OPERATORS WITH SINGULAR CONTINUOUS SPECTRUM, VI. GRAPH LAPLACIANS AND LAPLACE-BELTRAMI OPERATORS

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ABSTRACT. Examples are constructed of Laplace-Beltrami operators and graph Laplacians with singular continuous spectrum.

§1. Introduction

In previous papers in this series [6,3,5,2,8], we have explored the occurrence of singular continuous spectrum, particularly for Schrödinger operators and Jacobi matrices. In this note, we'll construct examples of graphs whose Laplacians and Riemannian manifolds whose Laplace-Beltrami operators have singular continuous spectrum. The idea will be to take models with considerable symmetry which reduce to Jacobi matrices or Schrödinger operators, and so reduce these two types of models to known cases.

The analysis is simple, almost trivial. However, since these are the first examples I know of graph Laplacians or Laplace-Beltrami operators with singular continuous spectrum, it seems worthwhile to make it explicit.

§2. Graph Laplacians

By a graph $G = (V, I)$, we will mean a countable set of vertices V, and an incidence matrix I_{ij} of zeros and ones obeying

(i) $I_{ij} = I_{ji}$; $\quad I_{ii} = 0$.
(ii) $N_i = \{j \mid I_{ij} = 1\}$ is finite for each i. Let $r(i) = \#(N_i) = \sum_j I_{ij}$, the coordination number of r.
(iii) $\sup_i r(i) < \infty$.

The graph is called regular if $r(i)$ is a constant.

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Given a graph G, we can define two bounded operators on $\ell_2(V) = \bigl\{\{u(n)\}_{n\in V} \mid \sum_{n\in V} |u(n)|^2 < \infty\bigr\}$:

$$(L_1 u)(i) = \sum_j I_{ij} u(j) = \sum_{j\in N_i} u(j)$$
$$(L_2 u)(i) = \sum_j I_{ij}(u(i) - u(j)) = \sum_{j\in N_i} (u(i) - u(j)).$$

L_1 and L_2 define quadratic forms by

$$(u, L_1 u) = \sum_{ij} I_{ij}\, \overline{u(i)}\, u(j)$$
$$(u, L_2 u) = \frac{1}{2} \sum_{ij} I_{ij}\, |u(i) - u(j)|^2.$$

They are related by

$$L_2 = R - L_1$$

where $(Ru)(i) = r(i)u(i)$. In particular if G is regular, L_1 and L_2 differ by a constant and sign and their spectral types are the same.

Our first example is a ladder with missing rungs. Let $V = \mathbb{Z} \times \mathbb{Z}_2 = \{(n,\alpha) \mid n \in \mathbb{Z},\ \alpha = 0 \text{ or } 1\}$.

Figure 1. A ladder with missing rungs

The incidence matrix will depend on the choice of a single sequence $s \in \mathbb{Z}_2^{\mathbb{Z}}$; that is, $s = \{s_n\}_{n\in\mathbb{Z}}$ with each s_n 0 or 1. Given s, we define an incidence matrix so $I_{ij} = 1$ if and only if one of

(1) $i = (n,\alpha)$ $\quad j = (n \pm 1, \alpha)$
(2) $i = (n,0)$ $\quad j = (n,1)$ or vice-versa where $s_n = 1$.

Thus, as Figure 1 shows, the graph is a ladder with those rungs with $s_n = 0$ missing.

Let $X = \mathbb{Z}_2^{\mathbb{Z}}$, a compact metric space in the product topology (and so complete). $s^{(m)} \to s^{(\infty)}$ if and only if $s_n^{(m)} \to s_n^{\infty}$ for each n. For each $s \in X$, we can define $L_1(s)$, $L_2(s)$, the two Laplacians on $\ell_2(V)$. We say a set $S \subset X$ is generic if and only if S is a dense G_δ.

Theorem 2.1. *For a generic $s \in X$, $L_1(s)$ has purely singular continuous spectrum and $L_2(s)$ has a spectrum with a non-zero purely singular component.*

As a preliminary, we'll need:

Lemma 2.2. *Fix $c \neq 0$. Given $s \in X$, let $J(s)$ be the Jacobi matrix on $\ell^2(\mathbb{Z})$:*

$$(J(s)u) = u(n+1) + u(n-1) + cs_n u(n).$$

For each fixed c, $\{s \in X \mid J(s)$ has spectrum $[-2,2] \cup [c-2, c+2]$ and it is purely singular continuous$\}$ *is a dense G_δ.*

Proof. Because a countable intersection of dense G_δ's is again a dense G_δ (by the Baire category theorem), we need only show that each of

(i) $A_1 = \{s \mid [-2,2] \subset \mathrm{spec}(J(s))$ and has no eigenvalues in $[-2,2]\}$
(ii) $A_2 = \{s \mid [c-2, c+2] \subset \mathrm{spec}(J(s))$ and has no eigenvalues in $[c-2, c+2]\}$
(iii) $A_3 = \{s \mid J(s)$ has no a.c. spectrum$\}$

is a dense G_δ. For writing $J = J_0 + V$ where $(Vu)(n) = cs_n(u(n))$, and noting $\mathrm{spec}(V) \subset \{0, c\}$ and $\|J_0\| \equiv 2$, we have $\mathrm{spec}(J(s)) \subset [-2,2] \cup [c-2, c+2]$.

Each of the sets in (i)–(iii) is a G_δ. That this is so for point and a.c. spectrum is a result of [6,7]. The argument that a given interval is in $\mathrm{spec}(A)$ on a G_δ is found in the appendix.

To see (i) is dense, note that $\{s \mid s_n = 0$ for $|n|$ large$\} \subset A_1$, and similarly, $\{s \mid s_n = 1$ for $|n|$ large$\} \subset A_2$. But those sets are dense.

For A_3, put any non-trivial product measure on X. By the result of Carmona, Klein, and Martinelli [1], for a.c. $s \in X$ with respect to this measure, $J(s)$ has pure point spectrum and, in particular, has a.c. spectrum. But the support of the product measure is all of X, so A_3 is dense. □

Proof of Theorem 2.1. We'll use reflection symmetry about the middle of the ladder. Let $\mathcal{H}_\pm = \{u \in \ell^2(V) \mid u(n,1) = \pm u(n,0)\}$ and let $W_\pm : \ell^2(\mathbb{Z}) \to \mathcal{H}_\pm$ by

$$(W_\pm f)(n,0) = f(n)/\sqrt{2}$$
$$(W_\pm f)(n,1) = \pm f(n)/\sqrt{2}.$$

Then $\mathcal{H}_\pm$ is left invariant by both $L_1(s)$ and $L_2(s)$ and

$$W_\pm^{-1} L_1(s) W_\pm = J(\pm s)$$
$$W_+^{-1} L_2(s) W_+ = 2 - J(0)$$
$$W_-^{-1} L_2(s) W_- = 2 - J(-2s_n).$$

The result now follows by the lemma. □

Note that $L_2(s)$ always has an a.c. component of the spectrum since $J(0)$, the free Schrödinger Laplacian, has a.c. spectrum. This argument also shows that generically $\mathrm{spec}(L_1(s)) = [-3,3]$ and $\mathrm{spec}(L_2(s)) \equiv [0,6]$ with a.c. spectrum on $[0,4]$ and singular spectrum on $[2,6]$.

We were able to find a cheap trick to reduce the first example to the well-studied Jacobi case. That will not be true in the second example, and so we'll settle for a partial result. A complete analysis should be possible (I expect purely singular continuous spectrum for a generic s) but will require some effort to extend Carmona et al. [1] to this example.

FIGURE 2. Rungs in Example 2

Our purpose in this second example is to find a regular lattice whose Laplacian has singular spectrum; the coordination number in Example 1 is sometimes 2 and sometimes 3. Our lattice will again depend on $s \in \mathbb{Z}_2^{\mathbb{Z}}$ and again be a basic ladder whose rungs are of the two types shown in Figure 2 if $s_n = 0$ or $s_n = 1$.

The corresponding lattices are regular with coordination number 3, so we'll only look at L_1, which we'll call $L(s)$. As before, the spaces $\mathcal{H}_\pm$ are invariant and we'll look only at the operator on $\mathcal{H}_-$ (call the restriction of $L(s)$ to $\mathcal{H}_-$, $B(s)$). The space on which $B(s)$ acts is pairs of function $u : \mathbb{Z} \to \mathbb{C}$ and $\varphi : \{n \mid s_n \equiv 1\} \equiv Y(s) \to \mathbb{C}$ with $\sum\limits_{n\in\mathbb{Z}} |u(n)|^2 + \sum\limits_{n\in Y(s)} |\varphi(n)|^2 < \infty$, and with

$$\begin{aligned}
(B(s)u)(n) &= u(n-1) + u(n+1) - u(n) \qquad &\text{if} \quad s_n = 0 \\
&= u(n+1) + u(n-1) + \varphi(n) \qquad &\text{if} \quad s_n = 1 \\
(B(s)\varphi)(n) &= u(n).
\end{aligned}$$

We want to compute $\operatorname{spec}(B(s))$ for the two degenerate cases $s_n \equiv 0$ and $s_n \equiv 1$. If $s_n \equiv 0$, we get a Jacobi matrix whose spectrum is well known.

$B(s_n \equiv 0)$ has spectrum $[-3, 1]$ with purely a.c. spectrum. For $s_n \equiv 1$, one looks at plane waves $u(n) = ce^{ikn}$, in which case $\varphi(n) = cE^{-1}e^{ikn}$ and E obeys $E = 2\cos k + E^{-1}$, which leads to

$$E = \cos k \pm \sqrt{1 + \cos^2 k}\,.$$

It is not hard to see that these continuous eigenfunctions are complete and so

$$\operatorname{spec}(B(s_n \equiv 1)) = [-1-\sqrt{2}, 1+\sqrt{2}\,] \text{ and is purely absolutely continuous.}$$

Theorem 2.3. *For a generic $s \in X$, $B(s)$ contains $[-3, -1-\sqrt{2}\,)$ in its spectrum and the spectrum is purely singular continuous there.*

Proof. The spaces on which $B(s)$ act are s-dependent, but there is still a natural sense in which $s^{(n)} \to s^{(\infty)}$ means $(B(s^{(n)}) - z)^{-1} \to (B(s^{(\infty)}) - z)^{-1}$ strongly (apply to vectors of compact support) and this notion is such that the results of [6,7] still apply. Let

$$A_1 = \{s \mid [-3, -1-\sqrt{2}\,) \subset \operatorname{spec}(B(s)) \text{ and has no point eigenvalues there}\}$$

$$A_2 = \{s \mid B(s) \text{ has no a.c. spectrum on } [-3, -1-\sqrt{2}\,)\}.$$

Each is a G_δ by [6,7].

We claim that if $s_n \equiv 0$ near infinity, $s \in A_1$; but if $s_n \equiv 1$ near infinity, then $s \in A_2$. This implies A_1 and A_2 are dense so their intersection is a dense G_δ and Theorem 2.3 is proven.

The claim for $s_n \equiv 0$ near infinity follows by using plane waves near infinity as a Weyl sequence (to see that $[-3, 1] \subset \operatorname{spec}(B(s))$) and then noting that eigenvalues must have compact support and then be zero by a simple argument. The claim for $s_n \equiv 1$ near infinity follows because $B(s)$ differs from $B(s_n \equiv 1 \text{ all } n)$ by a finite rank operator, so $\operatorname{spec}(B(s))$ is thus discrete outside $[-1-\sqrt{2}, 1+\sqrt{2}\,]$. $\square$

§3. Some Laplace-Beltrami Operators

In this section, we'll find metrics on $\mathbb{R} \times S^1 = M$ for which the Laplace-Beltrami operator has purely singular spectrum. Use coordinates (x, θ) on M and consider metrics of the form $dx^2 + f(x)^2\, d\theta$. We'll consider $1 \le f(x) \le 2$ and is C^∞ with the metric topology associated to $\{ \sup\limits_{|x| \le N} \|\frac{d^m f}{dx^m}\|_\infty \}_{\substack{m=0,1,\dots \\ N=1,2,\dots}}$. Denote by X the set of such f's. Then:

Theorem 3.1. *For a generic f in X, the Laplacian operator H_f has spectrum $[0, \infty)$ and is purely singular continuous.*

Proof. By a simple calculation (see, e.g., [4]), under the decomposition $L^2(\mathbb{R} \times S^1) \cong \bigoplus\limits_{n=-\infty}^{\infty} L^2(\mathbb{R})$ (with $g(z) = \sum g_n(x) e^{in\theta}$) $H_f \cong \oplus H_{f;n}$ and $H_{f;n}$ is unitarily equivalent to the operator $L_{f;n}$ on $L^2(\mathbb{R}, dx)$ with

$$L_{f;n} = -\frac{d^2}{dx^2} + V(x) + \frac{n^2}{f^2(x)} \equiv -\frac{d^2}{dx^2} + V_n(x)$$

where

$$V(x) = \frac{1}{4}\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\left(\frac{f'}{f}\right)'.$$

We'll show for each n that $L_{f;n}$ generically has purely singular spectrum on $[\frac{n^2}{4}, \infty)$ and that implies the full result. If $f \equiv 2$ near infinity, $L_{f;n}$ is $\frac{n^2}{4}$ plus a Schrödinger operator with potential of compact support. This operator has no point spectrum in $[\frac{n^2}{4}, \infty)$. Since $L_{f;n=0} \ge 0$ and $f \le 2$, in general $L_{f;n} \ge \frac{n^2}{4}$. Moreover, the functions $f \equiv 2$ near ∞ are dense in the topology on X. Thus, generically $L_{f;n}$ has spectrum $[\frac{n^2}{4}, \infty)$ with no point spectrum. We'll need to show that generically there is no a.c. spectrum either.

Consider a metric of the form $f(x) = \alpha + \lambda \cos(kx)$ where $1 < \alpha < 2$ and λ is small. Then

$$V_n(x) = \frac{n^2}{\alpha^2} + \lambda\left[\frac{k^2}{2\alpha} - \frac{n^2}{\alpha^2}\right] \cos(kx) + 0(\lambda^2).$$

By the standard theory of periodic Schrödinger operators, for λ small, $-\frac{d^2}{dx^2} + V_n(x)$ has a gap of order λ about every $\frac{n^2}{\alpha^2} + \left(\frac{k}{2}\right)^2$. If $f(x) \equiv \alpha + \lambda \cos(kx)$ near infinity, $-\frac{d^2}{dx^2} + V_n(x)$ has only point spectrum in that interval. Thus, for any $E_0 \in [\frac{n^2}{4}, \infty)$, we can find δ and a dense set of f's with no a.c. spectrum in $(E_0 - \delta, E_0 + \delta)$. By a compactness argument, we conclude that generically there is no a.c. spectrum. $\square$

Is it true that for any non-compact manifold, a generic metric leads to purely singular continuous spectrum?

Appendix: G_δ's and the Spectrum of Operators

Recall that a regular space of operators is a family of self-adjoint operators indexed by $x \in X$ a complete metric space so that $x_n \to x$ in the metric topology implies that $A_{x_n} \to A_x$ in strong resolvent sense.

Our purpose here is to prove that

Theorem A.1. *Let $[a,b]$ be a fixed closed interval and X a regular space of operators. Then $\{x \mid [a,b] \subset \operatorname{spec}(A_x)\}$ is a G_δ.*

Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for the underlying Hilbert space and let $d\mu_n(x)$ be the spectral measure defined by

$$(\varphi_n, e^{-itA}\varphi_n) = \int e^{-ixE}\, d\mu_n(E)$$

and $d\mu = \sum\limits_{n=1}^{\infty} 2^{-n}\, d\mu_n$. Then $\operatorname{spec}(A) = \operatorname{supp}(d\mu)$ so Theorem A.1 is a consequence of

Lemma A.2. *Let $M_{+,1}^R$ denote the family of probability measures on $[-R,R] \subset \mathbb{R}$. Fix $[a,b] \subset [-R,R]$. Then $\{\mu \in M_{+,1}^R \mid [a,b] \subset \operatorname{supp}(d\mu)\}$ is a G_δ in the topology of weak convergence.*

Remark. Thus, if $S \subset M_{+,1}^R$ is closed, $S \cap \{\mu \mid [a,b] \subset \operatorname{supp}(d\mu)\}$ is a G_δ in S.

Proof of lemma. Let I_n be a counting of open intervals $I_n \subset [a,b]$ with rational endpoints. Let $f_n(x) = \operatorname{dist}(x, \mathbb{R}\backslash I_n)$. Then $[a,b] \subset \operatorname{supp}\mu$ if and only if for all n, $\int f_n(x)\, d\mu > 0$, that is,

$$\{\mu \mid [a,b] \subset \operatorname{supp}(d\mu)\} = \bigcap_{n=1}^{\infty} \left\{\mu \mid \int f_n(x)\, d\mu > 0\right\}$$

is clearly a G_δ. $\square$

Remark. Using $\lim\limits_{\epsilon\downarrow 0} \frac{1}{\pi} \int\limits_c^d \operatorname{Im} F_\mu(x+i\epsilon)\, dx = \frac{1}{2}\, \mu((c,d)) + \frac{1}{2}\mu([c,d])$ with $F_\mu(z) = \int d\mu(x)/(x-z)$, it is easy to show that $\{\mu \mid [a,b] \cap \operatorname{supp} d\mu = \emptyset\}$ is also a G_δ.

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