DYNAMICAL ZETA FUNCTIONS AND CORRELATION FUNCTIONS FOR NON-UNIFORMLY HYPERBOLIC TRANSFORMATIONS

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Abstract. We consider a class of maps f of $[0,1]$ which are expanding everywhere but at a fixed point, which we allow to be neutral. The analytic properties of the weighted dynamical zeta function and those of the Fourier transform of correlation functions are shown to be related to one another via the spectral properties of a suitable operator-valued power series associated to an induced version g of the map f. One result is that when the fixed point is neutral these functions are holomorphic in the unit disk and have a non-polar singularity (branch point) in $z=1$. Moreover, they have a power series expansion in a neighbourhood of the singular point which can be determined from the behaviour of the map near the fixed point.

The decay rate of correlations of a σ-finite absolutely continuous invariant measure ν, when the latter is finite, as well as other relevant ergodic quantities when ν is infinite, are then obtained straightforwardly from these expansions.

1. Introduction.

Weighted dynamical zeta functions as invariant objects associated to dynamical systems are receiving an increasing attention since the early work of Ruelle [R1]. For dynamical systems satisfying uniform hyperbolicity, like Axiom A systems, the zeta function usually turns out to have poles in some region of the complex plane, and these poles contain ergodic and geometric information about the dynamics ([B],[PP],[R2]).

On the other hand, counterexamples have been constructed (for suspensions of shifts) where the zeta function has a non-polar singularity arbitrarily close to its circle of convergence ([G],[Po1]), even though a dynamical interpretation of this fact seems not to be available.

Consider now a transformation $f : M \to M$ of a compact metric space and set $\mathrm{Fix} f^n = \{x : f^n x = x\}$. Let $\phi : M \to \mathbb{C}$ be a weight function (in the sequel we shall consider only the case $\phi(x) = 1/|f'(x)|$). The formal weighted dynamical zeta function of f is then given by

$$\zeta(f, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} \prod_{k=0}^{n-1} \phi(f^k(x))$$

When f satisfies uniform hyperbolicity and some smoothness assumptions, and ϕ is sufficently regular, one can prove that $\zeta(f, z)$ is analytic in some domain and extends meromorphically to a larger domain, where its poles are in bijection with the eigenvalues of a transfer operator acting on a suitable Banach space of functions. The most striking applications then concern the equidistribution of closed orbits (see [PP]) and the determination of the decay rate of correlation functions ([Po2],[R3],[R4],[H2]).

One expects that if the assumptions of uniform hyperbolicity is relaxed then the situation may change considerably.

In this paper we consider a class of smooth transformations $f : [0, 1] \to [0, 1]$ with a fixed point at the origin which we allow to be neutral or repelling. In the former case, we obtain the Pomeau-Manneville type 1 intermittency model at the tangent bifurcation point. In Section 2 we introduce an induced version g of the map f, which has a countable number of expanding monotone branches. Using the invariant probability measure ρ for the map g, one can construct a σ-finite absolutely continuous measure ν invariant under f (see, e.g., [CI2]). The main result is that, when the fixed point at 0 is neutral, the zeta function, as well as the Fourier transform of correlation functions of ν, is analytic in the unit disk and has a non-polar singular point (branch point) at $z = 1$ whose precise characterisation only depends on the behaviour of the map near the origin. We also discuss the conditions under which the zeta function can be analytically continued in the cut plane. More importantly, these functions have a power series expansion in a neighbourhood of $z = 1$, from which one can extract several ergodic properties of the measure ν (Theorems 3.1 and 3.2). In particular, when ν is finite, this procedure yields the decay rate of correlations for locally Hölder observables (Corollary 3.1(a)). If, instead, ν is infinite, one is able to extract the scaling rate (defined below, Corollary 3.1(b)) and, from this, the wandering rate and the return sequence of the map (Section 4).

These results are obtained by means of a two-variables zeta function, which relates $\zeta(f,z)$ to $\zeta(g,z)$ (Proposition 7.1), and an operator-valued power series which relates the transfer operator of f to that of g (Propositions 7.2 and 7.3). The analytic properties of the zeta functions and of the Fourier tranform of correlation functions are then obtained from the spectral properties of this operator-valued power series (Theorems 7.1 and 7.2).

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2. Assumptions.

Our assumptions on the transformation $f : [0,1] \to [0,1]$ are the following:

(1) $f(0) = 0,\ f(1) = 1$;

(2) there is a number $q \in]0,1[$ such that, setting $I_0 = [0,q[$ and $I_1 = [q,1]$, we have

$f(I_0) = [0, 1[,\ f(I_1) = [0, 1];$

(3) the restriction $f_{|I_0}$ extends to a C^1-diffeomorphism f_0 on the closure of I_0 and $f_1 = f_{|I_1}$ is a C^1-diffeomorphism on I_1; the inverses are denoted as $\psi_i = f_i^{-1}$, $i = 0, 1$;

(4) f_0 and f_1 have Lipschitz derivative;

(5) there are two numbers $0 < \beta < \alpha \le 1$ such that $f' > 1/\alpha$ on $]0, q[$ and $f'(0) = 1/\alpha$; whereas $f' > 1/\beta$ on I_1;

(6) f has the following asymptotic behaviour when $x \to 0_+$

$$f(x) = \frac{x}{\alpha} + \gamma x^{1+s}(1 + u(x))$$

with constant $\gamma > 0$ and exponent $s + 1 > 1$, and where $u(x)$ is a C^1 function such that $u'(x) = \mathcal{O}(x^{t-1})$ as $x \to 0_+$, for some $t > 0$.

Remark. In what is to follow we shall consider the two situations corresponding to $\alpha = 1$ (neutral fixed point) and $\alpha < 1$ (uniform hyperbolicity) in a parallel way so as to emphasize the generality of the approach here considered and at the same time provide a direct comparison between the two cases.

We also introduce the sequence c_n, given by

$$c_0 = 1, \quad c_n = \psi_0^n(c_0), \quad (n \ge 1). \tag{2.1}$$

We shall also deal with the sequence

$$\tilde{c}_n = \sum_{l>n} c_l \tag{2.2}$$

The c_n's generate a countable partition of $[0, 1]$ into the intervals $A_n = [c_n, c_{n-1}]$, $n \ge 1$, which is a countable Markov partition. In particular, setting $A_0 = [0, 1]$, we have $f(A_n) = A_{n-1}$ for any $n \ge 1$.

In the sequel we shall need some information about the asymptotic behaviour of the c_n's.

Lemma 2.1. *Under the hypotheses (1)-(6) on the map f we have the following behaviour of the c_n when $n \to \infty$:*

i) if $\alpha < 1$ then $c_n = \mathcal{O}(1)\,\alpha^n$;

ii) if $\alpha = 1$ then $c_{n-1} = (s\gamma n)^{-1/s}(1 + \mathcal{O}(n^{-1}))$.

Proof. The last property of f gives for the inverse function:

$$\psi_0(x) = \alpha x - \gamma\alpha^{2+s}x^{1+s}(1 + v(x))$$

where $v(x)$ is some C^1 function such that $v'(x) = \mathcal{O}(x^{t-1})$ as $x \to 0_+$. We write this expression in a more manageable form, that is:

$$\psi_0(x) = \Big((\alpha x)^{-s} + \gamma s\alpha(1 + \tilde{v}(x))\Big)^{-1/s}$$

where $\tilde{v}(x)$ is another C^1 function such that $v'(x) - \tilde{v}'(x) = \mathcal{O}(x^{q-1})$ where $q = \min\{s, t\}$. It is then easy to check that

$$\psi_0^n(x) = \Big((\alpha^n x)^{-s} + \gamma s\alpha^{1-s(n-1)}\sum_{l=0}^{n-1}\alpha^{ls}(1 + \tilde{v}(\psi_0^l(x)))\Big)^{-1/s}$$

On the other hand for $\alpha < 1$ we have

$$\sum_{l=0}^{n-1}\alpha^{ls}(1 + \tilde{v}(\psi_0^l(x))) = \mathcal{O}(1)$$

and this concludes the proof of the first part. For $\alpha = 1$ we find

$$\psi_0^n(x) = \Big(x^{-s} + \gamma sn(1 + \frac{1}{n}\sum_{l=0}^{n-1}\tilde{v}(\psi_0^l(x)))\Big)^{-1/s}$$

whence

$$c_{n-1} = (s\gamma n)^{-1/s}(1 + \frac{\mathcal{O}(1)}{n})^{-1/s}$$

and the assertion follows. $\diamondsuit$

Remark. When $\alpha = 1$ the ergodic properties of the map f have different features according whether the series $\sum_n c_n$ is convergent or not. In the former case f has an absolutely continuous invariant probability measure which is ergodic and mixing. In the latter case the only invariant non-singular measure is a σ-finite measure with anomalous ergodic properties. We refer the reader to the following literature: [A1],[ADU],[CI1],[CI2],[CI3],[Th2].

2.1. The induced version g.

The 'first passage map' (on the interval A_1), is the map $g : [0,1] \to [0,1]$ induced by f in the following way:

$$x \to g(x) = f^{p(x)}(x) \quad \text{where} \quad p(x) = 1 + \min\{n \geq 0 \ : \ f^n(x) \in A_1 \}$$

Equivalently, the map g is given by

$$x \to g(x) = g_k(x) = f^k(x) \quad \text{if} \quad x \in A_k, \ \ k \geq 1.$$

and defined arbitrarily on the set $\{c_k\}_{k \geq 0}$.

The map g is expanding and surjective, i.e.: $g_k'(x) \geq 1/\beta > 1$, for any $k \geq 1$, and $g_k(A_k) = [0,1]$. We denote by $\phi_k = g_k^{-1}$ the inverse branches of g. We shall see later on how the Lipschitz property of f' is reflected on g'.

Remark. Notice that the usual return time function $\tau(x)$ in the interval A_1 is given by

$$\tau(x) = \min\{n \geq 1 \ : \ f^n(x) \in A_1 \} = p \circ f(x).$$

2.2. Invariant measures.

We now recall some known facts and prove some preliminar result about these maps[1].

Lemma 2.2. *There is an unique absolutely continuous probability measure $\rho(dx) = h(x)\, dm(x)$, where m denotes the Lebesgue measure on $[0,1]$, which is*

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[1] In the sequel the symbol C will always denote an arbitrary positive constant, which may assume different values within the same formula.

invariant for the dynamical system $([0,1], g)$ *and whose density* h *is Lipschitz continuous and satisfies* $d^{-1} < h < d$ *for some* $d > 0$. *Furthermore,* $\rho(A_n) \sim \mathrm{C}\,|A_n|$ *as* $n \to \infty$.

Proof. See [CI1], Theorem 2.1, and [CI2], Lemma 2.4. $\diamondsuit$

We then have,

Lemma 2.3. *The* σ*-finite absolutely continuous measure* $\nu(dx) = e(x)\,dm(x)$ *defined for any Borel subset* E *of* $[0,1]$ *by*

$$\nu(E) = \sum_{n\geq 1} \rho(f^{-n+1}E \cap \bigcup_{l\geq n} A_n)$$

is invariant for the dynamical system $([0,1], f)$. *Moreover, its density* e *is related to* h *by:*

$$e = h + \sum_{k=1}^{\infty} h \circ \psi_0^k \cdot (\psi_0^k)'$$

Proof. See [CI2], Lemmata 2.1 and 2.2. $\diamondsuit$

In particular we have $\nu(A_n) = \sum_{l\geq n} \rho(A_l)$.

Lemma 2.4. *Under the assumptions (1)-(6) with* $\alpha = 1$, *there are two constants* $C_1, C_2 > 0$ *such that*

$$\frac{C_1}{x^s} \leq e(x) \leq \frac{C_2}{x^s}$$

Proof. See [Th1]; also [CI2], Lemma 2.3. $\diamondsuit$

We also introduce the sequences d_n and $\tilde{d}_n$ given by

$$d_n = \sum_{l>n} \rho(A_l) = \nu(A_{n+1}), \qquad \tilde{d}_n = \sum_{l>n} d_l, \quad (n \geq 0), \tag{2.3}$$

and the series

$$\begin{aligned} \frac{1}{c(f,z)} &:= (1-z)\sum_{n=0}^{\infty} c_n z^n \\ \frac{1}{d(f,z)} &:= (1-z)\sum_{n=0}^{\infty} d_n z^n \end{aligned} \tag{2.4}$$

Lemma 2.5. *Under the assumptions (1)-(6), we have*

$$d_n \sim \mathrm{C}\, c_n \quad \textit{and} \quad \tilde{d}_n \sim \mathrm{C}\, \tilde{c}_n \quad \textit{as} \quad n \to \infty,$$

and

$$d(f,z) \sim \mathrm{C}\, c(f,z) \quad \text{as} \quad z \to 1_- .$$

Proof. The first statement easily follows from the last part of Lemma 2.2. Moreover, under the assumptions (1)-(6) for the map f, we have two possibilities: either the invariant measure ν is finite, but then the series $\sum_{n=0}^\infty d_n z^n$ and $\sum_{n=0}^\infty c_n z^n$ converge absolutely at $z = 1$; or ν is infinite. In the latter case, the first statement entails $\sum_{n=0}^\infty d_n z^n \sim \mathrm{C} \sum_{n=0}^\infty c_n z^n$ as $z \to 1_-$ ([Ti], p.224). $\diamondsuit$

¿From the above it follows that the σ-finite invariant measure ν is normalizable when either $\alpha < 1$ or $\alpha = 1$ and $s < 1$, whereas is infinite when $\alpha = 1$ and $s \geq 1$. Moreover, by Kac's formula, one finds

$$\nu([0,1]) = \int_{A_1} \tau(x)\, \nu(dx) = \int_0^1 p(x)\, \rho(dx) = \sum_{n=1}^\infty n\rho(A_n) \tag{2.5}$$

where $\tau(x) = p \circ f(x)$ is the return time function in the interval A_1. In other words, the number $\nu([0,1])$ is but the mean return time in the interval A_1 with respect to the invariant measure ν and we will denote it in the sequel with the symbol $< \tau >_\nu$. Thus, when either $\alpha < 1$ or $\alpha = 1$ and $s < 1$, one can obtain an f-invariant probability measure μ defined for any Borel subset E of $[0,1]$ by

$$\mu(E) = \frac{\nu(E)}{< \tau >_\nu}.$$

Finally, we introduce a space of piecewise Hölder functions. Let $x, x' \in I \subseteq A_n$, for some $n \geq 1$, and write

$$\mathrm{var}_I\, u = \sup \{\, |u(x) - u(x')| \;:\; x, x' \in I \,\}$$

Let $\mathcal{F}^\gamma([0,1])$ be the space of functions $u : [0,1] \to I\!\!R$, compactly supported in $]0,1]$, whose restriction to $[0,1] \setminus \{c_k\}_{k\geq 0}$ is continuous and verifies:

$$\sup_n \sup_{I \subseteq A_n} \left(\frac{\mathrm{var}_I\, u}{|I|^\gamma} \right) \leq M < \infty$$

for some $0 < \gamma \leq 1$ and $M > 0$. In particular, any function which assumes constant values over the intervals A_n is in $\mathcal{F}^\gamma$. In the sequel we shall need the following result.

Lemma 2.6. *If $u \in \mathcal{F}^\gamma$ then also $u \cdot e \in \mathcal{F}^\gamma$, where e is given in Lemma 2.3.*

Proof. We first observe that $u \in \mathcal{F}^\gamma$ entails that u is compactly supported in $]0,1]$. Therefore we have to estimate

$$\sup_{n<N} \sup_{I\subseteq A_n} \left(\frac{\mathrm{var}_I\,(u\cdot e)}{|I|^\gamma} \right)$$

where $N = N(u)$ is a fixed positive integer. Moreover,

$$\mathrm{var}_I\,(u\cdot e) \leq \sup_I u \cdot \mathrm{var}_I\, e + \sup_I e \cdot \mathrm{var}_I\, u$$

Now, for $I \subseteq A_n$, for some $n > 0$, we can write, by Lemma 2.3,

$$\mathrm{var}_I\, e \leq \mathrm{var}_I\, h + \sum_{k=1}^{\infty} \left(\mathrm{var}_{\psi_0^k(I)}\, h \cdot \sup_I\, (\psi_0^k)' + \mathrm{var}_I\,(\psi_0^k)' \cdot \sup_{\psi_0^k(I)} h \right)$$

where we have used the identity $\mathrm{var}_I\,(v \circ f) = \mathrm{var}_{f(I)}\, v$. Furthermore, the density h being Lipschitz continuous, we have

$$\sum_{k=1}^{\infty} \mathrm{var}_{\psi_0^k(I)}\, h \cdot \sup_I\, (\psi_0^k)' \leq \mathrm{C} \sum_{k=1}^{\infty} |\psi_0^k(I)|$$

On the other hand, for the class of transformations here considered one has the following property of uniform distorsion ([CI1], Lemma 2.1): there is a constant $R > 0$ such that for any $n > 0$ and $I \subseteq A_n$

$$R^{-1} \frac{|I|}{|A_n|} \leq \frac{|\psi_0^k(I)|}{|\psi_0^k(A_n)|} \leq R \frac{|I|}{|A_n|} \quad \text{for any} \quad k > 0$$

so that

$$\sum_{k=1}^{\infty} |\psi_0^k(I)| \leq R\, \frac{|I|}{|A_n|} \sum_{k=1}^{\infty} |\psi_0^k(A_n)| \leq R\, \frac{|I|}{|A_n|}$$

Moreover, since $d^{-1} \leq h \leq d$, for some $d > 0$, we have

$$\sum_{k=1}^{\infty} \mathrm{var}_I\,(\psi_0^k)' \cdot \sup_{\psi_0^k(I)} h \leq d \sum_{k=1}^{\infty} \mathrm{var}_I\,(\psi_0^k)'$$

We now estimate the last term as follows: Take $x, y \in A_n$ and let $I \subseteq A_n$ be the interval with endpoints x, y. We have

$$|\,(\psi_0^k)'(x) - (\psi_0^k)'(y)\,| \le \sup_I (\psi_0^k)' \cdot \left| \frac{(\psi_0^k)'(x)}{(\psi_0^k)'(y)} - 1 \right|$$

Now, using the Lipschitz property of ψ_0 and the above property of uniform distortion, we get,

$$\begin{aligned} \frac{(\psi_0^k)'(x)}{(\psi_0^k)'(y)} &\le \prod_{i=0}^{k-1} (1 + \mathrm{C}|\,\psi_0^i(x) - \psi_0^i(y)\,|) \\ &= \exp \sum_{i=0}^{k-1} \log(1 + \mathrm{C}|\psi_0^i(I)|) \\ &\le \exp \sum_{i=0}^{k-1} \log(1 + \mathrm{C}\, R\, \frac{|I|}{|A_n|} |\psi_0^i(A_n)|) \\ &\le 1 + \mathrm{C}\, \frac{|I|}{|A_n|}. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} \mathrm{var}_I\, (\psi_0^k)' \le \mathrm{C}\, \frac{|I|}{|A_n|} \sum_{k=1}^{\infty} \sup_I (\psi_0^k)' \le \mathrm{C}\, \frac{|I|}{|A_n|^2}$$

where the last inequality follows from the fact that, for $I \subseteq A_n$ one has, using the mean value theorem,

$$\sup_I (\psi_0^k)' \le \prod_{i=0}^{k-1} \sup_{A_{n+i}} \psi_0' = \frac{\prod_{i=1}^{n+k-1} \sup_{A_i} \psi_0'}{\prod_{i=1}^{n-1} \sup_{A_i} \psi_0'} \le \frac{\prod_{i=1}^{n+k-1} \sup_{A_i} \psi_0'}{|A_n|}$$

and

$$\sum_{k=1}^{\infty} \prod_{i=1}^{n+k-1} \sup_{A_i} \psi_0' \le \sum_{k=1}^{\infty} \prod_{i=1}^{k-1} \sup_{A_i} \psi_0' < \infty$$

by assumption. Finally, putting togheter the above estimates we have, for $I \subseteq A_n$,

$$\mathrm{var}_I\, (u \cdot e) \le \mathrm{C} \sup_I u \cdot |I| \cdot \left(1 + \frac{1}{|A_n|} + \frac{1}{|A_n|^2} \right) + \mathrm{C} \sup_I e \cdot \mathrm{var}_I\, u$$

and the assertion follows recalling that the integer n takes values not exceeding $N(u)$ and using Lemma 2.4. $\diamondsuit$

3. Statement of the main results.

Zeta functions.

We first consider the dynamical zeta function $\zeta(f,z)$ associated to the map f and defined by the following formal series:

$$\zeta(f,z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} \prod_{k=0}^{n-1} |f'(f^k(x))|^{-1}$$

Theorem 3.1. *Under the hypotheses (1)-(6) on the map f we have:*

$$(1-\alpha z) \cdot \zeta(f,z) = d(f,z) \cdot L(z)$$

where $d(f,z)$ is defined in (2.4) and $L(z)$ is analytic in $|z| \leq 1/\alpha$.

Remark.

1. If $\alpha < 1$ then $\zeta(f,z)$ is analytic in the unit disk and has an analytic continuation to the disk $|z| < 1/\alpha$, with one simple pole at $z = 1$ whose residue equals $L(1)/(1-\alpha) < \tau >_\nu$.
2. If $\alpha = 1$ then $\zeta(f,z)$ is analytic in the unit disk and has a (non-polar) singular point at $z = 1$. In particular, if $s < 1$, so that $< \tau >_\nu < \infty$, then

$$(1-z)^2 \zeta(f,z) \to L(1) < \tau >_\nu^{-1} \quad \text{when} \quad z \to 1_-.$$

If, instead, $s \geq 1$, so that $< \tau >_\nu = \infty$, then

$$(1-z)^2 \, \zeta(f,z) \to 0 \quad \text{but} \quad (1-z) \, \zeta(f,z) \to \infty \quad \text{when} \quad z \to 1_-.$$

One also realizes that if $\zeta(f,z)$ has an analytic continuation into some larger domain, then the point $z = 1$ is a branch point (for the analytically continued $\zeta(f,z)$) whose multiplicity only depends on the value of s. We shall now briefly discuss this circumstance.

Analytic continuation into the cut plane.

Using standard techniques of analytic continuation of power series based on the use of the Mellin transform (see, e.g., [E], Theorem 6.1), one can show that whenever it is possibile to determine a function $d(x)$ which is analytic in the half-plane $\mathrm{Re}\, x \geq 0$ and satisfies (see Lemmata 2.1 and 2.5):

$$d(x) \to 0, \quad d'(x) = \mathcal{O}((x+1)^{-1-\frac{1}{s}}) \quad \text{as} \quad x \to \infty, \quad \mathrm{Re}\, x \geq 0,$$

$$\text{and} \qquad d(n) = d_n \quad \text{for} \quad n \geq 0.$$

then the function $d(f, z)$ is holomorphic in the unit disk and can be continued analytically to the entire z-plane with a branch cut along the ray $(1, +\infty)$. The analytic continuation is given by the formula, valid for any $\delta \geq 0$,

$$1/d(f, z) = \frac{(1-z)}{2\pi i} \int_1^{+\infty} \int_{\delta - i\infty}^{\delta + i\infty} d(x) \frac{t^{-x}}{t-z} dx dt \tag{3.1}$$

Under these circumstances, Theorem 3.1 would tell us that $\zeta(f, z)$ has an analytic (or, better, meromorphic) continuation into the cut plane (see below, Example 5.2).

Correlation functions.

We now turn to correlation functions. Let $([0,1], \nu, f)$ be as above. For any pair of functions u, $v \in \mathcal{F}^\gamma([0,1])$ such that $0 < \nu(u)\nu(v) < \infty$, define the formal power series

$$s_{uv}(f, z) = \sum_{n=0}^{\infty} z^n \nu(u \cdot v \circ f^n)$$

Theorem 3.2. *Under the hypotheses (1)-(6) on the map f we have:*

$$s_{uv}(f, z) = \nu(u)\nu(v)\, d(f, z) + R(z)$$

where $R(z)$ is analytic in $|z| \leq 1/\alpha$.

Remark. As before, we have the analytic properties:

1. if $\alpha < 1$ then $s_{uv}(f, z)$ is analytic in the unit disk and has an analytic continuation to the disk $|z| < 1/\alpha$, with one simple pole at $z = 1$ whose residue equals $\nu(u)\nu(v)/ < \tau >_\nu$;

2. if $\alpha = 1$ then $s_{uv}(f, z)$ is analytic in the unit disk and has a non-polar singular point at $z = 1$. In addition, we could repeat word by word the remarks made above on the analytic continuation into the cut plane.

Corollary 3.1. *Let u_n be such that*

$$\sum_{n=0}^{\infty} u_n z^n = \frac{1}{(1-z)\sum_{n=0}^{\infty} d_n z^n},$$

then, under the hypotheses (1)-(6) on the map f we have:

a) assume that either $\alpha < 1$ or $\alpha = 1$ and $s < 1$, then the dynamical system $([0,1], f)$ is mixing and for any pair $u,\ v \in \mathcal{F}^\gamma$ such that $\mu(u)\mu(v) > 0$, we have

$$\mu(u \cdot v \circ f^n) - \mu(u)\mu(v) \sim \mu(u)\mu(v)\, \varrho_n \quad \text{as} \quad n \to \infty$$

where

$$\varrho_n := < \tau >_\nu\ u_n - 1.$$

Moreover, if the constants ϱ_n are not identically zero for $n > 0$ then

$$\varrho_n \sim \mathrm{C}\, \tilde{d}_n,$$

where $\tilde{d}_n$ is defined in (2.3);

b) assume that $\alpha = 1$ and $s \geq 1$, then for any pair $u,\ v \in \mathcal{F}^\gamma$ such that $0 < \nu(u)\nu(v) < \infty$, we have

$$\lim_{n\to\infty} \nu(u \cdot v \circ f^n) = 0$$

but

$$\lim_{n\to\infty} \frac{1}{u_n} \nu(u \cdot v \circ f^n) = \nu(u)\nu(v).$$

Moreover,

$$\sum_{k=0}^{n} u_k \cdot \sum_{k=0}^{n} d_k \sim n.$$

Remark.

1. The decay rate of correlations when the measure is finite can then be obtained using Lemma 2.1 and Lemma 2.5:

$$\tilde{d}_n \sim \begin{cases} \mathrm{C}\,\alpha^n & \text{if } \alpha < 1 \\ \mathrm{C}\,n^{1-1/s} & \text{if } \alpha = 1,\ s < 1. \end{cases}$$

   Related results have already been obtained in [LSV] and [Mo] (see also [Ch]) for the piecewise affine model we shall discuss in Section 5.

2. In the infinite case, using a Tauberian theorem (e.g., [F], p.445), one shows that:

$$1/u_n \sim \begin{cases} \mathrm{C}\,n^{1-1/s} & \text{if } s > 1 \\ \mathrm{C}\,\log n & \text{if } s = 1. \end{cases}$$

   We shall call the sequence u_n the *scaling rate* of the map f (preserving an infinite measure). By Corollary 3.1 it follows that from the scaling rate one can obtain information on the behaviour of the partial sums $\sum_{k=0}^{n-1} d_k$, which, in turn, is closely related to some natural objects arising in the ergodic theory of transformations preserving infinite measures. We now briefly dwell upon these relationships.

4. Scaling rate, wandering rate and return sequence when the invariant measure is infinite.

Let $([0,1],\mathcal{R},\nu,f)$ be as above (with $\alpha = 1$ and $s \geq 1$) and define

$$B_+ := \cup_{\epsilon>0}\{E \in \mathcal{R} :\ m(E) > 0,\ E \subseteq [0,1] \setminus (0,\epsilon)\,\}.$$

For any measurable set $E \in B_+$ with $0 < \nu(E) < \infty$ set $E_1 = E$ and

$$E_k = f^{-k+1}E \setminus \cup_{j=0}^{k-2} f^{-j}E, \qquad k \geq 2.$$

The *wandering rate* of E is then defined by

$$L_E(n) := \nu(\cup_{k=0}^{n-1} f^{-k}E) = \sum_{k=1}^{n} \nu(E_k) \tag{4.1}$$

An interesting result is given by the following

Theorem. ([Th2], Theorem 3) *For all* $E, F \in B_+$,

$$L_E(n) \sim L_F(n) \quad \text{as} \quad n \to \infty. \tag{4.2}$$

Hence, one defines the wandering rate $w_n(f)$ of the map f as the rate of growth of the sequences $\{L_E(n)\}$, $E \in B_+$. To obtain the explicit expression of $w_n(f)$ we then consider the particular set $E = A_1$. ¿From the definition (2.3) one finds $L_{A_1}(n) = \sum_{k=0}^{n-1} d_k$ and thus, using Lemma 2.5 and Lemma 2.1,

$$w_n(f) = \begin{cases} \mathrm{C}\, n^{1-1/s} & \text{if } s > 1 \\ \mathrm{C}\, \log n & \text{if } s = 1. \end{cases} \tag{4.3}$$

Intuitively, $w_n(f)$ measures the amount of $[0,1]$ visited by f-iterates of points of B_+ up to time n.

We now introduce a quantity which plays a central role in computing ergodic averages. Let $\mathcal{L}_f$ be the transfer operator associated to f, defined by

$$\int_0^1 \mathcal{L}_f u(x) \cdot v(x)\, m(dx) = \int_0^1 u(x) \cdot v \circ f(x)\, m(dx)$$

which satisfies

$$\mathcal{L}_f e = e.$$

Under the assumptions we have made on the transformation f and for any $u \in L^1([0,1], \mathcal{R}, \nu)$, one can show that there exist constants $a_n(f)$, $(n \geq 1)$, such that:

$$\lim_{n\to\infty} \frac{1}{a_n(f)} \sum_{k=0}^{n-1} \mathcal{L}_f^k\, u = e \cdot m(u) \tag{4.4}$$

uniformly on B_+ (see [A1]; also [CF]). The sequence $a_n(f)$ is uniquely determined up to asymptotic equality and is called the *return sequence* of f. When such a sequence exists the transformation f is called *pointwise dual ergodic*. Notice however that this property does not imply that the partial averages

$$\frac{1}{a_n(f)} \sum_{k=0}^{n-1} u \circ f^k(x)$$

converges m-almost everywhere to the number $\nu(u)$. On the contrary, it can be proved ([A1],[CI2]) that this cannot hold, not even for one particular sequence of constants a_n. Nevertheless, if the sequence a_n is as in (4.4), then one shows that the above partial averages converge in measure to $\nu(u)$ ([A2],[CI2]).

Now, from the asymptotic renewal equation and Karamata's theorem it follows that the return sequence can be identified from the wandering rate through the following result (whose formulation is here adapted to the present contex):

Theorem. ([A1], Theorem 3) *Suppose that* $([0,1],\mathcal{R},\nu,f)$ *is pointwise dual ergodic and* $w_n(f)$ *is regularly varying with index* $\delta\in[0,1)$. *Then,*

$$\lim_{n\to\infty}\frac{w_n\cdot a_n}{n}=\frac{1}{\Gamma(1+\delta)\Gamma(2-\delta)}. \tag{4.5}$$

Hence, from Corollary 3.1, equation (4.3) and Lemma 2.5 we obtain

$$a_n(f)=\sum_{k=0}^{n-1}u_k\sim\begin{cases}\mathrm{C}\,n^{1/s}, & \text{if } s>1\\ \mathrm{C}\,n/\log n & \text{if } s=1.\end{cases} \tag{4.6}$$

5. A piecewise affine approximation $\hat{f}$.

Before introducing the general set up which will enable us to prove the main results, we shall consider a simpler situation. For $k\geq 1$ we set (with $|A_0|=1$),

$$\alpha_k=\frac{|A_k|}{|A_{k-1}|}\quad\text{and}\quad \beta_k=\prod_{j=1}^{k}\alpha_j=|A_k|.$$

Clearly we have

$$\overline{\lim}\,\alpha_k\;=\;\overline{\lim}(\beta_k)^{1/k}\;=\;\alpha.$$

We then define the map $\hat{f}$ as follows:

$$\hat{f}(x)=\begin{cases}(x-c_1)/\alpha_1, & \text{if } x\in A_1\\ c_{k-1}+(x-c_k)/\alpha_k, & \text{if } x\in A_k,\ k\geq 2\end{cases}$$

and the induced version $\hat{g}$ as

$$\hat{g}(x)=\frac{x-c_k}{\beta_k},\qquad x\in A_k,\ k\geq 1.$$

The corresponding invariant measures $\hat{\nu}$ and $\hat{\rho}$ are such that

$$\hat{\nu}(A_k) = c_{k-1} \quad \text{and} \quad \hat{\rho}(A_k) = \beta_k, \quad (k \geq 1),$$

so that $\hat{d}_n = \hat{\nu}(A_{n+1}) = c_n$. The main advantage of this approximation scheme is that it allows for exact calculations, and it has been used by several authors (e.g. [LSV],[Mo],[W]). In particular one finds:

Proposition 5.1. *Under the hypotheses (1)-(6) on the map f we have:*

$$(1 - \alpha z)\, \zeta(\hat{f}, z) = c(f, z)$$

where $c(f, z)$ is defined in (2.4).

Remark. From Lemma 2.5 it follows that the functions $\zeta(f, z)$ and $\zeta(\hat{f}, z)$ have the same asymptotic behaviour when $z \to 1_-$.

Proof. We shall follow ([PP], chap.10). For any positive integer N consider the matrix A_N given by

$$A_N = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_N & \alpha \\ \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha_N & \alpha \end{pmatrix}.$$

It is then easy to realize that

$$\sum_{x \in \mathrm{Fix} \hat{f}^n} \prod_{k=0}^{n-1} |\hat{f}'(\hat{f}^k(x))|^{-1} = \mathrm{tr}\, A_N^n$$

provided $N > n$. Therefore,

$$\frac{1}{\zeta(\hat{f}, z)} = \lim_{N \to \infty} \det (1 - z A_N),$$

On the other hand, one easily gets the expression

$$\begin{aligned} \det (1 - z A_N) &= (1 - \alpha z)(1 - \sum_{n=1}^{N} z^n \alpha_1 \dots \alpha_n) + z^{N+1} \alpha \alpha_1 \dots \alpha_N \\ &= (1 - \alpha z)(1 - \sum_{n=1}^{N} z^n \beta_n) + z^{N+1} \alpha \beta_N \end{aligned}$$

Hence, taking $|z| < 1/\alpha$, so that $z^N \beta_N \to 0$ as $N \to \infty$, we get

$$\frac{1}{\zeta(\hat{f}, z)} = (1 - \alpha z)(1 - \sum_{n=1}^{\infty} \beta_n z^n)$$

and the assertion follows from the identity

$$1 - \sum_{n=1}^{\infty} \beta_n z^n = (1 - z) \sum_{n=0}^{\infty} c_n z^n \qquad \diamondsuit$$

Example 5.1. For some $0 < q < 1$ set

$$f(x) = \hat{f}(x) = \begin{cases} x/q & \text{if } x \leq q, \\ (x - q)/(1 - q) & \text{if } x > q. \end{cases}$$

Then $\alpha = q$ and $c_n = q^n$ so that

$$\zeta(f, z) = \frac{1}{1 - z}$$

which is analytic in the entire z-plane except for a simple pole at $z = 1$. In this example the constants ϱ_n of Corollary 3.1 vanish identically for $n > 0$ and the decay of correlations is faster than any exponential.

Example 5.2. Set

$$f(x) = \begin{cases} x/(1 - x) & \text{if } x \leq 1/2, \\ 2x - 1 & \text{if } x > 1/2. \end{cases}$$

Then $\alpha = 1$ and $c_n = 1/(n + 1)$ so that the invariant measure ν is infinite. For the piecewise affine approximation we find the zeta function

$$\zeta(\hat{f}, z) = \frac{z}{(1 - z)^2 \log 1/(1 - z)}$$

which has a logarithmic branch point at $z = 1$. Moreover, it is analytic in the entire z-plane with a branch cut along the ray $(1, +\infty)$. In a neighbourhood of $z = 1$ we have $(1 - z)\zeta(\hat{f}, z) \to \infty$ but $(1 - z)^2 \zeta(\hat{f}, z) \to 0$.

We point out that in this case one can obtain the analytic continuation in the cut plane also for $\zeta(f, z)$. Indeed, one can easily check that

$$e(x) = \frac{d\nu}{dm}(x) = \frac{1}{\log 2} \cdot \frac{1}{x}$$

so that $d_n = \log(n + 2/n + 1) / \log 2$ and the analytic continuation is given by Theorem 3.1 and formula (3.1) with

$$d(x) = \frac{\log(x + 2/x + 1)}{\log 2}.$$

Finally, using the above zeta functions and Karamata's theorem, one shows that both f and $\hat{f}$ have wandering rate $w_n = \mathrm{C} \log n$ and return sequence $a_n \sim \mathrm{C}\, n/ \log n$.

6. The coding.

We now construct a coding. Let Ω be the set of one-sided sequences $\omega = (\omega_0 \omega_1 \ldots)$, $\omega_i \in \{1, 2, \ldots\}$ satisfying the compatibility condition: given ω_i then either $\omega_{i-1} = \omega_i + 1$ or $\omega_{i-1} = 1$. Then, the map $\xi : \Omega \to [0, 1]$ defined by

$$\xi(\omega) = x \quad \text{according to} \quad f^j(x) \in A_{\omega_j}, \ \ j \geq 0$$

is a bijection between Ω and the points of $[0, 1]$ which are not preimages of the origin. In other words, to any sequence $\omega \in \Omega$ corresponds, via the map ξ, a point $x \in [0, 1] \setminus \{c_k\}_{k \geq 0}$, and viceversa.

Moreover, ξ conjugates the map f with the shift T on Ω.

Let us consider the infinite sequence $\{t_j\}_{j \geq 1}$ of successive entrance times in the state 1: $t_1(\omega) = \inf\{i \geq 0 \ : \ \omega_i = 1\}$ and, for $j \geq 2$, $t_j(\omega) = \inf\{i > t_{i-1} \ : \ \omega_i = 1\}$. Furthermore, we define a sequence of integer valued random variables by

$$\sigma_j(\omega) = t_{j+1} - t_j, \ \ j \geq 0$$

with the convention that $t_0 = -1$.

It is then easy to realize that for any $\omega \in \Omega$, we have $g^j(x) \in A_{\sigma_j}$, $j \geq 0$, where $x = \xi(\omega)$ and the integers $\sigma_j = \sigma_j(\omega)$ are defined above.

Let Σ be the set of *all* one-sided sequences σ of the form $\sigma = (\sigma_0\sigma_1 \dots)$, $\sigma_j \in \{1, 2, \dots\}$. Then, the map $\pi : \Sigma \to [0, 1]$ defined by

$$\pi(\sigma) = x \quad \text{according to} \quad g^j(x) \in A_{\sigma_j}, \;\; j \geq 0$$

is a bijection between Σ and the points of $[0, 1]\setminus\{c_k\}_{k\geq 0}$. Moreover, π conjugates the map g with the shift T on Σ. Observe that the map $\eta := \pi^{-1} \circ \xi$ determines a bijection between Ω and Σ.

Remark 1. We can represent the dependence of the σ_j's on ω recursively as follows:

$$\sigma_0 = \omega_0 \quad \text{and} \quad \sigma_j = \omega_{s_j} \quad \text{for} \quad j > 0, \quad \text{where} \quad s_j = \sum_{i=0}^{j-1} \sigma_i.$$

Notice however that this rule may associate to a periodic sequence an eventually periodic one. More precisely, let $\omega \in \Omega$ be a periodic sequence of period n for the shift T. We write it in the form $\omega = (\overline{\omega_0\omega_1 \dots \omega_{n-1}})$. If $\omega_0 = 1$ then we may write, for some $k \geq 0$,

$$\omega_0\omega_1 \dots \omega_{n-1} \; = \underbrace{1 \dots 1}_{r_0 \geq 1} \underbrace{l_1 l_1 - 1 \dots 1}_{l_1 \geq 1} \underbrace{1 \dots 1}_{r_1 \geq 0} \; \dots \; \underbrace{l_k l_k - 1 \dots 1}_{l_k \geq 1} \underbrace{1 \dots 1}_{r_k \geq 0}$$

where $r_0 + l_1 + \dots + r_k = n$. Hence, the above rule gives

$$\sigma_0\sigma_1 \dots \sigma_{m-1} \; = \underbrace{1 \dots 1}_{r_0 \geq 1} \, l_1 \, \underbrace{1 \dots 1}_{r_1 \geq 0} \; \dots \; l_k \, \underbrace{1 \dots 1}_{r_k \geq 0}$$

so that $\sigma(\omega) = (\overline{\sigma_0\sigma_1 \dots \sigma_{m-1}})$ is periodic of period $m = k + r_0 + \dots + r_k$ and satisfies $n = \sigma_0 + \dots + \sigma_{m-1}$.

On the other hand, if $\omega_0 > 1$, it may happen that, for some $k \geq 1$,

$$\omega_0\omega_1 \dots \omega_{n-1} \; = \underbrace{l_0 l_0 - 1 \dots 1}_{l_0 \geq 1} \underbrace{1 \dots 1}_{r_1 \geq 0} \underbrace{l_1 l_1 - 1 \dots 1}_{l_1 \geq 1} \; \dots \; \underbrace{1 \dots 1}_{r_k \geq 0} \underbrace{l_k + l_0 \dots l_0 + 1}_{l_k \geq 0}$$

and, according to the above rule, one would find the eventually periodic sequence $\sigma(\omega) = (\sigma_0\overline{\sigma_1\sigma_2\ldots\sigma_m})$ where

$$\sigma_0 = l_0 \quad \text{and} \quad \sigma_1\sigma_2\ldots\sigma_m \;=\; \underbrace{1\ldots1}_{r_1\geq 0}\,l_1\;\ldots\;\underbrace{1\ldots1}_{r_k\geq 0}\,l_k + l_0$$

whose ultimate period is $m = k+r_1+\ldots+r_k$ and which satisfies $n = \sigma_1+\ldots+\sigma_m$. Thus, in the latter case, in order to obtain a correspondence between periodic sequences it is necessary to apply the aformentioned rule to some iterate of the original sequence. By the way, this operation does not modify the weight associated to the periodic sequence (see below) and thus it has no influence in what follows.

Remark 2. For every integer $j \geq 0$, denote by x_j the projection on the j^{th} symbol, i.e. $x_j(\omega) = \omega_j$. Then the stochastic process on Ω given by $x_j(\omega) = \omega_j$, $j \geq 0$, is a Markov chain with transition probabilities $p_{ij} = |\psi_0(A_j)\bigcap A_i|/|A_i|$. An easy consequence is that the random variables σ_j are independent and identically distributed. Their common law is given by: $\mathrm{Prob}(\sigma_j = k) = \beta_k$ for any $j \geq 0$ and $k \geq 1$. Notice that, for $\alpha = 1$, the state 1 of the Markov chain is positive-recurrent when $s < 1$ and null-recurrent when $s \geq 1$.

7. Interactions, zeta functions and transfer operators.

We now use the maps ξ and π to lift the functions $\log f'$ and $\log g'$, respectively, up to some 'interactions' on the symbol spaces. More precisely, given $\omega \in \Omega$ and $\sigma \in \Sigma$, we set

$$V(\omega) = \begin{cases} V_0(\omega) = \log[(\psi_0)'(\xi(\omega_1\omega_2\ldots))], & \text{if } \omega_0 > 1 \\ V_1(\omega) = \log[(\psi_1)'(\xi(\omega_1\omega_2\ldots))], & \text{if } \omega_0 = 1 \end{cases} \tag{7.1}$$

and

$$W(\sigma) = \log[(\phi_{\sigma_0})'(\pi(\sigma_1\sigma_2\ldots))]. \tag{7.2}$$

Given $0 < \theta < 1$ we define a metric on Σ by setting $d_\theta(\sigma,\sigma') = \theta^n$ where n is such that: $\sigma_j = \sigma'_j$ for $0 \leq j \leq n$. Moreover, for any continuous function

$\Phi : \Sigma \to \mathbb{C}$ and integer $n \geq 0$, set:

$$\mathrm{var}_n \Phi = \sup\{\, |\Phi(\sigma) - \Phi(\sigma')| \, : \, \sigma_j = \sigma'_j, \, 0 \leq j \leq n \,\}$$

and

$$|\Phi|_\theta = \sup\{ \frac{\mathrm{var}_n \Phi}{\theta^n}, n \geq 0 \} \tag{7.3}$$

Finally, we denote by $\mathcal{F}_\theta$ the space of all Lipschitz functions on Σ with respect to the metric d_θ, that is all continuous function Φ on Σ satisfying $\mathrm{var}_n \Phi \leq C\theta^n$ for some constant $C > 0$ (so $|\Phi|_\theta$ is the least Lipschitz constant). With the norm $\|\Phi\|_\theta = |\Phi|_\theta + |\Phi|_\infty$, one makes $\mathcal{F}_\theta$ a Banach space.

A direct consequence of the assumptions made above on the map f is the following result:

Lemma 7.1. *$W(\sigma) \in \mathcal{F}_\theta$ for any $\theta \geq \beta$.*

Proof. The proof is a trivial adaptation to the present context of the argument given in [CI1], Lemma 2.1. $\diamondsuit$

We now write the dynamical zeta functions for the map f and g in the following way:

$$\zeta(f, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(f), \qquad \zeta(g, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(g) \tag{7.4}$$

where

$$Z_n(f) = \sum_{\substack{\omega \in \Omega \\ T^n \omega = \omega}} \exp \sum_{j=0}^{n-1} V(T^j \omega) + \alpha^n \tag{7.5}$$

and

$$Z_n(g) = \sum_{\substack{\sigma \in \Sigma \\ T^n \sigma = \sigma}} \exp \sum_{j=0}^{n-1} W(T^j \sigma) \tag{7.6}$$

The term α^n in (7.5) accounts for the contribution of the fixed point in 0. Using the functions $\hat{V}$ and $\hat{W}$ one obtains the corresponding quantities for the affine model.

Let us now examine how $\zeta(f, z)$ and $\zeta(g, z)$ are related to one another.

First, let $\omega \in \Omega$ be a periodic sequence of period n for the shift T. Moreover, let $\sigma(\omega)$ be a periodic sequence of period m in the space Σ corresponding to ω or to some iterate of it (see Section 6, Remark 1). Thus $s_m(\sigma) = \sum_{j=0}^{m-1} \sigma_j = n$. ¿From (7.1) and (7.2) we then have

$$\sum_{j=0}^{n-1} V(T^j\omega) = \sum_{j=0}^{m-1} W(T^j\sigma)$$

Using this fact we write $Z_n(f)$ as follows:

$$Z_n(f) = \alpha^n + \sum_{m=1}^{n} \frac{n}{m} \sum_{\substack{\sigma\in\Sigma,\, T^m\sigma=\sigma \\ s_m(\sigma)=n}} \exp \sum_{j=0}^{m-1} W(T^j\sigma) \tag{7.7}$$

The second sum ranges over the $\binom{n-1}{m-1}$ ways to write the integer n as a sum of m positive integers, counting all permutations.

Therefore,

$$\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(f) = \log(\frac{1}{1-\alpha z}) + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{m} \sum_{\substack{\sigma\in\Sigma,\, T^m\sigma=\sigma \\ s_m(\sigma)=n}} z^n \exp \sum_{j=0}^{m-1} W(T^j\sigma)$$

$$= \log(\frac{1}{1-\alpha z}) + \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{\sigma\in\Sigma \\ T^m\sigma=\sigma}} z^{s_m(\sigma)} \exp \sum_{j=0}^{m-1} W(T^j\sigma)$$

Putting together these observations we have the following

Proposition 7.1. *Consider the zeta function of two variables given by*

$$Z(f,w,z) = \exp \sum_{n=1}^{\infty} \frac{w^n}{n} \sum_{\substack{\sigma\in\Sigma \\ T^n\sigma=\sigma}} z^{s_n(\sigma)} \exp \sum_{j=0}^{n-1} W(T^j\sigma)$$

where $s_n(\sigma) = \sum_{j=0}^{n-1} \sigma_j$. *Then*

$$Z(f,1,z) = \zeta(f,z)(1-\alpha z) \quad \textit{and} \quad Z(f,w,1) = \zeta(g,w) \tag{7.8}$$

wherever the series expansions converge absolutely.

We shall now study the relationships between the zeta functions we have introduced above, in particular the two-variables zeta function defined in Proposition 7.1, and some transfer operators acting on the space $\mathcal{F}_\theta(\Sigma)$.

For $W \in \mathcal{F}_\theta$ and $z \in \mathbb{C}$, let $\mathcal{L}_{W,z} : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ be the operator-valued power series defined by

$$\mathcal{L}_W(z) = \sum_{k=1}^{\infty} z^k \, \mathcal{L}_{W,k} \tag{7.9}$$

where

$$(\mathcal{L}_{W,k}\, v)(\sigma) = e^{W(k\sigma)}\, v(k\sigma) \tag{7.10}$$

and $k\sigma$ denotes the sequence $(k\sigma_0\sigma_1\ldots)$.

Notice that for $z = 1$ one recovers the (symbolic version of the) Ruelle transfer operator $\mathcal{L}_W$ corresponding to the first passage map g.

Lemma 7.2. *The radius of convergence of $\mathcal{L}_W(z)$ is bounded below by $1/\alpha$.*

Proof. According to (7.9), the radius of convergence of $\mathcal{L}_W(z)$ is given by $\lim_{k\to\infty} \|\mathcal{L}_{W,k}\|_\theta^{-1/k}$. We have

$$|\mathcal{L}_{W,k} v(\sigma)| \le e^{W(k\sigma)} |v|_\infty$$

and also

$$|\mathcal{L}_{W,k} v(\sigma) - \mathcal{L}_{W,k} v(\sigma')| \le \; e^{W(k\sigma)} \; (\, \theta\, |v|_\theta + \mathrm{C}\, \theta\, |v|_\infty \,)\,.$$

Hence,

$$\|\mathcal{L}_{W,k}\|_\theta \le \mathrm{C}\, \sup_\sigma e^{W(k\sigma)}$$

On the other hand, from Lemmata 2.1 and 7.1 it follows that for $\alpha < 1$ one has $\sup_\sigma e^{W(k\sigma)} \le \mathrm{C}\, \alpha^k$; whereas, for $\alpha = 1$, one finds $\sup_\sigma e^{W(k\sigma)} \le \mathrm{C}\, k^{-(1+1/s)}$. $\diamondsuit$

The (simbolic version of the) transfer operator associated to the map f is given by

$$\mathcal{L}_V u(\omega) = (\mathcal{L}_0 + \mathcal{L}_1) u(\omega)$$

with

$$\begin{cases} \mathcal{L}_0 u(\omega) = e^{V_0((\omega_0+1)\omega)} u((\omega_0+1)\omega) \\ \mathcal{L}_1 u(\omega) = e^{V_1(1\,\omega)} u(1\,\omega) \end{cases}$$

We now compare the action of $\mathcal{L}_V$ and $\mathcal{L}_W(z)$ by means of the bijection $\eta = \pi^{-1} \circ \xi : \Omega \to \Sigma$. Let $\mathcal{I}_\eta : C(\Sigma) \to C(\Omega)$ be defined by $\mathcal{I}_\eta \psi = \psi \circ \eta$. If $\psi \in \mathcal{F}_\theta(\Sigma)$ and $|z| \leq 1/\alpha$, it is then easy to check that $\mathcal{L}_W(z)\psi \in \mathcal{F}_\theta(\Sigma)$ and also $\mathcal{I}_\eta^{-1} \mathcal{L}_V \mathcal{I}_\eta \psi \in \mathcal{F}_\theta(\Sigma)$. Set moreover

$$\mathcal{B}_\theta = \{ \phi \in C(\Omega) \, : \, \mathcal{I}_\eta^{-1} \phi \in \mathcal{F}_\theta(\Sigma) \}$$

and

$$\mathcal{M}_W(z) = \mathcal{I}_\eta \, \mathcal{L}_W(z) \, \mathcal{I}_\eta^{-1}. \tag{7.11}$$

Now observe that if $\omega = \eta^{-1}(\sigma)$ then we can write

$$W(\sigma) = \sum_{j=0}^{\sigma_0 - 2} V_0(T^j \omega) + V_1(T^{\sigma_0 - 1} \omega).$$

This yields the representation

$$\mathcal{M}_W(z) = \sum_{k=1}^{\infty} z^k \, \mathcal{L}_1 \mathcal{L}_0^{k-1} = z \mathcal{L}_1 (1 - z\mathcal{L}_0)^{-1} \tag{7.12}$$

from which one deduces an algebraic relation between the operators $\mathcal{M}_W(z)$ and $\mathcal{L}_V$, which can be viewed as the counterpart of Proposition 7.1 (see [Pr] for related results):

Proposition 7.2. *For any $0 < |z| \leq 1/\alpha$ we have*

$$(\, 1 - \mathcal{M}_W(z) \,)\,(\, 1 - z\mathcal{L}_0 \,) = 1 - z\,\mathcal{L}_V.$$

Remark. In [CI1] we have constructed the absolutely continuous invariant probability measure $\rho(dx) = h(x)\, m(dx)$ for the dynamical system $([0,1], g)$ as a Gibbs state on Σ for the function $W(\sigma)$, whose density $h(\sigma)$ is in $\mathcal{F}_\theta(\Sigma)$ and satisfies $d^{-1} < h < d$ for some $d > 0$. Taking $z = 1$ in Proposition 7.2 one deduces that if $\mathcal{M}_W h = h$ and $\mathcal{L}_V e = e$ then h and e are related by $h = (1 - \mathcal{L}_0)\, e$ or else $e = \sum_{k=0}^{\infty} \mathcal{L}_0^k h$ (with a slight abuse of notation we have denoted by the same symbols h and e the function $h \circ \xi^{-1}$ and $e \circ \xi^{-1}$). This correspondence has been used in [CI2] and [CI3] to study some ergodic properties of the σ-finite absolutely continuous measure $\nu(dx) = e(x)\, m(dx)$ invariant for the dynamical system $([0,1], f)$ with $\alpha = 1$ and $s = 1$.

We can actually say more.

Proposition 7.3. *Let $0 < |z| \le 1/\alpha$. Then $1 \in \mathrm{sp}\,(\mathcal{M}_W(z))$ if and only if $1/z \in \mathrm{sp}\,(\mathcal{L}_V)$, and they have the same geometric multiplicity. Furthermore, the corresponding eigenfunctions e_z of $\mathcal{L}_V$ and h_z of $\mathcal{M}_W(z)$ are related by $h_z = (1 - z\mathcal{L}_0)e_z$ or else $e_z = \sum_{k=0}^{\infty} z^k \mathcal{L}_0^k h_z$.*

Proof. Assume that $\mathcal{M}_W(z)h_z = h_z$. Then, from Proposition 7.2 it follows that $(1 - z\mathcal{L}_V) \sum_{k=0}^{\infty} z^k \mathcal{L}_0^k h_z = 0$. Conversely, assume that $z\mathcal{L}_V e_z = e_z$, then $(1 - \mathcal{M}_W(z))(1 - z\mathcal{L}_0)e_z = 0$. $\diamondsuit$

Now set

$$R_n(W, z) = \sup_{\sigma \in \Sigma} \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} z^{k_1 + \dots + k_n} \exp \sum_{i=1}^{n} W(k_i \dots k_n \sigma)$$

and

$$P(W, z) = \lim_{n \to \infty} \frac{1}{n} \log R_n(W, z). \tag{7.13}$$

For any fixed $0 < |z| \le 1/\alpha$ the quantity $P(W, z)$ is the *pressure* associated to the interaction $W(\sigma) + \sigma_0 \log z$. In particular $P(W, 1) = 0$.

Lemma 7.3. *The following formal identity holds:*

$$1 - \exp P(W, z) = (1 - z) \sum_{n=0}^{\infty} d_n z^n$$

Proof. We first notice that the derivative

$$DP(W, 1) = \frac{d}{dz} P(W, z) \bigg|_{z=1}$$

gives the mean return time in the state 1 with respect to the Gibbs measure on Σ for the function W ([R5], Chapter 5). In other words we have

$$DP(W, 1) = \sum_{n=1}^{\infty} n \rho(A_n) = < \tau >_\nu \tag{7.14}$$

where (2.5) and the subsequent comment have been used. Therefore, by (2.3), one finds

$$1 - \exp P(W,z) = 1 - \sum_{n=1}^{\infty} \rho(A_n) z^n = (1-z) \sum_{n=0}^{\infty} d_n z^n. \qquad \diamondsuit$$

Remark. We know from Section 2 that $<\tau>_\nu$ is finite in the two cases: $\alpha < 1$ and $\alpha = 1$, $s < 1$. If $\alpha = 1$ and $s \geq 1$ then $<\tau>_\nu = \infty$. According to (7.14), this fact can be interpreted as a phase transition characterized by the coexistence, for $\alpha = 1$ and $s \geq 1$, of two equilibrium states for $(f, [0,1])$: the σ-finite measure ν and the Dirac delta measure concentrated at the neutral fixed point (see [Me]).

We now get some more information about the spectrum of $\mathcal{L}_W(z)$. Let us recall that the spectrum $\mathrm{sp}\,(P)$ of a bounded linear operator P can be decomposed into a discrete part, made up of isolated eigenvalues of finite multiplicity, and its complement, the essential spectrum, denoted by ess(P) (see, e.g., [DS]). The essential spectral radius is then defined as $r_{\mathrm{ess}}(P) = \sup\{\, |\lambda| \,:\, \lambda \in \mathrm{ess(P)}\,\}$. Moreover, we shall denote by $r(P)$ the spectral radius of P.

Theorem 7.1. *For any $0 < |z| \leq 1/\alpha$, the spectrum of $\mathcal{L}_W(z) : \mathcal{F}_\theta \to \mathcal{F}_\theta$, as well as that of $\mathcal{M}_W(z) : \mathcal{B}_\theta \to \mathcal{B}_\theta$, consists of two disjoint parts:*

1) every point in the disk $\{\, \lambda \,:\, |\lambda| \leq \theta \, \exp P(W, |z|)\,\}$;

2) isolated eigenvalues in the annulus $\{\, \lambda \,:\, \theta \, \exp P(W, |z|) < |\lambda| \leq \exp P(W, |z|)\,\}$.

If z is real and positive the spectral radius coincides with $\exp P(W,z)$ which is also a maximal simple eigenvalue.

Remark. According to Lemma 4.1, the smallest essential spectral radius is attained by taking $\theta = \beta$.

Proof. It is easy to check that for all $v \in \mathcal{F}_\theta$

$$|\mathcal{L}_W(z)^n v|_\infty \leq R_n(|z|)\, |v|_\infty \leq R_n(|z|)\, \|v\|_\theta$$

and

$$|\mathcal{L}_W(z)^n v|_\theta \leq R_n(|z|)\, (\mathrm{C}\, |v|_\infty + \theta^n |v|_\theta) \leq R_n(|z|)\, (\mathrm{C} + 1) \|v\|_\theta.$$

Therefore $\|\mathcal{L}_W(z)^n\|_\theta \leq (\mathrm{C}+2)R_n(|z|)$, where we have also denoted by $\| \ \|_\theta$ the operator norm. Thus, the spectral radius formula implies that

$$r(\mathcal{L}_W(z)) \leq \exp P(W, |z|)$$

Moreover, repeating the argument used in ([Po1], p.151; see also [K]), one finds

$$r_{\mathrm{ess}}(\mathcal{L}_W(z)) \leq \theta \exp P(W, |z|)$$

and also shows that the disk of radius $\theta \exp P(W, |z|)$ is included in $\mathrm{sp}\,(\mathcal{L}_W(z))$.

Now, if z is real and positive we have

$$r(\mathcal{L}_W(z)) = \lim_{n\to\infty} \left(\|\mathcal{L}^n_{W,z}\|_\theta \right)^{1/n} \geq \lim_{n\to\infty} \left(|\mathcal{L}^n_{W,z} 1|_\infty \right)^{1/n} = \exp P(W, z)$$

and therefore $r(\mathcal{L}_W(z)) = \exp P(W, z)$.

It remains to show that $\exp P(W, z)$ is a maximal simple eigenvalue of $\mathcal{L}_W(z)$. Let Ψ_z be such that $\mathcal{L}_W(z)\Psi_z = \lambda_z \Psi_z$ where λ_z is the (simple) eigenvalue with largest modulus of $\mathcal{L}_W(z)$. Reasoning as in ([CI1], Theorem 2.1) it is not difficult to show that $\Psi_z \in \mathcal{F}_\theta(\Sigma)$ and $d_z^{-1} \leq \Psi_z \leq d_z$ for some positive constant d_z. Then,

$$\begin{aligned} \log \lambda_z &= \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathcal{L}_W(z)^n \Psi_z \\ &= \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathcal{L}_W(z)^n 1 = P(W, z). \qquad \diamondsuit \end{aligned}$$

Now choose $\Theta > \theta$ and write $\mathcal{P}_z$ for the projection corresponding to the part of the spectrum of $\mathcal{L}_W(z)$ in the disk of radius $\Theta \exp P(W, z)$. Let moreover $\lambda_1(z), \ldots, \lambda_M(z)$ be the eigenvalues outside this disk, ordered with decreasing modulus, and $m_1(z), \ldots, m_M(z)$ their multiplicites. Then, for any $n > 0$, we have the following decomposition:

$$\mathcal{L}^n_W(z)v = \sum_{i=1}^{M} \lambda_i^n(z) \cdot \Psi_{z,i} L^n_{z,i} \Psi^*_{z,i} v + \mathcal{P}_z \mathcal{L}^n_W(z)\, v \tag{7.15}$$

where the row vector $\Psi_{z,i}$ and the column vector $\Psi^*_{z,i}$ span the generalized eigenspaces of $\mathcal{L}_W(z)$ and $\mathcal{L}^*_W(z)$ corresponding to the eigenvalue $\lambda_i(z)$, and the matrices $L_{z,i}$ can be assumed in Jordan normal form, so that $m_i(z) = \operatorname{tr} L_{z,i}$. In the same manner, for the operator $\mathcal{M}_W(z) : \mathcal{B}_\theta \to \mathcal{B}_\theta$ we have

$$\mathcal{M}_W^n(z)\, v = \sum_{i=1}^{M} \lambda_i^n(z) \cdot h_{z,i} L_{z,i}^n m_{z,i} v + \mathcal{P}_z \mathcal{M}_W^n(z)\, v \tag{7.16}$$

where $h_{z,i} = \mathcal{I}_\eta \circ \Psi_{z,i}$ and $m_{z,i} = \Psi_{z,i}^* \circ \mathcal{I}_\eta^{-1}$.

¿From the above spectral properties of $\mathcal{L}_W(z)$ we get the following result:

Theorem 7.2. *The two-variables zeta function defined in Proposition 7.1 has the following analytic properties:*

(i) for any $0 < |z| \le 1/r(\mathcal{L}_0)$, $1/Z(f, w, z)$, considered as a function of the variable w, is holomorphic in the disk of radius $1/\theta \exp P(W, |z|)$. Its zeroes in this disk, counted with multiplicity, are the inverses of the eigenvalues of $\mathcal{L}_W(z) : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ in the corresponding annulus. Moreover, the zero of smallest modulus is simple and located at $1/\exp P(W, z)$;

(ii) for any $0 < |w| \le 1$, $1/Z(f, w, z)$, considered as a function of the variable z, is holomorphic in the disk of radius $1/|w| r(\mathcal{L}_0)$. Its zeroes in this disk are located at those values of z such that $\mathcal{L}_W(z) : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ has $1/w$ as an eigenvalue.

Proof. We shall follow Haydn ([H1], Theorem 4; see also [BK] and [R2]). Fixing $n > 0$ we let $\sum_\nu$ be the sum over words ν of length n, i.e. words of the form $\nu = (\sigma_0, \ldots, \sigma_{n-1})$, and denote by $\sigma^{(\nu)}$ the periodic concatenation $(\overline{\sigma_0 \sigma_1 \ldots \sigma_{n-1}})$. Let moreover $\chi_\nu \in \mathcal{F}_\theta(\Sigma)$ be such that $\chi_\nu(\sigma) = 1$ if σ begins with the word ν, $\chi_\nu(\sigma) = 0$ otherwise. Then we have the following key relation:

$$\Lambda_n(f, z) := \sum_{\substack{\sigma \in \Sigma \\ T^n \sigma = \sigma}} z^{s_n(\sigma)} \exp \sum_{j=0}^{n-1} W(T^j \sigma) = \sum_\nu (\mathcal{L}_W^n(z) \chi_\nu)(\sigma^{(\nu)}) \tag{7.17}$$

Inserting (7.15) into (7.17) we get

$$\begin{aligned}
\Lambda_n(f, z) &= \sum_\nu \sum_{i=1}^{M} \lambda_i^n(z) \cdot \Psi_{z,i}(\sigma^{(\nu)}) L_{z,i}^n \Psi_{z,i}^* \chi_\nu + \sum_\nu (\mathcal{P}_z \mathcal{L}_W^n(z) \chi_\nu)(\sigma^{(\nu)}) \\
&= \sum_{i=1}^{M} \lambda_i^n(z) \cdot (L_{z,i}^n \Psi_{z,i}^*)^{\mathrm{tr}} (\sum_\nu \Psi_{z,i}(\sigma^{(\nu)}) \cdot \chi_\nu)^{\mathrm{tr}} + \sum_\nu (\mathcal{P}_z \mathcal{L}_W^n(z) \chi_\nu)(\sigma^{(\nu)}) \\
&= \Lambda_n^{(0)}(f, z) + \Lambda_n^{(1)}(f, z) + \Lambda_n^{(2)}(f, z)
\end{aligned}$$

where

$$\Lambda_n^{(0)}(f,z) = \sum_{i=1}^{M} m_i(z)\,\lambda_i^n(z)$$

$$\Lambda_n^{(1)}(f,z) = \sum_{i=1}^{M} \lambda_i^n(z)\cdot(L_{z,i}^n \Psi_{z,i}^*)^{\mathrm{tr}}(\sum_\nu \Psi_{z,i}(\sigma^{(\nu)})\cdot\chi_\nu - \Psi_{z,i})^{\mathrm{tr}}$$

$$\Lambda_n^{(2)}(f,z) = \sum_\nu (\mathcal{P}_z \mathcal{L}_W^n(z)\chi_\nu)(\sigma^{(\nu)})$$

and Ψ^{tr} denotes transposition. The first term gives the contribution:

$$\exp -\sum_{n=1}^{\infty}\frac{w^n}{n}\Lambda_n^{(0)}(f,z) = \prod_{i=1}^{M}(1-w\lambda_i(z))^{m_i}.$$

Now, according to Theorem 7.1, if z is real and positive $\lambda_1(z)$ is simple and equals $\exp P(W,z)$. On the other hand, for any $w\in\mathbb{C}$, the function

$$Q(W,z,w) := 1 - w\exp P(W,z) \tag{7.18}$$

extends uniquely to a function holomorphic in the disk $\{z : |z|\le 1/\alpha\}$. Therefore, in the domain $\{z : |z|\le 1/\alpha\}\times\{w : |w|\le 1\}$ we have

$$\exp -\sum_{n=1}^{\infty}\frac{w^n}{n}\Lambda_n^{(0)}(f,z) = Q(W,z,w)\cdot\prod_{i=2}^{M}(1-w\lambda_i(z))^{m_i}. \tag{7.19}$$

Finally, proceeding as in [H1], one can show the existence of two constants $R_1, R_2 > 0$ such that

$$|\Lambda_n^{(1)}(f,z)| \le R_1\Theta^n e^{nP(W,|z|)} \quad\text{and}\quad |\Lambda_n^{(2)}(f,z)| \le R_2\Theta^n e^{nP(W,|z|)}$$

and the Theorem is proved. ♢

Putting together Proposition 7.1, Proposition 7.3 and Theorem 7.2 (with the subsequent Remark) we obtain the following,

Corollary 7.1.

(i) *$1/\zeta(g,z)$ is holomorphic in the disk of radius $1/\beta$. Its zeroes in this disk, counted with multiplicity, are the inverses of the eigenvalues of $\mathcal{L}_W$ in the annulus $\{\,\lambda : \beta < |\lambda| \le 1\,\}$.*

(ii) $1/\zeta(f,z)$ *is holomorphic in the disk of radius* $1/\alpha$*. In this disk,* $1/\zeta(f,z) = 0$ *if and only if* $1/z \in \mathrm{sp}(\mathcal{L}_V)$.

Remark. When $\alpha = 1$ the above result yields no zeroes of $1/\zeta(f,z)$ but the point $z = 1$. In this case one expects that the eigenvalue 1 of $\mathcal{L}_V$ is not isolated (i.e. there is no 'gap'). Nevertheless, one may consider a generalised transfer operator corresponding to the interaction βV, for some $\beta \in \mathbb{C}$, and then study the spectral gap and the analytic properties of the pressure $P(\beta V)$ as functions of β (see [Pr],[L]).

8. Proofs of the main results.

Proof of Theorem 3.1. Set

$$Q(W,z) \equiv Q(W,z,1) = 1 - \exp P(W,z)$$

According to Lemma 7.3 we have

$$Q(W,z) = (1-z)\sum_{n=0}^{\infty} d_n z^n$$

and the assertion follows putting together (7.19), with $w = 1$, and Proposition 7.1. $\diamondsuit$

Proof of Theorem 3.2. We first notice that having fixed $0 < \gamma \le 1$ one can find $0 < \theta < 1$ such that

$$u \circ \pi^{-1} \in \mathcal{F}_\theta \quad \text{and} \quad u \circ \xi^{-1} \in \mathcal{B}_\theta \quad \text{if} \quad u \in \mathcal{F}^\gamma$$

and the linear maps $\mathcal{F}^\gamma \to \mathcal{F}_\theta$ and $\mathcal{F}^\gamma \to \mathcal{B}_\theta$ thus defined are continuous. This enables us to study the problem in the spaces $\mathcal{F}_\theta$ or $\mathcal{B}_\theta$. The property of having compact support on $]0,1]$ of a given $u \in \mathcal{F}^\gamma([0,1])$ translates, for instance, into that of having compact support on $\Omega \setminus \Omega_\infty$, where $\Omega_\infty := \cup_{K>0}\{\,\omega \in \Omega \,:\, \omega_i > K,\ \forall i \ge 0\,\}$, for $u \circ \xi^{-1} \in \mathcal{B}_\theta$.

We now use the results of Section 7. In particular, we shall consider the decomposition (7.16) where, for the simplicity of formulae, we assume that the eigenvalues λ_i of $\mathcal{M}_W(z) : \mathcal{B}_\theta \to \mathcal{B}_\theta$ with modulus $> \Theta \exp P(W, z)$ are simple.

For z real and positive, let $h_z \equiv h_{z,1}$ and $m_z \equiv m_{z,1}$ be normalized so that $m_z(h_z) = 1$ and set $\rho_z = h_z\, m_z$.

Moreover, for $z = 1$, set $m \equiv m_1$ and $h \equiv h_1$. Then $\rho = h\, m$ is the symbolic version of the unique absolutely continuous probability measure invariant for the dynamical system $([0,1], g)$ (denoted with the same symbol with slight abuse of language). With the same abuse of language we denote by the symbol e the eigenvector of $\mathcal{L}_V$ to the eigenvalue 1 and $\nu = e\, m$. According to Proposition 7.3, we also define $e_{z,i} = (1 - z\mathcal{L}_0)^{-1}\, h_{z,i}$, for $i > 1$.

Let now $u, v \in \mathcal{F}^\gamma([0,1])$. For simplicity of notation (and without fear of confusion) we shall also denote with the symbols u, v the functions $u \circ \xi^{-1}, v \circ \xi^{-1} \in \mathcal{B}_\theta$. ¿From Lemma 2.6 and the above observations it follows that if $u \in \mathcal{B}_\theta$ then $u \cdot e \in \mathcal{B}_\theta$ as well. Then, using Proposition 7.2, Proposition 7.3 and the decomposition (7.16), we have the following calculation:

$$
\begin{aligned}
s_{uv}(z) &= \sum_{n=0}^{\infty} z^n\, \nu(u \cdot v \circ T^n) \\
&= \sum_{n=0}^{\infty} z^n\, m(v \cdot \mathcal{L}_V^n(u \cdot e)\,) \\
&= m(v \cdot (1 - z\mathcal{L}_V)^{-1}(u \cdot e)\,) \\
&= m(v \cdot (1 - z\mathcal{L}_0)^{-1}(1 - \mathcal{M}_W(z))^{-1}(u \cdot e)\,) \\
&= \frac{m_z(u \cdot e)\, m(e_z \cdot v)}{1 - \lambda_1(z)} + \sum_{i=2}^{M} \frac{m_{z,i}(u \cdot e)\, m(e_{z,i} \cdot v)}{1 - \lambda_i(z)} \\
&\qquad + m(v\,(1 - z\mathcal{L}_0)^{-1}(1 - \mathcal{M}_W(z))^{-1}\mathcal{P}_z(u \cdot e)\,)
\end{aligned}
$$

For $0 < z < 1$ the first term in the last expression reads, by Lemma 7.3,

$$
\frac{m_z(u \cdot e)\, m(e_z \cdot v)}{(1 - z) \sum_{n=0}^{\infty} d_n z^n}
$$

so that

$$
s_{uv}(z) \sim m(u \cdot e)\, m(e \cdot v)\, d(f, z) \quad \text{as} \quad z \to 1_-
$$

and the theorem follows. $\diamondsuit$

Proof of Corollary 3.1. The first assertion, both of a) and b), is immediate. The second assertion of b) easily follows from Karamata's theorem and Lemma 2.5. We then prove the second statement of a). We have

$$\sum_{n=0}^{\infty} u_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = \sum_{n=0}^{\infty} z^n$$

so that, putting $\varrho_n =< \tau >_\nu \; u_n - 1$, we get

$$\sum_{n=0}^{\infty} \varrho_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = \sum_{n=0}^{\infty} \tilde{d}_n z^n \tag{$*$}$$

with $\tilde{d}_n = \sum_{l>n} d_l$. Now, for $\alpha < 1$, the sequences c_n, $\tilde{c}_n$, d_n and $\tilde{d}_n$ have all the same asymptotic behaviour, as $\mathrm{C}\,\alpha^n$, so that the assertion is plainly true. For $\alpha = 1$ and $s < 1$, using Lemmata 2.1 and 2.5 we get

$$d_n \sim \mathrm{C}\, n^{-1/s} \quad \text{and} \quad \tilde{d}_n \sim \mathrm{C}\, n^{1-1/s}$$

and an elementary calculation shows that (*) implies $\varrho_n \sim \mathrm{C}\, n^{1-1/s}$. $\diamondsuit$

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