Stochastic Process Corresponding to the Photon Field in the Non - Positive Gauge*

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Abstract

The properties of generalized stochastic process without reflection positivity corresponding to the photon field are studied in the context of the reconstruction of algebraic quantum system on the Krein space.

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I Introduction

In the quantum field theory, positivity of Wightman functions corresponds to the reflection (or Osterwalder - Schrader) positivity of the Euclidean Schwinger functions [1]. In contrast to the Wightman functions which are the expectation values of quantum fields, Schwinger functions are not in general the expactation values of Euclidean fields [2]. The existence of Euclidean fields requires the additional condition on the Schwinger functions, namely so called Nelson - Symanzik positivity [3].

It is well known [4], that gauge quantum field theory is not a standard Wightman field theory : locality, covariance, and positivity can not be satisfied at the same time. From the mathematical point of view, it is better to keep locality and covariance, so we are led to consider quantum field theory without positivity of Wightman functions. As it was shown in [5], quantum field theory without positivity but satisfying so called Hilbert space structure condition, still allows the Euclidean continuation. The corresponding Schwinger functions satisfy Euclidean covariance and symmetry, but instead of reflection positivity, they satisfy Euclidean version of Hilbert space structure condition.

In this note, we consider the Schwinger functions corresponding to the Euclidean formulation of the photon field in the covariant (i.e. non positive) gauge [6], [7]. As in general case studied in [5]), considered Schwinger functions are not reflection positive, but they satisfy Nelson - Symanzik positivity. So they are expectation values of some generalized stochastic process. We study the properties of this process in the context of reconstruction of algebraic quantum system corresponding to the photon field. In particular, starting with the process, we construct the Krein space [8] of quantum states, the algebra of bounded operators on the Krein space which is closed with respect to Krein space involution and the non - positive state on the algebra, invariant under the time evolution. (Notice, that in the relativistic formulation on the Krein space, the field operator are essentially self - adjoint in the Krein space sence, and we are not able to construct bounded operators out of them. In particular, the Weyl operator are unbounded [9], [10]). We discuss also the Markov property of the process and its consequences for the reconstruction procedure.

II Euclidean theory for photon field

a. Photon field in covariant gauge.

The theory is defined by two point function

$$W^{(2)}(f \otimes g) = \sum_{\mu,\nu} \int d^4 p \, \widetilde{W}^{\mu\nu}(p) \, \overline{\widetilde{f}_\mu}(p) \, \widetilde{g}_\nu(p) \tag{2.1}$$

with $f, g \in \mathcal{S}(I\!\!R^4)^{\otimes 4}$ and

$$\widetilde{W}^{\mu\nu}(p) = (-G^{\mu\nu})\widetilde{W}(p) \tag{2.2}$$

where $\widetilde{W}(p)$ is a Fourier transform of two point function of massless scalar field. Obviously (2.1) is not Wightman positive, but there exists a semi - norm p_W on $\mathcal{S}(I\!\!R^4)^{\otimes 4}$ which is $\mathcal{S}$ - continuous and non - degenerate in the sense that $\ker p_W = \ker W^{(2)}$, and such that

$$|W^{(2)}(f \otimes g)| \leq p_W(f) \, p_W(g) \tag{2.3}$$

b. Analytic continuation to Euclidean points[6].

Let $z^S = (iz^0, \vec{z})$; $z^0, z^k \in I\!\!R$ be the point in the extended permuted tube $\mathcal{T}^{\mathrm{ext,p}}$ [11]. The corresponding Euclidean point x^E is defined by $x^E = I^{-1} z^S$, where $I = \mathrm{diag}(i, 1, 1, 1)$. The Schwinger function $S^{(2)}$ is defined by

$$S^{(2)}_{\mu\nu}(x^E, y^E) = \sum_{\rho,\lambda} (I^{-1})_{\mu\rho} \, (I^{-1})_{\nu\lambda} W_{\rho\lambda}(I\, x^E, \, I\, y^E) \tag{2.4}$$

Since $W^{(2)}$ analytically continued to $\mathcal{T}^{\mathrm{ext,p}}$ is $L_+(\mathbb{C})$ - covariant, $S^{(2)}$ defined by (2.4) is $SO(4)$ - covariant. In the case of photon field, (2.4) gives

$$S^{(2)}_{\mu\nu}(x^E, y^E) = \delta^{\mu\nu} \, S(x^E, y^E) \tag{2.5}$$

with

$$S(\xi) = \int d^4\, p\, e^{-(\xi^0 p^0 - i\vec{\xi}\cdot\vec{p})}\, \widetilde{W}(p) \tag{2.6}$$

Let $r(x^0, \vec{x}) = (-x^0, \vec{x})$ be the time reflection. In the vector case r is represented by

$$(u(r)f)_\mu(x) = \sum_\nu M_{\mu\nu}\, f_\nu(rx) \tag{2.7}$$

with $M = \mathrm{diag}(-1,1,1,1)$. Moreover, for $f, g \in \mathcal{S}(I\!R^4_+)^{\otimes 4}$

$$S^{(2)}(\overline{u(r)f} \otimes g) = \widetilde{W^{(2)}}(\overline{\breve{f}} \otimes \breve{g}) \tag{2.8}$$

where

$$\breve{f}(p) = \int d^4\, \xi e^{-(\xi^0 p^0 - i\vec{\xi}\cdot\vec{p})} \quad |_{\{p^0 \geq 0\}}$$

¿From (2.8) we see that $S^{(2)}$ is not reflection positive. On the other hand, it is Nelson - Symanzik positive.

c. Stochastic process.

By Nelson - Symanzik positivity of $S^{(2)}$

$$< f, g >= S^{(2)}(f \otimes g) \tag{2.9}$$

defines the Hilbert space inner product on $\mathcal{S}_{I\!R}(I\!R^4)^{\otimes 4}$. Let $\mathcal{H}$ be the Hilbert space completion of $\mathcal{S}_{I\!R}(I\!R^4)^{\otimes 4}$ with respect to the inner product (2.9). Then $(\mathcal{H}, <\,,\,>)$ generates a unique generalized Gaussian stochastic process φ over $\mathcal{S}_{I\!R}(I\!R^4)^{\otimes 4}$ [12]. Thus, there exist a probability space (Q, Σ, μ) and Gaussian random variables $\varphi(f)$ with covariance

$$E(\varphi(f)\varphi(g)) = S^{(2)}(f \otimes g) \tag{2.10}$$

Let Σ_0 denotes the sub- σ- algebra of Σ generated by random variables $\varphi(f)$ with supp $\subset \{(0, \vec{x}) \in I\!R^4\}$. From the properties of $S^{(2)}$ we can easily deduce the following properties of the stochastic process. There exist :

- a one - parameter group $U(t)$ of measure preserving automorphisms of $L^\infty(Q, \Sigma, \mu)$ representing Euclidean time translations,

- a measure preserving automorphism Θ of $L^\infty(Q, \Sigma, \mu)$ (representing time reflection) such that $\Theta^2 = 1\!\!1$, $\Theta U(t) = U(-t)\Theta$ and Θ maps $L^\infty(Q, \Sigma_0, \mu)$ into itself.

Since reflection positivity is not satisfied, the inner product

$$< F, G >= (\Theta F, G)_{L^2} \tag{2.11}$$

defined for $F, G \in L^2(Q, \Sigma_+, \mu)$ (Σ_+ is generated by $\bigcup_{t \geq 0} U(t)\Sigma_0$) is in general indefinite. Using the operator corresponding to the multiplication by M (equation (2.7)), we can define a measure preserving automorphism Γ of $L^\infty(Q, \Sigma, \mu)$ mapping $L^\infty(Q, \Sigma_0, \mu)$ into itself, and such that $\Gamma^2 = 1\!\!1$, $\Gamma\Theta = \Theta\Gamma$ and

$$< F, F >_\Gamma= (\Gamma\Theta F, F)_{L^2} \geq 0 \tag{2.12}$$

for $F \in L^2(Q, \Sigma_+, \mu)$.

III Reconstruction of Krein space algebraic quantum system

a. Krein space.

On the space $L^2(Q, \Sigma_+, \mu)$ consider the inner products

$$< F, G >= (\Theta F, G)_{L^2} \tag{3.1}$$

and

$$< F, G >_\Gamma= (\Gamma\Theta F, G)_{L^2} \tag{3.2}$$

Since $\ker < , >= \ker < , >_\Gamma$, on the quotient space

$$\mathcal{D} = L^2(Q, \Sigma_+, \mu)/\ker < , >$$

we can define

$$< [F], [G] >=< F, G > \tag{3.1a}$$

and

$$< [F],[G] >_\Gamma = < F,G >_\Gamma \tag{3.2a}$$

Obviously

$$| < [F],[G] > |^2 \le < [F],[F] >_\Gamma < [G],[G] >_\Gamma \tag{3.3}$$

Let $\mathcal{K}_\Gamma$ be the Hilbert space completion of $\mathcal{D}$ with respect to (3.2a). One can check that $\mathcal{K}_\Gamma$ is the Krein space [8] i.e. admits a fundamental decomposition

$$\mathcal{K}_\Gamma = \mathcal{K}_\Gamma^+ \oplus \mathcal{K}_\Gamma^-$$

into positive and negative subspaces, which are Hilbert spaces.

b. Time evolution.

Let

$$P(t)[F] = [U(t)F] \; ; t \ge 0 \tag{3.4}$$

Since

$$\begin{aligned} < U(t)F,G > &= (\Theta U(t)F,G)_{L^2} = (U(-t)\Theta F,G)_{L^2} \\ &= (\Theta F, U(t)G)_{L^2} = < F, U(t)G > \end{aligned}$$

$U(t)$ maps ker $< \, , >$ into itself, thus $P(t)$ is well defined on $\mathcal{D}$ and we have:

PROPOSITION 1 *The family of operators $\{P(t)\}_{t\ge 0}$, defined on $\mathcal{D}$, has the following properties:*

1. $P(t) \,:\, \mathcal{D} \to \mathcal{D} \; ; t \ge 0$,
2. $P(t_1+t_2) = P(t_1)P(t_2) \; ; t_1, t_2 \ge 0$,
3. $< P(t)\Phi_0, \Psi_0 > = < \Phi_0, P(t)\Psi_0 > \; ; \Phi_0, \Psi_0 \in \mathcal{D}$,
4. $t \to < \Phi_0, P(t)\Psi_0 >$ *is continuous,*
5. $P(t)\Omega = \Omega \; ; \Omega = [\mathbb{1}]$.

Remark 1 *In the case of the stochastic process defined by photon field, $U(t)$ commutes with Γ, thus $\{P(t)\}_{t\geq 0}$ form a strongly continuous semi - group of contractions on the Hilbert space completion of $\mathcal{D}$ and $P(t) = e^{-tH}$ for some positive self - adjoint operator H [13].*

c. Quantum algebra.

Now we consider the representation of the algebra $L^\infty(Q, \Sigma_0, \mu)$ on the space $\mathcal{K}_\Gamma$. For $F_0 \in L^\infty(Q, \Sigma_0, \mu)$ define

$$\pi(F_0)[G] = [F_0 G] \; ; \; G \in L^2(Q, \Sigma_+, \mu) \tag{3.5}$$

$\pi(F_0)$ is well defined on $\mathcal{D}$ since

$$\begin{aligned} < F_0 F, F_0 G > &= (\Theta(F_0 F), F_0 G)_{L^2} = \\ (\Theta F, |F_0|^2 G)_{L^2} &= < F, |F_0|^2 G > \end{aligned}$$

so $F_0(\ker <,>) \subset \ker <,>$. Moreover, using the positive inner product $<,>_\Gamma$ and the norm $|| \, ||_\Gamma^2 = <,>_\Gamma$ one can check that

$$||\pi(F_0)^n[G]||_\Gamma = < G, |F_0|^{2n} G >^{1/2} \leq ||F_0||_\infty^n \, ||G||_{L^2}$$

for $n = 1, 2, \ldots$. Applying the arguments from [13], we obtain that $||\pi(F_0)||_\Gamma = ||F_0||_\infty$. Now take

$$\mathcal{A} = \{\pi(F_0) \; : \; F_0 \in L^\infty(Q, \Sigma_0, \mu)\} \tag{3.6}$$

$\mathcal{A}$ defined in (3.6) is a commutative von Neumann algebra of operators on $\mathcal{K}_\Gamma$. Since Σ_+ is generated by $\bigcup_{t\geq 0} U(t)\Sigma_0$, $\mathcal{D}_+$ defined as

$$\mathcal{D}_+ = \mathrm{Lin}\{P(t_1)A_1 P(t_2) \cdots P(t_n) A_n \Omega \; ; \; A_1, \ldots, A_n \in \mathcal{A}\} \tag{3.7}$$

is dense in $\mathcal{K}_\Gamma$. The algebra $\mathcal{A}$ is also closed with respect to the involution defined by the Krein space adjoint of bounded operators. Indeed, since Γ leaves $L^\infty(Q, \Sigma_0, \mu)$ invariant, $\hat{\Gamma}\pi(F_0)^*\hat{\Gamma} \in \mathcal{A}$, where $\hat{\Gamma}[G] = [\Gamma G]$.

Let $\mathcal{B}$ be the W* - algebra of operators on $\mathcal{K}_\Gamma$ generated by

$$\bigcup_t e^{itH}\, \mathcal{A}\, e^{-itH}$$

$\mathcal{B}$ is closed with respect to Krein space involution and carries $*$ - automorphisms $\{\alpha_t\}$ representing time evolution of quantum system

$$\alpha_t(B) = e^{itH}\, B\, e^{-itH}\ ;\ B \in \mathcal{B} \tag{3.8}$$

Finally, let us define a functional ω on $\mathcal{B}$

$$\omega(B) = <\Omega,\, B\Omega> \tag{3.9}$$

Obviously, $\omega(\mathbb{1}) = 1$ and $\omega \circ \alpha_t = \omega$, but $\omega(B^*B)$ is not necessary non - negative. On the other hand, ω is *Krein - positive* i.e. is positive with respect to the involution

$$B \to B^{*_\Gamma} = \hat{\Gamma} B^* \hat{\Gamma}$$

because

$$\omega(\hat{\Gamma} B^* \hat{\Gamma} B) = < B\Omega, \hat{\Gamma} B\Omega > \geq 0$$

Thus, we obtain the following:

THEOREM 1 *Starting from the stochastic process corresponding to the photon field, we construct the algebraic quantum system on the Krein space consisting of*

1. *the algebra $\mathcal{B}$ of bounded operators on $\mathcal{K}_\Gamma$ which is closed with respect to the Krein space involution,*

2. *the group $\{\alpha_t\}$ of automorphisms of $\mathcal{B}$ representing time evolution,*

3. *the vector Ω which is cyclic for $\mathcal{B}$ i.e. the set $\mathcal{D}_+$ is dense in $\mathcal{K}_\Gamma$,*

4. *the state $\omega(B) = <\Omega, B\Omega>$ on $\mathcal{B}$ which is Krein - positive and α_t - invariant.*

REMARK 2 *It is worth to stress, that the Krein space of states obtained from the pair $(\mathcal{B}, \omega)$ by generalized GNS construction [10] is unitarily equivalent to the Krein space $\mathcal{K}_\Gamma$.*

IV Markov property

The stochastic process defined by $S^{(2)}$ satisfies the Markov property [12]. Now we examine the consequences of Markov property for the construction of quantum system. Let E_+, E_- and E_0 be the conditional expectations with respect to Σ_+, Σ_- (σ - algebra generated by $\bigcup_{t\leq 0} U(t)\Sigma_0$) and Σ_0. The Markov property of φ can be formulated as follows [13]

$$E_+E_- = E_+E_0E_- \tag{4.1}$$

In the case of photon field, the stochastic process is not reflection positive, so instead of reflection property [12], we have

$$E_0\Theta = \Theta E_0\Gamma \tag{4.2}$$

Thus for every $F, G \in L^2(Q, \Sigma_+, \mu)$

$$\begin{aligned}
< F, G >= (\Theta F, G)_{L^2} &= (\Theta E_+F, E_+G)_{L^2} = \\
(E_+\Theta E_+F, G)_{L^2} &= (E_+E_-\Theta E_+F, G)_{L^2} = \\
(E_+E_0E_-\Theta E_+F, G)_{L^2} &= (E_+E_0\Theta E_+F, G)_{L^2} = \\
(\Gamma E_0F, G)_{L^2} &= (E_0F, \Gamma E_0G)_{L^2}
\end{aligned}$$

and similarly

$$< F, G >_\Gamma = (E_0F, E_0G)_{L^2}$$

So we arrive at the following conclusion:

PROPOSITION 2 *1. The Krein space $\mathcal{K}_\Gamma$ can be identified with the space $L^2(Q, \Sigma_0, \mu)$ and the indefinite inner product is given by*

$$< F, G >= (F, \Gamma G)_{L^2}$$

2. Since $\mathcal{A} \simeq L^\infty(Q, \Sigma_0, \mu)$ and $P(t) \simeq E_0U(t)E_0$, Ω is cyclic also for $\mathcal{A}$.

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