \tag{4.10}$$

where

$$g = \frac{\eta}{\eta^{(1)}} \tag{4.11}$$

By the first two equations of (4.10),

$$\left(\frac{d}{dt} + \kappa + \gamma_{\perp}\right)Im(e_t\bar{c}_t) = 0$$

which signifies that $Im(e_t\bar{c}_t)$ is a transient quantity, which decays exponentially to zero. Hence, for the long-time dynamics, we may take $e_t\bar{c}_t$ to be real, i.e. we may assume that $e_t,\ c_t$ have the same phase:-

$$e_t = e_t^{(0)}\exp(i\beta_t);\ c_t = c_t^{(0)}\exp(i\beta_t) \tag{4.12}$$

where $e_t^{(0)},\ c_t^{(0)},\ \beta_t$ are real. Further, it is a simple consequence of the uniqueness Proposition 3.4(1) that β_t must be constant. Hence, equns. (4.10) reduce to the following form for the real-valued variables $e_t^{(0)},\ c_t^{(0)},\ f_t$.

$$\frac{de_t^{(0)}}{dt} = -\kappa(e_t^{(0)} - c_t^{(0)}) \tag{4.13a}$$

$$\frac{dc_t^{(0)}}{dt} = -\gamma_\perp(c_t^{(0)} - e_t^{(0)} f_t) \tag{4.13b}$$

$$\frac{df_t}{dt} = -\gamma_\parallel(f_t - g + (g-1)(e_t^{(0)} c_t^{(0)})) \tag{4.13c}$$

These are the single-mode Maxwell-Bloch equations. As shown by Haken[16], these transform, under a simple linear transformation to the equations of motion for a Lorenz attractor, which is known to be undergo a transition to chaos for appropriate values of the control variables. Specifically, these Maxwell-Bloch equations support such a transition if (cf. ref. 16)

$$\kappa > \gamma_\perp + \gamma_\parallel;\ and\ \frac{\eta}{\eta^{(1)}} - 1 > \frac{(\gamma_\perp + \gamma_\parallel + \kappa)(\gamma_\perp + \kappa)}{\gamma_\perp(\kappa - \gamma_\perp - \gamma_\parallel)} \tag{4.14}$$

Since these conditions are compatible with the demands of equns. (2.5), we have the following result.

Proposition 4.2. *The single-mode model, satisfying the resonance condition $\epsilon = \omega$, exhibits chaotic behaviour when the control parameters $\gamma_\perp,\ \gamma_\parallel,\ \kappa,\ \lambda$ lie in a certain domain.*

Comment. The point to emphasise here is that, *even in the single-mode case,* there is a chaotic regime. In the multi-mode case, the prevalence of additional degrees of freedom would be expected to be favour chaos still further[29,30].

Semantic Note. There is a terminological paradox here, since all the states arising in these scenarios, including the chaotic ones, are coherent in the sense that they satisfy Glauber's[32] factorisation condition in the limit $N \to \infty$.

V. Construction, via Quantum Stochastics, of the Semigroup T

We shall now present our construction of the dynamical semigroup T. Here, in accordance with Stinespring's representation[19] of CP transformations of $C^\star-$algebras, we construct T as the reduction to $\mathcal{A}$ of a semigroup of isomorphic mappings of this algebra into a larger one. Thus, we start by dilating Σ_{mat} and Σ_{rad} to quantum stochastic processes, in the sense defined by Accardi, Frigerio and Lewis[9]. These have dynamics corresponding to isomorphisms $\mathcal{J}_{mat}(t),\ \mathcal{J}_{rad}(t)$ of $\mathcal{A}_{mat},\ \mathcal{A}_{rad}$ into larger algebras, $\hat{\mathcal{A}}_{mat},\ \hat{\mathcal{A}}_{rad}$, whose reductions to the latter algebras are $T_{mat}(t),\ T_{rad}(t)$, respectively. We couple these processes via the interaction Hamiltonian H_{int}, and thereby obtain a process whose dynamics corresponds to isometries $\mathcal{J}(t)$ of $\mathcal{A}$ into $\hat{\mathcal{A}} = \hat{\mathcal{A}}_{mat} \otimes \hat{\mathcal{A}}_{rad}$. We then define $T(t)$ to be the reduction of $\mathcal{J}(t)$ to $\mathcal{A}$ and prove, in Prop. 5.4, that $T(\mathbf{R}_+)$ is indeed a CP semigroup.

Comment. Since $\mathcal{J}$ represents the conservative evolution of a composite of Σ and another system, $\tilde{\Sigma}$, one might think of regarding the latter as a heat bath. From a physical point of view, however, this is rather unnatural, since some limit proceedure, e.g. that of Van Hove (cf. ref. 4), is generally required in order that a system may evolve according to a Markovian semigroup as a result of its coupling to a thermal reservoir; and, even in the appropriate limit, this reservoir is generally quite different from the one arising in the above-described dilation scheme. We therefore regard that scheme as a mathematical one, designed for the construction of the semigroup T.

Since, in this Section, we shall deal only with the structure of $\Sigma^{(N)}$ for fixed N, we shall lighten the notation here by dropping all the superscripts (N) referring to this system.

A. The Material Process. By Kümmerer's construction[10], the system Σ_{mat}, formulated in §2B, has a minimal dilation to a conservative $W^{\star}-$dynamical system, $\hat{\Sigma}_{mat} = (\hat{\mathcal{A}}_{mat}, \hat{T}_{mat}, \hat{\phi}_{mat})$, where $\hat{\mathcal{A}}_{mat}$ is a $W^{\star}-$algebra, $\hat{\phi}_{mat}$ is a faithful normal state on $\hat{\mathcal{A}}_{mat}$ and $\hat{T}_{mat}$ is a weakly continuous representation of $\mathbf{R}$ in $\hat{\mathcal{A}}$, such that the following conditions are fulfilled.

(1) $\hat{\mathcal{A}}_{mat}$ is the $W^{\star}-$tensor product, $\mathcal{A}_{mat}\otimes\tilde{\mathcal{A}}_{mat}$, of $\mathcal{A}_{mat}$ and another $W^{\star}-$algebra, $\tilde{\mathcal{A}}_{mat}$. We define the injection $\mathcal{I}_{mat}$ and, for $t\in\mathbf{R}$, the isomorphism $\mathcal{J}(t)$ of $\mathcal{A}_{mat}$ into $\hat{\mathcal{A}}_{mat}$ by the formulae

$$\mathcal{I}_{mat}A = A\otimes\tilde{I}\ \forall A\in\mathcal{A}_{mat} \tag{5.1}$$

and

$$\mathcal{J}_{mat}(t) = \hat{T}_{mat}(t)\circ\mathcal{I}_{mat} \tag{5.2}$$

We shall be concerned exclusively with the restrictions of $\mathcal{J}$ and $\hat{T}_{mat}$ to $\mathbf{R}_{+}$.

(2) $\hat{\phi}_{mat}$ is a $\hat{T}_{mat}-$invariant state on $\hat{\mathcal{A}}$, which takes the form $\phi_{mat}\otimes\tilde{\phi}_{mat}$, where $\tilde{\phi}_{mat}$ is a normal state on $\tilde{\mathcal{A}}$. We define the projection $\mathcal{P}_{mat} : \hat{\mathcal{A}}_{mat}\to\mathcal{A}_{mat}$ by the formula

$$\mathcal{P}_{mat}(A\otimes\tilde{A}) = \tilde{\phi}_{mat}(\tilde{A})A\ \forall A\in\mathcal{A}_{mat}, \tilde{A}\in\tilde{\mathcal{A}}_{mat} \tag{5.3}$$

(3) T_{mat} is the reduction of $\hat{T}_{mat}$ to $\mathcal{A}$, given by

$$T_{mat}(t) = \mathcal{P}_{mat}\circ\hat{T}_{mat}(t)\circ\mathcal{I}_{mat}\equiv\mathcal{P}_{mat}\circ\mathcal{J}_{mat}(t) \tag{5.4}$$

(4) The triple $\Pi_{mat} = (\hat{\mathcal{A}}_{mat}, \mathcal{J}_{mat}, \hat{\phi}_{mat})$ is a stationary Markov process, with conditional expectations (CE's), over $\mathcal{A}$. Thus, defining $\mathcal{A}_{t,mat}$ to be $\mathcal{J}_{mat}(t)\mathcal{A}_{mat}$, and $\mathcal{A}_{t\pm,mat}$ to be the $W^{\star}-$algebras generated by $\{\mathcal{A}_{u,mat}|u\in[t,\infty)\ (resp.\ [0,t])\}$, Π_{mat} is equipped with CE's $\{E_{t,mat} : \hat{\mathcal{A}}_{mat}\to\mathcal{A}_{t-,mat}|t\in\mathbf{R}_{+}\}$, that are compatible with $\hat{\phi}_{mat}$ (i.e. $\hat{\phi}_{mat} = \hat{\phi}_{mat}\circ E_{t,mat}$) and satisfy the Markov condition

$$E_{t,mat}(\mathcal{A}_{t+,mat})\subset\mathcal{A}_{t,mat}$$

Since $\tilde{\phi}_{mat}$ is faithful and normal, we may assume, without loss of generality, that $\tilde{\mathcal{A}}_{mat}$ acts in a Hilbert space, $\tilde{\mathcal{H}}_{mat}$, and that the state $\tilde{\phi}_{mat}$ is that of a cyclic and separating vector,

$\tilde{\Phi}_{mat}$, in this space. Thus, $\tilde{\mathcal{A}}_{mat}\subset\mathcal{L}(\tilde{\mathcal{H}}_{mat})$ and $\tilde{\phi}_{mat} = (\tilde{\Phi}_{mat}, .\tilde{\Phi}_{mat})$. Correspondingly, $\hat{\mathcal{A}}_{mat}\subset\mathcal{L}(\hat{\mathcal{H}}_{mat})$, where $\hat{\mathcal{H}}_{mat} = \mathcal{H}_{mat}\otimes\tilde{\mathcal{H}}_{mat}$.

B. The Quantum Wiener Process. We assume that the radiative modes are subjected to the damping and fluctuating forces imposed by a Bose field, $\tilde{w}$, corresponding to a quantum Wiener process. To formulate $\tilde{w}$ (cf. ref. 33), we start by defining H to be the complex ('single-particle') Hilbert space, $\{f : \mathbf{R}_+\to\mathbf{C}^n \vert \int_0^\infty \Vert f(t)\Vert_n^2 dt < \infty\}$, with inner product

$$(f,g)_H = \int_0^\infty dt(f(t),g(t))_n \tag{5.5}$$

where, as in §2, $(.,.)_n$ is the $\mathbf{C}^n$ inner product. We define the field $\tilde{w}$ and the Fock space $\tilde{\mathcal{H}}_{rad}$ by the following standard conditions (cf.(1)-(3) of §2C). $\tilde{\mathcal{H}}_{rad}$ is a complex Hilbert space and $\tilde{w}$ is a map of H into the closed, densely-defined operators in $\tilde{\mathcal{H}}_{rad}$, such that

(1) $\tilde{\mathcal{H}}_{rad}$ contains a normalised vector $\tilde{\Phi}_{rad}$, which is annihilated by the operators $\tilde{w}(f)$;

(2) $\tilde{\Phi}_{rad}$ is cyclic w.r.t. the polynomial algebra in $\{\tilde{w}^\star(f)\vert f\in H\}$; and

(3) $\tilde{w}$ satisfies the CCR,

$$[\tilde{w}(f),\tilde{w}(g)^\star]_- = (g,f)_H I;\ [\tilde{w}(f),\tilde{w}(g)] = 0 \tag{5.6}$$

We define the algebra $\tilde{\mathcal{A}}_{rad}$ to be $\mathcal{L}(\tilde{\mathcal{H}}_{rad})$, and $\tilde{\phi}$ to be the state on this algebra represented by the vector $\tilde{\Phi}_{rad}$, i.e., $\tilde{\phi}_{rad} = (\tilde{\Phi}_{rad}, .\tilde{\Phi}_{rad})$.

By our definition of H, this space may be canonically identified with $L^2(\mathbf{R}_+)^n$, each element f of H being given by a sequence $(f_0, ., f_{n-1})$ of $L^2(\mathbf{R}_+)$ vectors. Under this identification,

$$(f,g)_H = \sum_{l=0}^{n-1}(f_l,g_l) \tag{5.7}$$

where $(.,.)$ is the $L^2(\mathbf{R}_+)$ inner product. Further, by linearity, the application of $\tilde{w}$ to $f = (f_0,..,f_{n-1})$ serves to define linear maps $\tilde{w}_0,..,\tilde{w}_{n-1}$ from $L^2(\mathbf{R}_+)$ into the operators in $\tilde{\mathcal{H}}_{rad}$ by the formula

$$\tilde{w}(f_0,..,f_{n-1}) = \sum_{l=0}^{n-1}\tilde{w}_l(f_l)\ \forall f_0,..,f_{n-1}\in L^2(\mathbf{R}_+) \tag{5.8}$$

It follows immediately from this equation and (5.6) that these maps, $\tilde{w}_l$, satisfy the CCR

$$[\tilde{w}_k(p),\tilde{w}_l(q)^\star]_- = (q,p)\delta_{kl};\ [\tilde{w}_k(p),\tilde{w}_l(q)]_- = 0\ \forall p,q\in L^2(\mathbf{R}_+) \tag{5.9}$$

We formulate structure of $\tilde{\mathcal{A}}_{rad}$ in terms of the Weyl map $\tilde{W} : H\to\mathcal{L}(\tilde{\mathcal{H}}_{rad})$, defined by the formula

$$\tilde{W}(f) = \exp i(\tilde{w}(f)+\tilde{w}(f)^\star)\ \forall f\in H \tag{5.10}$$

Thus, by (5.6), we may express the CCR as the Weyl relations

$$\tilde{W}(f)\tilde{W}(g) = \tilde{W}(f+g)\exp i(Im(f,g)_H)\ \forall f,g\in H \tag{5.11}$$

Hence, the algebra of polynomials in the Weyl operators $\tilde{W}(f)$ is simply their linear span. Further, by standard arguments, its weak closure is $\tilde{\mathcal{A}}$, and the state $\tilde{\phi}_{rad}$ is completely specified by the formula

$$\tilde{\phi}_{rad}(\tilde{W}(f)) = \exp(-\frac{1}{2}\|f\|_H^2)\ \forall f\in H \tag{5.12}$$

Note. It follows easily from equns. (5.7), (5.11) and (5.12) that, if f, g are elements of H, the intersection of whose supports is of Lebesgue measure zero, then $\tilde{W}(f)$ and $\tilde{W}(g)$ intercommute and are uncorrelated in the state $\tilde{\phi}_{rad}$, i.e. $\tilde{\phi}_{rad}(\tilde{W}(f)\tilde{W}(g)) = \tilde{\phi}_{rad}(\tilde{W}(f))\tilde{\phi}_{rad}(\tilde{W}(g))$.

In order to specify the temporally local structure of $\tilde{\mathcal{A}}_{rad}$, we introduce the set, Γ, of closed intervals in $\mathbf{R}_+$, and, for $J\in\Gamma$, we define J_c to be the closure of $\mathbf{R}_+\backslash J$. We then define H_J and H_{J_c} to be the subspaces of H given by $\{f\in H|supp(f)\in J\ (resp.\ J_c)\}$, and correspondingly, for $K = J$ or J_c, we define $(\tilde{\mathcal{H}}_{K,rad},\ \tilde{\Phi}_{K,rad},\ \tilde{\phi}_{K,rad},\ \tilde{\mathcal{A}}_{K,rad})$ to be the objects obtained by replacing H by H_K in the above definitions of $(\tilde{\mathcal{H}}_{rad},\ \tilde{\Phi}_{rad},\ \tilde{\phi}_{rad},\ \tilde{\mathcal{A}}_{rad})$. Thus, $\tilde{\mathcal{H}}_{rad},\ \tilde{\Phi}_{rad},\ \tilde{\mathcal{A}}_{rad},\ \tilde{\phi}_{rad}$ may be canonically identified with the tensor products $\tilde{\mathcal{H}}_{J,rad}\otimes\tilde{\mathcal{H}}_{J_c,rad},\ \tilde{\Phi}_{J,rad}\otimes\tilde{\Phi}_{J_c,rad},\ \tilde{\mathcal{A}}_{J,rad}\otimes\tilde{\mathcal{A}}_{J_c,rad},\ \tilde{\phi}_{J,rad}\otimes\tilde{\phi}_{J_c,rad}$, respectively.

Now, it follows from the Note after equn. (5.12) that, if the interiors of J, K are mutually disjoint and A, B belong to $\tilde{\mathcal{A}}_{J,rad}$ and $\tilde{\mathcal{A}}_{K,rad}$, respectively, then $\tilde{\phi}_{rad}(AB)$ factorises into the product $\tilde{\phi}_{rad}(A)\tilde{\phi}_{rad}(B)$. Hence, by our specifications of its local structure, the process $\tilde{w}$ is equipped with conditional expectations (CE's) $\{\tilde{E}_{J,rad} : \tilde{\mathcal{A}}_{rad}\rightarrow\tilde{\mathcal{A}}_{J,rad}|J\in\Gamma\}$, defined by the formula

$$(\tilde{\Psi}_1, \tilde{E}_{J,rad}(A)\tilde{\Psi}_2) = (\tilde{\Psi}_1\otimes\tilde{\Phi}_{J_c}, A(\tilde{\Psi}_2\otimes\tilde{\Phi}_{J_c})),\ \forall A\in\tilde{\mathcal{A}}_{rad},\ \tilde{\Psi}_1, \tilde{\Psi}_2\in\tilde{\mathcal{H}}_{J,rad} \tag{5.13}$$

or, equivalently,

$$\tilde{E}_{J,rad}(\tilde{W}(f)) = \tilde{W}(\chi_J f)\exp(-\frac{1}{2}\|(1-\chi_J)f\|_H^2)\ \forall f\in H,\ J\in\Gamma \tag*{(5.13)'}$$

where χ_J is the index function for J, acting multiplicatively on H. These CE's evidently satisfy the projective condition

$$\tilde{E}_{J,rad}\tilde{E}_{K,rad} = \tilde{E}_{J\cap K,rad} \tag{5.14}$$

and are compatible with $\tilde{\phi}_{rad}$, i.e., $\tilde{\phi}_{rad}\equiv\tilde{\phi}_{rad}\circ\tilde{E}_{J,rad}$.

C. The Radiation Process. To couple the field $\tilde{w}$ to the radiation modes of §2C, we define $\hat{\mathcal{H}}_{rad} = \mathcal{H}_{rad}\otimes\tilde{\mathcal{H}}_{rad},\ \hat{\mathcal{A}}_{rad} = \mathcal{A}_{rad}\otimes\tilde{\mathcal{A}}_{rad}\equiv\mathcal{L}(\hat{\mathcal{H}}_{rad})$, and identify the operators $a, W(z)$, in $\mathcal{H}_{rad}$, and $\tilde{w}(f), \tilde{W}(f)$, in $\tilde{\mathcal{H}}_{rad}$, with $a\otimes\tilde{I},\ W(z)\otimes\tilde{I},\ I\otimes\tilde{w}(f), I\otimes\tilde{W}(f)$, respectively, in $\hat{\mathcal{H}}_{rad}$. We then define the injection $\mathcal{I}_{rad} : \mathcal{A}_{rad}\rightarrow\hat{\mathcal{A}}_{rad}$ and the projection $\mathcal{P}_{rad} : \hat{\mathcal{A}}_{rad}\rightarrow\mathcal{A}_{rad}$ by the equations

$$\mathcal{I}_{rad}A = A\otimes\tilde{I}\ \forall A\in\mathcal{A}_{rad} \tag{5.15}$$

and

$$\mathcal{P}_{rad}(A\otimes\tilde{A}) = \tilde{\phi}_{rad}(\tilde{A})A \ \forall A\in\mathcal{A}_{rad}, \tilde{A}\in\tilde{\mathcal{A}}_{rad} \tag{5.16}$$

We assume the following Langevin equation of motion for the oscillators, under the action of the Wiener field $\tilde{w}$.

$$a_l(t) - a_l + (i\omega_l + \kappa_l)\int_0^t ds a_l(s) = (2\kappa_l)^{1/2}\tilde{w}_l(\chi_{[0,t]}) \tag{5.16}$$

The solution of this equation is

$$a_l(t) = a_l \exp(-(\kappa_l + i\omega_l)t) + (2\kappa)^{1/2}\tilde{w}_l(h_{t,l}) \tag{5.17}$$

where

$$h_{t,l}(s) = \chi_{[0,t]}(s)\exp(-(\kappa_l + i\omega_l)(t-s)) \tag{5.18}$$

Hence, defining $h_t = (h_{t,0}, . ., h_{t,n-1})$, it follows from (2.9), (2.12), (5.10) and (5.16) that the transformation $a \rightarrow a(t)$ sends $W(z)$ to $W_t(z)$, as defined by the formula

$$W_t(z) = W(\xi(t)z)\otimes\tilde{W}([z.h_t]) \ with \ [z.h_t]_l = z_l h_{t,l} \tag{5.19}$$

We define $\mathcal{A}_{t,rad}$ and $\hat{\mathcal{A}}_{rad}$ to be the $W^\star-$algebras generated by $\{W_t(z)|z\in\mathbf{C}^n\}$ and $\{W_t(z)|z\in\mathbf{C}^n, \ t\in\mathbf{R}_+\}$ $(\equiv\{\mathcal{A}_{t,rad}|t\in\mathbf{R}_+\})$, respectively. In fact, $\hat{\mathcal{A}}_{rad}$ is the tensor product $\mathcal{A}_{rad}\otimes\tilde{\mathcal{A}}_{rad}$, since the linear span of $\{[z.h_t]|z\in\mathbf{C}^n, t\in\mathbf{R}_+\}$ is dense in H, and consequently $\{\tilde{W}([z.h_t])|z\in\mathbf{C}^n, t\in\mathbf{R}_+\}'' = \tilde{\mathcal{A}}_{rad}$.

We define $\mathcal{J}_{rad}(t)$ to be the mapping of $\mathcal{A}_{rad}$ onto $\mathcal{A}_{t,rad}$ given by

$$\mathcal{J}_{rad}(t)W(z) = W_t(z) \tag{5.20}$$

Thus, by (5.17) and (5.18), $\mathcal{J}_{rad}(t)$ is an isomorphism. By equns. (2.11), (5.12), (5.14), (5.15) and (5.20), its relation to $T_{rad}(t)$ is given, analogously with (5.4), by the formula

$$T_{rad}(t) = \mathcal{P}_{rad}\circ\mathcal{J}_{rad}(t) \tag{5.21}$$

We define the semigroup, $\hat{T}_{rad}(\mathbf{R}_+)$, of isomorphisms of $\hat{\mathcal{A}}_{rad}$ by the formula

$$\hat{T}_{rad}(t)[(\mathcal{J}_{rad}(t_1)A_1).. .(\mathcal{J}_{rad}(t_m)A_m)] =$$

$$(\mathcal{J}_{rad}((t_1+t)A_1).. .(\mathcal{J}_{rad}(t_m+t)A_m) \ \forall A_1, . ., A_m\in\mathcal{A}_{rad}, \ t_1, . ., t_m\in\mathbf{R}_+ \tag{5.22}$$

Thus, the dynamical system $\hat{\Sigma}_{rad} = (\hat{\mathcal{A}}_{rad}, \hat{T}_{rad}, \hat{\phi}_{rad})$ is a minimal dilation of Σ_{rad}. One checks easily from our specifications that $\hat{T}_{rad}(t)$ is weakly continuous in t and that $\hat{\phi}_{rad}$ is $\hat{T}_{rad}-$invariant.

To specify the local properties of $\hat{\Sigma}_{rad}$, we define $\mathcal{A}_{J,rad}$ to be $\mathcal{A}_{rad}\otimes\tilde{\mathcal{A}}_{J,rad}$, for $J\in\Gamma$, and $\{E_{J,rad} : \hat{\mathcal{A}}_{rad}\rightarrow\mathcal{A}_{J,rad}|J\in\Gamma\}$ to be the conditional expectations given by the formula

$$E_{J,rad} = I\otimes\tilde{E}_{J,rad} \tag{5.23}$$

To lighten the notation a little, we shall denote $\mathcal{A}_{[0,t],rad}$ and $\mathcal{A}_{[t,\infty),rad}$ by $\mathcal{A}_{t\mp,rad}$, respectively, and put

$$E_{t,rad}\equiv E_{[0,t],rad} \tag{5.24}$$

It follows now from equations (5.13), (5.14), (5.19), (5.23) and (5.24) that these CE's are compatible with $\hat{\phi}_{rad}$.

Proposition 5.1. *The CE's $E_{t,rad}$ possess the Markov property*

$$E_{t,rad}(\mathcal{A}_{t+,rad})\subset\mathcal{A}_{t,rad} \tag{5.25}$$

Thus, by the $\hat{T}_{rad}-$invariance of $\hat{\phi}_{rad}$, the process $\Pi_{rad}=(\hat{\mathcal{A}}_{rad},\mathcal{J}_{rad},\hat{\phi}_{rad})$ is stationary and Markovian.

Proof. For $t_1,..,t_m\geq t$ and $z^{(1)},..,z^{(m)}\in\mathbf{C}^n$, it follows from equns. (5.11), (5.13)$'$, (5.19) and (5.23) that $E_{t,rad}[W_{t_1}(z^{(1)})...W_{t_m}(z^{(m)})]$ is a scalar multiple of $W_t(\sum_{j=1}^m\xi(t_j-t)z_j)$ and therefore lies in $\mathcal{A}_{t,rad}$. Hence, as the algebra $\mathcal{A}_{t_j,rad}$ is generated by $W_{t_j}(\mathbf{C}^n)$, the normality of $E_{t,rad}$ ensures that the Markov condition (5.25) is fulfilled.

Note. Since the vectors Φ_{rad} and $\tilde{\Phi}_{rad}$ lie in the domains of the polynomials in $a,a^\star$ and $\tilde{w}(f),\tilde{w}(f)^\star$, respectively, it follows from (5.13), (5.15), (5.16), (5.23) and (5.24) that $\mathcal{I}_{rad}$, $\mathcal{P}_{rad}$ and $E_{t,rad}$ have canonical extensions to polynomials in $a,a^\star$ and their images under $\{\mathcal{J}(t)\}$.

D. The Uncoupled Matter-cum-Field Process. Let $\Sigma^{(0)}=(\mathcal{A},T^{(0)},\phi)$ be the composite of Σ_{mat} and Σ_{rad} when these latter systems are uncoupled. Thus,

$$(\mathcal{A}=\mathcal{A}_{mat}\otimes\mathcal{A}_{rad},\ T^{(0)}=T_{mat}\otimes T_{rad},\ \phi=\phi_{mat}\otimes\phi_{rad}),$$

and hence, by Sections VA,B, $\Sigma^{(0)}$ has a minimum dilation to

$$\hat{\Sigma}^{(0)}=(\hat{\mathcal{A}}=\hat{\mathcal{A}}_{mat}\otimes\hat{\mathcal{A}}_{rad},\ \hat{T}^{(0)}=\hat{T}_{mat}\otimes\hat{T}_{rad},\ \hat{\phi}=\hat{\phi}_{mat}\otimes\hat{\phi}_{rad}).$$

Evidently, $\hat{T}^{(0)}$ is a weakly continuous one-parameter semigroup of isomorphisms of $\hat{\mathcal{A}}$. It follows from the definitions of §2 and the present Section that $\hat{\mathcal{A}}=\mathcal{A}\otimes\tilde{\mathcal{A}}$ and $\hat{\phi}=\phi\otimes\tilde{\phi}$, where $\tilde{\mathcal{A}}=\tilde{\mathcal{A}}_{mat}\otimes\tilde{\mathcal{A}}_{rad}$ and $\tilde{\phi}=\tilde{\phi}_{mat}\otimes\tilde{\phi}_{rad}$. Thus, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ are $W^\star-$algebras of operators in the Hilbert spaces $\tilde{\mathcal{H}}=\tilde{\mathcal{H}}_{mat}\otimes\tilde{\mathcal{H}}_{rad}$ and $\hat{\mathcal{H}}=\hat{\mathcal{H}}_{mat}\otimes\hat{\mathcal{H}}_{rad}$, respectively.

We define the injection $\mathcal{I}:\mathcal{A}\rightarrow\hat{\mathcal{A}}$, the projection $\mathcal{P}:\hat{\mathcal{A}}\rightarrow\mathcal{A}$ and the isomorphisms $\mathcal{J}^{(0)}(t)$ of $\mathcal{A}$ into $\hat{\mathcal{A}}$ to be $\mathcal{I}_{mat}\otimes\mathcal{I}_{rad}$, $\mathcal{P}_{mat}\otimes\mathcal{P}_{rad}$ and $\mathcal{J}_{mat}(t)\otimes\mathcal{J}_{rad}(t)$, respectively. Hence, by these definitions and those of $\S's$ 5A,C,

$$\mathcal{I}A=A\otimes\tilde{I}\ \forall A\in\mathcal{A} \tag{5.26}$$

$$\mathcal{P}(A\otimes\tilde{A})=\tilde{\phi}(\tilde{A})A\ \forall A\in\mathcal{A},\tilde{A}\in\tilde{\mathcal{A}} \tag{5.27}$$

and

$$T^{(0)}(t)=\mathcal{P}\circ\mathcal{J}^{(0)}(t) \tag{5.28}$$

For $t\in\mathbf{R}_+$, we define $\mathcal{A}_t^{(0)}$ to be $\mathcal{J}^{(0)}(t)\mathcal{A}$ and $\mathcal{A}_{t\pm}^{(0)}$ to be the $W^\star-$algebras generated by $\{\mathcal{A}_u^{(0)}|u\geq(resp.\ \leq t)\}$. For $s,t(>s)\in\mathbf{R}_+$, we define $\mathcal{A}_{[s,t]}^{(0)}$ to be the $W^\star-$algebra generated by $\mathcal{A}_u|s\leq u\leq t\}$.

The dynamics of $\hat{\Sigma}^{(0)}$ is given by the stochastic process $\Pi^{(0)}=(\hat{\mathcal{A}},\mathcal{J}^{(0)},\hat{\phi})$ over $\mathcal{A}$, which inherits the stationarity property from its components Π_{mat} and Π_{rad}. Evidently, $\Pi^{(0)}$ is equipped with conditional expectations $\{E_t^{(0)}(=E_{t,mat}\otimes E_{t,rad}):\hat{\mathcal{A}}\rightarrow\mathcal{A}_{t-}^{(0)}|t\in\mathbf{R}_+\}$, possessing the following standard properties.

(CE1)

$$E_t^{(0)}(AB)=E_t^{(0)}(A)B\ \forall A\in\hat{\mathcal{A}},B\in\mathcal{A}_{t-}^{(0)}$$

(CE2)

$$E_t^{(0)}E_s^{(0)}=E_{s\wedge t}^{(0)},\ with\ s\wedge t=min\{s,t\}\ (projectivity)$$

(CE3)

$$\hat{\phi}\circ E_t^{(0)}=\hat{\phi}\ (compatibility)$$

Furthermore, the process $\Pi^{(0)}$ inherits the Markov property from its constituents Π_{mat} and Π_{rad}, i.e.

(CE4)

$$E_t^{(0)}(\mathcal{A}_{t+}^{(0)})\subset\mathcal{A}_t^{(0)}$$

Note. It follows from Def. 2.1 and the observation at the end of §5C that $\mathcal{I}$, $\mathcal{P}$, $\mathcal{J}^{(0)}(t)$, $T(t)$ and $E_t^{(0)}$ have canonical extensions to the algebra $\mathcal{F}(\mathcal{A})$. In particular, by (2.18), the time-translate of the element H_{int} of this algebra is

$$H_{int}(t)=\mathcal{J}^{(0)}(t)H_{int}=$$

$$iN^{-1/2}\sum_{r=1}^{N}\sum_{l=0}^{n-1}\lambda_l(\sigma_{-,r}(t)a_l^\star(t)-h.c.) \tag{5.29}$$

with

$$\sigma_{+,r}(t)=\mathcal{J}_{mat}(t)\sigma_{+,r}\ and\ a_l(t)=\mathcal{J}_{rad}(t)a_l \tag{5.29$'$}$$

this last quantity being given by (5.17).

E. The Interactive Dynamics. We shall now formulate the dynamics of the system $\hat{\Sigma}$, formed by coupling $\hat{\Sigma}_{mat}$ and $\hat{\Sigma}_{rad}$ by the interactions H_{int}. For this, we employ an interaction representation, in the form of a two-parameter family $\{V(t,s)|t,s\in\mathbf{R}_+\}$ of unitary transformation of $\hat{\mathcal{H}}$, which implement a Markovian cocycle of automorphisms of $\hat{\mathcal{A}}$, corresponding to this coupling.

Definition 5.2. (1) For $L\in\mathbf{N}$, we define $\tilde{\mathcal{D}}_{L,rad}$ to be the subspace of $\tilde{\mathcal{H}}_{rad}$ generated by application to $\tilde{\Phi}_{rad}$ of all polynomials of degree L in $\{\tilde{w}(f)^\star|f\in H\}$. This is simply the subspace in which the number of quanta of the field $\tilde{w}$ does not exceed L.

(2) We define $\tilde{\mathcal{D}}_L$ to be the subspace $\tilde{\mathcal{H}}_{mat}\otimes\tilde{\mathcal{D}}_{L,rad}$ of $\tilde{\mathcal{H}}$, and $\hat{\mathcal{D}}_L$ ($\subset\hat{\mathcal{H}}$) to be $\mathcal{D}\otimes\tilde{\mathcal{D}}_L$, respectively, where $\mathcal{D}$ is the domain of $\mathcal{H}$ specified in Def. 2.1.

(3) We define $\hat{\mathcal{D}}$ to be the dense domain $\cup_{L\in\mathbf{N}}\hat{\mathcal{D}}_L$ of $\hat{\mathcal{H}}$.

Proposition 5.3. *Let*

$$V_m(t,s)=\int_s^t dt_1..\,.\int_s^t dt_m\mathcal{T}[H_{int}(t_1).\;.H_{int}(t_m)],\ \forall m\in\mathbf{N} \tag{5.30}$$

with $\mathcal{T}$ the time-ordering operator, and let

$$V(t,s)=\sum_{m=0}^{\infty}\frac{(-i)^m V_m(t,s)}{m!} \tag{5.31}$$

Then this sum is strongly convergent on $\hat{\mathcal{D}}$, and defines $V(t,s)$ as an isometry of $\hat{\mathcal{D}}$ into $\hat{\mathcal{H}}$. Moreover, this operator extends by continuity to a unitary element of $\mathcal{A}^{(0)}_{[s,t]}$, that is strongly continuous in both its arguments and satisfies the conditions

$$V(u,t)V(t,s)=V(u,s)\ for\ s\le t\le u \tag{5.32}$$

and

$$\hat{T}^{(0)}(u)V(t,s)=V(t+u,s+u)\ for\ t,s(\le t),u\in\mathbf{R}_+ \tag{5.33}$$

Definition 5.4. (1) We define $\{\mathcal{V}(t,s)|t,s(\le t)\in\mathbf{R}_+\}$ to be the two-parameter family of automorphisms of $\hat{\mathcal{A}}$, implemented by V according to the formula

$$\mathcal{V}(t,s)=V(t,s)^{\star}(.)V(t,s) \tag{5.34}$$

(2) We define $\{\hat{T}(t)|t\in\mathbf{R}_+\}$ to be the family of isomorphisms of $\hat{\mathcal{A}}$ given by the formula

$$\hat{T}(t)=\mathcal{V}(t,0)\hat{T}^{(0)}(t)\ \forall t\in\mathbf{R}_+ \tag{5.35}$$

(3) We define $\{T(t)|t\in\mathbf{R}_+\}$ to be the family of transformations of $\mathcal{A}$ by the formula

$$T(t)=\mathcal{P}\circ\hat{T}(t)\circ\mathcal{I}\equiv\mathcal{I}^{-1}\circ E_0^{(0)}\circ\hat{T}(t)\circ\mathcal{I}\ \forall t\in\mathbf{R}_+ \tag{5.36}$$

Proposition 5.5. *(1) The automorphisms $\mathcal{V}$ are weakly continuous in both arguments and satisfy the relation*

$$\mathcal{V}(t,s)\mathcal{V}(u,t)=\mathcal{V}(u,s)\ for\ s\le t\le u \tag{5.37}$$

and the cocycle condition

$$\mathcal{V}(t+s,t)\hat{T}^{(0)}(t)=\hat{T}^{(0)}(t)\mathcal{V}(s,0)\ \forall s,t\in\mathbf{R}_+ \tag{5.38}$$

(2) The isomorphisms $\hat{T}$ form a weakly continuous one-parameter semigroup, i.e.,

$$\hat{T}(s)\hat{T}(t)=\hat{T}(s+t)\ \forall s,t\in\mathbf{R}_+ \tag{5.39}$$

(3) $T(\mathbf{R}_+)$ is a weakly continuous, one-parameter semigroup of contractions of $\mathcal{A}$.

Proof of Prop. 5.5, assuming Prop. 5.3. (1) The relation (5.37) is an immediate consequence of equns. (5.32) and (5.34), while (5.38) follows from (5.33)-(5.35). Since $V(t,s)$ is continous, it follows from (5.34) that $\mathcal{V}$ is weakly continuous, in both its arguments.

(2) Since $\hat{T}^{(0)}$ is a one-parameter semigroup, it follows from equns. (5.35) and (5.36) that so too is $\hat{T}$. Further, by (5.34) and (5.35), the weak continuity of $\hat{T}(t)$ follows from that of $\hat{T}^{(0)}(t)$, and the strong continuity of the unitary $V(t,0)$.

(3) Since $\hat{T}(t)$ is an isomorphism of $\hat{\mathcal{A}}$, it follows immediately from (5.27) and (5.36) that $T(t)$ is a CP contraction of $\mathcal{A}$. To prove the semigroup property of T, we note that, by (5.35) and (5.36),

$$T(t)T(s)A=\mathcal{I}^{-1}E_0^{(0)}\mathcal{V}(t,0)\hat{T}^{(0)}(t)E_0^{(0)}\mathcal{V}(s,0)\hat{T}^{(0)}(s)(A\otimes\tilde{I})\ \forall A\in\mathcal{A} \tag{5.40}$$

Further, by the stationarity of the process $\Pi^{(0)}$, together with the compatibility condition (CE3), $\hat{T}^{(0)}(t)E_0^{(0)}\equiv E_t^{(0)}\hat{T}^{(0)}(t)$, and therefore, by equns. (5.33) and (5.34) and the semigroup property of $\hat{T}^{(0)}$,

$$\hat{T}^{(0)}(t)E_0^{(0)}\mathcal{V}(s,0)\hat{T}^{(0)}(s)(A\otimes\tilde{I})\equiv E_t^{(0)}\mathcal{V}(s+t,t)\hat{T}^{(0)}(s+t)(A\otimes\tilde{I})$$

and this belongs to $\mathcal{A}_t^{(0)}$, by the Markov property of $\Pi^{(0)}$. Hence, as $V(t,0)\in\mathcal{A}_{[0,t]}$, it follows from the fact that $E_t^{(0)}$ is a CE from $\hat{\mathcal{A}}$ onto $\mathcal{A}_{[0,t]}$ that

$$\mathcal{V}(t,0)E_t^{(0)}\mathcal{V}(s+t,t)\hat{T}^{(0)}(s+t)(A\otimes\tilde{I})\equiv E_t^{(0)}\mathcal{V}(t,0)\mathcal{V}(s+t,t)\hat{T}^{(0)}(s+t)(A\otimes\tilde{I})$$

$$\equiv E_t^{(0)}\mathcal{V}(s+t,0)\hat{T}^{(0)}(s+t)(A\otimes\tilde{I}),\ \textit{by}\ (5.37)$$

Consequently, the r.h.s. of (5.40) is

$$\mathcal{I}^{-1}E_0^{(0)}E_t^{(0)}\mathcal{V}(s+t,0)\hat{T}^{(0)}(s+t)(A\otimes\tilde{I})$$

and since $E_0^{(0)}E_t^{(0)}=E_0^{(0)}$, by the projectivity of the CE's, it follows from (5.26) that this is equal to $T(t+s)A$, as required.

Proof of Prop. 5.3. By equns. (5.17) and (5.29), Defs. 2.1 and 5.4, together with the boundedness of the spins σ_r and the elementary properties of creation and annihilation operators $a^\star,\tilde{w}(f)^\star,a$ and $\tilde{w}(f)$, the action of the operator $H_{int}(t_1).\ .H_{int}(t_m)$ on $\hat{\mathcal{D}}$ is continuous in all the $t's$, and hence, by (5.30), $V_m(t,s)$ is continuous there in both s and

t. In particular, the application of $V_m(t,s)$ to vectors $\Psi\otimes\tilde{\Psi}$, with $\Psi\in\mathcal{D}$ and $\tilde{\Psi}\in\tilde{\mathcal{D}}_L$, yields the estimate

$$\|V_m(t,s)(\Psi\otimes\tilde{\Psi})\| < B^m \sum_{r=0}^{m} \frac{m!}{(m-r)!r!} M_r(\Psi)[L(L+1)..(L+m-r)]^{1/2}(t-s)^m$$

where $M_r(\Psi)$ is defined in Def. 2.1 and B is a finite constant, whose value depends on N and the $\lambda's$. Hence,

$$\sum_{m=0}^{\infty} \frac{\|V_m(t,s)(\Psi\otimes\tilde{\Psi})\|}{m!} < \sum_{r=0}^{\infty} M_r(\Psi)(B(t-s))^r/r! \sum_{m=0}^{\infty} [(L+1)..(L+m)]^{1/2}/m!$$

and, by Def.2.1 (3), this converges, uniformly w.r.t. s and t, on the compacts in $\mathbf{R}_+^2$. In view of (5.31), this implies that the continuity of $V_m(t,s)$, in both its arguments, on the vectors $\Psi\otimes\tilde{\Psi}$, implies that of $V(t,s)$. Hence, by linearity, this operator is continuous w.r.t. s and t on $\hat{\mathcal{D}}$. Further, it is a straightforward matter to establish the same result for the operators $H_{int}(t)V(t,s)$ and $V(t,s)H_{int}(s)$.

Having established the convergence and continuity properties of V on $\hat{\mathcal{D}}$, we infer from (5.30) and (5.31) that

$$V(t,s) = I - i\int_s^t du H_{int}(u)V(u,s) = I + i\int_s^t du V(t,u)H_{int}(u)$$

on this domain. Hence, by the continuity of the integrands here,

$$s: \frac{d}{dt}V(t,s) = -iH_{int}(t)V(t,s);\ and\ s: \frac{d}{ds}V(t,s) = iV(t,s)H_{int}(s) \tag{5.41}$$

on $\hat{\mathcal{D}}$. Consequently,

$$\frac{d}{dt}(V(t,s)\hat{\Psi}_1, V(t,s)\hat{\Psi}_2) = 0\ \forall\hat{\Psi}_1, \hat{\Psi}_2\in\hat{\mathcal{D}}$$

which implies that

$$(V(t,s)\hat{\Psi}_1, V(t,s)\hat{\Psi}_2) = (\hat{\Psi}_1, \hat{\Psi}_2)\ \forall\hat{\Psi}_1, \hat{\Psi}_2\in\hat{\mathcal{D}}$$

since $V(s,s) = I$. Thus, $V(t,s)$ maps $\hat{\mathcal{D}}$ isometrically into $\hat{\mathcal{H}}$, and therefore extends by continuity to an isometry of $\hat{\mathcal{H}}$.

To show that this extension, which we also denote by $V(t,s)$, is unitary, it suffices to prove that its adjoint, $V^\star(t,s)$, is also isometric. Now it follows from (5.30) and (5.31) that the restriction of $V^\star(t,s)$ to $\hat{\mathcal{D}}$ is

$$\sum_{m=0}^{\infty} \frac{i^m}{m!}\int_s^t dt_1 ..\int_s^t dt_m \tilde{\mathcal{T}}[H_{int}(t_1)..H_{int}(t_m)]$$

where $\tilde{\mathcal{T}}$ is the anti-chronological ordering operator. Thus, proceeding as in the passage from (5.31) to (5.40), we obtain the equations

$$s: \frac{d}{dt}V^{\star}(t,s) = iV^{\star}(t,s)H_{int}(t);\ and\ s: \frac{d}{ds}V^{\star}(t,s) = -iH_{int}(s)V^{\star}(t,s)\ on\ \hat{\mathcal{D}} \quad (5.42)$$

The isometric property of $V^{\star}(t,s)$ now follows from these equations, by the same argument that we used to establish it for $V(t,s)$.

To show that V satisfies the condition (5.32), we note that, by (5.41) and (5.42)

$$\frac{d}{dt}(V^{\star}(u,t)\hat{\Psi}_1, V(t,s)\hat{\Psi}_2) = 0\ \forall\hat{\Psi}_1, \hat{\Psi}_2\in\hat{\mathcal{D}}$$

Hence, as $V^{\star}(u,u) = I$,

$$(V^{\star}(u,t)\hat{\Psi}_1, V(t,s)\hat{\Psi}_2) = (\hat{\Psi}_1, V(u,s)\hat{\Psi}_2)$$

for $\hat{\Psi}_1, \hat{\Psi}_2$ in $\hat{\mathcal{D}}$, and hence, by continuity, in $\hat{\mathcal{H}}$. This is equivalent to (5.32).

To obtain (5.33), we first infer from equns. (5.30) and (5.31), together with the definition of $H_{int}(t)$ as $\mathcal{J}^{(0)}(t)H_{int}\equiv\hat{T}^{(0)}(t)[H_{int}\otimes\tilde{I}]$, that

$$\hat{T}^{(0)}(u)V(t,s) = V(t+u, s+u)$$

on $\hat{\mathcal{D}}$ and hence, by continuity, of $\hat{\mathcal{H}}$.

Further, since, as noted above, the restriction of $V(s,t)$ to $\hat{\mathcal{D}}$ is continuous in both s and t, the same is true for this operator throughout the space $\hat{\mathcal{H}}$.

Finally, as $H_{int}(t)(=\hat{T}^{(0)}(t)H_{int})$ is affiliated to $\hat{\mathcal{A}}^{(0)}_{[s,t]}$, it follows from (5.29) and (5.30) that, since $V(t,s)$ is bounded, it must belong to this algebra.

F. Proof of Proposition 2.3. Let $\tilde{\Phi} = \tilde{\Phi}_{mat}\otimes\tilde{\Phi}_{rad}$, where the two components of this tensor product are as defined in $\S's$ 5A,B. Then, since $\psi = (\Psi.,.\Psi)$, it follows from equns. (2.20), (5.27) and (5.36) that

$$\psi_t(A) = \langle V(t,0)(\Psi\otimes\tilde{\Phi}), [\hat{T}^{(0)}(t)(A\otimes\tilde{I})]V(t,0)(\Psi\otimes\tilde{\Phi})\rangle\ \forall A\in\mathcal{A} \quad (5.43)$$

where $\langle .,.\rangle$ denotes the $\hat{\mathcal{H}}$ inner product. By a simple adaptation of the argument, employed in the proof of Prop. 5.3, to establish that the restriction of $V(t,s)$ to $\hat{\mathcal{D}}$ is continuous in both s and t, one readily establishes the same thing for $(\hat{T}^{(0)}(t)(Q\otimes\tilde{I})V(t,0))$, where $Q\in\mathcal{F}(\mathcal{A})$. Thus, by (5.43), ψ_t extends to a state on $\mathcal{F}(\mathcal{A})$, and $\psi_t(Q)$ is continuous in t for all $Q\in\mathcal{F}(\mathcal{A})$.

To derive the equation of motion (2.22), we note that, for such Q, it follows from (5.43) that

$$h^{-1}[\psi_{t+h}(Q)-\psi_t(Q)]=$$

$$\langle[\frac{V(t+h,0)-V(t,0)}{h}](\Psi\otimes\tilde{\Phi}),[\hat{T}^{(0)}(t+h)(Q\otimes\tilde{I})]V(t+h,0)(\Psi\otimes\tilde{\Phi})\rangle+ \tag{5.44a}$$

$$\langle V(t,0)(\Psi\otimes\tilde{\Phi}),[\hat{T}^{(0)}(t+h)(Q\otimes\tilde{I})](\frac{V(t+h,0)-V(t,0)}{h})(\Psi\otimes\tilde{\Phi})\rangle+ \tag{5.44b}$$

$$\langle V(t,0)(\Psi\otimes\tilde{\Phi}),[(\frac{\hat{T}^{(0)}(t+h)-\hat{T}^{(0)}(t)}{h})(Q\otimes\tilde{I})]V(t,0)(\Psi\otimes\tilde{\Phi})\rangle \tag{5.44c}$$

In our treatment of these three terms, as in the proof of Prop. 5.3, we shall repeatedly utilise the fact that, on $\hat{\mathcal{D}}$, the polynomials in the operators $a^\sharp(t)$ and $V(t,s)$ are continuous in all their arguments.

Thus, these continuity properties, together with (5.41), imply that, as $h\to0$,

$$Term\ (a)\to-\langle iH_{int}(t)V(t,0)(\Psi\otimes\tilde{\Phi}),[\hat{T}^{(0)}(t)(Q\otimes\tilde{I})]V(t,0)(\Psi\otimes\tilde{\Phi})\rangle$$

$$\equiv i\langle V(t,0)(\Psi\otimes\tilde{\Phi}),[\hat{T}^{(0)}(t)(H_{int}Q\otimes\tilde{I})]V(t,0)(\Psi\otimes\tilde{\Phi})\rangle$$

since $H_{int}(t)=T^{(0)}(t)(H_{int}\otimes\tilde{I})$, and, by (2.20) and (5.36), this expression is equal to $i\psi_t(H_{int}Q)$. Similarly, the term (b) tends to $-i\psi_t(QH_{int})$ as $h\to0$, and consequently, by (2.19),

$$(a)+(b)\to\psi_t(L_{int}Q)\ as\ h\to0 \tag{5.45}$$

Further, by our definitions of the conditional expectations $E_t^{(0)}$, and in view of the continuity properties, on $\hat{\mathcal{D}}$, of polynomials in the operators $a^\sharp(t)$ and $V(t,s)$, term (c) may be re-expressed as the $\mathcal{H}$ inner product

$$\langle\Psi,\mathcal{I}^{-1}E_0^{(0)}[V(t,0)^\star(\frac{\mathcal{J}^{(0)}(t+h)Q-\mathcal{J}^{(0)}(t)Q}{h})V(t,0)]\Psi\rangle$$

Further, in view of the properties (CE1-3) of $E_t^{(0)}$, specified in §$5D$, this is identical to

$$\langle\Psi,\mathcal{I}^{-1}E_0^{(0)}E_t^{(0)}[V(t,0)^\star(\frac{\mathcal{J}^{(0)}(t+h)Q-\mathcal{J}^{(0)}(t)Q}{h})V(t,0)]\Psi\rangle$$

$$\equiv\langle\Psi,\mathcal{I}^{-1}E_0^{(0)}[V(t,0)^\star E_t^{(0)}[\frac{\mathcal{J}^{(0)}(t+h)Q-\mathcal{J}^{(0)}(t)Q}{h}]V(t,0)]\Psi\rangle$$

$$\equiv\langle\Psi,\mathcal{I}^{-1}E_0^{(0)}[V(t,0)^\star[\hat{T}^{(0)}(t)E_0^{(0)}(\frac{\mathcal{J}^{(0)}(h)Q-Q}{h})]V(t,0)]\Psi\rangle$$

and, by (5.36), this is equal to

$$\langle\Psi,T(t)\mathcal{P}(\frac{\mathcal{J}^{(0)}(h)Q-Q}{h})\Psi\rangle$$

Hence, as $\mathcal{P}\circ\mathcal{J}^{(0)}(h) = T_{mat}(h)\otimes T_{rad}(h)$, it follows that

$$term\ (c)\to\langle\Psi, [T(t)(L_{mat} + L_{rad})Q]\Psi\rangle\equiv\psi_t((L_{mat} + L_{rad})Q)\ as\ h\to 0 \tag{5.46}$$

The required result (2.22) is an immediate consequence of equns. (5.44)-(5.46).

VI. The Macroscopic Classical Limit.

We begin with two lemmas, that will be needed for the proofs of Props. 3.2 and 3.4; and we shall then prove those Propositions in reverse order, in accordance with their logical relationship.

Lemma 6.1. *Let ξ be a continuous, positive- valued function on $\mathbf{R}_+$, which satisfies the equation*

$$\xi(t) = \xi(0)\exp(-kt) + \int_0^t ds\theta(s)\exp(-k(t-s)) \tag{6.1}$$

where k is a positive constant and $\xi(0)$, θ satisfy the conditions

$$\xi(0)\leq B_1 \tag{6.2}$$

and

$$|\theta(s)|\leq B_2[\xi(s)]^{1/2} \tag{6.3}$$

and B_1, B_2 are finite constants. Then there are finite positive constants B, C, whose values depend on k, B_1, B_2 only, such that

(1) the function ξ is majorised by B; and

(2)

$$\xi(0)\leq(\xi(t) + C)\exp(kt)\ \forall t\in\mathbf{R}_+ \tag{6.4}$$

Proof. By (6.1)-(6.3),

$$\xi(t)\leq B_1\exp(-kt) + B_2\int_0^t ds\exp(-k(t-s))[\xi(s)]^{1/2} \tag{6.5}$$

and

$$\xi(0)\exp(-kt) - B_2\int_0^t ds\exp(-k(t-s))[\xi(s)]^{1/2}\leq\xi(t) \tag{6.6}$$

Hence, defining M_t to be $max\{\xi(s)|s\in[0,t]\}$, it follows from (6.5) that

$$M_t\leq B_1\exp(-kt) + k^{-1}B_2M_t^{1/2}(1-\exp(-kt))\leq B_1 + k^{-1}B_2M_t^{1/2}$$

and consequently that

$$M_t^{1/2}\leq\frac{1}{2}(k^{-1}B_2 + [k^{-2}B_2^2 + B_1]^{1/2})$$

This implies the result (1), with B equal to the square of the r.h.s. of this last inequality. It follows now from this result and (6.6) that

$$\xi(0)\exp(-kt)\leq\xi(t)+k^{-1}B_2B^{1/2}(1-\exp(-kt))\leq\xi(t)+k^{-1}B_2B^{1/2}$$

This implies the required result (2).

Lemma 6.2. *Assuming the equation of motion (2.22) and the initial condition (3.11), the time-dependent expectation value of* $\alpha^{(N)\star}\alpha^{(N)}$ *is uniformly bounded w.r.t.* N *and* t*, i.e., for some finite constant* D,

$$\psi_t^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)})<D\ \forall t\in\mathbf{R}_+;N\in\mathbf{N};l=0,..,n-1 \tag{6.7}$$

Proof. By equns. (2.14), (2.19), (2.22), (3.3) and (3.8),

$$\frac{d}{dt}\psi_t^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)})=-2\kappa_l\psi_t^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)})+2Re(\lambda_l\psi_t^{(N)}(\alpha_l^{(N)}s_l^{(N)\star}))$$

Hence,

$$\psi_t^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)})=\psi^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)})\exp(-2\kappa_lt)$$

$$+2\lambda_l\int_0^t ds\exp(-2\kappa_l(t-s))Re(\psi_s^{(N)}(\alpha_l^{(N)}s_l^{(N)\star})) \tag{6.8}$$

Further, by (3.6) and the Schwarz inequality,

$$|Re(\psi_s^{(N)}(\alpha_l^{(N)}s_l^{(N)\star}))|^2\leq\psi_s^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)})$$

In view of this inequality and the initial condition (3.11), the application of Lemma 6.1(1) to equation (6.8) yields the required result.

Proof of Proposition 3.4. In order to treat equns. (3.20), we first render them amenable to standard fixed point methods by applying a cut-off to their coupling terms of (3.19), and then remove the cut-off. To this end, we define the function $g_.$ on $\mathbf{R}_+$, by the formula[34]

$$g_t=\sum_{l=0}^{n-1}(4\overline{s}_{t,l}s_{t,l}+\overline{p}_{t,l}p_{t,l}) \tag{6.9}$$

We then introduce a map $(R,x)\rightarrow f_R(x)$ of $\mathbf{R}^2$ into $[0,1]$, such that $f_R(.)$ is a smooth function, which takes the value unity on $[0,R]$, vanishes on $[R+2,\infty)$, and whose first derivative is confined to $[-1,0]$. Correspondingly, we define $\mathcal{K}_R$ to be the modification of the dynamical system $\mathcal{K}$ obtained by multiplying the coupling terms of (3.19) by the factor $f(g_t)$. The equations of motion for $\mathcal{K}_R$ are therefore

$$\frac{d\alpha_{t,l}}{dt}=-(i\omega_l+\kappa_l)\alpha_{t,l}+\lambda_ls_{t,l}f_R(g_t) \tag{6.10a}$$

$$\frac{ds_{t,l}}{dt} = -(i\epsilon + \gamma_{\perp})s_{t,l} + \sum_{m=0}^{n-1} \lambda_m p_{t,[l-m]} \alpha_{t,m} f_R(g_t) \tag{6.10b}$$

and

$$\frac{dp_{t,l}}{dt} = -\gamma_{\|}(p_{t,l} - \eta\delta_{l,0}) + 2\sum_{m=0}^{n-1} \lambda_m (\overline{\alpha}_{t,m} s_{t,[l+m]} + \alpha_{t,m} \overline{s}_{t,[m-l]}) f_R(g_t) \tag{6.10c}$$

On re-expressing (6.10a) in the integral form

$$\alpha_{t,l} = \alpha_{0,l} \exp - (\kappa_l + i\omega_l)t + \lambda_l \int_0^t du \exp(-(\kappa_l + i\omega_l)(t-u)) s_{u,l} f_R(g_u) \tag{6.11}$$

and substituting this formula into (6.10b) and (6.10c), it follows from standard fixed point analysis that, for given initial data (α_0, s_0, p_0), equations (6.10) have a unique global solution.

To show that this solution becomes independent of R, when this parameter is sufficiently large, we introduce a positive constant γ, whose value does not exceed either $\gamma_{\perp}$ or $\gamma_{\|}$, and note that, by equns. (6.9) and (6.10b,c),

$$\frac{dg_t}{dt} + 2\gamma g_t = \sum_{l=0}^{n-1} [8(\gamma - \gamma_{\perp})\overline{s}_{t,l} s_{t,l} + 2(\gamma - \gamma_{\|})\overline{p}_{t,l} p_{t,l}] + 2\gamma_{\|}\eta p_{t,0}$$

and therefore

$$g_t = g_0 \exp(-2\gamma t) + 2\sum_{l=0}^{n-1} \int_0^t du \exp(-2\gamma(t-u))[\gamma_{\|} p_{u,0} + 4(\gamma - \gamma_{\perp})|s_{u,l}^2|^2 + (\gamma - \gamma_{\|})|p_{u,l}|^2] \tag{6.12}$$

Hence, as γ does not exceed either $\gamma_{\perp}$ or $\gamma_{\|}$, and as $p_{0,u} \le g_u^{1/2}$, by (6.9),

$$g_t \le g_0 \exp(-2\gamma t) + 2\gamma_{\|}\eta \int_0^t du \exp(-2\gamma(t-u)) g_u^{1/2}$$

Consequently, by Lemma 6.1(1), g_t is majorised by a finite g_0−dependent constant, G, i.e.

$$g_t \le G \ \forall t, R \in \mathbf{R}_+ \tag{6.13}$$

Hence, by (6.9),

$$|\alpha_{t,l}| \le \alpha_{0,l} + \frac{|\lambda_l| G^{1/2}}{2\kappa_l}$$

It follows immediately from this estimate, together with (6.9) and (6.13), that, if the phase point $x_t (= (\alpha_t, s_t, p_t))$ lies in some compact domain K of X, at $t = 0$, then x_t will subsequently be confined to a bounded region K' of this space.

We now note that, since $f_R = 1$ on $[0, R]$, the solution of equns. (6.10) becomes independent of R when $R > G$; and, in this case, by (3.19), it also satisfies (3.20). This establishes the existence of a solution of the latter equations. To prove its uniqueness, we proceed as in our treatment of (6.10), and show that, for $\mathcal{K}$, too, the evolution $x \to x_t$ maps compacts, K, into compacts, K', for all positive t. Therefore, since, by (3.19), the functions A, S, P and their derivatives of all orders are uniformly bounded on the compacts, it follows from standard fixed point methods that the solution x_t of (3.20) is both unique and differentiable w.r.t. the initial data x_0. Evidently, this solution defines a one- parameter semigroup $\tau(\mathbf{R}_+)$ of transformations of X according to the prescription $x_t = \tau(t)x_0$.

(2) On applying Lemma 6.1(2) to equn. (6.12), and then using equns. (6.9) and (6.11), we see that, for $t \in \mathbf{R}_+$, the inverse image, under $\tau(t)$, of any compact region of X is itself compact. Hence, as we have seen that $\tau(t)x$ is differentiable w.r.t. x, it follows that $\mathcal{C}_0^{(1)}(X)$ is stable under the transformations $f \to f \circ \tau(t)$. We define

$$f_t = f \circ \tau(t) eqno(6.14)$$

and

$$F(s,t) = \int_X dm_s f_t \tag{6.15}$$

In view of the semi-group property of $\tau(\mathbf{R}_+)$, it follows from equns. (3.17), (3.18), (6.14) and (6.15) that

$$\frac{\partial}{\partial t} F(s,t) = \frac{\partial}{\partial s} F(s,t) = \int_X dm_s \mathcal{L} f_t \ \forall f \in \mathcal{C}_0^{(1)}(X); \ s, t \in \mathbf{R}_+$$

from which it follows that $F(s,t) \equiv F(t+s, 0)$, and therefore that $F(t,0) \equiv F(0,t)$. Thus, by (6.14) and (6.15),

$$\int_X dm_t f = \int_X dm_0 f \circ \tau(t)$$

for $f \in \mathcal{C}_0^{(1)}(X)$ and hence, by continuity, for $f \in \mathcal{C}_0(X)$.

Our proof of Prop. 3.2 will be based on the method devised in refs. 20, 21 for other models. Here, we shall omit details of some straightforward, but rather tedious, manipulations employed in the proof.

Proof of Prop. 3.2. (1) Since, by equations (3.5) and (3.6), the norms of all the commutators of the Lie algebra $\mathbf{M}^{(N)}$ are $O(N^{-1})$, it follows easily from equations (2.22), (3.9), (3.10), (3.14) and (3.15) that, for (z, w, v) in a compact region of X,

$$\frac{\partial \mu_t^{(N)}}{\partial z} = i\psi_t^{(N)}(U^{(N)}\alpha^{(N)}) + O(N^{-1}) \tag{6.16a}$$

$$\frac{\partial \mu_t^{(N)}}{\partial w} = i\psi_t^{(N)}(U^{(N)}s^{(N)}) + O(N^{-1}) \tag{6.16b}$$

$$\frac{\partial \mu_t^{(N)}}{\partial v} = i\psi_t^{(N)}(U^{(N)}p^{(N)}) + O(N^{-1}) \tag{6.16c}$$

and

$$\frac{\partial \mu_t^{(N)}}{\partial t} = i\psi_t^{(N)}(U^{(N)}([z.A^{(N)} + w.S^{(N)} + h.c.] + v.P^{(N)})) + O(N^{-1}) \tag{6.17}$$

where the arguments z, w, v of $U^{(N)}$ and $\mu_t^{(N)}$ have been omitted. In view of Lemma 6.2 and the unitarity of $U^{(N)}$, it follows from equns. (3.10), (6.16) and (6.17) that both $\mu_t^{(N)}$ and its derivatives w.r.t. z, w, v and t are uniformly bounded on the $X-$compacts. Hence, by the Arzela-Ascoli theorem, $\mu_t^{(N)}$ converges pointwise to a function μ_t on X, as N tends to infinity over some sequence of integers, the convergence being uniform on the compacts. We shall generalise this result to one of convergence over $\mathbf{N}$ at the end of the proof of part (2) of this Proposition.

It also follows from (3.3), (3.6), (3.14) and (3.15) that

$$\mu_t^{(N)}(z - z', w - w', v - v') = \psi_t^{(N)}(U^{(N)\star}(z', w', v')U^{(N)}(z, w, v)) + O(N^{-1}) \tag{6.18}$$

Further, the positivity of $\psi_t^{(N)}$ ensures that, for any complex numbers $c_1, . ., c_m$, and elements $(z_1, w_1, v_1), . ., (z_m, w_m, v_m)$ of X,

$$\sum_{j,k=1}^{m} \bar{c}_j c_k \psi_t^{(N)}(U^{(N)\star}(z_j, w_j, v_j)U^{(N)}(z_k, w_k, v_k)) \geq 0$$

By equation (6.18), this inequality reduces to the following one in the limit $N \to \infty$.

$$\sum_{j,k=1}^{m} \bar{c}_j c_k \mu_t(z_k - z_j, w_k - w_j, v_k - v_j) \geq 0$$

Therefore, since μ_t is a continuous function on X, which, by (3.14) and (3.15), reduces to unity when its argument is zero, it follows from Bochner's theorem that μ_t is the characteristic function of a probability measure, m_t, on X, in accordance with (3.16).

(2) We now re-employ the method we have just used to establish the convergence of $\mu_t^{(N)}$, in order to show that the derivatives of this function w.r.t. t, z, w, v converge subsequentially to the corresponding ones of mu_t. The argument runs as follows. By equns. (2.22), (3.5), (3.9), (3.10), (3.14) and (3.15), the derivatives w.r.t. z, w, v, t of the r.h.s.'s of equns. (6.16) and (6.17), barring the $O(N^{-1})$ terms, all take the form

$$\psi_t^{(N)}(U^{(N)}Q) + O(N^{-1}) \tag{6.19}$$

on the $X-$compacts, where Q is a polynomial in the elements of $\mathbf{M}^{(N)}$, whose coefficients are $N-$ independent functions of z, w, v, and which is of second degree in $\alpha^{(N)}$, $\alpha^{(N)\star}$. Hence, as the commutators $[U, \alpha^{(N)\sharp}]$ are $O(N^{-1})$, by (2.8), (3.3) and (3.14), the expression (6.19) reduces, up to $O(N^{-1})$, to a finite sum of terms of the forms

$$\psi_t^{(N)}(\alpha_l^{(N)\sharp}U^{(N)}R_1\alpha_m^{(N)\sharp}),\ \psi_t^{(N)}(U^{(N)}R_2\alpha_l^{(N)\sharp})\ \textit{and}\ \psi_t^{(N)}(U^{(N)}R_3)$$

where $R_1,\ R_2,\ R_3$ are polynomials in $s^{(N)},\ s^{(N)\star}$ and $p^{(N)}$ only. Hence, by Lemma 6.2 and the Arzela-Ascoli theorem, it converges pointwise to a limit, uniformly so on the compacts, as N tends to infinity over some sequence of integers.

On combining this result with that of (1), we see that $\mu_t^{(N)}$ and its derivatives w.r.t. z, w, v, t converge to μ_t and its corresponding derivatives, uniformly on the compacts, as N tends to infinity over some sequence of integers; and further, $\mu_t(y)$ and its first derivatives are continuous in y and t. Thus, by (6.16),

$$\frac{\partial\mu_t}{\partial z} = \lim_{N\to\infty} i\psi_t^{(N)}(U^{(N)}\alpha^{(N)}) \tag{6.20a}$$

$$\frac{\partial\mu_t}{\partial w} = \lim_{N\to\infty} i\psi_t^{(N)}(U^{(N)}s^{(N)}) \tag{6.20b}$$

and

$$\frac{\partial\mu_t}{\partial v} = \lim_{N\to\infty} i\psi_t^{(N)}(U^{(N)}p^{(N)}) \tag{6.20c}$$

Similarly, we obtain the following formulae for the second derivatives of μ_t that we shall require.

$$\frac{\partial^2\mu_t}{\partial z_l\partial v_m} = -\lim_{N\to\infty}\psi_t^{(N)}(U^{(N)}\alpha_l^{(N)}p_m^{(N)}) \tag{6.21a}$$

$$\frac{\partial^2\mu_t}{\partial \overline{z}_l\partial w_m} = -\lim_{N\to\infty}\psi_t^{(N)}(U^{(N)}\alpha_l^{(N)\star}s_m^{(N)}) \tag{6.21b}$$

$$\frac{\partial^2\mu_t}{\partial z_l\partial \overline{w}_m} = -\lim_{N\to\infty}\psi_t^{(N)}(U^{(N)}\alpha_l^{(N)}s_m^{(N)\star}) \tag{6.21c}$$

We now use these formulae to derive the equation of motion (3.17) by integrating (6.17) over the time interval [0,t] and against suitable test-functions on X, and then passing to a (subsequential) limit $N\to\infty$. Thus, denoting by $\mathcal{Z}(X)$ the space of Fourier transforms of the Schwartz space $\mathcal{D}(X)$, we infer from equns. (3.13), (3.16) and (6.14) that

$$\int_X dm_t(x)f(x) - \int_X dm_0(x)f(x) =$$

$$\lim_{N\to\infty} i\int_0^t du\int_X dy\hat{f}(y)\psi_u^{(N)}(U^{(N)}([z.A^{(N)}+w.S^{(N)}+h.c.]+p.P^{(N)}))\ \forall f\in\mathcal{Z}(X) \tag{6.22}$$

Hence, as the Fourier transformation $\hat{f}\to f$ of $\mathcal{Z}(X)$ onto $\mathcal{D}(X)$ converts the multipliers $iz, i\overline{z}, iw, i\overline{w}, iv$ into the differential operators $\partial/\partial\alpha,\ \partial/\partial\overline{\alpha}, \partial/\partial s,\ \partial/\partial\overline{s}, \partial/\partial p$, it follows from (3.16), (3.18), (3.19) and (6.20)-(6.22) that

$$\int_X dm_t(x)f(x) - \int_X dm_0(x)f(x) = \int_0^t du\int_X dy\mu_u\widehat{\mathcal{L}f}\equiv\int_0^t du\int_X dm_u\mathcal{L}f \tag{6.23}$$

for $f\in\mathcal{Z}(X)$. This result can be extended by continuity to functions f in the Schwartz space $\mathcal{S}(X)$ and thence to those of class $\mathcal{C}_0^{(1)}(X)$. Further, for $f\in\mathcal{C}_0^{(1)}$, $\int_X dm_t\mathcal{L}f$ is continuous in t, for the following reasons. On the one hand, by Def. 3.1(3) and equns. (3.19), $\mathcal{L}f$ is the limit, in the $\mathcal{C}_0(X)$ topology, of a sequence $\{\mathcal{L}f_n\}$, with $f_n\in\mathcal{S}(X)$; while, on the other hand, the uniform boundedness and continuity of μ_t in t ensure that $\int_X dm_t f_n$ is continuous in this variable. Therefore, when $f\in\mathcal{C}_0^{(1)}(X)$, we may differentiate equn. (6.22) w.r.t. t and thereby obtain the required result (3.17).

Finally, we note that since, by Prop. 3.4, the solutions of equns. (3.18) and (3.20) are unique, the above compactness arguments, which led to the convergence of $\mu_t^{(N)}$ and its derivatives over subsequences of integers, can now be extended to establish their convergence over $\mathbf{N}$.

VII. Conclusion

We have cast the theory of the multi-mode Dicke laser model within the framework of quantum dynamical semigroups and stochastic processes, and have thereby obtained a number of new results. On the physical side, these are encapsulated by the generalisation to the HL theory provided by Props. 3.4, 3.5, 4.1 and 4.2. In particular, we see from this last proposition and the discussion of §4B that the present model admits the phenomena of both chaotic and polychromatic laser radiation.

On the mathematical side, we have extended the theory of one-parameter CP semigroups to a regime where the generators are perturbed by unbounded *-derivations (Prop. 2.3).

We shall conclude by noting two outstanding problems. The first is that of generalising the theory to a continuum of radiation modes, in the limit $N\to\infty$. This could be formulated by putting $n = N$, and scaling the interaction Hamiltonian by N^{-1}, as in refs. 11-13, rather than $N^{-1/2}$. In this case, the theory of $\S's$ 2 and 5 would still be applicable, and the remaining problem would be to carry through the limit procedures of §6.

The second outstanding problem, of course, is to obtain a general characterisation of the conditions that favour the onset of chaotic or polychromatic laser radiation. This is evidently a very deep problem, similar to that of turbulence.

Acknowledgments

The authors would like to thank John Lewis and Hans Maassen for some clarifying remarks on quantum stochastic process. G. A. wishes to thank the SERC for financial support, and the work of both authors was partially supported by European Capital and Mobility Contract No. CHRX-Ct. 92-0007.

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