THE LYAPUNOV AND DIMENSION SPECTRA OF EQUILIBRIUM MEASURES FOR CONFORMAL EXPANDING MAPS.

HOWARD WEISS

The Pennsylvania State University

Abstract. In this note, we find an explicit relationship between the dimension spectrum for equilibrium measures and the Lyapunov spectrum for conformal repellers. We explicitly compute the Lyapunov spectrum and show that it is a delta function. We observe that while the Lyapunov exponent exists for almost every point with respect to an ergodic measure, the set of points for which the Lyapunov exponent does not exist has positive Hausdorff dimension if the SRB measure does not coincide with the measure of maximal entropy. It follows that for such conformal repellers, the set of points for which the pointwise dimension of the measure of maximal entropy does not exist has positive Hausdorff dimension.

In [EP], Eckmann and Procaccia suggested an analysis of the spectrum of Lyapunov exponents for chaotic dynamical systems which is similar to the multifractal analysis of measures invariant under chaotic dynamical systems. This suggestion was further investigated on a physical level by Szépfalusy and Tél [ST] and by Tél [T].

In this note, we find an explicit relationship between the dimension spectrum for equilibrium measures (see Appendix) and the Lyapunov spectrum for conformal expanding maps. We also explicitly compute the Lyapunov spectrum and show that it is a delta function. Using the multifractal analysis of equilibrium measures for conformal repellers in [PW1], we show that while the Lyapunov exponent exists for almost every point with respect to an ergodic measure, the set of points for which the Lyapunov exponent does not exist has positive Hausdorff dimension if the SRB measure (Sinai-Ruelle-Bowen) [B] does not coincide with the measure of maximal entropy. It follows that for such conformal repellers, the set of points for which the pointwise dimension of the measure of maximal entropy does not exist has positive Hausdorff dimension.

The dimension spectrum is one of the principle components in the multifractal analysis of measures. In [PW1] the authors effected a complete multifractal analysis of equilibrium measures for conformal expanding maps. Examples of conformal expanding maps include Markov one-dimensional maps, rational maps whose Julia sets are hyperbolic, and conformal endomorphisms of the torus. See [PW1] for definitions and examples.

Let M be a smooth manifold and $g\colon M \to M$ a C^2 map. Suppose that Λ is a compact invariant subset of M and consider the map g restricted to Λ. We say that g is a *conformal*

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expanding map on Λ if there exists a Riemannian metric on Λ and a function $a(x)$ such that for all $x \in \Lambda$, $dg_x = a(x)\mathrm{Isom}(x)$, where $\mathrm{Isom}(x)$ denotes an isometry, dg_x is the differential of g at x, and $|a(x)| > 1$ for all $x \in \Lambda$. Clearly $\|dg_x\| = |a(x)|$. We define the *Lyapunov exponent of g at x* by

$$\chi(x) = \lim_{n\to\infty} \chi_n(x), \quad \text{where} \quad \chi_n(x) = \frac{1}{n}\log\|dg_x^n\| = \frac{1}{n}\log\prod_{k=0}^{n-1}|a(g^k(x))|.$$

When the map g is expanding, *if* the above limit exists, then it must be strictly positive. Let ν be an invariant Borel probability measure for g which is supported on Λ. It follows from the Subadditive Ergodic Theorem that $\chi(x)$ exists for $\nu-$almost every x and defines a $\nu-$measurable function. Furthermore, it follows that there exists a subset $\Lambda' \subset \Lambda$ with $\nu(\Lambda') = 1$ such that for every $x \in \Lambda'$ and every tangent vector $v \in T_x\Lambda$ we have

$$\chi(x) = \lim_{n\to\infty}\frac{1}{n}\log\|dg_x^n(v)\|.$$

The function $\chi(x)$ is clearly $g-$invariant. Hence, if ν is ergodic (i.e., if ν is an equilibrium measure), then $\chi(x) = \chi_\nu$ for $\nu-$almost every $x \in \Lambda$, where χ_ν is a constant. It is not at all clear whether $\chi(x)$ attains any value besides $\chi(x) = \chi_\nu$ in this case. To study this question Eckmann and Procaccia defined the *Lyapunov (exponent) spectrum for the map* g by

$$l(\beta) = \dim_H\{x \in \Lambda \,:\, \chi(x) = \beta\},$$

where $\dim_H F$ denotes the Hausdorff dimension of the set F.

We turn to the second ingredient in our analysis and define the spectrum for dimensions. Let ν be an invariant Borel probability measure on Λ for g. Given $x \in \Lambda$ we consider the *upper* and *lower pointwise dimensions of ν at x*,

$$\overline{d}_\nu(x) = \limsup_{r\to 0}\frac{\log\nu(B(x,r))}{\log r} \quad \text{and} \quad \underline{d}_\nu(x) = \liminf_{r\to 0}\frac{\log\nu(B(x,r))}{\log r},$$

where $B(x,r)$ denotes the ball of radius r centered at x. If $\underline{d}_\nu(x) = \overline{d}_\nu(x)$ we call the common value the *pointwise dimension* at x and denote it by $d_\nu(x)$. We call ν exact dimensional if $\overline{d}_\nu(x) = \underline{d}_\nu(x) = d_\nu(x) = c$ for $\nu-$almost every x where c is a non-negative constant. For a general dynamical system one does not expect the pointwise dimension of an invariant measure ν to exist at a typical point, even for *nice* measures [LM, PW2, PW3]. Even when the pointwise dimension of ν does exist ν need not be exact dimensional [C, PW2].

Nevertheless, measures which are invariant under smooth dynamical systems with hyperbolic behavior often turn out to be exact dimensional. Eckmann and Ruelle have conjectured that hyperbolic measures (i.e., ergodic measures with non-zero Lyapunov exponents almost everywhere) are exact dimensional. This has been established for hyperbolic measures in the two dimensional case in [Y] and for hyperbolic BRS-measures and equilibrium measures for Axiom A diffeomorphisms in [Le, PY]. The analog for conformal expanding maps was established in [PW1].

The multifractal analysis is a description of the fine-scale geometry of the Λ whose constituent components are the sets $\{x \in \Lambda \mid \underline{d}_\nu(x) = \alpha\}$ for $\alpha \in \mathbb{R}$. The $f_\nu(\alpha)$ *spectrum for dimensions* is defined by

$$f_\nu(\alpha) = \dim_H\{x \in \Lambda \mid d_\nu(x) = \alpha\}.$$

If ν is exact dimensional of dimension c, then $\nu(\{x \in \Lambda \mid d_\nu(x) = c\}) = 1$. Shereshevsky [Sh] showed that for a special class of C^2 Axiom A surface diffeomorphisms, the Hausdorff dimension of the set of points where the pointwise dimension does not exist is positive for any equilibrium measure ν. This result can be easily extended to conformal expanding maps.

We now describe another characterization of the Lyapunov exponent for conformal expanding maps which allows us to apply some results in [PW1] and relate the Lyapunov exponent at a point to the pointwise dimension at that point. It is well known that (conformal) expanding maps have Markov Partitions consisting of partition elements (rectangles) $\mathfrak{R} = \{R_1, \dots, R_m\}$ with arbitrarily small diameter such that each rectangle R is the closure of its interior $int(R)$, $\Lambda = \cup_i R_i$, $int(R_i) \cap int(R_j) = \emptyset$ unless $i = j$, and each $g(R_i)$ is a union of sets R_j [Ru]. See [PW1] for an explicit geometric construction of Markov partitions for expanding maps.

The Markov partition generates a symbolic model of g on Λ by a subshift of finite type (Σ_A, σ) where A is the incidence matrix of the Markov partition and $\sigma\colon \Sigma_A \to \Sigma_A$ is the shift map. This gives a coding map $\pi : \Sigma_A \to \Lambda$ which is Hölder continuous, surjective, and injective on the set of points whose trajectories never hit the boundary of any element of the Markov partition such that $\pi \circ \sigma = g \circ \pi$. Furthermore, the cardinality of $\pi^{-1}(x)$ is uniformly bounded for all $x \in \Lambda$. The pullback by π of any Hölder continuous function on Λ is Hölder continuous on Σ_A. Furthermore, the pushforward of any Gibbs measure (see Appendix) on Σ_A is an equilibrium measure on Λ and the pullback of any equilibrium measure on Λ is a Gibbs measure on Σ_A.

Define the *basic sets*

$$\Delta_{i_1 \dots i_n} = R_{i_1} \cap g^{-1} R_{i_2} \cap \dots \cap g^{-n} R_{i_n}.$$

By the Markov property, every basic set $\Delta_{i_1 \dots i_n} = g^{-n}(R_{i_n}) \cap R_{i_1}$. Let $\Delta_{i_1 \dots i_n}(x)$ denote a basic set at level n that contains the point x.

We now show that the Lyapunov exponent at a point x is the exponential decay rate of the diameter of the basic set that contains x. We pass from the infinitesimal characterization of Lyapunov exponents to a local characterization using the famous *Jacobian estimate* (Proposition 2) in hyperbolic dynamics.

Proposition 1. *The Lyapunov exponent of g at x satisfies*

$$\chi(x) = \lim_{n\to\infty} \frac{1}{n} \log \|dg_x^n\| = -\lim_{n\to\infty} \frac{1}{n} \log diam(\Delta_{i_1 \dots i_n}(x)),$$

where $diam(E)$ denotes the diamter of the set E.

Proof. An upper estimate for the diameter of basic sets is obtained by

$$\mathrm{diam}(\Delta_{i_1 \dots i_n}(x)) \le \mathrm{diam}(R_{i_n}) \max_{y \in R_{i_n}} \|dg^{-n}(y)\| = \mathrm{diam}(R_{i_n}) \max_{y \in R_{i_n}} |Jac\, g^{-n}(y)|$$

$$= \operatorname{diam}(R_{i_n}) \left(\frac{\max_{y \in R_{i_n}} |Jac\, g^{-n}(y)|}{|Jac\, g^{-n}(g^n(x))|} \right) |Jac\, g^{-n}(g^n(x))|,$$

where $Jac\, g$ denotes the Jacobian of g.

We now require the following Jacobian estimate (see [M]) to uniformly bound the ratio of Jacobians from above and below, independent of n.

Proposition 2: (Jacobian Estimate). *There exist positive constants C_1, C_2 such that for all $x, y \in R_k, k = 1, \dots, m$ and any branch of g^{-n}*

$$C_1 \le \frac{|Jac\, g^{-n}(y)|}{|Jac\, g^{-n}(x)|} \le C_2.$$

It immediately follows from the Jacobian estimate that

$$\operatorname{diam}(\Delta_{i_1 \dots i_n}(x)) \le C_2 \lambda(\Delta_{i_1}) \, \|dg_x^{-n}\|,$$

where $\lambda(F)$ denotes the Lebesgue measure of the set F. With the analogous lower estimate

$$C_1 \lambda(\Delta_{i_1}) \, \|dg_x^{-n}\| \le \operatorname{diam}(\Delta_{i_1 \dots i_n}(x)),$$

we immediately obtain that

$$\chi(x) = \lim_{n\to\infty} \frac{1}{n} \log \|dg_x^n\| = - \lim_{n\to\infty} \frac{1}{n} \log \operatorname{diam}(\Delta_{i_1 \dots i_n}(x)).$$

■

Remark. One can use Proposition 1 to define the Lyapunov exponent for some non-smooth "conformal" expanding maps including the induced map on the limit set for some Moran geometric constructions [PW1]. The results in this note also hold for such non-smooth conformal expanding maps.

We also require a characterization of pointwise dimension, a priori defined using balls, which uses the basic sets. This is desirable since the set Λ can be naturally viewed as a limit set for a geometric construction using the basic sets. Also, the measure of the basic sets can be well estimated using symbolic dynamics and thermodynamic formalism. This was a key ingredient in the multifractal analysis in [PW1]. The idea is to replace balls containing a point x with the basic set containing x. Let

$$\overline{\delta}_\nu(x) \equiv \limsup_{n\to\infty} \frac{\log \nu(\Delta_{i_1 \dots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1 \dots i_n}(x))} \quad \text{and} \quad \underline{\delta}_\nu(x) \equiv \liminf_{n\to\infty} \frac{\log \nu(\Delta_{i_1 \dots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1 \dots i_n}(x))}.$$

If $\overline{\delta}_\nu(x) = \underline{\delta}_\nu(x)$ we denote the common value by $\delta_\nu(x)$. We need the following result from [PW1] that describes the relations between $\overline{\delta}_\nu(x)$, $\underline{\delta}_\nu(x)$ and the lower and upper pointwise dimensions at x.

Proposition 3.

(1) $\underline{\delta}_\nu(x) \le \underline{d}_\nu(x) \le \overline{d}_\nu(x) \le \overline{\delta}_\nu(x)$ *for all* $x \in \Lambda$.

(2) *If* $\underline{\delta}_\nu(x) = \overline{\delta}_\nu(x)$ *for* $\nu-$*almost every* $x \in \Lambda$, *then* $\underline{d}_\nu(x) = \overline{d}_\nu(x) = d(x)$ *for* $\nu-$*almost every* $x \in \Lambda$.

Hence, if $\delta_\nu(x)$ *exists for some point* $x \in \Lambda$, *then* $\delta_\nu(x)$ *coincides with the pointwise dimension* $d_\nu(x)$ *at* x.

The following theorem establishes a relation between the pointwise dimension at a point and the Lyapunov exponent at the point. We stress that the following formula holds for *every* $x \in \Lambda$ and not just almost every x.

Theorem 1. *Let* $g\colon \Lambda \to \Lambda$ *be a smooth conformal expanding map and let* ν_ξ *be the equilibrium measure corresponding to the Hölder continuous potential* ξ. *If the Birkhoff average* $\lim_{n\to\infty} S_n\xi(x)/n = \frac{1}{n}\sum_{i=0}^{n-1}\xi(g^k(x)) = \bar{\xi}(x)$, *then*

$$d_{\nu_\xi}(x) = \frac{P(\xi) - \bar{\xi}(x)}{\chi(x)} = \frac{h_{\nu_\xi}(g) + \int \xi \, d\nu_\xi - \bar{\xi}(x)}{\chi(x)},$$

where $P(\xi)$ *denotes the thermodynamic pressure of the function* ξ *(see Appendix) and* $h_{\nu_\xi}(g)$ *denotes the Kolmogorov-Sinai entropy of the map* g *with respect the the measure* ν_ξ.

Proof. Let $\mathfrak{R}$ be a Markov partition for g and let μ_{ξ^*} be the pullback of ν_ξ under the coding map π. Then μ_{ξ^*} is the Gibbs measure for the pullback potential $\xi^* = \xi \circ \pi$ and hence

$$C_1 \le \frac{\mu_{\xi^*}(C_n(\omega))}{\exp(-nP(\xi^*) + S_n\xi^*(\omega))} = \frac{\nu_\xi(\Delta_{i_1\ldots i_n}(x))}{\exp(-nP(\xi^*) + S_n\xi(x))} \le C_2,$$

where $\pi(\omega) = x$ and $C_n(\omega)$ denotes the $n-$cylinder that contains ω. Since the coding map π is uniformly bounded-to-one, we have $P(\xi^*) = P(\xi)$ (see Appendix), and hence

$$\frac{\log \nu_\xi(\Delta_{i_1\ldots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1\ldots i_n}(x))} \asymp \frac{-nP(\xi) + S_n\xi(x)}{\log \operatorname{diam}(\Delta_{i_1\ldots i_n}(x))} = \frac{h_\nu(g) + (\int \xi \, d\nu - \frac{S_n\xi(x)}{n})}{-\frac{1}{n}\log \operatorname{diam}(\Delta_{i_1\ldots i_n}(x))}.$$

The theorem immediately follows from Proposition 2. ∎

We present four applications of this simple theorem. The first application establishes the link between the Lyapunov spectrum and the dimension spectrum for the measure of maximal entropy and explicitly computes the Lyapunov spectrum for conformal expanding maps. In short, the Lyapunov spectrum is a delta function with a spike at $\alpha = h_{TOP}(g)/d$, where d is the Hausdorff dimension of Λ and $h_{TOP}(g)$ is the topological entropy of the map g.

Proposition 4. *Let* $g\colon \Lambda \to \Lambda$ *be a smooth conformal expanding map. Then*

$$l(\alpha) = f_{\nu_{\max}}\left(\frac{h_{TOP}(g)}{\alpha}\right) = \begin{cases} \frac{h_{TOP}(g)}{d}, & \text{for } \alpha = \frac{h_{TOP}(g)}{d} \\ 0, & \text{for } \alpha \neq \frac{h_{TOP}(g)}{d}, \end{cases}$$

where $\nu_{\max}$ denotes the measure of maximal entropy for g.

Proof. If $\nu = \nu_{\max}$ is the measure of maximal entropy, then it immediately follows from Theorem 1 that

$$d_{\nu_{\max}}(x) = \frac{h_{TOP}(g)}{\chi(x)} \quad \text{and hence} \quad l(\alpha) = f_{\nu_{\max}}\left(\frac{h_{TOP}(g)}{\alpha}\right).$$

In [PW1], the authors observe that the multifractal analysis of the measure of maximal entropy is trivial in the sense that the spectrum for dimensions is a delta function, i.e.,

$$f_{\nu_{\max}}(\alpha) = \begin{cases} d, & \text{for } \alpha = d \\ 0, & \text{for } \alpha \neq d. \end{cases}$$

Proposition 4 follows immediately. ■

We say that a smooth conformal expanding map $g : \Lambda \to \Lambda$ is a *conformal repeller* for g if there exists an open set V such that $\Lambda \subset V \subset M$ and $\Lambda = \{x \in V : g^n(x) \in V \text{ for all } n \geq 0\}$ and that g has a dense orbit. Markov maps of an interval, rational maps whose Julia sets are hyperbolic, and conformal endomorphims of the torus are all examples of conformal repellers.

The next proposition is the analog of Shereshevsky's result [Sh] for the Lyapunov exponent.

Proposition 5. *Let $g\colon \Lambda \to \Lambda$ be a conformal repeller and let ν_{SRB} be the SRB measure. Then $d_{\nu_{SRB}}(x) = 1$ for $\nu_{SRB}-$almost every $x \in \Lambda$. Furthermore, although the Lyapunov exponent $\chi(x) = h_{SRB}(g)$ (K-S entropy with respect to the SRB measure) for $\nu_{SRB}-$almost every x, the Lyapunov exponent $\chi(x)$ does not exist for a set of positive Hausdorff dimension if the SRB measure does not coincide with the measure of maximal entropy.*

Proof. It immediately follows from Proposition 1 that

$$d_{\nu_{SRB}}(x) = \lim_{n\to\infty} \frac{\frac{1}{n}\log\prod_{k=0}^{n-1}|a(g^k x)|}{-\frac{1}{n}\log \operatorname{diam}(\Delta_{i_1\ldots i_n}(x))} = \frac{\chi(x)}{\chi(x)} = 1,$$

provided that the limits exist. It follows from the Subadditive Ergodic Theorem that the limit exists for $\nu_{SRB}-$almost every point x.

In [PW1], the authors show that there exists an open interval, $\alpha \in (1-\eta, 1+\eta)$ for some $\eta > 0$ such that the function $f_{\nu_{SRB}}(\alpha)$ is real analytic and strictly convex if the SRB measure does not coincide with the measure of maximal entropy. It immediately follows that the Lyapunov exponent $\chi(x)$ does not exist for a set of positive Hausdorff dimension. Moreover, the value of η can be crudely estimated from a version of the inverse function theorem with estimates [K]. This would supply a lower estimate for the Hausdorff dimension of the set of points for which the Lyapunov exponent does not exist. ■

The following proposition about the non-existence of pointwise dimension for the measure of maximal entropy complements Shereshevsky's result and immediately follows from

Proposition 5 and the fact that $d_{\nu_{\max}}(x) = h_{TOP}(g)/\chi(x)$. We stress that this proposition holds for *most* conformal repellers, i.e., those conformal repellers for which the SRB measure does not coincide with the measure of maximal entropy, while (the extension of) Shereshevsky's result holds only for a very special class of conformal repellers. It is also interesting to note that we applied the analysis of the dimension spectrum in [PW1] to yield information about the Lyapunov spectrum, which in turn yields new information about the dimension spectrum.

Proposition 6. *Let $g\colon \Lambda \to \Lambda$ be a conformal repeller such that the SRB measure does not coincide with the measure of maximal entropy. Then the pointwise dimension for the measure of maximal entropy $d_{\nu_{\max}}(x)$ does not exist for a set of positive Hausdorff dimension.*

The final application is a Young-type formula [Y] for the Hausdorff dimension of an equilibrium measure.

Proposition 7. *Let $g\colon \Lambda \to \Lambda$ be a smooth conformal expanding map and ν_ξ the equilibrium measure corresponding to the Hölder potential ξ. Then*

$$\dim_H \nu_\xi \stackrel{\text{def}}{\equiv} \inf_{\substack{F \subset \Lambda \\ \nu_\xi(F)=1}} \dim_H(F) = \frac{h_{\nu_\xi}(g)}{\chi_\xi}.$$

Proof. Since ν_ξ is ergodic, the Birkhoff Ergodic Theorem applied to Theorem 1 yields that $d_{\nu_\xi}(x) = h_{\nu_\xi}(g)/\chi_\xi$ for ν_ξ−almost every $x \in \Lambda$. This easily implies that $\dim_H(\nu_\xi) = h_{\nu_\xi}(g)/\chi_\xi$. ∎

Appendix

This Appendix contains some essential definitions and facts from symbolic dynamics and thermodynamic formalism. For details consult [B, Ru] and [W]. Let X denote a compact metric space and let $C(X)$ denote the space of real valued continuous functions on X.

1. Let $g\colon X \to X$ be a continuous map. We define the *pressure function* $P\colon C(X) \to \mathbb{R}$ defined by

$$P(\phi) = \sup_{\mu \in \mathfrak{M}(X)} \left(h_\mu(g) + \int_X \phi \, d\mu \right),$$

where $\mathfrak{M}(X)$ denotes the set of shift-invariant probability measures on X and $h_\mu(f)$ denotes the Kolmogorov-Sinai entropy of the map g with respect to the measure μ. A Borel probability measure $\mu = \mu_\phi$ on X is called an *equilibrium measure* for the potential $\phi \in C(X)$ if

$$P(\phi) = h_\mu(g) + \int_X \phi \, d\mu.$$

3. Let X and Y be compact metric spaces and suppose $\psi\colon X \to Y$ is a continuous surjection such that the cardinality of $\psi^{-1}(y)$ is uniformly bounded for all $y \in Y$. Then for any $\phi \in C(Y)$ we have that $P_Y(\phi) = P_X(\phi \circ \psi)$.

4. Let $\phi \in C(\Sigma_A^+)$. A Borel probability measure $\mu = \mu_\phi$ on Σ_A^+ is called a *Gibbs measure* for the potential ϕ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \leq \frac{\mu\{\kappa : \kappa_i = \omega_i,\ i = 0, \cdots, n-1\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k \omega))} \leq D_2$$

for all $\omega = (\omega_1 \omega_2 \cdots) \in \Sigma_A^+$ and $n \geq 0$.

5. For subshifts of finite type, Gibbs measures exist for any Holdër continuous potential ϕ, are unique, and coincide with the equilibrium measure for ϕ.

References

[B] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, SLN # 470, Springer Verlag, 1975.

[C] C. D. Cutler, *Connecting Ergodicity and Dimension in Dynamical Systems*, Ergod. Th. and Dynam. Systems **10** (1990), 451–462.

[EP] J. P. Eckmann and I. Procaccia, *Fluctuations of Dynamical Scaling Indices in Nonlinear Systems*, Physical Review A **34** (1986), 659–661.

[K] H. Keller, *Numerical Methods in Bification Problems*, Tata Institute of Fundamental Research, Bombay, 1987.

[Le] F. Ledrappier, *Proceedings of the Conference on Ergodic Theory and Related Topics*, vol. 11, Teubner, 1986, pp. 116–124.

[LM] F. Ledrappier and M. Misiurewicz, *Dimension of Invariant Measures for Maps with Exponent Zero*, Ergod. Th. and Dynam. Systems **5** (1985), 595–610.

[M] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer Verlag, 1987.

[PW1] Y. Pesin and H. Weiss, *A Multifractal Analysis of Gibbs Measures for Conformal Expanding Maps and Markov Moran Geometric Constructions*, Submitted for publication (1995).

[PW2] Y. Pesin and H. Weiss, *On the Dimension of Deterministic and Random Cantor-like Sets*, Math. Research Letters **1** (1994), 519–529.

[PW3] Y. Pesin and H. Weiss, *On the Dimension of Deterministic and Random Cantor-like Sets, Symbolic Dynamics, and the Eckmann-Ruelle Conjecture*, Submitted for publication (1994).

[PY] Y. Pesin and C. B. Yue, *Hausdorff Dimension of Measures with Non-zero Lyapunov Exponents and Local Product Structure*, PSU preprint (1993).

[Ru] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, 1978.

[Sh] M. Shereshevsky, *A Complement to Young's Theorem on Measure Dimension: The Difference Between Lower and Upper Pointwise Dimension*, Nonlinearity **4** (1991), 15–25.

[ST] P. Szépfalusy and T. Tél, *Dynamical Fractal Properties of One-dimensional Maps*, Physical Review A **35** (1987), 477–480.

[T] T. Tél, *Dynamical Spectrum and Thermodynamics Functions of Strange Sets from an Eigenvalue Problem*, Physical Review A **36** (1987), 2507–2510.

[W] P. Walters, *Introduction to Ergodic Theory*, Springer Verlag, 1982.

[Y] L. S. Young, *Dimension, Entropy, and Lyapunov Exponents*, Ergod. Th. and Dynam. Systems **2** (1982), 109–124.

Department of Mathematics, University Park, PA 16802, USA
E-mail address: weiss@math.psu.edu