BOUNDED EIGENFUNCTIONS AND ABSOLUTELY CONTINUOUS SPECTRA FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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Abstract. We provide a short proof of that case of the Gilbert-Pearson theorem that is most often used: That all eigenfunctions bounded implies purely a.c. spectrum. Two appendices illuminate Weidmann's result that potentials of bounded variation have strictly a.c. spectrum on a half-axis.

§1. Introduction and Reduction to m-functions

In this note, I want to consider Schrödinger operators and Jacobi matrices on a half-line. Specifically, we'll consider the operator h on $\ell^2(\mathbb{Z}_+)$ (with $\mathbb{Z}_+ = \{1, 2, \dots\}$) given by

$$(hu)(n) = u(n+1) + u(n-1) + v(n)u(n) \tag{1.1a}$$

$$u(0) = 0 \tag{1.1b}$$

and the self-adjoint operator on $L^2(0, \infty)$

$$(Hu)(x) = -u''(x) + V(x)u(x) \tag{1.2a}$$

$$u(0) = 0 \tag{1.2b}$$

where we suppose

$$\Gamma(V) \equiv \sup_x \left(\int_{x-1}^{x+1} |V(y)|^2 \right) < \infty. \tag{1.3}$$

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For any $E \in \mathbb{C}$, define two solutions u_1, u_2 of the formal difference (resp. differential) equation $hu = Eu$ (resp. $Hu = Eu$) with boundary conditions:

$$u_1(0,E) = 0 \qquad u_1(1,E) = 1$$
$$u_2(0,E) = 1 \qquad u_2(1,E) = 0$$

in the discrete case and

$$u_1(0,E) = 0 \qquad u_1'(0,E) = 1$$
$$u_2(0,E) = 1 \qquad u_2'(0,E) = 0$$

in the continuous case.

Let $S = \{E \in \mathbb{R} \mid u_1 \text{ and } u_2 \text{ are bounded on } [0,\infty)\}$. Then our purpose here is to prove

Theorem 1. *On S, the spectral measure ρ for h (resp. H) is purely absolutely continuous in the sense that*

(i) $\rho_{\mathrm{ac}}(T) > 0$ *for any $T \subset S$ with $|T| > 0$ (where $|\cdot|$ = Lebesgue measure)*
(ii) $\rho_{\mathrm{sing}}(S) = 0$.

This theorem is not new. In [9,8,11,13], Gilbert, Khan, and Pearson proved a complete characterization of the essential support of ρ_{ac} in terms of mutually subordinate solutions. Their approach has the advantage of not requiring (1.3). Behncke [2] and Stolz [16] have noted that V uniformly L^1_{loc} with bounded eigenfunctions allows one to use the Gilbert-Pearson theory. Virtually all applications of [16,12] use the weaker Theorem 1. There seems to be some point in the short proof I'll present here which avoids some of their tricky calculations and which makes the result transparent. In addition, we'll obtain explicit bounds on m-functions.

I should mention earlier work of Carmona [4] (which is weaker than Theorem 1) and related work of Briet-Mourre [3].

As with Gilbert-Pearson, our proof uses the theory of Weyl m-functions. For $E \in \mathbb{C}_+ = \{z \mid \operatorname{Im} z > 0\}$, we can find a unique solution $u_+(n,E)$ (resp. $u_+(x,E)$) of (1.1a)/(1.2a) with $u_+ \in \ell^2$ (resp. L^2) at infinity, normalized by

$$u_+(0,E) = 1. \tag{1.4}$$

Then one defines the m function by

$$m_+(E) = u_+(1,E) \tag{1.5}$$

in the discrete case and

$$m_+(E) = u_+'(0,E) \tag{1.6}$$

in the continuous case.

By looking at the Wronskian of u_+ and $\bar{u}_+$, one gets the well-known formula:

$$\operatorname{Im} m_+(E) = \operatorname{Im} E \sum_{n=1}^{\infty} |u_+(n,E)|^2 \tag{1.7}$$

in the discrete case and

$$\operatorname{Im} m_+(E) = \operatorname{Im} E \int_0^\infty |u_+(x,E)|^2\, dx \tag{1.8}$$

in the continuous case.

It is known (see [5,15]) that

$$d\rho(E) = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} m_+(E+i\epsilon)\, dE. \tag{1.9}$$

It follows [1,7] by the de la Vallée-Poussin theorem that

$$\rho_{\rm sc} \text{ is supported on } \Big\{ E \mid \lim_{\epsilon\downarrow 0} \operatorname{Im} m_+(E+i\epsilon) = \infty \Big\}$$

and

$$d\rho_{\rm ac}(E) = \frac{1}{\pi} \operatorname{Im} m_+(E+i0)\, dE.$$

Thus, Theorem 1 is an immediate consequence of

Theorem 2. *If $E \in S$, then*

$$\text{(i)} \qquad \underline{\lim} \operatorname{Im} m_+(E+i0) > 0 \tag{1.10}$$

$$\text{(ii)} \qquad \overline{\lim}\, |m_+(E+i0)| < \infty. \tag{1.11}$$

Remark. While the results are stated for the half-line with Dirichlet boundary conditions, Theorem 2 immediately implies the result for any fixed boundary condition and for the whole line. For it is known [1,15] that the essential support $d\rho_{{\rm ac},\theta}$ for θ boundary conditions (given by $\sin(\theta)u'(0) + \cos(\theta)u(0) = 0$) is θ independent and that $d\rho_{{\rm sc},\theta}$ is supported on the set where $m_+(E+i0) = -\cot(\theta)$, which cannot happen if (1.10)/(1.11) holds. For the whole line, we can define S via the right half-line condition from which (1.10)/(1.11) and the formula (for the continuous case; the discrete case is similar)

$$d\rho_1(E) = -\lim_{\epsilon\downarrow 0} \frac{1}{\pi} \operatorname{Im}\left(\frac{1}{m_+(E+i\epsilon) + m_-(E+i\epsilon)} \right) dE$$

$$d\rho_2(E) = \lim_{\epsilon\downarrow 0} \frac{1}{\pi} \operatorname{Im}\left(\frac{1}{m_+(E+i0)^{-1} + m_-(E+i0)^{-1}} \right) dE$$

imply $\rho_{i,{\rm sc}}(S) = 0$.

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§2. The Jacobi Matrix Case

In this section, we'll prove Theorem 2 in the discrete case. Define the fundamental or transfer matrix by

$$T(E,n,0)=\begin{pmatrix} u_1(n+1,E) & u_2(n+1,E) \\ u_1(n,E) & u_2(n,E) \end{pmatrix}$$

and then

$$T(E,n,m)=T(E,n,0)T(E,m,0)^{-1} \tag{2.1}$$

T is defined so that if u obeys $hu=Eu$, then $\Phi(n)=\binom{u(n+1)}{u(n)}$ obeys

$$\Phi(n)=T(E,n,m)\Phi(m).$$

Constancy of the Wronskian implies det $T=1$ so $\|T^{-1}\|=\|T\|$ and thus by (2.1)

$$C(E)=\sup_{n,m}\|T(E,n,m)\|\leq \sup_n \|T(E,n,0)\|^2 \tag{2.2}$$

is finite if and only if $E\in S$. We'll prove Theorem 2 in the following explicit form:

Theorem 2J. *If $E\in S$, then*

$$\varliminf \operatorname{Im} m_+(E+i\epsilon)\geq \frac{1}{4}C^{-3} \tag{2.3}$$

$$\varlimsup |m_+(E+i\epsilon)|\leq 4C^3 \tag{2.4}$$

where $C(E)$ is given by (2.2).

Proof. Let

$$A(E,n)\equiv \begin{pmatrix} E-V(n) & -1 \\ 1 & 0 \end{pmatrix}$$

so $T(E,n,0)=A(E,n)T(E,n-1,0)$. It follows (as a telescoping sum) that

$$T(E+i\epsilon,n,0)=T(E,n,0)+\sum_{j=0}^{n-1}(i\epsilon)T(E,n,j+1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}T(E+i\epsilon,j,0)$$

so by iteration, we get

$$\|T(E+i\epsilon,n,0)\|\leq \sum_{k=0}^{n}\binom{n}{k}C^{k+1}\epsilon^k=C(1+C\epsilon)^n\leq Ce^{\epsilon Cn} \tag{2.5}$$

By $\|T^{-1}\|=\|T\|$, we see that

$$\left\|\begin{pmatrix} u_+(E+i\epsilon,n+1) \\ u_+(E+i\epsilon,n) \end{pmatrix}\right\|\geq C^{-1}e^{-\epsilon Cn}(|m_+(E+i\epsilon)|^2+1)^{1/2}$$

since $\binom{u_+(E+i\epsilon,1)}{u_+(E+i\epsilon,0)}=\binom{m_+(E+i\epsilon)}{1}$.

Squaring and summing over $n = 1, 3, \dots$ we see that

$$\sum_{n=1}^{\infty} |u_+(E+i\epsilon, n)|^2 \geq C^{-2} e^{-2\epsilon C}(1-e^{-4\epsilon C})^{-1}(|m_+(E+i\epsilon)|^2 + 1).$$

Thus by (1.7)

$$\operatorname{Im} m_+(E+i\epsilon) \geq \frac{1}{4}\, C^{-3} e^{-2\epsilon C}(4\epsilon C)(1-e^{-4\epsilon C})^{-1}[1+|m_+(E+i\epsilon)|^2]$$

or

$$\underline{\lim} \left[\operatorname{Im} m_+(E+i\epsilon)/[1+|m_+(E+i\epsilon)|^2]\right] \geq \frac{1}{4}\, C^{-3} \tag{2.6}$$

Noting that $(1+|m_+|^2)^{-1} \leq 1$, we see that (2.6) immediately implies (2.3). And since $(1+|m_+|^2)/\operatorname{Im} m_+ \geq |m_+|$, it also implies (2.4). $\square$

With only minor changes, the theorem extends to the general Jacobi matrix (tridiagonal self-adjoint) matrix:

$$(hu)(n) = a_{n+1}u(n+1) + a_n u(n-1) + b_n u(n) \tag{2.7}$$

so long as there is α finite with

$$\alpha^{-1} < |a_n| < \alpha \tag{2.8}$$

for all n. If $d\rho$ is the spectral measure for $u(n) = \delta_{1n}$, then

$$\int \frac{d\rho(E)}{z-E} = m_+(z)$$

where $m_+(z)$ is defined to be $a_1^{-1}u_+(1)$ (if u_+ is normalized by $u_+(0) = 1$). (1.7) becomes

$$\operatorname{Im} m_+(E) = a_1^{-2}(\operatorname{Im} E) \sum_{n=1}^{\infty} |u_+(n,E)|^2.$$

It is no longer true that $\|T(E,n,0)^{-1}\| = \|T(E,n,0)\|$ since $\det(T(E,n,0))$ may not be 1. Rather $\det(T(E,n,0)) = \frac{a_1}{a_n+1}$ so using (2.8), (2.2) becomes $C(E) \leq \alpha^2 \sup_n \|T(E,n,0)\|^2$. (2.5) becomes

$$\|T(E+i\epsilon, n, 0)\| \leq C(1+C\alpha\epsilon)^n \leq Ce^{\epsilon C n\alpha}$$

and (2.6) becomes

$$\underline{\lim}\, [\operatorname{Im} m_+(E+i\epsilon)]/[1+a_1^2|m_+(E+i\epsilon)|^2] \geq \frac{1}{4}, a_1^{-2}C^{-3}\alpha^{-1}.$$

§3. The Schrödinger Case

To carry the proof through from the discrete case, we must use (1.3) to bound u' locally by u. This is a standard Sobolev-type estimate; we haven't tried to optimize constants.

Lemma 3.1. *If u obeys $-u''+Vu=Eu$, then*

$$|u'(x)|^2 \leq \left[4+\frac{3}{4}\,\Gamma(|V-E|)\right]\int_{x-1}^{x+1}|u(y)|^2\,dy \tag{3.1}$$

where Γ is given by (1.3).

Proof. By Taylor's theorem with remainder,

$$f'(0)=\frac{1}{2}\,\frac{f(x)-f(-x)}{x}-\frac{1}{2x}\int_0^x (x-y)[f''(y)+f''(-y)]\,dy.$$

Integrate this from $\frac{1}{2}$ to 1 to get

$$|f'(0)|\leq \int_{-1}^{1}|f(x)|\,dx+\frac{3}{8}\int_{-1}^{1}|f''(0)|\,dx.$$

Let $f(y)=u(y+x)$ and use $u''=(V-E)u$ and the Schwarz inequality to get (3.1). $\square$

By (3.1), if $E\in S$, u' is also bounded and thus the transfer matrix $T(E,x,y)$ defined by

$$T(E,x,y)\begin{pmatrix}u'(y)\\ u(y)\end{pmatrix}=\begin{pmatrix}u'(x)\\ u(x)\end{pmatrix}$$

is bounded. Let

$$C(E)\equiv\sup_{x,y}\,\|T(E,x,y)\|.$$

Theorem 2S. *Let $E\in S$ and define $A(E)=\frac{1}{2}C(E)^{-3}/(9+\frac{3}{2}\Gamma(|E-V|))$. Then*

$$\begin{aligned}\underline{\lim}\ \operatorname{Im} m_+(E+i\epsilon)&\geq A\\ \overline{\lim}\ |m_+(E+i\epsilon)|&\leq A^{-1}.\end{aligned}$$

Proof. By mimicking the proof of (2.5), using integrals in place of sums

$$\|T(E+i\epsilon,x,0)\|\leq Ce^{\epsilon C|x|} \tag{3.2}$$

By (3.1)

$$\int_1^\infty |u'(x)|^2\,dx\leq\left[8+\frac{3}{2}\,\Gamma(|V-E-i\epsilon|)\right]\int_0^\infty |u(y)|^2\,dy$$

so if $\beta = 1/(9 + \frac{3}{2}\Gamma)$, then

$$\int_0^\infty |u(y)|^2\,dy \geq \beta \int_1^\infty [|u(x)|^2 + |u'(x)|^2]\,dx$$
$$\geq C^{-2}\beta(1 + |m_+|^2) \int_1^\infty e^{-2\epsilon Cx}\,dx$$

so by (1.8),

$$\operatorname{Im} m_+ \geq \frac{1}{2}\,C^{-3}\beta e^{-\epsilon C}(1 + |m_+|^2)$$

and the result follows as in the discrete case. □

Appendix 1: A Discrete Version of Weidmann's Theorem

One of the more interesting applications of Theorem 2 is the result of Weidmann [17,18,19] that if $V = V_1 + V_2$ where $V_1 \in L^1$ and V_2 is of bounded variation with $V_2(x) \to 0$ at infinity, then $-\frac{d^2}{dx^2} + V(x)$ has purely a.c. spectrum on $(0, \infty)$. A key to his argument is a proof that for any $E > 0$, solutions are bounded. He does this by noting one can suppose V_2 is C^1 with $V_2' \in L^1$ (by adjusting the breakup) and that if $K(x) = (u')^2 + (E - V_2)u^2$, then $K'(x) = 2V_1 u'u - 2V_2' u^2 \leq C(|V_1| + |V_2'|)K(x)$ for x large. Here we'll prove a discrete analog:

Theorem A.1. *Let v_n be a sequence on $\{1, 2, \dots\}$ so that $v_n \to 0$ and*

$$\sum_{n=1}^\infty |v_{n+1} - v_n| < \infty. \tag{A.1}$$

Then, the operator h of (1.1) *has purely absolutely continuous spectrum on* $(-2, 2)$.

Remarks. 1. (A.1) implies $\lim v_n$ exist so by adding a constant, it is no loss to suppose $v_n \to 0$.

2. If $v_n \in \ell^1$, then (A.1) holds so we don't need to consider sums as Weidmann does in the continuous case.

Proof. Given a solution of $hu = Eu$, let

$$K_n = u_{n+1}^2 + u_n^2 + (v_n - E)u_n u_{n+1}.$$

Then

$$(K_{n+1} - K_n) = (u_{n+2} - u_n)(u_{n+2} + u_n + (v_{n+1} - E)u_{n+1}) + (v_n - v_{n+1})u_n u_{n+1}$$

so

$$|K_{n+1} - K_n| \leq |v_n - v_{n+1}|\,|u_n u_{n+1}|. \tag{A.2}$$

Suppose now $E \in (-2,2)$. Then for $n \geq$ some N_0, $2 - |v_n - E| \geq \delta > 0$. For such n,

$$\begin{aligned} K_n &\geq \frac{\delta}{2}\,({u_{n+1}}^2 + {u_n}^2) + \left(1 - \frac{\delta}{2}\right)(|u_{n+1}| - |u_n|)^2 \\ &\geq \frac{\delta}{2}\,({u_{n+1}}^2 + {u_n}^2) \end{aligned}$$

so (A.2) becomes

$$K_{n+1} \leq \left(1 + \frac{2}{\delta}\,|v_n - v_{n+1}|\right) K_n$$

and for all $n \geq N_0$:

$$K_n \leq \prod_{m=N_0}^{\infty} \left(1 + \frac{2}{\delta}\,|v_m - v_{m+1}|\right) K_{N_0}.$$

The product is convergent by (A.1). $\square$

By using the remark at the end of Section 1, Theorem A.1 extends to the operator (2.7) so long as (2.8) holds and

$$\begin{aligned} b_n \to 0, &\qquad \sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty \\ a_n \to 1, &\qquad \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty. \end{aligned}$$

We merely define K_n by

$$K_n = a_{n+1} u_{n+1}^2 + a_n u_n^2 + (b_n - E) e_n u_{n+1}.$$

This is related to results of [6].

Appendix 2: Eigenfunctions for Weidmann's Theorem

We want to further elucidate Weidmann's theorem by showing how to actually find the asymptotics of the eigenfunctions. We'll suppose $V(x) = V_1(x) + V_2(x)$ with $V_1 \in L^1$ and V_2 a C^1 function with $V_2' \in L^1$ and $V_2 \to 0$ at infinity. We claim:

Theorem B.1. *Fix $E = k^2 > 0$ with $k > 0$. Then every solution of $(-\frac{d^2}{dx^2} + V(x))u = Eu$ is bounded; indeed, there exist a, b so that*

$$\begin{aligned} |u(x) - au_+(x) - bu(x)| \to 0 \\ |u'(x) - iaku_+(x) + ibku_-(x)| \to 0 \end{aligned}$$

where

$$u_\pm(x) = \exp\left(\pm i \int_{x_0}^{x} \sqrt{k^2 - V_2(x)}\,dx\right) \tag{B.1}$$

where x_0 is chosen so large that $V_2(x) < k^2$ for $x > x_0$.

Remarks. 1. Since $(k^2 - V_2(x))^{-1/4} \to k^{-1/2}$, we could use the WKB form instead of (B.1), but the form (B.1) is what enters naturally.

2. This theorem and proof can be regarded as specializations of arguments in Hinton-Shaw [10].

Proof. Define $u_\pm$ by (B.1). Note that $u_\pm$ are C^2 and

$$-u_\pm'' + (V(x) - E)u_\pm = F_\pm u_\pm \tag{B.2a}$$

where

$$F_\pm(x) = V_1(x) \pm \frac{i}{2} V_2'(x)(k^2 - V_2(x))^{-1/2} \tag{B.2b}$$

is in L^1 near infinity.

Let $W(x)$ be the Wronskian of u_+ and u_-. Clearly, $W(x) = 2ik + o(1)$. Define $a(x), b(x)$ by the equations (variation of parameters)

$$\begin{aligned} u(x) &= a(x)u_+(x) + b(x)u_-(x) \\ u'(x) &= a(x)u_+'(x) + b(x)u_-'(x). \end{aligned}$$

A straightforward and standard calculation (see prob. 98 on pg. 395 of [14]) shows that a, b obey the equations

$$\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}' = M(x) \begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$$

where

$$M(x) = W(x)^{-1} \begin{pmatrix} -F_+ & -u_-^2 F_- \\ F_+ u_+^2 & F_- \end{pmatrix}.$$

Since this is in L^1, standard arguments show that $\lim_{k\to\infty} \binom{a(x)}{b(x)} = \binom{a}{b}$ exists. $\square$

If, moreover, V obeys (1.3) (a mild restriction), this and Theorem 1 implies that $\sigma_{\text{ac}}(H) = [0, \infty)$, $\sigma_{\text{sing}} \cap (0, \infty) = \emptyset$.

References

[1] N. Aronszajn, *On a problem of Weyl in the theory of Sturm-Liouville equations*, Am. J. Math. **79** (1957), 597–610.

[2] H. Behncke, *Absolute continuity of Hamiltonians with von Neumann Wigner potentials, II*, Manu-scripta Math. **71** (1991), 163–181.

[3] P. Briet and E. Mourre, *Some resolvent estimates for Sturm-Liouville operators*, preprint.

[4] R. Carmona, *One-dimensional Schrödinger operators with random or deterministic potentials: New spectral types*, J. Funct. Anal. **51** (1983), 229–258.

[5] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[6] J. Dombrowksi and P. Nevai, *Orthogonal polynomials, measures, and recurrence relations*, Siam J. Math. **17** (1986), 752–759.

[7] W. Donoghue, *On the perturbation of the spectra*, Commun. Pure Appl. Math. **18** (1965), 559–579.

[8] D.J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*, Proc. Roy. Soc. Edinburgh Sect. A **112** (1989), 213–229.

[9] D.J. Gilbert and D.B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), 30–56.

[10] D.B. Hinton and J.K. Shaw, *Absolutely continuous spectra of second order differential operators with short and long range potentials*, Siam J. Math. Anal. **17** (1986), 182–196.

[11] S. Kahn and D.B. Pearson, *Subordinacy and spectral theory for infinite matrices*, Helv. Phys. Acta **65** (1992), 505–527.

[12] A. Kiselev, in preparation.

[13] D.B. Pearson, *Quantum Scattering and Spectral Theory. Techniques of Physics*, vol. 9, Academic Press, London, 1988.

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III. Scattering Theory*, Academic Press, New York, 1979.

[15] B. Simon, *Spectral analysis of rank one perturbations and applications*, Proc. Mathematical Quantum Theory II: Schrödinger Operators (J. Feldman, R. Froese, and L.M. Rosen, eds.), Amer. Math. Soc., Providence, RI (to appear).

[16] G. Stolz, *Bounded solutions and absolute continuity of Sturm-Liouville operators*, J. Math. Anal. Appl. **169** (1992), 210–228.

[17] J. Weidmann, *Zur Spektraltheorie von Sturm-Liouville-Operatoren*, Math. Z. **98** (1967), 268–302.

[18] ———, *Absolut stetiges Spektrum bei Sturm-Liouville-Operatoren und Dirac-Systemen*, Math. Z. **180** (1982), 423–427.

[19] ———, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics Vol. 1258, Springer-Verlag, Berlin/Heidelberg, 1987.