QUANTUM ERGODICITY OF C^* DYNAMICAL SYSTEMS

STEVEN ZELDITCH*

0. Introduction

The purpose of this paper is to generalize some basic notions and results on quantum ergodicity ([Sn], [CV], [Su], [Z.1], [Z.2]) to a wider class of C^* dynamical systems (A, G, α) which we call *quantized Gelfand-Segal systems* (Definition 1.1). The key feature of such a system is an invariant state ω which in a certain sense is the barycenter of the normal invariant states. By the Gelfand-Segal construction, it induces a new system $(\mathcal{A}_\omega, G, \alpha_\omega)$, which will play the role of the classical limit. Our main abstract result (Theorem 1) shows that if $(\mathcal{A}, G, \alpha)$ is a quantized GS system, if the classial limit is abelian (or if $(\mathcal{A}, \omega)$ is a "G-abelian" pair), and if ω is an ergodic state, then "almost all" the ergodic normal invariant states ρ_j of the system tend to ω as the "energy" $E(\rho_j) \to \infty$. This leads to an intrinsic notion of the quantum ergodicity of a quantized GS system in terms of operator time and space averages (Definition 0.1), and to the result that a quantized GS system is quantum ergodic if its classical limit is an ergodic abelian system (or if $(\mathcal{A}, \omega)$ is an ergodic G-abelian pair) (Theorem 2). Concrete applications include a simplified proof of quantum ergodicity of the wave group of a compact Riemannian manifold with ergodic geodesic flow, as well as extensions to manifolds with concave boundary and ergodic billiards, to quotient Hamiltonian systems on symplectic quotients and to ergodic Hamiltonian subsystems on sympletic subcones. More elaborate applications will appear in forthcoming articles: to manifolds with general piecewise smooth boundary and ergodic billiards in [Z.Zw] , and in [Z.5] to quantized ergodic contact (or contactible) transformations acting on powers of a line bundle (including quantized hyperbolic toral automorphisms acting on spaces of theta functions).

To state our results, we will need to introduce some terminology and notation. We will also briefly review some relevant background on quantum ergodicity and on C^* dynamical systems, with the aim of clarifying the connections between the two.

Quantum ergodicity, in the sense of this paper, is the study of quantum dynamical systems whose underlying classical dynamical systems are ergodic. For instance, the wave group $U_t = \exp it\sqrt{\Delta}$ of a compact Riemannian manifold (M, g) is the quantization of the geodesic flow G^t on S^*M. The basic problem is to determine the asymptotic properties of various invariants of the spectrum $\{\lambda_j\}$ and eigenfunctions $\{\phi_j\}$ in the limit $\lambda_j \to \infty$, under the condition that G^t acts ergodically with respect to the normalized Liouville measure $d\mu$ on S^*M. For some of the many heuristic and numerical results we refer to the recent survey of Sarnak [Sa].

¿From the C^* algebra point of view, a quantum dynamical system is a C^* dynamical system $(\mathcal{A}, G, \alpha)$ where $\mathcal{A}$ is a C^*-algebra, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a representation of G by automorphisms of $\mathcal{A}$. We will always assume $\mathcal{A}$ is unital and separable, that G is amenable and that the system is covariantly represented on a Hilbert space $\mathcal{H}$. That is, we will assume there is a representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of $\mathcal{A}$ as bounded operators on $\mathcal{H}$, and a unitary representation $U : G \to U(\mathcal{H})$ such that $\alpha_g(A) = U_g^* \pi(A) U_g$. Representations are understood to be continuous. Henceforth we will denote $\pi(A)$ simply by A. For terminology regarding C^* algebras we follow [B.R] and [R].

As is evident, the notion of quantum ergodicity which we intend to generalize is a semi-classical one. Hence we must define a class of C^* dynamical systems for which it makes sense to speak of the semi-classical limit. To this end, we introduce in §1 the class of *quantized Gelfand-Segal* systems. For such systems, there

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will be a well-defined "energy"

$$E : \mathcal{N}_A \cap \mathcal{E}(E^G_{\mathcal{A}}) \to \mathbb{R}^+$$

on the set of normal ergodic states; roughly speaking, to each such state ρ will correspond an irreducible $\sigma \in \hat{U}$ and the energy will be defined by

$$E(\rho) = \delta(\sigma, 1)$$

with $\delta(\sigma, 1)$ more or less the distance of σ from the trivial representation of G. Above $\hat{U}$ is the spectrum of U, i.e. the set of irreducibles σ in the unitary dual $\hat{G}$ of G which occur in U. Moreover, there will exist for each $E > 0$ a well-defined *microcanonical ensemble* ω_E at energy level E, which will essentially be the average of all normal ergodic states ρ of energy $E(\rho) \leq E$. Enough (in fact, more than enough) will be assumed about $\hat{G}$ and $\hat{U}$ to make the definitions of E and ω_E run smoothly.

The key property of quantized Gelfand-Segal systems will be the following:

- there exists a unique "classical limit" state ω such that $\omega_E \to \omega$ weakly as $E \to \infty$.

By the Gelfand-Segal construction (§1; [B.R]), ω gives rise to a a cyclic representation π_ω of $\mathcal{A}$, and a unitary representation U_ω of G, on a Hilbert space $\mathcal{H}_\omega$. As mentioned above, the induced system $(\pi_\omega(\mathcal{A}), G, \alpha_\omega)$ will play the role of the classical limit, and $(\mathcal{A}, G, \alpha)$ will be regarded as its quantization. Of course, the classical limit need not be abelian; if it is, the original system will be called *quantized abelian*. For the proofs of Theorems 1 and 2 it is in fact sufficient that the pair $(\mathcal{A}, \omega)$ be "G-abelian" (see [B.R] or §1 for the definition). In this case the original system will be called *quantized G-abelian*.

To illustrate the notion of quantized abelian system, consider the example above with $G = \mathbb{R}$, $\mathcal{H} = L^2(M)$ and $U_t = \exp it\sqrt{\Delta}$. The relevant algebra is $\mathcal{A} = \Psi^\circ(M)$, the algebra of zero-th order pseudodifferential operators on M (or its C^* closure, to be perfectly precise). The action of $\mathbb{R}$ is given by $\alpha_t(A) = U_t^* A U_t$. The spectrum of U is of course the set of characters $\{\exp it\sqrt{\lambda_j}\}$, and $\delta(\exp it\sqrt{\lambda_j}, 1) = \sqrt{\lambda_j}$. The normal ergodic states are given by $\rho_j(A) = (A\varphi_j, \varphi_j)$, and the energy $E(\rho_j) = \sqrt{\lambda_j}$. The microcanonical ensemble is

$$\omega_E = \frac{1}{N(E)} \sum_{\sqrt{\lambda_j} \leq E} \rho_j$$

and as is well-known it tends to the state

$$\omega(A) = \int_{S^*M} \sigma_A d\mu \ .$$

The classical limit system is then G^t acting on $L^2(S^*M, d\mu)$; hence the original system is quantized abelian. For further discussion, see §3.

Postponing the precise definitions until §1, we can state our main abstract result as follows:

Theorem 1 . *Let $(\mathcal{A}, G, \alpha)$ be a quantized abelian (or G-abelian) system and suppose that the classical limit state ω is an ergodic state.*

Then, for any admissible density D^* on the set $\mathcal{N}_A \cap \mathcal{E}(E^G_{\mathcal{A}})$ of normal ergodic states, there exists a subset $\mathcal{S} \subset \mathcal{N}_A \cap \mathcal{E}(E^G_{\mathcal{A}})$ such that:

(a) $D^*(\mathcal{S}) = 1$

(b) $\underset{\substack{E(\rho)\to\infty \\ \rho \in \mathcal{S}}}{\text{weak}^*\text{-}\lim} \rho = \omega.$

For the previous example of $(\Psi^\circ(M), \mathbb{R}, \alpha)$, the theorem shows that

$$(A\varphi_j, \varphi_j) \to \int_{S^*M} \sigma_A d\mu \qquad (\lambda_j \in \mathcal{S})$$

where $\mathcal{S}$ is a subset of full counting density in the spectrum $\{\lambda_j\}$, and $A \in \Psi^\circ(M)$. Hence Theorem 1 gives a rather abstract version of the quantum ergodicity theorem that eigenstates of quantizations of classical ergodic systems become uniformly distributed on energy surfaces in the high energy limit ([Sn],[CV],[Z.1]).

The proof of Theorem 1 is quite simple, and indeed simplifies the previous proofs. The underlying idea (which is perhaps not visible in the proof) is even simpler: By assumption, the limit state ω is an extreme point of the compact convex set of invariant states. The condition $\omega_E \to \omega$ states more or less that ω is the barycenter of the set of pure normal invariant states. This is a contradiction unless these pure states tend individually to ω. This idea suggests that Theorem 1 may admit a more general formulation. In the actual proof, the additional fact is used that for ergodic states of abelian systems,or for ergodic G-abelian pairs, there is "uniqueness of the vacuum" in the associated classical limit (i.e. rank $E_\omega = 1$; see [R, p. 155] for terminology).

The conclusion of Theorem 1 may be taken as a definition of the quantum ergodicity of a quantized abelian or G-abelian system. To obtain a better understanding of it, we reformulate it in terms of the operator averages

$$\langle A \rangle_\alpha = \int_G \chi_\alpha(g) \alpha_g(A) dg$$

where $\{\chi_\alpha\}$ is an "M-net" for the amenable group $G[R]$. For instance, if $G = \mathbb{R}^n \times T^m \times \mathbb{Z}^k \times K$ as above, then $\chi_\alpha(g)dg$ could be the product $\chi_\alpha^{\mathbb{R}^n}(x)dx \otimes d\theta \otimes \chi_\alpha^{\mathbb{R}^k}(n)dn \otimes d\mu$, where $d\mu$ (resp. $d\theta$) is the normalized Haar measure on K (resp. T^n), where $\chi_\alpha^{\mathbb{R}^n}$ is α^{-n} times the characteristic function of a cube of side α and where dx (resp. dn) is Lesbesque measure (resp. counting measure on $\mathbb{Z}^k$). The limit as $\alpha \to \infty$ of $\langle A \rangle_\alpha$ does not exist in $\mathcal{A}$, but it does exist in the W^* (von Neumann) closure of $\pi(\mathcal{A})$, i.e. the closure in the strong operator topology of $\mathcal{L}(\mathcal{H})$. We will denote this closure of $\mathcal{A}$ by $\mathcal{M}$, and set

$$\langle A \rangle = \text{w-}\lim_{\alpha \to \infty} \langle A \rangle_\alpha \; .$$

Following [Su] and [Z.3], we will say:

(0.1) Definition. Let $(\mathcal{A}, G, \alpha)$ be a quantized GS system. Say, $(\mathcal{A}, G, \alpha)$ is a *quantum ergodic* system, if there exists an (invariant) state $\omega \in E_{\mathcal{A}}^G$ such that for all $A \in \mathcal{A}$,

$$\langle A \rangle = \omega(A) I + K$$

where $K \in \mathcal{M}$ and where

$$\lim_{E \to \infty} \omega_E(K^* K) = 0 \; .$$

Thus, the time average of an observable equals its space average plus an asymptotically neglible error as $E \to \infty$. Note that ω_E is normal, so is well-defined on K^*K. We have

Theorem 2 . *Let $(\mathcal{A}, G, \alpha)$ satisfy the assumptions of Theorem 1. Then it is a quantum ergodic system.*

We remark that the state ω in the definition of quantum ergodicity is necessarily the weak* limit of ω_E. However, it is not clear that it has to be ergodic; there may exist quantum ergodic systems which are not classically ergodic. Regarding this converse direction we have the following result (cf. [Su][Z.2,5]):

Theorem 3 . *Let $(\mathcal{A}, G, \alpha)$ be a quantized abelian system, with G abelian.*

(a) Suppose ω is ergodic. Then

(0.2). $$\lim_{T \to \infty} \lim_{E \to \infty} \omega_E(< A >_T^* A) = \lim_{E \to \infty} \lim_{T \to \infty} \omega_E(< A >_T^* A) = |\omega(A)|^2$$

(b) Suppose conversely that $(\mathcal{A}, G, \alpha)$ is quantum ergodic and (0.2) holds. Then ω, hence the classical limit system, is ergodic.

The condition (0.2) is of course equivalent to

(0.3) $$\lim_{T \to \infty} \omega(< A >_T^* A) = \lim_{E \to \infty} \omega_E(< A >^* A) = \omega(< A >^* A)$$

at least when ω extends to the von-Neumann completion. Less obviously it is equivalent to

(0.4) $$\forall \epsilon \exists \delta \lim_{E \to \infty} \frac{1}{N(E)} \sum_{i \neq j, |\chi_i|, |\chi_j| \leq E, |\chi_i - \chi_j| \leq \delta} |(A\phi_i, \phi_j)|^2 \leq \epsilon$$

where $\{\phi_i\}$ is an orthonormal basis of joint eigenfunctions and where $\{\chi_i\}$ are the corresponding eigenvalues (characters) (cf. [Su][Z.2]). The proof of (b) is based on the following

Spectral measure Lemma. *Define the measure dm_A on $C_c(\hat{G})$ by*

$$\int_{\hat{G}} f(\chi) dm_A(\chi) := \lim_{E\to\infty} \omega_E(< A >^*_{\mathcal{F}f} A) \tag{0.3}$$

with $\mathcal{F}f$ the Fourier transform of f and with $< A >_h := \int_G h(g)\alpha_g(A)dg$. Then: dm_A is the spectral measure for the classical dynamical system corresponding to vector $\pi_\omega(A)$.

Theorems 1 and 2 have a number of applications to C^* dynamical systems $(\mathcal{A}, G, \alpha)$ where $\mathcal{A}$ is an algebra of pseudodifferential, Fourier Integral or Toeplitz operators. We will present some rather simple examples with $G = \mathbb{R}$ in §3; more elaborate examples will be presented in [Z.Zw] and [Z.5].

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1. Quantized Gelfand-Segal systems

In this section we state more precisely the conditions on $(\mathcal{A}, G, \alpha)$ which are assumed in the statements of Theorems 1 and 2.

As mentioned above, $\mathcal{A}$ will be assumed to be unital and separable and $(\mathcal{A}, G, \alpha)$ will be assumed to have an effective covariant representation $(\mathcal{H}, \pi, U)$ on a Hilbert space $\mathcal{H}$. $\mathcal{A}$ will also be assumed to contain a subalgebra $\mathcal{K}$ which gets represented as the compact operators on $\mathcal{H}$. We will further assume that the spectrum $\hat{U}$ of U is discrete, in particular that the multiplicity $m(\sigma)$ of each $\mathcal{H}_\sigma$ is finite. We then denote by

$$\mathcal{H} = \bigoplus_{\sigma\in\hat{U}} \mathcal{H}_\sigma$$

the isotypic decomposition of U and by

$$\Pi_\sigma : \mathcal{H} \to \mathcal{H}_\sigma$$

the orthogonal projection onto the isotypic summand H_σ.

For the sake of simplicity, we will assume that G is an amenable Lie group of the form

$$G = \mathbb{R}^n \times T^m \times \mathbb{Z}^k \times K ,$$

where T^m is the real m-torus and where K is a compact semi-simple Lie group. Hence the unitary dual $\hat{G}$ of G has the form

$$\hat{G} = \mathbb{R}^n \times \mathbb{Z}^m \times T^k \times \hat{K}$$

where in the usual way we identify

$$\hat{K} = I^* \cap \mathfrak{t}^*_+$$

with I^* the lattice of integral forms and $\mathfrak{t}^*_+$ a closed Weyl chamber in the dual of a Cartan subalgebra. We can then define the "distance" $\delta(1, \sigma)$ of a representation $\sigma \in \hat{G}$ from the trivial representation by

$$\delta(\sigma, 1) = |\bar{\sigma}|$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n \times \mathbb{R}^m \times \mathfrak{t}^*_+$, and where $\bar{\sigma}$ is the projection of σ to this space. We regard $\delta(\sigma, 1)$ as the semi-classical parameter, i.e. as the inverse Planck constant or energy level.

The numerical spectrum

$$spec(U) := \{\delta(\sigma, 1) : \sigma \in \hat{U}\}$$

of "energy levels" is then a discrete subset of $\mathbb{R}^+$,

$$0 = E_o < E_1 < E_2 < \ldots \uparrow \infty.$$

There are two natural notions of the multiplicity of an energy level. The first, given by

$$m(E_j) = \sum_{\sigma\in\hat{U}, \delta(\sigma,1)=E_j} rank \Pi_\sigma ,$$

counts the total dimension in the energy range, while the second

$$m^*(E_j) = \sum_{\sigma\in\hat{U}, \delta(\sigma,1)=E_j} m(\sigma)$$

counts the number of irreducibles. They give rise to the two spectral counting functions

$$N(E) := \sum_{j: E_j \leq E} m(E_j),$$

respectively

$$N^*(E) := \sum_{j: E_j \leq E} m^*(E_j).$$

In many applications, $N(E)$ has an asymptotic expansion as $E \to \infty$, and $m(E_j)$ is of strictly lower order than $N(E_j)$, but it does not seem natural in the rather general context of this section to introduce too many hypothesis on the spectrum. To avoid pathologies, however, we will assume that the spec(U) is *regular* in the sense that

$$m(E_{j+1}) \leq CN(E_j), \qquad\qquad m^*(E_{j+1}) \leq CN^*(E_j)$$

for some $C > 0$.

Corresponding to each isotypic summand $\mathcal{H}_\sigma, \sigma \in \hat{U}$ we define the normal invariant state

$$\omega_\sigma(A) = \frac{1}{\operatorname{rank} \Pi_\sigma} \operatorname{Tr} \Pi_\sigma A \ .$$

Note that ω_σ is not ergodic unless the multiplicity of σ in $\mathcal{H}_\sigma$ is one: In fact, the normal ergodic states are in one-one correspondence with projections P onto irreducible subspaces in $\mathcal{H}$. To see this, recall that a normal invariant state corresponds to a density matrix (positive trace-class operator) ρ which commutes with G. It is therefore a sum of scalar multiples of projections onto irreducibles, and is indecomposible if and only if it is a multiple of one such projection. Since it has unit mass, each normal ergodic state ρ must be of the form $\rho(A) = \frac{1}{d(\sigma)} \operatorname{Tr} P_\sigma A$ where $d(\sigma) = \operatorname{Tr} P_\sigma$ and P_σ is a projection onto an irreducible subspace of some type $\sigma \in \hat{U}$.

We then introduce the *microcanonical ensemble at energy level E*,

$$\omega_E := \frac{1}{N(E)} \sum_{\sigma: \delta(\sigma, 1) \leq E} (\operatorname{rank} \Pi_\sigma) \omega_\sigma .$$

It is the state corresponding to the usual microcanonical density matrix

$$\frac{1}{N(E)} \sum_{\sigma: \delta(\sigma, 1) \leq E} \Pi_\sigma$$

(see [T,(2.3.1)], and is the most mixed combination of the states of energy less than E. We also introduce the ensemble

$$\tilde{\omega}_E := \frac{1}{N^*(E)} \sum_{\sigma: \delta(\sigma, 1) \leq E} m(\sigma) \omega_\sigma$$

with

$$N^*(E) := \sum_{\delta(\sigma, 1) \leq E} m(\sigma).$$

which is the most mixed combination of the normal ergodic states of energy less than E. Both ensembles seem to be natural candidates for the microcanonical ensemble, and the statements of Theorems 1-3 are valid for both. Since ω_E differs from $\tilde{\omega}_E$ only in weighting the σth term by $d(\sigma)$, the two ensembles coincide if G is abelian, and in applications the two ensemble averages are often asymptotically equivalent, i.e. $\omega_E(A) \sim \tilde{\omega_E}(A)$.

Associated to the microcanonical ensemble ω_E (or $\tilde{\omega_E}$) is the corresponding collection of *admissible densities* on the set of normal ergodic states. To define these densities, we denote by S_σ the set of irreducible subspaces in $\mathcal{H}_\sigma$, and by $\mathcal{N}_A \cap \mathcal{E}(E^G_{\mathcal{A}})$ the set of normal ergodic states. We then have

$$\mathcal{N}_A \cap \mathcal{E}(E^G_{\mathcal{A}}) = \bigsqcup_{\sigma \in \hat{U}} S_\sigma \ ,$$

and define the *energy* of a normal ergodic state by

$$E(\rho) = \delta(\sigma, 1) \qquad (\rho \in S_\sigma) \ .$$

The admissible densities D^*_ν on $\mathcal{N}_A \cap \mathcal{E}(E^G_{\mathcal{A}})$ are constructed from families $\nu = \{\nu_\sigma : \sigma \in \hat{U}\}$ of unit mass measures on the S_σ's, with each ν_σ giving a barycentric decomposition

$$\omega_\sigma = \int_{S_\sigma} \omega_\phi d\nu_\sigma(\phi)$$

of ω_σ into ergodic states. To define the corresponding density, we note that a subset $\mathcal{S} \subset \mathcal{N}_A \cap \mathcal{E}(E^G_{\mathcal{A}})$ has the form

$$\mathcal{S} = \bigsqcup_{\sigma \in \hat{U}} \tilde{\mathcal{S}}_\sigma \qquad (\tilde{\mathcal{S}}_\sigma = \mathcal{S} \cap S_\sigma).$$

For the choice ω_E of microcanonical ensemble, we then set

$$D^*_\nu(\mathcal{S}) := \lim_{E\to\infty} \frac{1}{N(E)} \sum_{\delta(\sigma,1)\leq E} \nu_\sigma(\tilde{\mathcal{S}}_\sigma)\,\mathrm{rank}\,\Pi_\sigma.$$

In the case of $\tilde{\omega_E}$, we define the density $\tilde{D}^*_\nu$ analogously but with $m(\sigma)$ in place of rankΠ_σ. In the simplest case where G is abelian and U is multiplicity free, both densities coincide and are given by $D^*(\mathcal{S}) = \lim_{E\to\infty} \frac{1}{N(E)} \#\{\sigma \in \mathcal{S} : \delta(\sigma, 1) \leq E\}$.

We then say:

(1.1) Definition. $(\mathcal{A}, G, \alpha)$ is a *quantized Gelfand-Segal system* if it satisfies the following conditions:

(a) $G = \mathbb{R}^n \times T^m \times \mathbb{Z}^k \times K$;

(b) $\hat{U}$ is discrete and *spec*(U) is regular;

(c) There exists an invariant state ω such that $\lim_{E\to\infty} \omega_E = \omega$.

In (c), the limit is understood to be in the weak* sense. Corresponding to the choice of $\tilde{\omega}_E$, (c) is of course replaced by

(c') There exists an invariant state such that $\lim_{E\to\infty} \tilde{\omega}_E = \omega$.

Let us recall that by the Gelfand-Segal (or GS) construction [R; A.3.5, 6.2.2], [B.R], the invariant state ω gives rise to a covariant cyclic representation $(\mathcal{H}_\omega, \pi_\omega, U_\omega, \Omega_\omega)$ of $(\mathcal{A}, G, \alpha)$ with the properties

$$\alpha_\omega(g)\pi_\omega(A) := U_\omega(g)\pi_\omega(A)U_\omega(g)^{-1} = \pi_\omega(\alpha_g(A))$$
$$U_\omega(g)\Omega_\omega = \Omega_\omega$$
$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)\ .$$

We recall that the Hilbert space $\mathcal{H}_\omega$ is the closure of $\mathcal{A}/\mathcal{N}$ with respect to the inner product $\omega(AB^*)$, where $\mathcal{N}$ is the left ideal $\{A \in \mathcal{A} : \omega(A^*A) = 0\}$. Also, that the representation π_ω is defined by $\pi_\omega(A)(B + \mathcal{N}) = (AB + \mathcal{N})$; that $\Omega_\omega = I + \mathcal{N}$; and that $U_\omega(g)(B + \mathcal{N}) = (\alpha_g(B) + \mathcal{N})$. The new C^* dynamical system $(\pi_\omega(\mathcal{A}), G, \alpha_\omega)$ will be referred to as the classical limit of $(\mathcal{A}, G, \alpha)$.

In semi-classical analysis, it is natural to focus on the case where $\pi_\omega(\mathcal{A})$ is abelian, and hence isomorphic to $C(X)$ for a compact Hausdorff space X. We recall that X is the set of pure states of $\pi_\omega(\mathcal{A})$, and that the isomorphism is given by $A + \mathcal{N} \to \psi_A$, where $\psi_A(\rho) = \rho(A + \mathcal{N})$. As the notation suggests ψ_A will denote the element of $C(X)$ corresponding to A under the composition $A \to \pi_\omega(A) \to \psi_A$. Also, it is clear that the states of $\pi_\omega(A)$ determine states of $\mathcal{A}$ which annihilate $\mathcal{N}$. Under this isomorphism, the states of $\pi_\omega(A)$ correspond to the probability measures on X. In particular, ω induces the state $\pi_\omega(A) \to (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$. Let us denote by μ the corresponding measure. Then $\mathcal{H}_\omega \simeq L^2(X, \mu)$, and the automorphisms $\alpha_\omega(g)$ determine a group of measure preserving transformations of (X, μ) and the unitary group $U_\omega(g)$ of translations in $L^2(X, \mu)$. We will say:

(1.2) Definition). $(\mathcal{A}, G, \alpha)$ is a *quantized abelian* system if it is a quantized GS system and if the classical limit system is abelian.

It is potentially interesting to consider quantized GS systems with nonabelian classical limits. For the purposes of this paper, a second natural condition on the classical limit is the uniqueness of the vacuum state. We recall that this means that the projection E_ω onto the $U_\omega(G)-$ invariant vectors in $\mathcal{H}_\omega$ has rank one, i.e. that Ω_ω is the unique invariant vector up to scalar multiples. This is equivalent to ergodicity of

ω (or equivalently of μ) in the abelian case, or more generally in the case where the algebra generated by $E_\omega \pi_\omega(\mathcal{A}) E_\omega$ is abelian (i.e. if $(\mathcal{A}, \omega)$ is a "G-abelian pair", see [BR, Proposition 4.3.7 and Theorem 4.3.17]). Hence we also distinguish the following case:

(1.3) Definition. $(\mathcal{A}, G, \alpha)$ is a *quantized G-abelian* system if it is a quantized GS system and if $(\mathcal{A}, \omega)$ is a G-abelian pair.

We note that the usual terminology "G-abelian" applies to systems for which all invariant states define G-abelian pairs; while here quantized G-abelian refers only to the classical limit state ω.

2. Quantum ergodicity theorems

The purpose of this section is to prove Theorems 1-3. The following lemma provides a simple model for the somewhat more complicated situation of Theorem 1:

(2.1) Lemma. *Let (A, G, α) be a C^* dynamical system with G an amenable group. Let $\{\rho_j : j = 1, 2, 3, \dots\}$ be any sequence of G-invariant states on $\mathcal{A}$, and let $\rho_N = \frac{1}{N} \sum_{j=1}^{N} \rho_j$.*

Assume:

(a) $\text{weak}^*\text{-}\lim_{N\to\infty} \rho_N$ *exists.*

(b) The Gelfand-Segal system defined by the limit ω has a unique vacuum state.

Then, there exists a subsequence $\mathcal{S} \subset \mathbb{N}$ of counting density one such that

$$\text{weak}^* \lim_{\substack{j\to\infty \\ j\in\mathcal{S}}} \rho_j = \omega \ .$$

Proof. Let $A \in \mathcal{A}$, and consider the sums

$$S_2(N, A) = \frac{1}{N} \sum_{j=1}^{N} |\rho_j(A) - \omega(A)|^2 \ .$$

Since ρ_j is G-invariant,

$$S_2(N, A) = \frac{1}{N} \sum_{j=1}^{N} |\rho_j(\langle A \rangle_\alpha) - \omega(A)|^2 \ . \tag{2.2}$$

By the Schwartz inequality for positive linear functionals ([B.R.,Lemma 2.3.10]),

$$|\rho_j(\langle A \rangle_\alpha) - \omega(A)|^2 = |\rho_j(\langle A \rangle_\alpha - \omega(A))|^2 \le \rho_j(((\langle A \rangle_\alpha - \omega(A))^* (\langle A \rangle_\alpha - \omega(A))) \ .$$

Hence,

$$S_2(N, A) \le \rho_N[(\langle A \rangle_\alpha - \omega(A))^* (\langle A \rangle_\alpha - \omega(A))] \ . \tag{2.3}$$

Letting $N \to \infty$ we obtain

$$\overline{\lim_{N\to\infty}} S_2(N, A) \le \omega[(\langle A \rangle_\alpha - \omega(A))^* (\langle A \rangle_\alpha - \omega(A)))] \ . \tag{2.4}$$

We now claim:

$$\lim_{\alpha\to\infty} \omega[(\langle A \rangle_\alpha - \omega(A))^* (\langle A \rangle_\alpha - \omega(A))] = 0 \ . \tag{2.5}$$

Indeed, (2.5) is equivalent to the condition rank $E_\omega = 1$ if G is amenable, see [R, Proposition 6.3.5]. Hence, we have proved that for any $A \in \mathcal{A}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} |\rho_j(A) - \omega(A)|^2 = 0 \ . \tag{2.6}$$

By a standard lemma on averages of positive numbers [W,Theorem 1.20], (2.6) implies that for each $A \in \mathcal{A}$, there is a subsequence $\mathcal{S}_A \subset \mathbb{N}$ of counting density one such that

$$\lim_{\substack{k\to\infty \\ k\in\mathcal{S}_A}} \rho_k(A) = \omega(A) \ . \tag{2.7}$$

To obtain a density one subsequence $\mathcal{S}$ independent of A, we use a diagonalization argument ([CV], [Z.1]). Since $\mathcal{A}$ is separable, there exists a countable dense subset $\{A_j\}$ of the unit ball of $\mathcal{A}$. For each j, let $\mathcal{S}_j \subset \mathbb{N}$ be a density one subsequence such that (2.7) is correct for A_j. We may assume $\mathcal{S}_j \subset \mathcal{S}_{j+1}$. Then choose N_j so that

$$\frac{1}{N}\#\{k \in \mathcal{S}_j : k \leq N\} \geq 1 - 2^{-j} \text{ for } N \geq N_j \ .$$

Let $\mathcal{S}_\infty$ be the subsequence defined by

$$\mathcal{S}_\infty \cap [N_j, N_{j+1}] = \mathcal{S}_j \cap [N_j, N_{j+1}] \tag{2.8.}$$

Then $\mathcal{S}_\infty$ is of density one and

$$\lim_{\substack{k\to\infty \\ k\in\mathcal{S}_\infty}} \rho_k(A) = \omega(A) \tag{2.9}$$

for all $A \in \mathcal{A}$: as follows since (2.9) holds for the set $\{A_j\}$ and since $\{A_j\}$ is dense in the unit ball. □

(2.10) Remark 1. Uniqueness of the vacuum state implies that ω is an ergodic state [R, Theorem 6.3.3]. It is equivalent to ergodicity of ω if the pair $(\mathcal{A}, \omega)$ is G-abelian [loc.cit]. In particular, if the GS system is abelian, it is equivalent to ergodicity of the induced flow. Hence:

(2.11a) Corollary. *The conclusion of Lemma (1.2) is correct if we replace assumption (b) with the assumption that $(\mathcal{A}, \omega)$ is abelian and that ω is ergodic.*

(2.11b) Corollary. *The conclusion of Lemma (1.2) if in place of (b) we assume ω is ergodic and $(\mathcal{A}, \omega)$ is G-abelian.*

We now give the

Proof of Theorem 1. Let us consider first the special case

$$\nu_\sigma = \frac{1}{m(\sigma)} \sum_{j=1}^{m(\sigma)} \delta_{\omega\sigma j} \ . \tag{2.12}$$

Here $\delta_{\omega_{\sigma j}}$ is the point mass at the ergodic state

$$\omega_{\sigma j}(A) = \frac{1}{d(\sigma)} \mathrm{Tr}\, \Pi_{\sigma j} A$$

where we have chosen a decomposition

$$\Pi_\sigma = \bigoplus_{j=1}^{m(\sigma)} \Pi_{\sigma j}$$

corresonding to a decomposition $\mathcal{H}_\sigma = \bigoplus_{j=1}^{n(\sigma)} \mathcal{H}_{\sigma j}$ of $\mathcal{H}_\sigma$ into irreducibles. Also, we recall that $d(\sigma)$ is the dimension of the irreducible and $m(\sigma)$ is its multiplicity in $\hat{U}$. The associated density D^*_ν is then supported on the set $\{\omega_{\sigma j}\}$.

Conside first the choice of $\tilde{\omega}_E$ as microcanonical ensemble, since it is somewhat simpler to work with. We have, by (1.1c'),

$$\text{weak}^*\text{-}\lim_{E\to\infty} \frac{1}{N^*(E)} \sum_{\substack{\sigma: \\ \delta(\sigma,1)\leq E}} \sum_{j=1}^{m(\sigma)} \omega_{\sigma j} = \omega \tag{2.13a}$$

for some ergodic state ω. By the argument of Lemma 2.1 (leading to (2.6)), we then have for each $A \in \mathcal{A}$

$$\lim_{E\to\infty} \frac{1}{N^*(E)} \sum_{\substack{\sigma: \\ \delta(\sigma,1)\leq E}} \sum_{j=1}^{m(\sigma)} |\omega_{\sigma j}(A) - \omega(A)|^2 = 0. \tag{2.13b}$$

Our aim is then to construct a subset

$$\mathcal{S} \subset \{\omega_{\sigma j}\}$$

with $\tilde{D}^*_\nu(\mathcal{S}) = 1$ and such that

$$w - \lim_{\substack{E(\omega_{\sigma j})\to\infty \\ \omega_{\sigma j}\in\mathcal{S}}} \omega_{\sigma j} = \omega \ .$$

We will in fact construct such a subset of full counting density in a natural sense.

We begin by arranging the ergodic states $\omega_{\sigma j}$ in a sequence: First fix an ordering $\{\sigma_\ell : \ell = 1, 2, \dots\}$ of the irreducibles σ occurring in $\hat{U}$, with $\delta(\sigma_l, 1) \le \delta(\sigma_m, 1)$ if $l \le m$, and then arrange the states $\omega_{\sigma_\ell j}$ in lexicographic order, $\omega_{\sigma_\ell j} = \omega_{n(\ell,j,)}$. Henceforth we denote this sequence of states by $\{\omega_n\}$. We also define postive integers N^*_m by $N^*_m := N^*(E_m)$, where $\{E_m\} = spec(U)$. We then have:

$$\lim_{N^*_m\to\infty} \frac{1}{N^*_m} \sum_{n=1}^{N^*_m} |\omega_n(A) - \omega(A)|^2 = 0.$$

We note that

$$\frac{N^*_{m+1}}{N^*_m} = \frac{N^*_m + m^*(E_{m+1})}{N^*_m} \le (1 + C)$$

by regularity of the spectrum. It follows that for $N^*_m \le N \le N^*_{m+1}$

$$\frac{1}{N}\sum_{n=1}^{N} |\omega_n(A) - \omega(A)|^2 \le \frac{1}{N^*_m}\sum_{n=1}^{N^*_{m+1}} |\omega_n(A) - \omega(A)|^2$$

$$\le (1 + C)\frac{1}{N^*_{m+1}}\sum_{n=1}^{N^*_{m+1}} |\omega_n(A) - \omega(A)|^2$$

hence

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\omega_n(A) - \omega(A)|^2 = 0. \tag{2.13c}$$

As in the proof of Lemma 2.1, this implies the existence of a subsequence $\mathcal{S}_1 \subset \{\omega_n\}$ of counting density one such that

$$w - \lim_{\substack{n\to\infty \\ \omega_n\in\mathcal{S}_1}} \omega_n = \omega \ .$$

The choice of ω_E as microcanonical ensemble leads to the somewhat more complicated limit formulae

$$\lim_{E\to\infty} \frac{1}{N(E)} \sum_{\substack{\sigma: \\ \delta(\sigma,1)\le E}} d(\sigma) \sum_{j=1}^{m(\sigma)} |\omega_{\sigma j}(A) - \omega(A)|^2 = 0. \tag{2.14a}$$

Ordering the states as above, and letting $d(n)$ denote the dimension of the representation corresponding to ω_n we now have

$$\lim_{N^*_m\to\infty} \frac{1}{N_m}\sum_{n=1}^{N^*_m} d(n)|\omega_n(A) - \omega(A)|^2 = 0. \tag{2.14b}$$

where $N_m := N(E_m) = \sum_{n=1}^{N^*} d(n)$. The regularity of the spectrum then implies

$$\lim_{N\to\infty} \frac{1}{D(N)}\sum_{n=1}^{N} d(n)|\omega_n(A) - \omega(A)|^2 = 0, \tag{2.14c}$$

with $D(N) := \sum_{n=1}^{N} d(n)$. This leads to the conclusion that there exists a subsequence $\mathcal{S}_1$ of D^*-density one of the set $\{\omega_n\}$ which tends to ω in the sense that

$$\lim_{N\to\infty} \frac{1}{D(N)} \sum_{\substack{n\le N \\ \omega_n\in\mathcal{S}_1}} d(n) = 0.$$

With further hypotheses on the distribution of irreducibles of G in $\hat{U}$ and on the growth rate of the spectrum, this conclusion could be sharpened to give a subsequence of counting density one tending to ω as in the case of $\tilde{\omega}_E$. However, such hypotheses seem best left to arise naturally in applications.

We now turn to the case of a general admissible density D_ν^* for ω_E, for which we will prove the existence of a subset of normal ergodic states of D_ν^*-density one tending to ω. In this general case, we have

$$\omega_E = \frac{1}{N(E)} \sum_{\substack{\sigma: \\ \delta(\sigma,1)\le E}} \operatorname{rank} \Pi_\sigma \int_{S_\sigma} \omega_\phi d\nu_\sigma(\phi) \ .$$

Imitating the proof of Lemma 2.1 we let

$$S_2(E,A) = \frac{1}{N(E)} \sum_{\substack{\sigma: \\ \delta(\sigma,1)\le E}} \operatorname{rank} \Pi_\sigma \int_{S_\sigma} |\omega_\phi(A) - \omega(A)|^2 d\nu_\sigma(\phi) \ , \tag{2.15a}$$

and by a similar argument obtain,

$$\lim_{E\to\infty} S_2(E,A) = 0 \ . \tag{2.15b}$$

Our goal is to construct a subset $\mathcal{S} \subset \mathcal{N}_A \cap \mathcal{E}(E_{\mathcal{A}}^G)$ of D_ν^*-density one such that $\lim\limits_{\substack{E(\phi)\to\infty \\ \phi\in\mathcal{S}_\infty}} \omega_\phi(A) = \omega(A)$ (all $A \in \mathcal{A}$).

As in the proof of Lemma 2.1, we let $\{A_j\}$ denote a countable dense subset of the unit ball of $\mathcal{A}$ and begin by constructing for each A_j a subset $\mathcal{S}_j \subset \mathcal{N}_A \cap \mathcal{E}(E_{\mathcal{A}}^G)$ such that $D_\nu^*(\mathcal{S}_j) = 1$ and such that

$$\lim_{\substack{\phi\in\mathcal{S}_j \\ E(\phi)\to\infty}} \omega_\phi(A_j) = \omega(A_j) \ . \tag{2.16}$$

The construction is similar to that in the lemma on sequences. We let

$$J_{\sigma jk} = \{\omega_\phi \in S_\sigma : |\omega_\phi(A_j) - \omega(A_j)|^2 > \frac{1}{k}\} \ .$$

and let $J_{jk} = \bigsqcup_{\sigma\in\hat{U}} J_{\sigma jk}$. Since

$$\int_{S_\sigma} |\omega_\phi(A_j) - \omega(A_j)|^2 d\nu_\sigma(\phi) \ge \frac{1}{k} \nu_\sigma\{|\omega_\phi(A_j) - \omega(A_j)|^2 > \frac{1}{k}\} \tag{2.17}$$

it is clear that $D_\nu^*(J_{jk}) = 0$ for all jk. Hence there exist integers $0 = \ell_0 < \ell_1 \le \ell_2 < \cdots$ such that for $E \ge \ell_k$,

$$\frac{1}{N(E)} \sum_{\substack{\sigma\in\hat{U} \\ \delta(\sigma,1)\le E}} \operatorname{rank} \Pi_\sigma \nu_\sigma(J_{\sigma j(k+1)}) < \frac{1}{k+1} \ . \tag{2.18}$$

Let $J_j = \bigcup_{k=0}^\infty \bigcup\limits_{\substack{\sigma\in\hat{U} \\ \ell_k<\delta(\sigma,1)\le\ell_{k+1}}} J_{\sigma j(k+1)}$. We claim that $D_\nu^*(J_j) = 0$. Indeed, $J_{\sigma jk}$ increases with k, so if $\ell_k \le E \le \ell_{k+1}$, then

$$\begin{aligned} \frac{1}{N(E)} \sum_{\substack{\sigma\in\hat{U} \\ \delta(\sigma,1)\le E}} \operatorname{rank} \Pi_\sigma \nu_\sigma(J_j) &\le \frac{1}{N(E)} \sum_{\substack{\sigma\in\hat{U} \\ \delta(\sigma,1)\le \ell_k}} \operatorname{rank} \Pi_\sigma \nu_\sigma(J_{\sigma j(k)}) \\ &+ \frac{1}{N(E)} \sum_{\substack{\sigma\in\hat{U} \\ \ell_k\le\delta(\sigma,1)\le E}} \operatorname{rank} \Pi_\sigma \nu_\sigma(J_{\sigma j(k+1)}) \le \frac{1}{k} + \frac{1}{k+1} \ . \end{aligned} \tag{2.19}$$

Hence $\lim\limits_{E\to\infty} \dfrac{1}{N(E)} \sum\limits_{\substack{\sigma\in\hat{U} \\ \delta(\sigma,1)\le E}} \operatorname{rank} \Pi_\sigma \nu_\sigma(J_j) = 0$. Let $\mathcal{S}_j$ be the complement of J_j. We observe that

$$\lim_{\substack{E(\phi)\to\infty \\ \omega_\phi\in\mathcal{S}_j}} \omega_\phi(A_j) = \omega(A) \ . \tag{2.20}$$

Indeed, if $E(\phi) > \ell_k$ and $\omega_\phi \notin J_j$, then $\omega_\phi \notin J_{\sigma j(k+1)}$, and so $|\omega_\phi(A_j) - \omega(A_j)|^2 < \frac{1}{k}$.

Finally, we use a diagonal argument as in the proof of Lemma 2.1 to get rid of the dependence of $\mathcal{S}_j$ on A_j. We may again assume $\mathcal{S}_j \subset \mathcal{S}_{j+1}$, and choose N_j so that

$$\frac{1}{N(E)} \sum_{\substack{\sigma \in \hat{U} \\ \delta(\sigma,1) \leq E}} \operatorname{rank} \Pi_\sigma \nu_\sigma(\mathcal{S}_{\sigma j}) \geq 1 - 2^{-j} \quad (E \geq N_j)$$

where $\mathcal{S}_{\sigma j} = \mathcal{S}_j \cap S_\sigma$. We define $\mathcal{S} := \mathcal{S}_\infty$ by:

$$\mathcal{S}_\infty \cap \bigcup_{\substack{\sigma \in \hat{U} \\ N_j \leq \delta(\sigma,1) \leq N_{j+1}}} S_\sigma = \bigcup_{\substack{\sigma \in \hat{U} \\ N_j \leq \delta(\sigma,1) \leq N_{j+1}}} \mathcal{S}_{\sigma j} \ .$$

Then $D^*_\nu(\mathcal{S}_\infty) = 1$ and by a density argument $\lim\limits_{\substack{E(\phi) \to \infty \\ \omega_\phi \in \mathcal{S}_\infty}} \omega_\phi(A) = \omega(A)$ (all $A \in \mathcal{A}$). $\square$

(2.21) Remark. In the preceding, we let ω_E be the average over the whole "ball" of normal ergodic states of energy $\leq E$. But analogous results hold if we only average along a ray of representations. Such rays are frequently used to define semi-classical limits. So we include:

(2.22) Addendum to Theorem 1 (Localized version). *Let L be a ray of representations in $\hat{G}$, and let*

$$\mathcal{H}_L = \bigoplus_{\sigma \in \hat{U} \cap L} \mathcal{H}_\sigma \ .$$

Also let

$$\omega_E^L := \frac{1}{N(E,L)} \sum_{\substack{\rho(\sigma,1) \leq E \\ \sigma \in \hat{U} \cap L}} \operatorname{rank} \Pi_\sigma \omega_\sigma$$

with $N(E,L) = \sum_{\substack{\sigma \in \hat{U} \cap L \\ \rho(\sigma,1) \leq E}} \operatorname{rank} \Pi_\sigma$.

Suppose $\operatorname{weak}^*\text{-}\lim\limits_{E \to \infty} \omega_E^L$ *exists; let us denote it by ω^L. Let $\mathcal{N}_A^L \cap \mathcal{E}(E_A^G)$ denote the set of normal ergodic states which occur in the covariant representation $(\mathcal{H}_L, G, U|_{\mathcal{H}_L})$.*

Then we have: if $(\mathcal{A}, \omega^L)$ is G-abelian, and ω^L is ergodic, there is a subset $\mathcal{S}_L \subset \mathcal{N}_A^L \cap \mathcal{E}(E_A^G)$ of relative density one such that

$$\operatorname{weak}^*\text{-}\lim_{\substack{\rho \in \mathcal{S}_L \\ E(\rho) \to \infty}} \rho = \omega^L \ .$$

Here, relative density one is as above with a set $\nu = \{\nu_\sigma : \sigma \in \hat{U} \cap L\}$ of barycentric decompositions of ω_σ for $\sigma \in L$.

The proof is essentially the same as for the full set $\hat{U}$, so we omit it.

We now give

Proof of Theorem 2. We must show:

$$\lim_{E \to \infty} \omega_E[(\langle A \rangle - \omega(A))^*(\langle A \rangle - \omega(A))] = 0 \ . \tag{2.23}$$

We will see that this follows from a special case of Theorem 1. First, we observe that it is sufficient to prove it for A satisfying $A^* = A$ or $A^* = -A$. Indeed, we may express $A = B + C$ with $B^* = B$, $C^* = -C$, and eliminate the cross term with the Schwartz inequality $\omega(B^*C)^2 \leq \omega(B^*B)\omega(C^*C)$ for positive linear functionals.

Let us assume $A^* = A$ since the other case is similar. Let us also first assume that G is abelian. Then it is easily seen that

$$\langle A \rangle = \sum_{\sigma \in \hat{U}} \Pi_\sigma A \Pi_\sigma \tag{2.24}$$

so that

$$\begin{aligned}\omega_\sigma[(\langle A\rangle-\omega(A))^2] &= \tfrac{1}{\operatorname{rank}\Pi_\sigma}\operatorname{Tr}\Pi_\sigma(A-\omega(A))\Pi_\sigma(A-\omega(A)),\\ &= \tfrac{1}{\operatorname{rank}\Pi_\sigma}\|\Pi_\sigma(A-\omega(A))\Pi_\sigma\|^2_{\mathrm{HS}}\ ,\end{aligned}$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert Schmidt norm. Since $\Pi_\sigma(A-\omega(A))\Pi_\sigma$ is self-adjoint on $\mathcal{H}_\sigma$, there exists an orthonormal basis $\{\phi_{\sigma\ell}:\ell=1,\cdots,\dim\mathcal{H}_\sigma\}$ of its eigenvectors:

$$\Pi_\sigma(A-\omega(A))\Pi_\sigma\phi_{\sigma\ell}=\langle(A-\omega(A))\phi_{\sigma\ell},\phi_{\sigma\ell}\rangle\phi_{\sigma\ell}\ .$$

Hence,

$$\omega_\sigma[(\langle A\rangle-\omega(A))^2]=\frac{1}{\operatorname{rank}\Pi_\sigma}\sum_{\ell=1}^{\operatorname{rank}\Pi_\sigma}|\langle(A-\omega(A))\phi_{\sigma\ell},\phi_{\sigma\ell}\rangle|^2\ . \tag{2.25}$$

Let $\nu_\sigma=\frac{1}{\operatorname{rank}\Pi_\sigma}\sum_{\ell=1}^{\operatorname{rank}\Pi_\sigma}\delta_{\omega_{\sigma\ell}}$ where $\omega_{\sigma\ell}(B)=\langle B\phi_{\sigma\ell},\phi_{\sigma\ell}\rangle$. Note that $\omega_{\sigma\ell}$ is ergodic invariant state since G is assumed to be abelian. Hence

$$\omega_E[(\langle A\rangle-\omega(A))^*(\langle A\rangle-\omega(A))]=S_2(E,A)\qquad(\text{cf. }(2.14))\ , \tag{2.26}$$

The conclusion now follows from (2.14a).

Now let us consider $G=G_a\times K$ where G_a is abelian and K is a compact Lie group. We then have

$$\langle A\rangle=\sum_{\sigma\in\hat U}\Pi_\sigma\langle A\rangle\Pi_\sigma$$

where $\Pi_\sigma\langle A\rangle\Pi_\sigma$ is an intertwining operator from $\mathcal{H}_\sigma$ to itself. Hence each eigenspace of $\Pi_\sigma\langle A\rangle\Pi_\sigma$ is an invariant subspace, and we have a spectral decomposition

$$\Pi_\sigma\langle A\rangle\Pi_\sigma=\sum_{i=1}^{m(\sigma)}\lambda_{\sigma i}\Pi_{\sigma i}$$

where $\Pi_{\sigma i}$ projects to an irreducible subspace. The eigenvalue is obviously given by, $\lambda_{\sigma i}=\omega_{\sigma i}(A)$. Hence,

$$\omega_\sigma[(\langle A\rangle-\omega(A))^2]=\frac{1}{m(\sigma)}\sum_{i=1}^{m(\sigma)}|\omega_{\sigma i}(A)-\omega(A)|^2\ .$$

Now let

$$\nu_\sigma=\frac{1}{m(\sigma)}\sum_{i=1}^{m(\sigma)}\delta\omega_{\sigma i}\ .$$

and apply (2.14-2.14a) as above. □

Finally, we give

Proof of Theorem 3. Some general remarks before the proofs of (a) and (b) proper: Since the system is abelian, $U_\omega(g)$ is translation by an action of G by measure-preserving transformations on $L^2(X,\mu)$ (§1). By definition, the spectral measure for this action corresponding to the vector $\psi_A\in L^2(X,\mu)$ is the measure $d\mu_A$ on $\hat G$ defined by

(2.27). $$(U_\omega(g)\psi_A,\psi_A)=\int_{\hat G}\chi(g)d\mu_A(\chi)$$

Here we identify $\hat G$ with the dual group of characters χ of G. Ergodicity of the action is then equivalent to the condition

$$(d\mu_A-|\omega(A)|^2\delta_1)(\{1\})=0 \tag{2.28}$$

i.e. this measure has no point mass at the trivial character 1 for any A. We may rewrite this condition in terms of the invariant mean on G as follows:

(2.29). $$\lim_{T\to\infty}\int_G\int_{\hat G}\chi(g)(d\mu_A-|\omega(A)|^2\delta_1)(\chi)\chi_T(g)dg=0$$

Here as above χ_T denotes an M-net for the invariant mean on G, while χ alone denotes a character of G.

Temporarily assuming the Spectral measure Lemma, we now give the proofs of (a)-(b):
(a) Since ω is ergodic, (2.29) holds. From the Spectral measure Lemma, we get

$$(2.30). \qquad \lim_{T\to\infty}\int_G\int_{\hat{G}}\chi(g)(dm_A-|\omega(A)|^2\delta_1)(\chi)\chi_T(g)dg=0$$

By the definition of dm_A this gives

$$\lim_{T\to\infty}\lim_{E\to\infty}\omega_E(< A >_T^* A)=|\omega(A)|^2.$$

However, by Theorem 1, the right side is the same as the left side with the order of the limits reversed. Indeed, we have $< A >= \omega(A)I+K$ so

$$\lim_{E\to\infty}\lim_{T\to\infty}\omega_E(< A >_T^* A)=\lim_{E\to\infty}\omega_E(< A >^* A)$$
$$=\lim_{E\to\infty}\omega_E((\omega(A)I+K)^*(A))=|\omega(A)|^2$$

by the Schwartz inequality. □
(b) Conversely, if the system is quantum ergodic and if (2.30) holds, then by reversing the steps we conclude that (2.29) holds. Hence the classical system is ergodic. □

Last we give the

Proof of the Spectral measure Lemma. By definition of quantized abelian we have

$$\lim_{E\to\infty}\omega_E(\alpha_g(A)^*A)=(U_\omega(g)\psi_A,\psi_A)$$

Suppose now that $\mathcal{F}f\in L^1(G)$. Since $\omega_E(\alpha_g(A)^*A)\in C_b(G)$ we have

$$\int_G\mathcal{F}f(g)\ lim_{E\to\infty}\omega_E(\alpha_g(A)^*A)dg=\lim_{E\to\infty}\int_G\mathcal{F}f(g)\omega_E(\alpha_g(A)^*A)=\lim_{E\to\infty}\omega_E(< A >_f^* A)$$
$$=< U_\omega(\mathcal{F}f)\psi_A,\psi_A >=\int_{\hat{G}}f(d\mu_{\psi_A})$$

where $U_\omega(h)=\int_G h(g)U_\omega(g)dg$. □

3. Examples: Continuous time systems ($G=\mathbb{R}$)

In this section we will present four applications of Theorem 1 to quantum ergodic systems $(\mathcal{A},G,\alpha)$ with $G=\mathbb{R}$. The algebras $\mathcal{A}$ will be (C^* closures of) $*$ algebras of Fourier Integral operators, covariantly represented on $L^2(M,d\nu)$ for some compact manifold M. The automorphisms α_t will be of the form $\alpha_t^H(A)=U_t^*AU_t$ where $U_t=\exp\mathrm{i}tH$ for some positive elliptic pseudodifferential operator of order 1 on M.

In all these examples, the GS construction will come down to a symbol map

$$\sigma:\mathcal{A}\to C(SB)$$

where $B\subset T^*M\backslash 0$ is a symplectic cone, and where SB is a section of the cone of the form $\{\sigma_H=1\}$. The classical limit state ω will have the form

$$\omega(A)=\int_{SB}\sigma_A d\mu$$

where $d\mu$ is the normalized surface measure on SB induced by H and by the symplectic volume measure Ω^n, i.e. up to a scalar, $d\mu=dH\lrcorner\Omega^n$. Normalized will mean that $\int_{SB}d\mu=1$. We will refer to $d\mu$ as the Liouville measure on SB.

We will consider the following four algebras:
(A) $\mathcal{A}=\bar{\Psi}^\circ(M)$ (scalar pseudodifferential operators)
(B) $\mathcal{A}=\bar{\Psi}^\circ(M,E)$ (matrix pseudodifferential operators)
(C) $\mathcal{A}=\bar{\mathcal{A}}_\Sigma^\circ$ (co-isotropic operators)
(D) $\mathcal{A}=\bar{\mathcal{T}}_\Sigma^\circ$ (Toeplitz operators).

Here, the bar indicates the norm closure of the usual smooth subalgebras.

Example (A) has been discussed in detail in the articles [CV], [Sn], [Z.1-3], [Su] and others, and is only included here to illustrate the terminology and notation in a familiar context. The algebras $\mathcal{A}_\Sigma^\circ$ and $\mathcal{T}_\Sigma^\circ$ are probably less familiar, but we will have to assume the reader's familiarity with them: in particular, with their behaviour under composition with other types of Fourier Integral operators. For background on $\mathcal{A}^\circ_\Sigma$ we refer to Guillemin–Sternberg ([GS], [G.2]) and for $\mathcal{T}^\circ_\Sigma$ we refer to Boutet de Monvel–Guillemin ([B.G], [B]).

(A) $\mathcal{A} = \overline{\Psi^\circ(M)}$.

(a) $\partial M = \phi$.

Let $H \in \Psi^1(M)$ be positive elliptic, and let $H\phi_j = \lambda_j \phi_j$, $\langle \phi_i, \phi_j \rangle = \delta_{ij}$, denote its spectral data. We set

$$\omega_j(A) = \langle A\phi_j, \phi_j \rangle$$
$$\omega_\lambda = \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \omega_j$$
$$N(\lambda) = \#\{j : \lambda_j \le \lambda\}$$
$$\omega(A) = \int_{S^*M} \sigma_A d\mu$$

where $S^*M \subset T^*M$ is the level set $\{H = 1\}$.

We observe that $\{\omega_j\}$ are normal invariant ergodic states of $(\bar{\Psi}^\circ, \mathbb{R}, \alpha_t^H)$. Also, that ω is a (non-normal) invariant state by virtue of the Egorov theorem $\sigma(\alpha_t^H(A)) = \sigma_A \circ G^t$, where G^t is the Hamilton flow of σ_H on $\{H = 1\}$. Condition (c) of Definition (1.1),

$$\text{weak}^*\text{-}\lim_{E\to\infty} \omega_E = \omega$$

is well-known and can be proved by studying the principal term at $t = 0$ of the distribution trace, $\operatorname{Tr} AU_t$. Indeed, by the calculus of Fourier Integral operators $\operatorname{Tr} AU_t$ is a Lagrangean distribution on $\mathbb{R}$, whose symbol at $t = 0$ is, up to universal constants, essentially $\omega(A)$. Condition (c) then follows by a Tauberian argument. For further details we refer to [HoIV, §29] or [G.1]. The other conditions of Definition (1.1) are obvious, and it is straightforward that the classical limit system is the geodesic flow on S^*M. Hence the original system is quantized abelian.

By Theorems 1 and 2, ergodicity of G^t on S^*M will imply the quantum ergodicity of $(\bar{\Psi}^\circ, \mathbb{R}, \alpha_t^H)$. In particular, there is a subsequence $\{\varphi_{j_k}\}$ of eigenfunctions of density one such that

$$\lim_{k\to\infty} (A\varphi_{j_k}, \varphi_{j_k}) = \omega(A) \ ,$$

and $\langle A \rangle = \omega(A)I + K$, where $\|\Pi_\lambda K \Pi_\lambda\|_{\text{HS}} = o(N(\lambda))$; here $\Pi_\lambda = \sum_{\lambda_j \le \lambda} \varphi_j \otimes \varphi_j^*$.

(b) ∂M diffractive.

If $\partial M \ne \phi$, the proof that classical ergodicity (now of the billiard flow) implies quantum ergodicity becomes more complicated. The principal difficulty is that $\alpha_t(A) = U_t^* A U_t$ no longer necessarily defines an automorphism of the algebra of pseudodifferential operators on M. In the case of manifolds with diffractive boundary, Farris' extension of the Egorov Theorem (which is carried out from the C^* algebra point of view) is sufficient for the proof of quantum ergodicity using Theorem 1. For further discussion and a generalization to manifolds with piecewise smooth boundary and ergodic billiards, we refer to [Z.Zw].

(B) $\mathcal{A} = \bar{\Psi}^\circ(M, E)$, where $E \to M$ is a real rank n vector bundle.

The new feature is that symbols are now matrix valued.

As above, we let $H \in \Psi^1(M, E)$ be positive elliptic. Then

$$\sigma_H(x, \xi) : (\pi^* E)_{(x,\xi)} \to (\pi^* E)_{(x,\xi)}$$

where $\pi : T^*M \to M$ is the natural projection, $\pi^*E \to T^*M$ is the pulled back bundle, and $(\pi^*E)_{(x,\xi)}$ is its fiber over $(x, \xi) \in T^*M$. Unless $\sigma_H(x, \xi) = h(x, \xi)\,\text{Id}$ for some scalar symbol $h(x, \xi)$, conjugation by $U_t = \exp \text{i}tH$ will not define an automorphism of $\hat{\Psi}^\circ(M, E)$. This is of course the problem of generalizing Egorov's theorem to systems; see Cordes [C].

To obtain a C^* dynamical system satisfying the hypothesis of Theorem 1, we will need to place some conditions on σ_H and possibly restrict to a subalgebra of $\Psi^\circ(M,E)$. The condition on σ_H is that it have constant multiplicities as (x,ξ) varies over S^*M. Let us consider just the two extremes:

(i) $\sigma_H(x,\xi) = h(x,\xi)\,\mathrm{Id}$ (real scalar type)

(ii) $\sigma_H(x,\xi)$ has real distinct eigenvalues $\lambda_1(x,\xi) < \lambda_2(x,\xi) < \cdots < \lambda_m(x,\xi)$ with $\lambda_{j+1}(x,\xi) - \lambda_j(x,\xi) \geq C(1+|\xi|)$ for some $C > 0$ (strictly hyperbolic type).

In either case, let σ'_H denote the symbolic commutant of σ_H, i.e. the matrix valued symbols $\sigma(x,\xi)$ on T^*M such that $[\sigma_H(x,\xi), \sigma(x,\xi)] = 0$ for all (x,ξ). In case (i), σ'_H constants of all $\mathrm{End}(E)_{(x,\xi)}$-valued symbols. It follows as in the scalar case that α_t^H is an automorphism of the full $\bar{\Psi}^\circ(M,E)$ and that $\sigma(\alpha_t^H(A)) = \sigma_A \circ G^t$ where G^t is the Hamilton flow of σ_H. Analysis of $\mathrm{Tr}\, AU_t$ at $t=0$ leads as above to the formula

$$\operatorname*{weak^*\text{-}lim}_{E\to\infty} \omega_E = \omega$$

where $\omega(A) = \int_{S^*M} \mathrm{tr}\, \sigma_A d\mu$. Ergodicity of ω is then equivalent to ergodicity of G^t on S^*M. We have:

(3.1) Corollary. *If H is of real scalar type and G^t is ergodic, then $(\bar{\Psi}^\circ(M,E), \mathbb{R}, \alpha_t^H)$ is quantum ergodic.*

As special cases, one could let $E = \Lambda^k T^*M$, and $H = \sqrt{\Delta_k}$, where Δ_k is the Laplacian on k-forms. If $\{\eta_j\}$ is an orthonormal basis of eigenforms, one obtains $(A\eta_{j_k}, \eta_{j_k}) \to \omega(A)$ along a density one subsequence. In particular, $|\eta_{j_k}|^2(x) \stackrel{\omega}{\to} 1$ where $|\eta|(x)$ is the norm of η at x. Similarly for $H = (\not{\partial}^* \not{\partial})$ where $\not{\partial}$ is the Dirac operator on a spin bundle.

In the strictly hyperbolic case, σ'_H consists of matrix valued symbols of the form

$$\sigma(x,\xi) = \sum_{i=1}^n a_i(x,\xi)\pi_i(x,\xi)\ , \tag{3.2}$$

where $\pi_i(x,\xi)$ is the eigenprojection on $(\pi^*E)_{(x,\xi)}$ corresponding to $\lambda_i(x,\xi)$, and where $a_i(x,\xi)$ is a scalar symbol of order 0.

Let $G_i^t : T^*M \to T^*M$ denote the Hamilton flow of λ_i, and let

$$\sigma \circ G^t(x,\xi) = \sum_{i=1}^n a_i(G_i^t(x,\xi))\pi_i(x,\xi)\ .$$

The map $\sigma \mapsto \sigma \circ G^t$ defines an automorphism of σ'_H. By the Egorov theorem of Cordes (loc. cit.) for each $\sigma \in \sigma'_H$ one can construct an operator $A \in \Psi^\circ(M,E)$ such that

$$\begin{aligned} \sigma_A &= \sigma \\ \alpha_t^H(A) &\in \Psi^\circ(M,E) \\ \sigma(\alpha_t^H(A)) &= \sigma_A \circ G^t\ . \end{aligned} \tag{3.3}$$

Let us define $\Psi_H^\circ(M,E) \subset \Psi^\circ(M,E)$ by:

$$\Psi_H^\circ(M,E) = \{A : A \text{ satisfies } (3.3)\}\ .$$

Then Ψ_H° is a non-trivial $*$-subalgebra, and hence $(\bar{\Psi}_H^\circ, \mathbb{R}, \alpha_t^H)$ is a C^* dynamical system. Analysis of $\mathrm{Tr}\, AU_t$ leads to the limit formula:

$$\lim_{E\to\infty} \omega_E(A) = \omega(A) := \sum_{j=1}^n \int_{\{\lambda_j = 1\}} a_j d\mu_j \quad (A \in \Psi_H^\circ)$$

where $d\mu_j$ is the Liouville measure on $\{\lambda_j = 1\}$. Ergodicity of ω is equivalent to the ergodicity of each of the flows G_j^t. Therefore we have:

(3.4) Corollary. $(\bar{\Psi}^\circ_H, \mathbb{R}, \alpha^H_t)$ *is quantum ergodic if H is of strictly hyperbolic type and if all the flows G^t_j are ergodic.*

(C) $\mathcal{A} = \bar{\mathcal{A}}^\circ_\Sigma = \overline{\Pi R^\circ_\Sigma \Pi}$ (a corner of a co-isotropic or flowout algebra).

Here the algebra is (the C^* closure of) a "corner" of the $*$ algebra R_Σ associated to a co-isotropic cone $\Sigma \subset T^*M \backslash O$, in the sense of Guillemin–Sternberg ([G.S], [G.2]). These algebras arise in the reduction of quantum systems with symmetries, and have already been studied in connection with quantum ergodicity in ([Z.4], [S.T]). We briefly review the definition and properties of R_Σ. For more details, we refer to [G.S].

Let $\mathcal{N}$ denote the null foliation of Σ. The equivalence relation in $\Sigma \times \Sigma$ of belonging to the same leaf of $\mathcal{N}$ is a Lagrangean relation $\Gamma \subset T^*(M \times M \backslash O)$ such that $\Gamma \circ \Gamma = \Gamma$ and $\Gamma^t = \Gamma$. Hence the algebra $R_\Sigma := I^*(M \times M, \Gamma)$ of Fourier Integral operators associated to Γ is a $*$ algebra. We will assume that the foliation $\mathcal{N}$ defines a fibre bundle over a base B, necessarily a symplectic cone. The symbol algebra S_Σ corresponding to R_Σ can then be identified with functions $T(b, \cdot, \cdot)$ on B with values in the smoothing operators on the fibres F_b of $\Sigma \to B$.

We will assume further that the fibers F_b are compact. Then (the weak closures of) S_Σ and R_Σ are factors of type I_∞, i.e. contain minimal projections. If $\Pi \in R_\Sigma$ is such a projection, then its symbol $\sigma(\Pi)$ is for each b a rank one projection $\sigma(\Pi)(b, \cdot, \cdot)$ on $L^2(F_b)$.

Let us fix one minimal projection $\Pi \in R_\Sigma$ and consider the corner $\Pi R_\Sigma \Pi$ of R_Σ. By [G.S], [G.2] each element of $\Pi R_\Sigma \Pi$ can be presented in the form $\Pi A \Pi$, where $A \in \Psi^*_\Pi = \{A \in \Psi^* : [A, \Pi] = 0\}$. If $A \in \Psi^*_\Pi$, its symbol σ_A is constant on the fibers F_b and can be identified with a function $\hat{\sigma_A}$ on B. Also, $\sigma(\Pi A \Pi) = \hat{\sigma_A} \sigma_\Pi$; so that the symbol algebra of $\Pi R^\circ_\Sigma \Pi$ can be identified with homogeneous functions on B of order 0.

Now let $H \in \Psi^1_\Pi$ be positive elliptic. Then α^H_t defines an automorphism of $\Pi R^\circ_\Sigma \Pi$ for $t \in \mathbb{R}$. The faithful covariant representation in this example is on the Hilbert space $\mathcal{H}_\Pi = \text{range}(\Pi)$, which is heuristically the quantization of the symplectic quotient B. Since $H \in \Psi^1_H$, $U_t = \exp it H$ operates on $\mathcal{H}_\Pi$ with discrete spectrum. Let $\{\varphi^\Pi_j\}$ denote an orthonormal basis of $\mathcal{H}_\Pi$ of eigenvectors of H, and let $\{\omega^\Pi_j\}$ denote the corresponding invariant states. The trace $\mathrm{Tr}\, \Pi A U_t$ can be analyzed as a composition of Fourier Integral operators, which leads to the limit formula

$$\lim_{E \to \infty} \omega^\Pi_E(A) = \int_{\mathrm{SB}} \hat{\sigma}_A d\mu \qquad (A \in \Psi^\circ_\Pi)$$

where ω^Π_E is the microcanonical ensemble for $\mathcal{H}_\Pi$. In other words, if Π_E denotes the full spectral projection for H on the interval $[0, E]$, then

$$\omega^\Pi_E = \frac{1}{\mathrm{Tr}\, \Pi \cdot \Pi_E} \sum_{\lambda_j \le E} \omega^\Pi_j \ .$$

Ergodicity of the state $\omega_\Pi(A) = \int_{\mathrm{SB}} \hat{\sigma}_A d\mu$ is equivalent to ergodicity of the quotient flow G^t on (SB, μ). Hence we have:

(3.5) Corollary. $(\Pi \bar{R}^0_\Sigma \Pi, \mathbb{R}, \alpha^H_t)$ *is quantum ergodic if the quotient flow G^t on* SB *is ergodic.*

It would be interesting to study the non-fibrating case where the fibers are not compact.

(D) $\mathcal{A} = \bar{\mathcal{T}}_\Sigma$ (a Toeplitz algebra)

Here, $\Sigma \subset T^*M \backslash 0$ is a closed symplectic cone. By [B.G] it has a Toeplitz structure; that is, there is an associated projector Π_Σ on $L^2(M)$ with the microlocal properties of the Szegö projector of a strictly pseudo convex domain Ω. In this case, for instance, $\partial\Omega$ has a natural contact structure α, $\Sigma = \{(x, r dx) : x \in \partial\Omega, r > 0\} \subset T^*(\partial\Omega) \backslash 0$ and Π_Σ is the orthogonal projection $L^2(\partial\Omega) \to H^2(\partial\Omega)$ onto boundary values of holomorphic functions on $\partial\Omega$ which lie in $L^2(\partial\Omega)$. In general, the range of Π_Σ is a Hilbert space $\mathcal{H}_\Sigma$ which one thinks of as the quantization of Σ.

The Toeplitz algebra $\mathcal{T}_\Sigma$ is the algebra of elements $\Pi_\Sigma A \Pi_\Sigma$ with $A \in \Psi^*(M)$. One can again represent each element in the above form with $A \in \Psi^*_{\Pi_\Sigma} = \{B \in \Psi^* : [B, \Pi_\Sigma] = 0\}$ ([B.G,Proposition 2.13 and p. 82]). Hence $\mathcal{T}_\Sigma \simeq \Pi \Psi^*_\Pi \Pi / \vartheta(\Pi)$ where $\vartheta_\Pi = \{A \in \Psi^*_\Pi : A\Pi = 0\}$. Here (and henceforth) we write Π for Π_Σ. $\mathcal{T}_\Sigma$ has a faithful covariant representation on $\mathcal{H}_\Sigma$.

The principal symbol $\sigma(\Pi A\Pi)$ of an element of $\mathcal{T}_\Sigma$ may be identified with $\sigma|_\Sigma$, and the symbol algebra for $\bar{\mathcal{T}}_\Sigma^\circ$ with the algebra of continuous homogeneous functions of degree 0 on Σ. (See [B.G].)

Let $H \in \Psi_\Pi'$ be positive elliptic. Then $U_t = \exp \mathrm{i}t H|_{\mathcal{H}_\Sigma}$ defines a unitary representation of $\mathbb{R}$ with discrete spectrum. As before we let $\{\phi_j\}$ be an orthonormal basis of eigenfunctions of H in $\mathcal{H}_\Sigma$, let ω_j be the corresponding states, let Π_E project to span $\{\phi_j : \lambda_j \leq E\}$ and let

$$\omega_E = \frac{1}{\operatorname{rank}(\Pi_E \Pi)} \sum_{\lambda_j \leq \lambda}^{\operatorname{rank} \Pi_E \Pi} \omega_j \ .$$

Analysis of the trace $\operatorname{Tr} \Pi U_t A \quad (A \in \Psi_\Pi^\circ)$ shows that $\omega_E \overset{\text{weak}^*}{\to} \omega_\Pi$ with

$$\omega_\Pi(\Pi A \Pi) = \int_{S\Sigma} \sigma_A d\mu$$

where $S\Sigma = \{\sigma_H = 1\} \cap \Sigma$. The composition theorem for Fourier Integral and Hermite operators [B.G.,§7] shows that

$$\sigma(\alpha_t^H(\Pi A \Pi)) = \sigma_A \circ G^t|_\Sigma$$

where G^t is the Hamilton flow of σ_H on Σ. Ergodicity of ω_Π is equivalent to ergodicity of the sub-flow $G^t|_\Sigma$ with respect to μ. Hence,

(3.6) Corollary. *$(\bar{\mathcal{T}}_\Sigma^\circ, \mathbb{R}, \sigma_t^H)$ is quantum ergodic if G^t is ergodic on (Σ, μ).*

Let us note that if $[\Pi, H] = 0$, then $\Pi U_t \Pi = \Pi \exp \mathrm{i}t(\Pi H \Pi)\Pi$. Hence we may view the generator of the covariant representation of $\mathbb{R}$ as the Toeplitz operator $\Pi H \Pi$.

(3.7) Example. Suppose that H_1 is positive elliptic, for instance $H_1 = \sqrt{\Delta}$ for some Riemannian metric, and let γ be a closed orbit for the Hamilton flow of H_1 on S^*M. Then the cone $\Sigma = \mathbb{R}^+\gamma$ through γ is a symplectic submanifold of $T^*M\backslash 0$. Let Π_Σ be a Toeplitz structure for Σ. Ideally we would like $[H_1, \Pi_\Sigma] = 0$ but it is not generally possible to construct Π_Σ with this property unless the whole geodesic flow G_1^t of H_1 is periodic [B.G.,Appendix]. However, by [B.G.,Proposition 2.13] for any choice of Π_Σ we can find $H \in \Psi^1$, such that $[H, \Pi_\Sigma] = 0$, $\sigma_H|_\Sigma = \sigma_{H_1}|_\Sigma$, and $\Pi_\Sigma H_1 \Pi_\Sigma = \Pi_\Sigma H \Pi_\Sigma$. Since σ_{H_1} and σ_H generate the same Hamilton flows on Σ, γ is a periodic orbit of the flow of σ_H. Obviously, the uniform measure μ_γ is ergodic for this flow. It follows from (3.6) that $(\bar{\mathcal{T}}_\Sigma^\circ, \mathbb{R}, \alpha_t^H)$ is quantum ergodic, hence the eigenfunctions ϕ_j^γ of $\Pi_\Sigma H_1 \Pi_\Sigma$ concentrate on γ in the limit $j \to \infty$.

Note that the ϕ_j^γ are actual eigenfunctions of H_1 in $\mathcal{H}_\Sigma$. Since H_1 and H are close in a microlocal neighborhood of γ, the $\{\phi_j^\gamma\}$ may be viewed as a kind of quasi-mode for H associated to γ. The approximation here is very weak, of course; the ϕ_j^γ concentrate (or "scar") along γ, while it is doubtful that any sequence of H-eigenfunctions has this property (see however [H]).

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Johns Hopkins University, Baltimore, Maryland 21218