Preprint
May 1995

LOW TEMPERATURE PHASE DIAGRAMS FOR QUANTUM PERTURBATIONS OF CLASSICAL SPIN SYSTEMS

C. Borgs[1,†], R. Kotecký[2,‡]and and D. Ueltschi[3]

[1]*School of Mathematics, Institute for Advanced Study, Princeton*
[2]*Centre de Physique Théorique, CNRS, Marseille*
[3]*Institut de Physique Théorique, EPF Lausanne*

Abstract. We consider a quantum spin system with Hamiltonian

$$H = H^{(0)} + \lambda V,$$

where $H^{(0)}$ is diagonal in a basis $|s\rangle = \bigotimes_x |s_x\rangle$ which may be labeled by the configurations $s = \{s_x\}$ of a suitable classical spin system on $\mathbb{Z}^d$,

$$H^{(0)}\,|s\rangle = H^{(0)}(s)\,|s\rangle.$$

We assume that $H^{(0)}(s)$ is a finite range Hamiltonian with finitely many ground states and a suitable Peierls condition for excitations, while V is a finite range or exponentially decaying quantum perturbation. Mapping the d dimensional quantum system onto a *classical* contour system on a $d+1$ dimensional lattice, we use standard Pirogov-Sinai theory to show that the low temperature phase diagram of the quantum spin system is a small perturbation of the zero temperature phase diagram of the classical Hamiltonian $H^{(0)}$, provided λ is sufficiently small. Our method can be applied to bosonic systems without substantial change. The extension to fermionic systems will be discussed in a subsequent paper.

1. Introduction

1.1. General ideas.

Many models of classical statistical mechanics provide examples of first-order phase transitions and phase coexistence at low temperatures. It became clear already from the first proof of such a transition for the Ising model by the Peierls argument [Pei36, Gri64, Dob65] that a convenient tool for the study of phase coexistence and first-order phase transitions is a representation in terms of configurations of geometrical objects — contours. This has been systematically developed in Pirogov-Sinai theory [PS75, Sin82], see also [KP84, Zah84, BI89], which allows one

---

[†]Heisenberg Fellow, on leave from Institut für Theoretische Physik, Freie Universität Berlin

[‡]On leave of absence from Center for Theoretical Study, Charles University, Praha. Partly supported by the grants GAČR 202/93/0449 and GAUK 376

to prove these phenomena for a wide class of models, with or without symmetry assumptions on the coexisting phases.

For quantum spin systems, the theory of first-order phase transitions and phase coexistence is much less developed. While several papers deal with this problem in the presence of a symmetry relating the two phases, [Gin69 , Ken85], no general theory is known which provides a systematic approach to quantum spin systems once the symmetry constraint is relaxed.

In this paper we propose to develop such a theory for low temperature quantum spin systems which are small perturbations of suitable classical systems. To be more precise, we assume that the Hamiltonian of the system is of the form

$$H = H^{(0)} + \lambda V \tag{1.1}$$

where $H^{(0)}$ is diagonal in a basis $|s\rangle = \bigotimes_x |s_x\rangle$ which may be labelled by the configurations $s = \{s_x\}$ of a classical spin system with finite single spin space $S = \{1, \cdots, |S|\}$ and Hamiltonian

$$\langle s| H^{(0)} |s\rangle = H^{(0)}(s), \tag{1.2}$$

while V is a local or exponentially decaying quantum perturbation,

$$V = \sum_A V_A, \tag{1.3}$$

where the sum goes over connected sets A and V_A is an arbitrary operator on $\mathcal{H}_A = \bigotimes_{x\in A} \mathcal{H}_x$, except for the constraint that its norm $\|V_A\|$ is exponentially decaying with the size of A.

Assuming that the classical system has a contour representation in d dimensions that allows to apply the methods of Pirogov-Sinai theory for sufficiently low temperatures, we propose to study the quantum perturbation of this system by mapping it into a suitable contour system in $d + 1$ dimensions, which can again be analysed by the methods of Pirogov Sinai theory. Actually, our method is very similar to the method developed in [Bor88], were this strategy was used to develop weak coupling cluster expansions for lattice gauge theories with discrete gauge group and continuous time.

Our approach differs, however, from that used by Ginibre [Gin69] and Kennedy [Ken85]. In order to explain the main difference, let us first recall their method. It is based on the idea of developping the density matrix $e^{-\beta H}$ of the model (1.1) arround the unperturbed matrix $e^{-\beta H^{(0)}}$ using Trotter's formula

$$e^{-\beta H} = \lim_{n\to\infty} \Big(e^{-(\beta/n)H^{(0)}} \big(1 - \frac{\beta}{n} V\big)\Big)^n. \tag{1.4}$$

While the leading term of this expansion gives the partition function of the classical spin system at temperature β, the expansion of W according to (1.3) will introduce transitions between classical contours at various times, leading therefore to a representation in terms of "quantum contours" on $\mathbb{Z}^d \times [0, \beta]$.

In the symmetric case considered in [Ken85] and [Gin69], these quantum contours could be controlled with affordable effort using standard methods. The asymmetric situation considered here, however, obviously requires significant modifications involving something like a "Quantum Pirogov-Sinai" theory. Such an approach is currently being pursued by Datta, Fernandez and Fröhlich [DFF95], leading to results very similar to those presented in this paper.

Here, we follow an alternative approach, motivated by [Bor88]. The main idea is not to consider contours on $\mathbb{Z}^d \times [0,\beta]$, but to use a suitable blocked approach, which allows to map a d-dimensional quantum system onto a classical contour system on the $(d+1)$-dimensional block lattice. As a consequence, our results make it possible to apply directly the usual Pirogov-Sinai theory to quantum spin systems as well, thereby allowing to analyze questions concerning the low temperature phase structure, finite size scaling, analyticity properties, etc. using the well developped machinery of standard Pirogov-Sinai theory.

1.2. Contour representations of quantum lattice models.

In the remaining part of this introduction, we present the main ideas of our approach. In the first step, we rewrite the partition function $Z = \operatorname{Tr} e^{-\beta H}$ of the quantum spin system as

$$Z = \operatorname{Tr} T^M \quad \text{where} \quad T = e^{-\tilde\beta H} \quad \text{and} \quad \beta = M\tilde\beta \tag{1.5}$$

with an integer M to be chosen later. We then expand the partition function Z of the quantum system around the partition function $Z^{\text{class}} = \operatorname{Tr} e^{-\beta H^{(0)}}$ of the classical spin system using the Duhamel formula (a reference concerning the Duhamel formula is e.g. [SS76]) for the transfer matrix $T = e^{-\tilde\beta H}$. Introducing the family $\mathcal{A}_0$ of all sets A contributing to (1.3), the Duhamel expansion gives

$$T = e^{-\tilde\beta H^{(0)} - \tilde\beta\lambda \sum_{A\in\mathcal{A}_0} V_A} = \sum_{\mathbf{n}} \Big[\prod_{A\in\mathcal{A}_0} \frac{(-\lambda)^{n_A}}{n_A!} \int_0^{\tilde\beta} d\tau_A^1 \dots d\tau_A^{n_A} \Big] T(\boldsymbol{\tau}, \mathbf{n}), \tag{1.6}$$

where $\mathbf{n}$ is an multiindex, $\mathbf{n} : \mathcal{A}_0 \to \{0,1,2,\dots\}$, and $T(\boldsymbol{\tau},\mathbf{n})$ is obtained from $T^{(0)} = e^{-\tilde\beta H}$ by "inserting" the operator V_A at the times $\tau_A^1, \dots \tau_A^{n_A}$, see Section 2 for the precise definition. Next, we resum (1.6) to obtain the expansion

$$T = \sum_B T(B), \tag{1.7a}$$

where

$$T(B) = \sum_{\substack{\mathcal{A}=\{A_1,\dots,A_k\} \\ \cup_i A_i = B}} \tilde T(\mathcal{A}), \tag{1.7b}$$

with

$$\tilde T(\mathcal{A}) = \sum_{\boldsymbol{n}:\operatorname{supp}\boldsymbol{n}=\mathcal{A}} \Big[\prod_{A\in\mathcal{A}_0} \frac{(-\lambda)^{n_A}}{n_A!} \int_0^{\tilde\beta} d\tau_A^1 \dots d\tau_A^{n_A} \Big] T(\boldsymbol{\tau}, \boldsymbol{n}). \tag{1.7c}$$

Using the basis $|s\rangle$ to rewrite (1.5) as

$$Z = \sum_{s^{(1)},\dots,s^{(M)}} \langle s^{(1)}|\, T\, |s^{(2)}\rangle \dots \langle s^{(M)}|\, T\, |s^{(1)}\rangle \tag{1.8}$$

and inserting the formula (1.7) to expand T around $T^{(0)}$, we obtain

$$Z = \sum_{\substack{s^{(1)},\dots,s^{(M)} \\ B^{(1)},\dots,B^{(M)}}} \prod_{t=1}^{M} \langle s^{(t-1)}|\, \tilde{T}(B^{(t)})\, |s^{(t)}\rangle, \tag{1.9}$$

where we have identified $s^{(0)}$ and $s^{(M)}$.

At this point, the quantum spin system is easily mapped to a classical contour system in $d+1$ dimensions. Before doing so, let us discuss the expansion (1.9) from a more heuristic point of view. Starting with the leading term in $B^{(1)},\dots,B^{(M)}$, namely the term where all $B^{(t)}$ are empty, the matrices $T(B^{(t)})$ reduce to the unperturbed transfer matrix $T^{(0)}$, which implies that only the term with $s^{(1)} \equiv s^{(2)} \equiv \dots \equiv s^{(M)}$ contributes to the sum over $s^{(1)},\dots,s^{(M)}$, giving the partition function of the classical spin system at the inverse temperature β. A non empty set $B^{(t)}$, on the other hand, corresponds to the insertion of one or several operators V_A, $A \subset B^{(t)}$, inducing transitions between different classical states $s^{(t-1)}$ and $s^{(t)}$. It should be noted, however, that for a fixed set of $B^{(t)}$'s only those spin configurations $s^{(1)} \equiv s^{(2)} \equiv \dots \equiv s^{(M)}$ contribute to (1.9), for which $s^{(t-1)}$ and $s^{(t)}$ are identical on all points x for which $x \notin B^{(t)}$.

In order rewrite (1.9) in terms of contours, we assign contours to "configurations" specified by $s^{(1)},\dots,s^{(M)}$ and $B^{(1)},\dots,B^{(M)}$ Namely, we introduce elementary cubes as the unit closed cubes $C \subset \mathbb{R}^{d+1}$ with centers (x,t) where $t \in \{1,2,\dots,M\}$ and $x \in \mathbb{Z}^d$. We say that an elementary cube C with center (x,t), $x \in \mathbb{Z}^d$, lies in the t'th time slice; we say that it is in a quantum excited state if $x \in B^{(t)}$, while we say it is in a classical state if this is not the case. Consider now a cube C in the t'th time slice which is in a classical state. Then $s^{(t-1)}$ and $s^{(t)}$ must assume the same value $s_x \in S$ on the corresponding point $x \in \mathbb{Z}^d$, and we say that the cube C is in the (classical) state s_x.

In order to explain our definition of contours, let us assume for the purpose of this introduction that $H^{(0)}(s)$ is the Hamiltonian of a classical spin system with nearest neighbor interactions, and that the contours of the classical system correspond to bonds $\langle xy\rangle$ for which $s_x \neq s_y$ (see Section 2 for the more general case). We then say that a cube C is part of the ground state region W_m, $m \in S$, if it is in the classical state m, and if all neighboring cubes in the same time slice are in the classical state m as well. All cubes which are not part of a ground state region are called excited. As usual, we define a contour as a connected component of the set of excited cubes.

Resumming all terms in (1.9) which lead to the same set of ground state regions and contours, we finally obtain Z as a sum over sets $\{Y_1,\dots,Y_n\}$ of non-overlapping contours. Given our definition of excited cubes, it is an easy exercise to show that the weight of each such configuration factors into a product of contour activities

$\rho(Y_i)$ and ground state terms $e^{-\tilde{\beta} e_m |W_m|}$, where e_m is the classical groundstate energy of the ground state m, while $|W_m|$ is the number of cubes which are in the ground state m. This gives the representation

$$Z = \sum_{\{Y_1,\ldots,Y_n\}} \prod_i \rho(Y_i) \prod_m e^{-\tilde{\beta} e_m |W_m|}, \tag{1.10}$$

which is exactly of the same form as the contour representation of a classical spin system. We can therefore apply standard Pirogov-Sinai theory to analyze the quantum spin system considered here, provided we can prove a bound of the form

$$|\rho(Y)| \leq e^{-\gamma|Y|} e^{-\tilde{\beta} e_0 |Y|} \tag{1.11}$$

where γ is a sufficiently large constant, and $e_0 = \min_m e_m$.

Observing that cubes in a quantum excitation are supressed by a small factor proportional to $\lambda\tilde{\beta}$, while excited classical cubes are exponentially suppressed by a "classical" contour energy proportional to $\tilde{\beta}$, such a bound can easily be proven, see Section 4 for the details.

Notice that the weights $\rho(Y)$ are in general complex. The version of the Pirogov-Sinai theory to be used thus must deal with this fact. Actually, such a case has been discussed in [BI89] and we are closely following their approach.

A novel feature of the models considered here stems from the fact that the resulting classical model resides in a finite slab of thickness proportional to β; actually we should talk of a cylinder because of the periodic boundary conditions. It is therefore not possible to directly apply standard Pirogov-Sinai theory. Actually, this gives rise to an interesting problem in both quantum and classical spin systems: dimensional crossover for first order phase transitions. We will not discuss the physics of dimensional crossover in this paper, but the technical modifications needed to deal with finite temperature quantum spin systems do actually provide the necessary framework to deal with this problem as well.

The organization of this paper is as follows: In the next section, we state our main assumptions and results. In Section 3, we derive the contour representation (1.10), proving in particular the needed factorization of contour activities. In Section 4 we prove the exponential decay of the contour activities. Section 5 is devoted to the discussion of the resulting contour model including the discussion of modifications to Pirogov-Sinai theory on a finite slab. In Section 6 we discuss expectation values of local observables, and in Section 7 we combine the results of the preceeding section to prove the theorems stated in Section 2. Details of the necessary modifications to Pirogov-Sinai theory on a finite slab are deferred to an appendix.

We close this introduction with a discussion of possible extensions. We recall that we assumed that $H^{(0)}$ is diagonal in a basis $|s\rangle = \otimes_x |s_x\rangle$, where s_x lies in a finite spin space S. While this is a natural setting for quantum spin systems as, e.g. the anisotropic quantum Heisenberg model, it is not for the discussion of bosonic or fermionic lattice gases. In this situation, a typical choice for $H^{(0)}$ would be an operator which is diagonal in the usual bosonic or fermionic Fock representation where basis vectors are characterized by eigenvalues of the corresponding number operators n_x. While the correponding classical system still has a finite state space ($n_x = n_\uparrow + n_\downarrow = 0, 1, 2$) in the fermionic case, bosons now give rise to a state space which contains infinitely many classical states per site. In most application, however, this is not a serious problem, because the hamiltonian $H^{(0)}$ supresses high values of n_x (or, in the usual field representation, high values of the boson field ϕ_x). Our methods and results are therefore applicable to bosonic lattice gases without major modifications.

For fermions, on the other hand, the antisymmetrization of the wave function leads to sign problems in the contour representation (1.10) which have to be dealt with carefully. While this is *a priori* not obvious at all, it turns out, however, that fermion signs do not spoil the factorization properties needed to apply Pirogov-Sinai theory (see [BK95], where the methods developed here are used to prove the existence of staggered charge order in the narrow band extended Hubbard model).

2. Definition of the model, statements of results

2.1. Assumptions on the classical model.

We start this section by stating the precise assumptions for the classical model. We consider a classical spin system with finite spin space $S = \{1, \dots, |S|\}$, spin configurations $s : \mathbb{Z}^d \to S$, $x \mapsto s_x$, and finite range Hamiltonian $H^{(0)}(s)$ with translation invariant interactions, depending on a vector parameter $\mu \in \mathcal{U}$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^\nu$. We assume that $H^{(0)}(s)$ is given in the form

$$H^{(0)}(s) = \sum_x \Phi_x(s), \tag{2.1}$$

where $\Phi_x(s) \in \mathbb{R}$ depends on s only via the spins s_y for which $y \in U(x) = \{y \in \mathbb{Z}^d \mid \operatorname{dist}(x, y) \leq R_0\}$, where R_0 is a finite number. In our notation we supress the dependence of $H^{(0)}$ on Φ_x and μ.

As usually, a configuration g which minimises the Hamiltonian (2.1) is called a ground state configuration. For the purpose of this paper, we will assume that the number of ground states of the Hamiltonian (2.1) is finite, and that all of them are periodic. More precisely, we will assume that there is a finite number of periodic configurations $g^{(1)}, \dots, g^{(r)}$, with (specific) energies

$$e_m = e_m(\mu) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \Phi_x(g^{(m)}), \tag{2.2}$$

such that for each $\mu \in \mathcal{U}$, the set of ground states $G(\mu)$ is a subset of $\{g^{(1)}, \dots, g^{(r)}\}$. Obviously, $G(\mu)$ is given by those configurations $g^{(m)}$ for which $e_m(\mu)$ is equal to the "ground state energy"

$$e_0 = e_0(\mu) = \min_m e_m(\mu). \tag{2.3}$$

Note that we may assume, without loss of generality, that $\Phi_x(g^{(m)})$ is independent of the point x for all ground state configurations $g^{(m)}$, because this condition can always be achieved by averaging $\Phi_x(s)$ in (2.1) over the minimal common period of $g^{(1)}, \ldots, g^{(r)}$.

Our goal will be to prove that the low temperature phase diagram of the quantum model is a small perturbation of the classical ground state diagram provided the quantum perturbation is sufficiently small. In order to formulate and prove this statement, we need some assumptions on the structure of the ground state diagram. Here we assume that for some value of $\mu_0 \in \mathcal{U}$ all states in $\{g^{(1)}, \ldots, g^{(r)}\}$ are ground states,

$$e_m(\mu_0) = e_0(\mu_0) \qquad \text{for all} \qquad m = 1, \ldots, r, \tag{2.4}$$

that $e_m(\mu)$ are C^1 functions in $\mathcal{U}$, and that the matrix of derivatives

$$E = \Big(\frac{\partial e_m(\mu)}{\partial \mu_i}\Big) \tag{2.5}$$

has rank $r-1$ for all $\mu \in \mathcal{U}$, with uniform bounds on the inverse of the corresponding submatrices. We remark that this condition implies that the zero temperature phase diagram has the usual structure of a $\nu - (r-1)$ dimensional coexistence surface S_0 where all states $g^{(m)}$ are ground states, r different $\nu - (r-1) - 1$ dimensional surfaces S_n ending in S_0 where all states but the state $g^{(m)}$ are ground states, ...

Next, we formulate a suitable Peierls condition. In order to present it, we introduce, for a given configuration s, the notion of excited sites $x \in \mathbb{Z}^d$. We say that a site x is in the *state* $g^{(m)}$ if the configuration s coincides with the configuration $g^{(m)}$ on $U(x)$, i.e. on all sites y for which $\operatorname{dist}(x,y) \leq R_0$; a site is *excited*, if it is not in any of the states $g^{(1)}, \ldots, g^{(r)}$. Given this notation, the Peierls assumption used in this paper is that there exists a constant $\gamma_0 > 0$, independent of μ, such that

$$\Phi_x(s) \geq e_0(\mu) + \gamma_0 \qquad \text{for all excited sites } x \text{ of all configurations } s. \tag{2.6}$$

Finally, we assume that the derivatives of ϕ_x are uniformly bounded in $\mathcal{U}$. More explicitely, we assume that there is a constant $C_0 < \infty$, such that

$$\Big|\frac{\partial}{\partial \mu_i}\phi_x(s)\Big| \leq C_0 \tag{2.7}$$

for all $i = 1, \ldots, \nu$, $\mu \in \mathcal{U}$, $x \in \mathbb{Z}^d$, and all configurations s.

Remark. Given the assumptions stated in this subsection, standard Pirogov-Sinai theory implies that the low temperature phase diagram of the classical model has the same topological structure as the corresponding zero temperature phase diagram (see above).

2.2. Assumptions on the quantum perturbation.

As pointed out in the introduction, we propose to develop a theory which allows to control low temperature quantum spin systems that are small perturbations of the classical system introduced above. We consider quantum spin systems with Hamiltonians of the form

$$H = H^{(0)} + \lambda V \tag{2.8}$$

where $H^{(0)}$ is diagonal in a basis $|s\rangle = \bigotimes_x |s_x\rangle$ that may be labelled by the configurations $s = \{s_x\}$ of the classical spin system,

$$H^{(0)}\,|s\rangle = H^{(0)}(s)\,|s\rangle. \tag{2.9}$$

We assume that V is of the form

$$V = \sum_A V_A, \tag{2.10}$$

where the sum goes over connected sets A and V_A is a self-adjoint operator on $\mathcal{H}_A = \bigotimes_{x\in A}\mathcal{H}_x$. In addition to translation invariance, we assume that V_A and its derivatives, $\frac{\partial}{\partial\mu_i}V_A$, $i = 1,\dots,\nu$, are bounded operators, with a suitable decay constraint on the corresponding operator norms $\|V_A\|$ and $\left\|\dfrac{\partial}{\partial\mu_i}V_A\right\|$. In order to formulate this constraint, we introduce the Sobolev norm

$$|||V|||_\gamma = \sum_{A:x\in A}\Bigl(\|V_A\| + \sum_{i=1}^{\nu}\Bigl\|\frac{\partial}{\partial\mu_i}V_A\Bigr\|\Bigr)e^{\gamma|A|} \tag{2.11}$$

where $|A|$ is the number of points in A. Given this definition, our assumption on the decay of V is the assumption that

$$|||V|||_{\gamma_Q} < \infty \tag{2.12}$$

for a sufficiently large constant γ_Q.

Remarks.

i) For a finite range perturbation, where $V_A = 0$ if the diameter of A exceeds the range R_Q of the interaction, the assumption (2.12) is automatically fulfilled for arbitrary large $\gamma_Q < \infty$.

ii) If the quantum perturbation V is of infinite range, we need that $\|V_A\|$ and $\left\|\dfrac{\partial}{\partial\mu_i}V_A\right\|$ decay exponentially in the size $|A|$ of A. Assuming exponential decay with a sufficiently large decay constant γ, and observing that the number of connected sets A of size s that contain a given point $x\in\mathbb{Z}^d$ is bounded by $(2d)^{2s}$, the condition (2.12) can be satisfied provided $\gamma > \gamma_Q + 2\log(2d)$.

iii) Strictly speaking, an exponentially decaying pair potential, $V = \sum_{x,y}V_{x,y}$ where the norm $||V_{x,y}||$ decays exponentially with the distance between x and y, is not in the class considered in this section because V is not given as a sum over *connected* sets A. It is obvious, however, that such a potential can be *rewritten* in the required form, by artificially connecting the two points x and y by a nearest neighbor path. In (2.11), this effectively replaces the size of the set $A = \{x,y\}$ by its ℓ_1-diameter $\sum_i |x_i - y_i|$.

2.3. Finite volume states for the quantum system.

In order to discuss the phase diagram of the quantum spin system, we will consider suitable finite volume states $\langle\cdot\rangle_{q,\Lambda}$ which are analogues of the classical states with boundary condition q, where $q = 1, \ldots, r$. We first introduce, for any configuration s and any finite set A, the vector $|s_A\rangle = \bigotimes_{x\in A} |s_x\rangle$. Given a finite set $\Lambda \subset \mathbb{Z}^d$, we then define suitable finite volume Hamiltonians $H_\Lambda^{(0)}$ and H_Λ on the Hilbert space $\mathcal{H}_{\bar\Lambda} = \bigotimes_{x\in\bar\Lambda} \mathcal{H}_x$, where $\bar\Lambda = \cup_{x\in\Lambda} U(x)$. Namely, we introduce operators

$$H_x^{(0)} = \sum_{s_{\bar\Lambda}} \Phi_x(s_{\bar\Lambda})\, |s_{\bar\Lambda}\rangle\langle s_{\bar\Lambda}|\,, \tag{2.13}$$

$$H_\Lambda^{(0)} = \sum_{x\in\Lambda} H_x^{(0)}, \tag{2.14}$$

and

$$H_\Lambda = H_\Lambda^{(0)} + \lambda \sum_{A\subset\Lambda} V_A. \tag{2.15}$$

The Hamilton operator with boundary conditions q is then defined as the "partial expectation value"

$$H_{q,\Lambda} = \langle g_{\partial\Lambda}^{(q)}|\, H_\Lambda\, |g_{\partial\Lambda}^{(q)}\rangle, \tag{2.16}$$

where $\partial\Lambda$ is the set $\bar\Lambda \setminus \Lambda$. More precisely, $H_{q,\Lambda}$ is an operator on $\mathcal{H}_\Lambda$ whose matrix elements are

$$\langle s_\Lambda|\, H_{q,\Lambda}\, |s'_\Lambda\rangle = \langle s_\Lambda| \otimes \langle g_{\bar\Lambda\setminus\Lambda}^{(q)}|\, H_\Lambda\, |g_{\bar\Lambda\setminus\Lambda}^{(q)}\rangle \otimes |s'_\Lambda\rangle. \tag{2.17}$$

Given the Hamiltonian with boundary conditions q, we introduce the quantum state $\langle\cdot\rangle_{q,\Lambda}$ as

$$\langle\cdot\rangle_{q,\Lambda} = \frac{1}{Z_{q,\Lambda}} \operatorname{Tr}_{\mathcal{H}_\Lambda}(\cdot\, e^{-\beta H_{q,\Lambda}}), \tag{2.18}$$

where

$$Z_{q,\Lambda} = \operatorname{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H_{q,\Lambda}}. \tag{2.19}$$

Note that $H_\Lambda^{(0)}$ and H_Λ are operators on $\mathcal{H}_{\bar\Lambda}$, while $H_{q,\Lambda}$ and it's analogues,

$$H_{q,x}^{(0)} = \langle g_{\partial\Lambda}^{(q)}|\, H_x^{(0)}\, |g_{\partial\Lambda}^{(q)}\rangle \tag{2.20}$$

and

$$H_{q,\Lambda}^{(0)} = \langle g_{\partial\Lambda}^{(q)}|\, H_\Lambda^{(0)}\, |g_{\partial\Lambda}^{(q)}\rangle, \tag{2.21}$$

are operators on $\mathcal{H}_\Lambda$.

Remark. Following Ginibre [Gin69], it might seem more natural to implement the boundary conditions with the help of suitable projection operators $P_{\partial\Lambda}^{(q)}$. Here, this would amount to defining $P_{\partial\Lambda}^{(q)} = |g_{\partial\Lambda}^{(q)}\rangle\langle g_{\partial\Lambda}^{(q)}|$. With the help of this projection operator, one would then define

$$Z_{m,\Lambda} = \operatorname{Tr} P_{\partial\Lambda}^{(q)} e^{-\beta H_\Lambda}$$

and similarly for the finite volume states (2.18). Observing that

$$\mathrm{Tr}_{\mathcal{H}_\Lambda} W e^{-\beta H_{q,\Lambda}} = \mathrm{Tr}_{\mathcal{H}_{\bar\Lambda}} W P^{(q)}_{\partial\Lambda} e^{-\beta H_\Lambda}$$

for all operators W on $\mathcal{H}_\Lambda$, these two implementations of boundary conditions are actually equivalent.

2.4. Statement of results.

In order to state our results in the form of a theorem, we recall that a local observable is an operator which is a selfadjoint bounded operator on $\mathcal{H}_\Lambda$ for some finite set Λ. We also introduce, for each x in $\mathbb{Z}^d$ and any local obserbable Ψ, the translate $t_x(\Psi)$. Defining finally $\Lambda(L)$ as the box

$$\Lambda(L) = \{x \in \mathbb{Z}^d \mid |x_i| \le L \quad \text{for all} \quad i = 1, \dots, d\}, \tag{2.22}$$

our main results are stated in the following two theorems.

Theorem 2.1. *Let $d \ge 2$ and let $H^{(0)}$ be a Hamiltonian obeying the assumptions of Section 2.1. Then there are constants $0 < \beta_0 < \infty$ and $0 < \gamma_Q < \infty$, such that for all quantum perturbations V obeying the assumptions of Section 2.2, all $\beta \ge \beta_0$ and all $\lambda \in \mathbb{C}$ with*

$$|\lambda| \le \lambda_0 := \frac{1}{e\beta_0 |||V|||_{\gamma_Q}} \tag{2.23}$$

there are constants ξ_q and continuously differentible functions $f_q(\mu)$, $q = 1, \dots, r$, such that the following statements hold true whenever

$$a_q(\beta, \lambda, \mu) := \mathrm{Re}\, f_q(\mu) - \min_m \mathrm{Re}\, f_m(\mu) = 0\,. \tag{2.24}$$

i) The infinite volume free energy corresponding to $Z_{q,\Lambda(L)}$ exists and is equal to f_q:

$$f_q = -\frac{1}{\beta} \lim_{L\to\infty} \frac{1}{|\Lambda(L)|} \log Z_{q,\Lambda(L)} \tag{2.25}$$

ii) The infinite volume limit

$$\langle \Psi \rangle_q = \lim_{L\to\infty} \langle \Psi \rangle_{q,\Lambda(L)} \tag{2.26}$$

exists for all local observables Ψ.

iii) For all local observables Ψ and Φ, there exists a constant $C_{\Psi,\Phi} < \infty$, such that

$$\left| \langle \Psi t_x(\Phi) \rangle_q - \langle \Psi \rangle_q \langle t_x(\Phi) \rangle_q \right| \le C_{\Psi,\Phi} e^{-|x|/\xi_q}. \tag{2.27}$$

iv) The projection operators

$$P^{(q)}_{U(x)} = |g^{(q)}_{U(x)}\rangle\langle g^{(q)}_{U(x)}| \tag{2.28}$$

onto the "classical states" $g^{(q)}_{U(x)}$ *obey the bounds*

$$\big|\langle P^{(q)}_{U(x)}\rangle_q - 1\big| < \frac{1}{2} \tag{2.29}$$

and

$$\big|\langle P^{(m)}_{U(x)}\rangle_q\big| < \frac{1}{2} \tag{2.30}$$

for all $m \neq q$.

v) There exists a point $\tilde{\mu}_0 \in \mathcal{U}$ *such that* $a_m(\tilde{\mu}_0) = 0$ *for all* $m = 1, \dots, r$. *For all* $\mu \in \mathcal{U}$, *the matrix of derivatives*

$$F = \Big(\frac{\partial \mathrm{Re}\, f_m(\mu)}{\partial \mu_i}\Big) \tag{2.31}$$

has rank $r-1$, *and the inverse of the corresponding submatrix is uniformly bounded in* $\mathcal{U}$.

Remarks.

i) Following the usual terminology of Pirogov-Sinai theory, we call a phase with $a_q = 0$ *stable*. By the inverse function theorem, statement v) of the Theorem implies that the phase diagram of the quantum system has the same structure as the zero temperature phase diagram of the classical sytem, with a $\nu - (r-1)$ dimensional coexistence surface $\tilde{S}_0$ where all states are stable, r different $\nu - (r-1) - 1$ dimensional surfaces $\tilde{S}_n$ ending in $\tilde{S}_0$ where all states but the state m are stable, $\dots$.

ii) Choosing β sufficiently large and λ sufficiently small, the bounds (2.29) and (2.30) can be made arbitrary sharp. In this sense, the quantum states $\langle\cdot\rangle_q$ are small perturbations of the corresponding classical state whenever q is stable.

iii) While Theorem 2.1 is stated (and proven) for general complex λ, the physical situation corresponds, of course, to real values of λ, as required by the self-adjointness of the Hamiltonian H. As we will see in Section 5, the "meta-stable free energies" f_q are real in this case, making the real part in (2.24) and (2.31) superfluous.

In order to state the next theorem, we define states with periodic boundary conditons on $\Lambda(L)$. To this end, we consider the torus $\Lambda_{\mathrm{per}}(L) = \big(\mathbb{Z}/(2L+1)\mathbb{Z}\big)^d$ and the corresponding Hamiltonian

$$H_{\mathrm{per},\Lambda(L)} = \sum_{x\in\Lambda_{\mathrm{per}}(L)} H^{(0)}_x + \lambda \sum_{A\subset\Lambda_{\mathrm{per}}(L)} V_A\,, \tag{2.32}$$

where the second sum goes over all subsets $A \subset \Lambda_{\mathrm{per}}(L)$ which are do not wind around the torus $\Lambda_{\mathrm{per}}(L)$. With these definitions, we then introduce the quantum state with periodic boundary conditions as

$$\langle\cdot\rangle_{\mathrm{per},\Lambda(L)} = \frac{1}{Z_{\mathrm{per},\Lambda(L)}} \mathrm{Tr}_{\mathcal{H}_{\Lambda(L)}}\big(\cdot\, e^{-\beta H_{\mathrm{per},\Lambda(L)}}\big), \tag{2.33}$$

where

$$Z_{\mathrm{per},\Lambda(L)} = \mathrm{Tr}_{\mathcal{H}_{\Lambda(L)}} e^{-\beta H_{\mathrm{per},\Lambda(L)}}. \tag{2.34}$$

Theorem 2.2. *Let $H^{(0)}$, V, β and λ as in Theorem 2.1. Assume in addition that λ is real. Then the infinite volume state with periodic boundary conditions,*

$$\langle \Psi \rangle_{\text{per}} = \lim_{L\to\infty} \langle \Psi \rangle_{\text{per},\Lambda(L)} \tag{2.35}$$

exists for all local observables Ψ, and is a convex combination (with equal weights) of the stable states,

$$\langle \Psi \rangle_{\text{per}} = \sum_{q\in Q(\mu)} \frac{1}{|Q(\mu)|} \langle \Psi \rangle_q. \tag{2.36}$$

Here

$$Q(\mu) = \{q \in \{1,\dots,r\} \mid a_q(\mu) = 0\}. \tag{2.37}$$

3. Derivation of the contour representation

As explained in the introduction, we start with the Duhamel expansion for the transfer matrix. In this section, we will consider a fixed finite volume $\Lambda = \Lambda(L) = \{x \in \mathbb{Z}^d \mid |x_i| \le L \text{ for all } i = 1,\dots,d\}$, and a fixed value $q \in \{1,\dots,r\}$ for the boundary condition; further, we are not explicitely specifying this in our notation. Introducing the transfer matrices

$$T^{(0)} = e^{-\tilde\beta H^{(0)}_{q,\Lambda}} \tag{3.1}$$

and

$$T = e^{-\tilde\beta H_{q,\Lambda}}, \tag{3.2}$$

we rewrite the partition function $Z_{q,\Lambda}$ as

$$Z_{q,\Lambda} = \operatorname{Tr} T^M, \tag{3.3}$$

where $\tilde\beta$ and $M \in \mathbb{N}$ are related to the inverse temperature β by the equality

$$\beta = M\tilde\beta. \tag{3.4}$$

The Duhamel expansion (or Dyson series) for the operator T yields

$$T = \sum_{\boldsymbol{n}} \Big[\prod_{A\in\mathcal{A}_0} \frac{(-\lambda)^{n_A}}{n_A!} \int_0^{\tilde\beta} d\tau_A^1 \dots d\tau_A^{n_A} \Big] T(\boldsymbol{\tau},\boldsymbol{n}). \tag{3.5}$$

Here, $\mathcal{A}_0$ is the family of all sets A contributing to the sum (2.15), $\boldsymbol{n}$ is an multiindex $\boldsymbol{n}\colon \mathcal{A}_0 \to \{0,1,\dots,\}$ with finite $n = \sum_{A\in\mathcal{A}_0} n_A$, $\boldsymbol{\tau} = \{\tau_A^1,\dots,\tau_A^{n_A}, A \in \mathcal{A}_0\} \in [0,\tilde\beta]^n$, and the operator $T(\boldsymbol{\tau},\boldsymbol{n})$ is obtained from $T^{(0)}$ by "inserting" the operator V_A at the times $\tau_A^1,\dots,\tau_A^{n_A}$. Formally, it can be defined as follows. For a given $\boldsymbol{n}$ and $\boldsymbol{\tau}$, let $\operatorname{supp}\boldsymbol{n} \equiv \mathcal{A} = \{A_1,\dots,A_k\}$ be the set of all $A \in \mathcal{A}_0$ with $n_A \ne 0$, $n_i = n_{A_i}$, and $V_i = V_{A_i}$. Let

$$(s_1,\dots,s_n) = \pi(\tau_{A_1}^1,\dots,\tau_{A_1}^{n_1},\dots,\tau_{A_k}^1,\dots,\tau_{A_k}^{n_k})$$

be a permutation of the times $\boldsymbol{\tau}$ such that $s_1 \leq s_2 \leq \cdots \leq s_n$, and set

$$(\tilde{V}_1, \ldots, \tilde{V}_n) = \pi(V_1, \ldots, V_1, \ldots, V_k, \ldots, V_k),$$

where on the right-hand side each V_i appears exactly n_i times. Then $T(\boldsymbol{\tau}, \boldsymbol{n})$ is defined by

$$T(\boldsymbol{\tau}, \boldsymbol{n}) = e^{-s_1 H^{(0)}_{q,\Lambda}} \tilde{V}_1 e^{-(s_2 - s_1) H^{(0)}_{q,\Lambda}} \tilde{V}_2 \ldots e^{-(s_n - s_{n-1}) H^{(0)}_{q,\Lambda}} \tilde{V}_n e^{-(\tilde{\beta} - s_n) H^{(0)}_{q,\Lambda}}. \tag{3.6}$$

Next, we resum (3.5) to obtain the expansion

$$T = \sum_{B \subset \Lambda} T(B), \tag{3.7}$$

where

$$T(B) = \sum_{\substack{\mathcal{A} = \{A_1, \ldots, A_k\} \\ \cup_i A_i = B}} \tilde{T}(\mathcal{A}), \tag{3.8}$$

with

$$\tilde{T}(\mathcal{A}) = \sum_{\boldsymbol{n} : \operatorname{supp} \boldsymbol{n} = \mathcal{A}} \Big[\prod_{A \in \mathcal{A}} \frac{(-\lambda)^{n_A}}{n_A!} \int_0^{\tilde{\beta}} d\tau_A^1 \ldots d\tau_A^{n_A} \Big] T(\boldsymbol{\tau}, \boldsymbol{n}). \tag{3.9}$$

Before continuing with the expansion for the partition function as sketched in the introduction, we discuss the factorization properties of the operator $T(B)$. Given a subset Λ' of Λ, we introduce the operators $T_{\Lambda'}(\boldsymbol{\tau}, \boldsymbol{n})$, $\tilde{T}_{\Lambda'}(\mathcal{A})$, and $T_{\Lambda'}(B)$ that are obtained from $T(\boldsymbol{\tau}, \boldsymbol{n})$, $\tilde{T}(\mathcal{A})$, and $T(B)$, respectively, by replacing $H^{(0)}_{q,\Lambda} = \sum_{x \in \Lambda} H^{(0)}_{q,x}$ by the operator $\sum_{x \in \Lambda'} H^{(0)}_{q,x}$. Using the fact that

$$[H^{(0)}_{q,x}, H^{(0)}_{q,y}] = 0 \qquad \text{for all} \qquad x, y \in \Lambda, \tag{3.10}$$

while

$$[H^{(0)}_{q,x}, V_A] = 0 \qquad \text{if} \qquad \operatorname{dist}(x, A) > R_0, \tag{3.11}$$

one immediately obtains that

$$T(B) = T^{(0)}_{\Lambda \setminus \bar{B}} \, T_{\bar{B}}(B) = T_{\bar{B}}(B) \, T^{(0)}_{\Lambda \setminus \bar{B}}, \tag{3.12}$$

where $\bar{B}$ is the set

$$\bar{B} = \{x \in \Lambda \mid \operatorname{dist}(x, B) \leq R_0\}. \tag{3.13}$$

Let us now consider a set B that can be decomposed as $B = B_1 \cup B_2$, with $\bar{B}_1 \cap \bar{B}_2 = \emptyset$. Then

$$T_{\bar{B}}(B) = T_{\bar{B}_1}(B_1) T_{\bar{B}_2}(B_2) = T_{\bar{B}_2}(B_2) T_{\bar{B}_1}(B_1), \tag{3.14}$$

due to (3.10), (3.11), and the fact that

$$[V_A, V_{A'}] = 0 \qquad \text{if} \qquad A \cap A' = \emptyset. \tag{3.15}$$

For $B \subset \Lambda$, we therefore get the decompositions

$$T_{\bar{B}}(B) = \prod_{i=1}^{k} T_{\bar{B}_i}(B_i) \tag{3.16}$$

and

$$T(B) = T^{(0)}_{\Lambda\setminus\bar{B}} \prod_{i=1}^{k} T_{\bar{B}_i}(B_i) \tag{3.17}$$

provided $B = \cup_{i=1}^{k} B_i$, where $\bar{B}_1, \dots, \bar{B}_k$ are pairwise disjoint.

Deviating a little bit from the strategy explained in the introduction, we further expand the transfer matrix T. Using (3.17), we observe that

$$\begin{aligned} &\langle s_B| \otimes \langle s_{\Lambda\setminus B}|\, T(B)\, |s'_{\Lambda\setminus B}\rangle \otimes |s'_B\rangle = \\ &\quad = \delta_{s_{\Lambda\setminus B}, s'_{\Lambda\setminus B}} e^{-\tilde{\beta}\sum_{x\in\Lambda\setminus\bar{B}} \Phi_x(s_{U(x)})} \langle s_B| \otimes \langle s_{\Lambda\setminus B}|\, T_{\bar{B}}(B)\, |s'_{\Lambda\setminus B}\rangle \otimes |s'_B\rangle. \end{aligned} \tag{3.18}$$

Introducing thus the operator $T_B(s_{\overline{\partial B}})$ on $\mathcal{H}_B$ as the partial expectation value

$$T_B(s_{\overline{\partial B}}) = \langle s_{\Lambda\setminus B}|\, T_{\bar{B}}(B)\, |s_{\Lambda\setminus B}\rangle, \tag{3.19}$$

we get

$$T(B) = \sum_{s_{\Lambda\setminus B}} e^{-\tilde{\beta}\sum_{x\in\Lambda\setminus\bar{B}} \Phi_x(s_{U(x)})} \big(|s_{\Lambda\setminus B}\rangle\langle s_{\Lambda\setminus B}| \otimes T_B(s_{\overline{\partial B}})\big) \tag{3.20}$$

As before, $U(x) = \{y \in \mathbb{Z}^d \,|\, \mathrm{dist}\,(x,y) \le R_0\}$, while $\overline{\partial B}$ is the set

$$\overline{\partial B} = \{x \notin B \,|\, \mathrm{dist}\,(x,B) \le 2R_0\}. \tag{3.21}$$

Considering "configurations" $\Sigma = (B, s_{\Lambda\setminus B})$ on Λ specyfiyng the set B as well as the configuration $s_{\Lambda\setminus B}$ outside it, we can combine (3.7) and (3.20) to get the expansion

$$T = \sum_{\Sigma} K(\Sigma). \tag{3.22}$$

Here the operators $K(\Sigma)$ (on $\mathcal{H}_\Lambda$) are defined by

$$K(\Sigma) \equiv K(B, s_{\Lambda\setminus B}) = e^{-\tilde{\beta}\sum_{x\in\Lambda\setminus\bar{B}} \Phi_x(s_{U(x)})} \Big(|s_{\Lambda\setminus B}\rangle\langle s_{\Lambda\setminus B}| \otimes T_B(s_{\overline{\partial B}})\Big). \tag{3.23}$$

Remarks.

i) The operator $T_B(s_{\overline{\partial B}})$ inherits from $T_{\bar{B}}(B)$ the factorization property (3.16). Namely,

$$T_B(s_{\overline{\partial B}}) = \bigotimes_{i=1}^{n} T_{B_i}(s_{\overline{\partial B_i}}) \tag{3.24}$$

provided $B = \cup_{i=1}^{n} B_i$ with $\bar{B}_1, \dots, \bar{B}_n$ pairwise disjoint.

ii) For x and B near to the boundary of Λ, the spin configurations $s_{U(x)}$ and $s_{\overline{\partial B}}$ appearing in Φ_x and $T_B(s_{\overline{\partial B}})$ involve spins s_y with $y \notin \Lambda$. A more precise notation would therefore involve the spin configuration $s_{\Lambda\setminus B} \cup g^{(q)}_{\partial\Lambda}$ restricted to the sets $U(x)$ and $\overline{\partial B}$, respectively.

Next, we combine the representation (3.22) for the transfer matrix T with the formula (3.3) to rewrite $Z_{q,\Lambda}$ as

$$Z_{q,\Lambda} = \sum_{\Sigma_1,\ldots,\Sigma_M} w(\Sigma_1,\ldots,\Sigma_M) \tag{3.25}$$

with the weights

$$w(\Sigma_1,\ldots,\Sigma_M) = \operatorname{Tr}_{\mathcal{H}_\Lambda} \prod_{t=1}^{M} K(\Sigma_t). \tag{3.26}$$

For a given collection of configurations $\Sigma_1,\ldots,\Sigma_M$, $\Sigma_t = (B^{(t)}, s^{(t)}_{\Lambda\setminus B^{(t)}})$, on "time slices" $t = 1,\ldots,M$, we now assign a variable $\sigma_{(x,t)} \in \bar{S} = \{0,1,\ldots,q\}$ to each point in "space-time" lattice

$$\mathbb{L} = \mathbb{Z}^d \times \{1,\ldots,M\} \tag{3.27}$$

by defining

$$\sigma_{(x,t)} = \begin{cases} g^{(q)}_x & \text{if } x \notin \Lambda, \\ s^{(t)}_x & \text{if } x \in \Lambda \setminus B^{(t)}, \\ 0 & \text{if } x \in B^{(t)}. \end{cases} \tag{3.28}$$

Considering elementary cubes, i.e. the closed unit cubes $C(x,t)$ with center (x,t) in

$$\mathbb{L}_\Lambda = \Lambda \times \{1,\ldots,M\}, \tag{3.29}$$

we say that a cube $C(x,t)$ is *in the ground state* m, if the configuration $\sigma_{(y,t)}$ coincides with the configuration $g^{(m)}$ on all points $y \in U(x)$. Otherwise, the cube $C(x,t)$ is called an *excited cube*. Note that a cube $C(x,t)$ may be excited for two reasons (possibly both): either the R_0-neighborhood $U(x)$ of x contains a point $y \in B^{(t)}$, corresponding to the insertion of some operator V_A with $y \in A$, i.e. due to a *quantum excitation*, or it contains a point y for which the classical variable $s^{(t)}_y$ differs from the ground state value $g^{(m)}_y$, which corresponds to a *classical excitation*. Note also that a configuration where two succesive cubes $C(x,t)$ and $C(x,t+1)$ are in a classical state, $\sigma_{(x,t)} = s^{(t)}_x$ and $\sigma_{(x,t+1)} = s^{(t+1)}_x$, has weight zero unless $\sigma_{(x,t)} = \sigma_{(x,t+1)}$; indeed, otherwise one has $\langle s^{(t)}_x |\, K(\Sigma_t)K(\Sigma_{t+1})\, | s^{(t+1)}_x\rangle = 0$. This is true also for $t = M$ once we identify $t = M+1$ with $t = 1$ (periodic boundary conditions on $\mathbb{L}$).

Recalling the relation (3.23), we now extract a factor $e^{-\tilde\beta\Phi_x(g^{(m)})} \equiv e^{-\tilde\beta e_m}$ for each cube C in the ground state m, leading to an overall factor of

$$e^{-\tilde\beta \sum_m e_m |W_m|}, \tag{3.30}$$

where $|W_m|$ is the number of cubes in the ground state m. Considering, on the other hand, the union D of all excited cubes, we assign a label $\alpha_D(F)$ to all elementary faces F in the boundary of D, by defining $\alpha_D(F) = m$ if F is a common face for a cube C_D in D and a cube C_m outside D in the ground state m, $F = C_D \cap C_m$. Defining the reduced weight $\omega(\sigma_{\mathbb{L}})$ by

$$w(\Sigma_1, \ldots, \Sigma_M) = e^{-\tilde{\beta} \sum_m e_m |W_m|} \omega(\sigma_{\mathbb{L}}), \tag{3.31}$$

we observe that $\omega(\sigma_{\mathbb{L}})$ depends only on the configuration σ_D and the label α_D,

$$\omega(\sigma_{\mathbb{L}}) = \omega(\sigma_D, \alpha_D). \tag{3.32}$$

(The configuration outside D is entirely determined by the labels α_D.) The weight $\omega(\sigma_D, \alpha_D)$ inherits from (3.24) the factorization property

$$\omega(\sigma_D, \alpha_D) = \prod_{i=1}^{n} \omega(\sigma_{D_i}, \alpha_{D_i}). \tag{3.33}$$

Here $D_1, \ldots, D_n$ are the connectivity components of D.

At this point, the rest is standard. One considers the sets

$$\mathbb{T}_\Lambda = \bigcup_{(x,t) \in \mathbb{L}_\Lambda} C(x,t) \tag{3.34}$$

and

$$\mathbb{T} = \bigcup_{(x,t) \in \mathbb{L}_\infty} C(x,t), \tag{3.35}$$

imposing periodic bundary conditions in the "time direction", and defines a (labeled) contour Y as a pair $(\operatorname{supp} Y, \alpha)$, where $\operatorname{supp} Y \subset \mathbb{T}_\Lambda$ is a connected union of closed unit cubes with centers in $\mathbb{L}_\Lambda$ (considered as a subset of $\mathbb{T}$), while α is an asignment of a label $\alpha(F)$ to faces of $\partial \operatorname{supp} Y$ which is constant on the boundary of all connected components of $\mathbb{T} \setminus \operatorname{supp} Y$.

The contours $Y_1, \ldots, Y_n$ corresponding to a configuration $\sigma_{\mathbb{L}_\Lambda}$ are then defined by taking the connected components of the set D of excited cubes in $\mathbb{T}_\Lambda$ for their supports $\operatorname{supp} Y_1, \ldots, \operatorname{supp} Y_n$ and by taking the labels m of the ground states for the cubes C in $\mathbb{T}_\Lambda \setminus \operatorname{supp} Y_i$ that touch the face F, see above, for the corresponding labels $\alpha_i(F)$.

Resumming over all configurations in (3.25) that lead to the same set of contours, and taking into account the factorization property (3.33), it is an easy exercise to show that the resulting weight factors into a product of contour activities $\rho(Y_i)$ and ground state terms $e^{-\tilde{\beta} e_m |W_m|}$. This yields the representation

$$Z_{q,\Lambda} = \sum_{\{Y_1, \ldots, Y_n\}} \prod_i \rho(Y_i) \prod_m e^{-\tilde{\beta} e_m |W_m|}, \tag{3.36}$$

which is exactly of the same form as the contour representation of a classical spin system. We can therefore apply standard Pirogov-Sinai theory to analyze the quantum spin system considered here, provided we can verify its basic assumption — the Peierls condition. Namely, we should prove a bound of the form

$$|\rho(Y)| \leq e^{-\gamma|Y|} e^{-\tilde{\beta} e_0 |Y|} \tag{3.37}$$

where γ is a sufficiently large constant, $e_0 = \min_m e_m$, and $|Y|$ is the number of elementary cubes in $\operatorname{supp} Y$. This will be done in the next section.

Remark. The weights $\rho(Y)$ are in general complex, even if the coupling constant λ is real. Observing that the operators $K(\Sigma_t)$ in (3.26) are selfadjoint for real λ, we have, however, that

$$w(\Sigma_1, \ldots, \Sigma_M)^* = w(\Sigma_M, \ldots, \Sigma_1)\,. \tag{3.38}$$

Considering two contours Y and Y^* which can be obtained from each other by a reflection at a constant time plane, we therefore get

$$\rho(Y)^* = \rho(Y^*) \tag{3.39}$$

provided $\lambda \in \mathbb{R}$.

4. Exponential decay for contour activities

We first give an explicit expression for the weight $\rho(Y)$. Combining (3.25), (3.31), (3.32), and (3.33) we have

$$Z_{q,\Lambda} = \sum_{\Sigma_1,\ldots,\Sigma_M} e^{-\tilde{\beta}\sum_m e_m |W_m|} \prod_{i=1}^n \omega(\sigma_{D_i}, \alpha_{D_i}), \tag{4.1}$$

where $D_1, \ldots, D_n$ are the connectivity components of the union D of all excited cubes corresponding to $\Sigma_1, \ldots, \Sigma_M$. This expression is equivalent to (3.36) once we take for a contour $Y \equiv (D, \alpha_D)$ the weight

$$\rho(Y) = \sum_{\sigma_D \to Y} \omega(\sigma_D, \alpha_D). \tag{4.2}$$

Here the sum is over all configurations σ_D consistent with the contour Y, i.e. over all configurations σ_D on $D = \operatorname{supp} Y$ that, if extended outside $\operatorname{supp} Y$ by appropriate ground states determined by the labels α_D, yield the contour Y.

Proposition 4.1. *Let $\lambda \in \mathbb{R}$, $\tilde{\beta} > 0$, and $\gamma_Q \geq 1$ be such that, for all $x \in \mathbb{Z}^d$,*

$$(e-1)\tilde{\beta}|\lambda| \sum_{\substack{A \in \mathcal{A}_0:\\ x \in A}} \|V_A\| e^{\gamma_Q |A|} \leq 1. \tag{4.3}$$

Then

$$|\rho(Y)| \leq e^{-(\tilde{\beta} e_0 + \gamma)|Y|} \tag{4.4}$$

where

$$\gamma = \min\{\tilde{\beta}\gamma_0, R_0^{-d}(\gamma_Q - 1)\} - \log(2|S|). \tag{4.5}$$

The proof of the proposition relies on the following lemma.

Lemma 4.2. *Let $B \subset \Lambda$, and let s_B, $\tilde{s}_B$ and $s_{\overline{\partial B}}$ be arbitrary classical configurations on B and $\overline{\partial B} = \{x \notin B \mid \mathrm{dist}\,(x, B) \le 2R_0\}$, respectively. Let γ_Q, $\tilde{\beta}$ and λ be as in Proposition 4.1. Then*

$$\big|\langle s_B|\, T_B(s_{\overline{\partial B}})\, |\tilde{s}_B\rangle\big| \le e^{-\tilde{\beta} e_0 |\bar{B}|} e^{-(\gamma_Q - 1)|B|}. \tag{4.6}$$

Proof of Lemma 4.2. Let s_Λ be an arbitrary extension of the configuration $s_{B \cup \overline{\partial B}}$ to the full set Λ, and let $\tilde{s}_\Lambda$ be the configuration which agrees with s_Λ on $\Lambda \setminus B$, and with $\tilde{s}_B$ on B. Then

$$\big|\langle s_B|\, T_B(s_{\overline{\partial B}})\, |\tilde{s}_B\rangle\big| = \big|\langle s_\Lambda|\, T_{\bar{B}}(B)\, |\tilde{s}_\Lambda\rangle\big| \le \big\|T_{\bar{B}}(B)\big\|, \tag{4.7}$$

where, in agreement with (3.8) and (3.9), the operator $T_{\bar{B}}(B)$ is defined by

$$T_{\bar{B}}(B) = \sum_{\substack{\mathcal{A} = \{A_1, \dots, A_k\} \\ A_i \in \mathcal{A}_0,\, \cup_j A_j = B}} \tilde{T}_{\bar{B}}(\mathcal{A}) \tag{4.8}$$

and

$$\tilde{T}_{\bar{B}}(\mathcal{A}) = \sum_{\boldsymbol{n} : \mathrm{supp}\, \boldsymbol{n} = \mathcal{A}} \Big[\prod_{A \in \mathcal{A}} \frac{(-\lambda)^{n_A}}{n_A!} \int_0^{\tilde{\beta}} d\tau_A^1 \dots d\tau_A^{n_A}\Big] T_{\bar{B}}(\boldsymbol{\tau}, \boldsymbol{n}). \tag{4.9}$$

The time-ordered operator $T_{\bar{B}}(\boldsymbol{\tau}, \mathbf{n})$ is defined as in (3.6), with the Hamiltonian $\sum_{x \in \bar{B}} H_{q,x}^{(0)}$ replacing $H_{q,\Lambda}^{(0)}$. Observing that for all $s > 0$,

$$\|e^{-s \sum_{x \in \bar{B}} H_{q,x}^{(0)}}\| = e^{-s|\bar{B}| e_0}, \tag{4.10}$$

we now bound

$$\big\|T_{\bar{B}}(\boldsymbol{\tau}, \boldsymbol{n})\big\| \le e^{-\tilde{\beta}|\bar{B}| e_0} \prod_{A \in \mathcal{A}} \|V_A\|^{n_A}. \tag{4.11}$$

Combining (4.7) — (4.9) with the bound (4.11) and the fact that the assumption (4.3) implies that $\tilde{\beta}|\lambda|\|V_A\| \le 1$ for all $A \in \mathcal{A}_0$, we obtain

$$\begin{aligned}
\big|\langle s_B|\, T_B(s_{\overline{\partial B}})\, |\tilde{s}_B\rangle\big| &\le e^{-\tilde{\beta} e_0 |\bar{B}|} \sum_{\substack{\mathcal{A} = \{A_1, \dots, A_k\} \\ A_i \in \mathcal{A}_0,\, \cup_j A_j = B}} \prod_{A \in \mathcal{A}} \Big(\sum_{n_A = 1}^{\infty} \frac{(\tilde{\beta}|\lambda|)^{n_A}}{n_A!} \|V_A\|^{n_A}\Big) \\
&\le e^{-\tilde{\beta} e_0 |\bar{B}|} \sum_{\substack{\mathcal{A} = \{A_1, \dots, A_k\} \\ A_i \in \mathcal{A}_0,\, \cup_j A_j = B}} \prod_{A \in \mathcal{A}} \Big((e-1)\tilde{\beta}|\lambda|\|V_A\|\Big) \\
&\le e^{-\tilde{\beta} e_0 |\bar{B}|} e^{-\gamma_Q |B|} \sum_{\substack{\mathcal{A} = \{A_1, \dots, A_k\} \\ A_i \in \mathcal{A}_0,\, \cup_j A_j = B}} \prod_{A \in \mathcal{A}} \Big((e-1)\tilde{\beta}|\lambda|\|V_A\| e^{\gamma_Q |A|}\Big).
\end{aligned} \tag{4.12}$$

We proceed with the bound

$$\sum_{\substack{\mathcal{A}=\{A_1,\dots,A_k\}\\ A_i\in\mathcal{A}_0,\,\cup_j A_j=B}} \prod_{A\in\mathcal{A}} \Big((e-1)\tilde\beta|\lambda|\|V_A\|e^{\gamma_Q|A|}\Big) \le$$

$$\le \sum_{k=1}^{\infty}\frac{1}{k!}\prod_{i=1}^{k}\Big(\sum_{\substack{A_i\in\mathcal{A}_0:\\ A_i\cap B\neq\emptyset}}(e-1)\tilde\beta|\lambda|\|V_{A_i}\|e^{\gamma_Q|A_i|}\Big)$$

$$\le \sum_{k=1}^{\infty}\frac{1}{k!}\prod_{i=1}^{k}\Big(\sum_{x\in B}\sum_{\substack{A_i\in\mathcal{A}_0:\\ x\in A_i}}(e-1)\tilde\beta|\lambda|\|V_{A_i}\|e^{\gamma_Q|A_i|}\Big)$$

$$\le \sum_{k=1}^{\infty}\frac{1}{k!}|B|^k \le e^{|B|}, \tag{4.13}$$

where we have used the assumption (4.3) in the second to last step. Combining (4.12) and (4.13), we obtain the lemma. □

Proof of Proposition 4.1. In order to bound the sum in (4.2), we first derive a more explicit representation for the activity $\rho(Y)$. We decompose the torus $\mathbb{T}$ and the set of excited cubes D into time-slices, $\mathbb{T}=\cup_{t=1}^{M}\mathbb{T}^{(t)}$ and $D=\cup_{t=1}^{M}D^{(t)}$, and recall that $\sigma_{(x,t)}=0$ iff the cube $C(x,t)$ belongs to $B^{(t)}$, the set of sites where transitions between times t and $(t+1)$ may occur. Summing over all configurations σ_D consistent with the contour Y then corresponds to the following three restrictions:

1. The configuration σ_D is such that the R_0-neighbourhood of each (x,t) with $\sigma_{(x,t)}=0$ is included in D; i.e. $\overline{B}^{(t)}\subset D^{(t)}$ for each $t=1,\dots,M$.
2. All cubes $C(x,t)\subset D$ are excited; i.e. either $\sigma_{(x,t)}=0$ or $s^{(t)}_{U(x)}\neq g^{(m)}_{U(x)}$ for all $m=1,\dots,r$.
3. Let F be a vertical face in the boundary of D, let $m=\alpha_D(F)$ be the label of F, and let $\mathbb{T}^{(t)}$ be the time-slice containing F. Then $\sigma(x,t)=g^{(m)}_x$ for all $x\in D^{(t)}$ whose distance from F is less then R_0.

Let us observe that since $|s_{\Lambda\setminus B_1}\rangle\otimes|s_{B_1}\rangle\equiv|s_{\Lambda\setminus B_2}\rangle\otimes|s_{B_2}\rangle$ for any $B_1,B_2\subset\Lambda$, we may write the expansion of unity $\mathbf{1}=\sum_{s_\Lambda}|s_\Lambda\rangle\langle s_\Lambda|$ on $\mathcal{H}_\Lambda$ in the form

$$\mathbf{1}=\sum_{s_\Lambda}|s_{\Lambda\setminus B_1}\rangle\otimes|s_{B_1}\rangle\langle s_{\Lambda\setminus B_2}|\otimes\langle s_{B_2}|\,. \tag{4.14}$$

Inserting now (3.26) with (3.31) and (3.32) into (4.2), we may use the above observation to get the expression

$$\rho(Y)=\sum_{\substack{B^{(1)},\dots,B^{(M)}\\ \bar B^{(t)}\subset D^{(t)}}}\;\sum_{s^{(1)}_{D^{(1)}\setminus B^{(1)}},\dots,s^{(M)}_{D^{(M)}\setminus B^{(M)}}}\;\sum_{s^{(1)}_{I^{(1)}},\dots,s^{(M)}_{I^{(M)}}}\;\prod_{t=1}^{M}e^{-\tilde\beta\sum_{x\in D^{(t)}\setminus\bar B^{(t)}}\Phi_x(s^{(t)}_{U(x)})}\times$$

$$\times\langle s^{(t)}_{D^{(t)}\setminus B^{(t)}}|\,s^{(t+1)}_{D^{(t)}\setminus B^{(t)}}\rangle\langle s^{(t)}_{B^{(t)}}|\,T_{B^{(t)}}(s^{(t)}_{\overline{\partial B}^{(t)}})\,|s^{(t+1)}_{B^{(t)}}\rangle. \tag{4.15}$$

Here we defined $I^{(t)} = B^{(t-1)} \cap B^{(t)}$. The three succesive summations are equivalent to the sum in (4.2). Namely, the sum over $B^{(1)}, \dots, B^{(M)}$ in (4.15) obeys automatically the first restriction above; the second sum must respect the two others (and there is no restriction on the third sum). Observe that the choice of a contour Y (or of several contours), together with the choice of partial configurations $s^{(t)}_{D^{(t)} \setminus B^{(t)}}$, $s^{(t)}_{I^{(t)}}$, at each time t defines completely the configurations between each time slice.

In order to get a bound on $|\rho(Y)|$, we estimate the absolute values of the factors of the terms on the right hand side of (4.15). Taking into account the condition (2) above and the assumption (2.6) on the classical part of the Hamiltonian, we have

$$e^{-\tilde\beta \sum_{x \in D^{(t)} \setminus \bar B^{(t)}} \Phi_x(s^{(t)}_{U(x)})} \leq e^{-\tilde\beta (e_0 + \gamma_0) |D^{(t)} \setminus \bar B^{(t)}|}. \tag{4.16}$$

The scalar product between the two base vectors is equal to 0 or 1. In fact, we could use it to reduce the number of terms appearing in the second sum in (4.15); however, we just bound it by one. Combined with Lemma 4.2, we finally get

$$|\rho(Y)| \leq \sum_{\substack{B^{(1)}, \dots, B^{(M)} \\ \bar B^{(t)} \subset D^{(t)}}} \prod_{t=1}^{M} \sum_{s^{(t)}_{D^{(t)} \setminus B^{(t)}}} \sum_{s^{(t)}_{I^{(t)}}} e^{-\tilde\beta (e_0 + \gamma_0) |D^{(t)} \setminus \bar B^{(t)}|} e^{-\tilde\beta e_0 |\bar B^{(t)}|} e^{-(\gamma_Q - 1)|B^{(t)}|}. \tag{4.17}$$

The summands do not depend any more on the partial configurations $s^{(t)}_{D^{(t)} \setminus B^{(t)}}$ and $s^{(t)}_{I^{(t)}}$. Their number is bounded by

$$\prod_{t=1}^{M} r^{|D^{(t)} \setminus \bar B^{(t)}| + |I^{(t)}|} \leq |S|^{|Y|}. \tag{4.18}$$

To estimate the exponential in (4.17) we use the equation $|D^{(t)} \setminus \bar B^{(t)}| = |D^{(t)}| - |\bar B^{(t)}|$ and the bound $|\bar B^{(t)}| \leq R_0^d |B^{(t)}|$. Thus $e^{-(\gamma_Q - 1)|B^{(t)}|} \leq e^{-(\gamma_Q - 1) R_0^{-d} |\bar B^{(t)}|}$, and

$$|\rho(Y)| \leq \left[r\, e^{-\tilde\beta e_0} \max\{ e^{-\tilde\beta \gamma_0}, e^{-(\gamma_Q - 1) R_0^{-d}} \} \right]^{|Y|} \prod_{t=1}^{M} \sum_{\substack{B^{(t)} \\ \bar B^{(t)} \subset D^{(t)}}} 1. \tag{4.19}$$

The last sum can be easily bounded, yielding

$$\sum_{\substack{B^{(t)} \\ \bar B^{(t)} \subset D^{(t)}}} 1 \leq \sum_{B^{(t)} \subset D^{(t)}} 1 = 2^{|D^{(t)}|}. \tag{4.20}$$

Combined with (4.19), this completes the proof of Proposition 4.1. $\square$

We close this section with a proposition providing the necessary bounds on derivatives:

Proposition 4.3. *Let $\lambda \in \mathbb{R}$, $\tilde{\beta} > 0$, and $\gamma_Q \geq 1$ be such that, in addition to (4.3) we have*

$$e\tilde{\beta}|\lambda| \sum_{\substack{A \in \mathcal{A}_0: \\ x \in A}} \Big\| \frac{\partial}{\partial \mu_i} V_A \Big\| e^{\gamma_Q |A|} \leq 1. \tag{4.21}$$

Then

$$\Big| \frac{\partial}{\partial \mu_i} \rho(Y) \Big| \leq \big(\tilde{\beta} C_0 + 1\big) |Y| e^{-(\tilde{\beta} e_0 + \gamma)|Y|}. \tag{4.22}$$

Here C_0 is the constant from (2.7) *and γ is the constant defined in* (4.5).

Proof. We first derive an analogue of the bound (4.6) in Lemma 4.2. To this end, we have to bound the norm of $\partial T_{\bar{B}}(\boldsymbol{\tau}, \boldsymbol{n}) / \partial \mu_i$. Using the representation (3.6) in conjunction with the assumption (2.7) and the bound (4.10), we get

$$\begin{aligned} \Big\| \frac{\partial}{\partial \mu_i} T_{\bar{B}}(\boldsymbol{\tau}, \boldsymbol{n}) \Big\| \leq e^{-\tilde{\beta}|\bar{B}| e_0} \Big\{ & \tilde{\beta} C_0 |\bar{B}| \prod_{A \in \mathcal{A}} \|V_A\|^{n_A} + \\ & + \sum_{j=1}^{k} n_{A_j} \Big\| \frac{\partial V_{A_j}}{\partial \mu_i} \Big\| \|V_{A_j}\|^{n_{A_j} - 1} \prod_{A \in \mathcal{A} \setminus \{A_j\}} \|V_A\|^{n_A} \Big\}. \end{aligned} \tag{4.23}$$

Proceeding as before, we obtain

$$\begin{aligned} & \Big| \langle s_B | \frac{\partial}{\partial \mu_i} T_B(s_{\overline{\partial B}}) \, | \tilde{s}_B \rangle \Big| \leq \\ & \leq e^{-\tilde{\beta} e_0 |\bar{B}|} \sum_{\substack{\mathcal{A} = \{A_1, \ldots, A_k\} \\ A_i \in \mathcal{A}_0, \cup_j A_j = B}} \Big\{ \tilde{\beta} C_0 |\bar{B}| \prod_{A \in \mathcal{A}} \big((e-1) \tilde{\beta} |\lambda| \|V_A\| \big) + \\ & \qquad + \sum_{j=1}^{k} e\tilde{\beta}|\lambda| \Big\| \frac{\partial V_{A_j}}{\partial \mu_i} \Big\| \prod_{A \in \mathcal{A}} \big((e-1) \tilde{\beta} |\lambda| \|V_A\| \big) \Big\} \leq \\ & \leq e^{-\tilde{\beta} e_0 |\bar{B}|} e^{-\gamma_Q |B|} \sum_{k=1}^{\infty} \frac{1}{k!} |B|^k \big\{ \tilde{\beta} C_0 |\bar{B}| + k \big\} \\ & \leq e^{-\tilde{\beta} e_0 |\bar{B}|} e^{-\gamma_Q |B|} \big\{ \tilde{\beta} C_0 |\bar{B}| + |B| \big\} e^{|B|} \\ & \leq \big(\tilde{\beta} C_0 + 1 \big) |\bar{B}| e^{-\tilde{\beta} e_0 |\bar{B}|} e^{-(\gamma_Q - 1)|B|}. \end{aligned} \tag{4.24}$$

Inserted into (4.15), and continuing in the same way as before, we obtain the bound (4.21). □

5. Truncated free energies and the stable phases of the model

While the next section will be devoted to a detailed discussion of the mean values of general local variables, here we will anticipate the fact that the state $\langle\cdot\rangle_q$ can be linked with the "probability" – in the ensemble (3.36) of labeled contours – that a given site x is in W_q, the area outside contours and in the ground state $g^{(q)}$. To discuss the stability of phases in dependence on the "tuning" parameter μ (cf. Section 2.1), we can thus use the standard Pirogov-Sinai theory. There are only two features that are not entirely standard — the fact that the weights $\rho(Y)$ are in general *complex numbers* and the fact that our model is actually considered *on a slab* of thickness M (linked with the temperature β) with periodic boundary conditions in the "time" direction. The former was taken into account in [BI89] (also in [GKK88], but here we will base our discussion on [BI89] and later works based on it, in particular [BK90] and [BK94]). The latter is a novel feature of quantum models. Even though it leads only to small modifications, it is important to realize that the metastable free energies used to determine the phase diagram will have contributions comming from contours wrapped around $\mathbb{T}_\Lambda$ in the "time" direction. This leads to certain modifications in the definition of truncatd free energies which will be described in this section.

We start with some notation. As usual (see also Section 3) a *contour* is a pair $(\operatorname{supp} Y, \alpha)$, where $\operatorname{supp} Y \subset \mathbb{T}_\Lambda$ is a connected union of finitely many closed unit cubes with centers in $\mathbb{L}_\infty$ (we call those cubes *elementary cubes* in the sequel), while α is an asignment of a label $\alpha(F)$ to the faces F in $\partial \operatorname{supp} Y$ which is constant on the boundary of all connected components of $\mathbb{T} \setminus \operatorname{supp} Y$. Its interior $\operatorname{Int} Y$ is the union of all finite components of $\mathbb{T} \setminus \operatorname{supp} Y$ and $\operatorname{Int}_m Y$ the union of all components of $\operatorname{Int} Y$ whose boundary is labeled by m. Recalling that we assumed $d \geq 2$, we note that the set $\mathbb{T} \setminus (\operatorname{supp} Y \cup \operatorname{Int} Y)$ is a connected set, implying that the functions $\alpha(\cdot)$ is constant on the boundary of the set $V(Y) = \operatorname{supp} Y \cup \operatorname{Int} Y$. We say that Y is a q-contour, if $\alpha_Y = q$ on this boundary. Two contours Y and Y' are called *compatible* or not touching if $\operatorname{supp} Y \cap \operatorname{supp} Y' = \emptyset$, and *mutually external* if $V(Y) \cap V(Y') = \emptyset$. Given a finite set of mutually compatible contours $Y_1, \dots, Y_n$ we say that Y_i, $i = 1, \dots, n$, is an *external contour* in $\{Y_1, \dots, Y_n\}$, if $\operatorname{supp} Y_i \cap V(Y_j) = \emptyset$ for all $j \neq i$. Consider now a set of contours $\{Y_1, \dots, Y_n\}$ contributing to (3.36). The contours in $\{Y_1, \dots, Y_n\}$ are then mutually compatible, and all external contours are q-contours. In addition, the labels of these contours are *matching* in the sense that the boundary of each connected component of $\mathbb{T}_\Lambda \setminus (\operatorname{supp} Y_1 \cup \dots \cup \operatorname{supp} Y_n)$ has constant boundary conditions.

Given this setup, it is now standard to derive a second representation for $Z_{q,\Lambda}$ which does not involve such a matching condition. To this end, we first introduce partition functions $Z_q(V)$ for all volumes $V \subset \mathbb{T}_\Lambda$ for which $V^c = \mathbb{T}_\Lambda \setminus V$ is a (possibly empty) union of closed elementary cubes in $\mathbb{T}_\Lambda$. We say that Y is a contour in V, if $V(Y) \subset V$, and call a set $\{Y_1, \dots, Y_n\}$ of mutually compatible contours in V a set of *matching contours in V* if the boundary of each connected component of $V \setminus (\operatorname{supp} Y_1 \cup \dots \cup \operatorname{supp} Y_n)$ has constant boundary conditions. Denoting the union of those components which have boundary condition q by W_q, we define

$$Z_q(V) = \sum_{\{Y_1,\dots,Y_n\}} \prod_i \rho(Y_i) \prod_m e^{-\tilde\beta e_m |W_m|}, \tag{5.1}$$

where the sum runs over all sets of mutually compatible, matching contours in V for which all external contours are q-contours. Recalling the expression (3.36) for $Z_{q,\Lambda}$, we note that the partition function (5.1) is actually equal to $Z_{q,\Lambda}$ if $V = \mathbb{T}_\Lambda$,

$$Z_q(\mathbb{T}_\Lambda) = Z_{q,\Lambda}. \tag{5.2}$$

We now rewrite the partition function $Z_q(V)$, in a standard way, as a sum over contours without any matching condition, see e.g. [Zah84] or [BI89]. Introducing the weights

$$K_q(Y) := \rho(Y) e^{\tilde\beta e_q |Y|} \prod_{m=1}^{r} \frac{Z_m(\mathrm{Int}_m Y)}{Z_q(\mathrm{Int}_m Y)}, \tag{5.3}$$

one gets

$$Z_q(V) = e^{-\tilde\beta e_q |V|} \sum_{\{Y_1,\dots,Y_n\} \subset V^0} \prod_{k=1}^{n} K_q(Y_k). \tag{5.4}$$

where the sum runs over all sets of mutually disjoint q-contours in V.

The weights $K_q(Y)$ do not, necessarily, satisfy the bound

$$|K_q(Y)| \le \epsilon^{|Y|} \tag{5.5}$$

with a sufficiently small constant $\epsilon > 0$. While it turns out that such a bound can be proven for stable phases q, it is false for unstable phases. In order to circumvent this problem, we follow the standard strategy and construct truncated contour activities $K'_q(Y)$, truncated partition functions

$$Z'_q(V) = e^{-\tilde\beta e_q |V|} \sum_{\{Y_1,\dots,Y_n\} \subset V^0} \prod_{k=1}^{n} K'_q(Y_k), \tag{5.6}$$

and the corresponding free energies f_q in such a way that the weights $K'_q(Y)$ satisfy the bound (5.5) and, in the same time do not differ from $K_q(Y)$ whenever the phase q is stable, i.e. whenever the real part of the free energy f_q of the truncated model is minimal, $\mathrm{Re}\, f_q = \min_m \mathrm{Re}\, f_m$.

Our definition of the truncated weights K'_q follows closely the treatment from [BK94]. The main difference is that since the contours can have only a limited extension in the time direction, we take their "horizontal diameter" as the parameter to use in inductive definitions (and proofs). Namely, for a contour Y, we take the diameter $\delta(Y)$ defined as the diameter of the d-dimensional projection

$$\{y \in \mathbb{R}^d \text{ such that } (y,t) \in C(x,t), \text{for some } C(x,t) \subset \operatorname{supp} Y\} \tag{5.7}$$

We define $\delta(V)$ in the same way. Notice that $\delta(\tilde{Y}) < \delta(Y)$ whenever $\operatorname{supp}\tilde{Y} \subset \operatorname{Int}(Y)$.

To introduce the weight $K'_q(Y)$ in an inductive manner, assume that it has already been defined for all q and all contours Y with $\delta(Y) < n$, $n \in \mathbb{N}$, and that it obeys a bound of the form (5.5). Introduce $f_q^{(n-1)}$ as the free energy of a contour model with activities

$$K^{(n-1)}(Y^q) = \begin{cases} K'(Y^q) & \text{if } \delta(Y^q) \le n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Consider now a contour Y with $\delta(Y) = n$. Since $\delta(\tilde{Y}) < n$ for all contours $\tilde{Y}$ in $\operatorname{Int} Y$, the truncated partition functions $Z'_q(\operatorname{Int}_m Y)$ are well defined for all q and m. Their logarithm can be controlled by a convergent cluster expansion, and $Z'_q(\operatorname{Int}_m Y) \neq 0$ for all q and m. We therefore may define $K'_q(Y)$ for q-contours Y with $\delta(Y) = n$ by

$$K'_q(Y) = \chi'_q(Y)\rho(Y)e^{\tilde{\beta}e_q|Y|}\prod_m \frac{Z_m(\operatorname{Int}_m Y)}{Z'_q(\operatorname{Int}_m Y)}\ , \tag{5.9}$$

with

$$\chi'_q(Y) = \prod_{m\neq q} \chi\Big(\alpha - \tilde{\beta}(\operatorname{Re} f_q^{(n-1)} - \operatorname{Re} f_m^{(n-1)})\delta(Y)\Big)\ . \tag{5.10}$$

Here α is a constant that will be chosen later and χ is a smoothed characteristic function. We assume that χ is a C_1 function that has been defined in such a way that it obeys the conditions

$$0 \le \chi(x) \le 1, \quad 0 \le \frac{d\chi}{dx} \le 1, \tag{5.11}$$

and

$$\chi(x) = 0 \quad \textit{if} \quad x \le -1 \quad \textit{and} \quad \chi(x) = 1 \quad \textit{if} \quad x \ge 1\ . \tag{5.12}$$

As a final element of the construction of K'_q, we have to establish the bound (5.5) for contours Y with $\delta(Y) = n$. The proof of this fact, together with the proof of the following Lemma 5.1, follows closely [BK94]. However, since, on the one hand, the claims of Lemma 5.1 (as well as the definition of $K'_q(Y)$) slightly differ from similar claims in [BK94] and, on the other hand, there is certain number of complications in [BK94] that are not necessary for the present case, we present the proof in the appendix.

We use f_q to denote the free energy corresponding to the partition function $Z'_q(V)$,

$$f_q = -\frac{1}{\tilde{\beta}} \lim_{V\to\mathbb{T}} \frac{1}{|V|} \log Z'_q(V)\ , \tag{5.13}$$

and introduce f_0 and a_q as

$$f_0 = \min_m \operatorname{Re} f_m\ , \tag{5.14}$$

$$a_q = \tilde{\beta}(\operatorname{Re} f_q - f_0)\ . \tag{5.15}$$

Finally, we recall that volumes V as well as supports of contours, $\operatorname{supp} Y$, are unions of elementary cubes and $|Y|$ or $|\operatorname{supp} Y|$ denotes their $(d+1)$-dimensional volume. Similarly for the boundary ∂V of V we use $|\partial V|$ to denote its d-dimensional euclidean area.

Lemma 5.1. *Assume that $\rho(\cdot)$ obeys the conditions (4.4) and (4.22) and let*

$$\epsilon = e^{-\gamma+\alpha+2} \qquad \text{and} \qquad \bar{\alpha} = \alpha - 2. \tag{5.16}$$

Then there exists a constant $\epsilon_0 > 0$ (depending only on d and r) such that the following statements hold provided $\epsilon < \epsilon_0$ and $\overline{\alpha} \geq 1$.

i) The contour activities $K_q'(Y)$ are well defined for all Y and obey (5.5) and

$$\Big|\frac{\partial}{\partial \mu_i} K_q'(Y)\Big| \leq (3r\tilde{\beta} C_0 + 2)|V(Y)|\epsilon^{|Y|}. \tag{5.17}$$

ii) If $a_q\delta(Y) \leq \overline{\alpha}$, then $\chi_q(Y) = 1$ and $K_q(Y) = K_q'(Y)$.
iii) If $a_q\delta(V) \leq \overline{\alpha}$, then $Z_q(V) = Z_q'(V)$.
iv) For all volumes $V \subset \mathbb{T}_\Lambda$ for which $V^c = \mathbb{T}_\Lambda \setminus V$ is a union of closed elementary cubes, one has

$$|Z_q(V)| \leq e^{-\tilde{\beta} f_0 |V| + O(\epsilon)|\partial V|} \tag{5.18}$$

and

$$\Big|\frac{\partial}{\partial \mu_i} Z_q(V)\Big| \leq (2\tilde{\beta} C_0 + 1)|V| e^{-\tilde{\beta} f_0 |V| + O(\epsilon)|\partial V|}. \tag{5.19}$$

Remarks.

i) Here, as in the appendix, $O(\epsilon)$ stands for a bound $K\epsilon$, where $K < \infty$ is a constant that depends only on the dimension d and the number of classical states r.

ii) For real λ, the free energy f independent of boundary conditions and can be expressed as

$$f = -\frac{1}{\beta} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \operatorname{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H_{q,\Lambda}} = -\frac{1}{\beta} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_{q,\Lambda}. \tag{5.20}$$

Since $H_{q,\Lambda}$ is a selfadjoint operator in this case, f is real. Rewriting the partition function in terms of the $(d+1)$-dimensional contour model introduced in Section 3, see equation (5.2), we have

$$f = -\frac{1}{\tilde{\beta} M} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_q(\mathbb{T}_\Lambda) \equiv -\frac{1}{\tilde{\beta}} \lim_{V \nearrow \mathbb{T}} \frac{1}{|V|} \log Z_q(V). \tag{5.21}$$

iii) By contrast, the meta-stable free energies f_q defined in (5.13) do in general depend on q, even if λ is real. It is worth noting, however, that the meta-stable free energies are real-valued in this case. To see this, we first observe that by the symmetry property (3.39), $Z_q(V)$ is real for all V, implying that $K_q(Y)^* = K_q(Y^*)$. Using the definitions (5.8) through (5.10), one then establishes by induction that $Z_q'(V)$ is real while $K_q'(Y)^* = K_q'(Y^*)$. Given this symmetry for $K_q'(Y)$, the reality of f_q now easily follows from the cluser expansion of $\log Z_q'(V)$.

6. Expectation values of local variables

To distinguish between different phases we have to evaluate expectation values of local observables. Whenever we have a local bounded observable Ψ, represented by an operator acting on $\mathcal{H}_{\operatorname{supp}\Psi}$ with a finite $\operatorname{supp}\Psi \subset \mathbb{Z}^d$, we have

$$\langle\Psi\rangle_{q,\Lambda} = \frac{\operatorname{Tr}_{\mathcal{H}_\Lambda}(\Psi e^{-\beta H_{q,\Lambda}})}{\operatorname{Tr}_{\mathcal{H}_\Lambda}(e^{-\beta H_{q,\Lambda}})} = \frac{\operatorname{Tr}_{\mathcal{H}_\Lambda}(\Psi T^M)}{\operatorname{Tr}_{\mathcal{H}_\Lambda}(T^M)} = \frac{Z^\Psi_{q,\Lambda}}{Z_{q,\Lambda}}. \tag{6.1}$$

Retracing the steps leading to the contour representation (3.36) of $Z_{q,\Lambda}$, we can get in a straightforward manner a similar expression for $Z^\Psi_{q,\Lambda}$. Namely, introducing

$$w_\Psi(\Sigma_1,\dots,\Sigma_M) = \operatorname{Tr}_{\mathcal{H}_\Lambda}\big(\Psi \prod_{t=1}^M K(\Sigma_t)\big) \tag{6.2}$$

with $K(\Sigma)$ given as before by (3.23), we have

$$Z^\Psi_{q,\Lambda} = \sum_{\Sigma_1,\dots,\Sigma_M} w_\Psi(\Sigma_1,\dots,\Sigma_M). \tag{6.3}$$

Localizing the observable Ψ, *by definition*, in the first time slice, we define the $d+1$ dimensional support of Ψ as

$$\mathcal{S}(\Psi) := \bigcup_{x\in\operatorname{supp}\Psi} C(x,1)\,. \tag{6.4}$$

Assuming without loss of generality that $\operatorname{supp}\Psi$ and hence $\mathcal{S}(\Psi)$ is a connected set, we introduce the contours corresponding to a configuration $\sigma_{\mathbb{L}_\Lambda}$ in the same way as before, with the only difference that the supports of these contours are now the connected components of $D\cup\mathcal{S}(\Psi)$, where, as in Section 3, D is the set of excited cubes in $\mathbb{T}_\Lambda$. By this definition, one of the contours corresponding to $\sigma_{\mathbb{L}_\Lambda}$ will contain the set $\mathcal{S}(\Psi)$ as part of its support. We denote this contour by Y_Ψ. Continuing as before, we obtain the representation

$$Z^\Psi_{q,\Lambda} = \sum_{\substack{\{Y_\Psi,Y_1,\dots,Y_k\}\\ \operatorname{supp}Y_\Psi\supset\mathcal{S}(\Psi)}} \rho_\Psi(Y_\Psi)\prod_i \rho(Y_i)\prod_m e^{-\tilde\beta e_m|W_m|}, \tag{6.5}$$

where

$$\rho_\Psi(Y_\Psi) = \sum_{\sigma_D\to Y} \omega_\Psi(\sigma_D,\alpha_D)\,, \tag{6.6}$$

with the reduced weight ω_Ψ given by a formula of the type (3.31).

More generally, we introduce, for a volume V which contains the $d+1$ dimensional support $\mathcal{S}(\Psi)$ of Ψ, the diluted partition function

$$Z^\Psi_q(V) = \sum_{\substack{\{Y_\Psi,Y_1,\dots,Y_k\}\\ Y\supset\operatorname{supp}\Psi}} \rho_\Psi(Y_\Psi)\prod_i \rho(Y_i)\prod_m e^{-\tilde\beta e_m|W_m|}, \tag{6.7}$$

where the sum runs over all sets of mutually compatible, matching contours in V for which all external contours are q-contours. For $V = \mathbb{T}_\Lambda$, $Z_q^\Psi(V) = Z_{q,\Lambda}^\Psi$, which implies that

$$\langle\Psi\rangle_{q,\Lambda} = \frac{Z_q^\Psi(\mathbb{T}_\Lambda)}{Z_q(\mathbb{T}_\Lambda)}. \tag{6.8}$$

For every term contributing to (6.7) we consider the collection $\mathcal{Y}_\Psi$ consisting of the contour Y_Ψ as well as all contours Y among $\{Y_1, \cdots, Y_k\}$ encircling it ($\operatorname{Int} Y \supset \mathcal{S}(\Psi)$) and define

$$\rho(\mathcal{Y}_\Psi) = \rho_\Psi(Y_\Psi) \prod_{\substack{Y \in \mathcal{Y}_\Psi \\ Y \neq Y_\Psi}} \rho(Y). \tag{6.9}$$

Denoting further $\operatorname{supp} \mathcal{Y}_\Psi = \bigcup_{Y \in \mathcal{Y}_\Psi} \operatorname{supp} Y$, $\operatorname{Int} \mathcal{Y}_\Psi$ the union of all finite components of $\mathbb{T} \setminus \operatorname{supp} \mathcal{Y}_\Psi$, $\operatorname{Int}_m \mathcal{Y}_\Psi$ the union of all components of $\operatorname{Int} \mathcal{Y}_\Psi$ that are labeled by m, and $\operatorname{Ext} \mathcal{Y}_\Psi = \bigcap_{Y \in \mathcal{Y}_\Psi} \operatorname{Ext} Y$, we introduce, in addition to the weights $K_q(Y)$ defined in the preceeding section for an arbitrary q-contour Y, also the weight

$$K_{q,\Psi}(\mathcal{Y}_\Psi) := \rho_\Psi(\mathcal{Y}_\Psi) e^{\tilde{\beta} e_q |\operatorname{supp} \mathcal{Y}_\Psi|} \prod_{m=1}^{r} \frac{Z_m(\operatorname{Int}_m \mathcal{Y}_\Psi)}{Z_q(\operatorname{Int}_m \mathcal{Y}_\Psi)} \tag{6.10}$$

attributed to the collection $\mathcal{Y}_\Psi$. As a result we get the representation

$$Z_q^\Psi(V) = e^{-\tilde{\beta} e_q |V|} \sum_{\mathcal{Y}_\Psi, Y_1, \ldots, Y_n} K_{q,\Psi}(\mathcal{Y}_\Psi) \prod_{\ell=1}^{n} K_q(Y_\ell). \tag{6.11}$$

Here the sum goes over set of all collections $\mathcal{Y}_\Psi$ and all sets $\{Y_1, \ldots, Y_n\}$ of non-overlapping q contours in V, such that for all contours Y_i, $i = 1, \ldots, n$, the set $V(Y_i)$ does not intersect the set $\operatorname{supp} \mathcal{Y}_\Psi$.

Assuming for the moment that the weights $K_{q,\Psi}(\mathcal{Y}_\Psi)$ and $K_q(Y)$ decay sufficiently fast with the size of $\mathcal{Y}_\Psi$ and Y, respectively, we now use the standard Mayer cluster expansion for polymer systems to get

$$\frac{Z_q^\Psi(V)}{Z_q(V)} = \sum_{\mathcal{Y}_\Psi} K_{q,\Psi}(\mathcal{Y}_\Psi) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{Y_1, \ldots, Y_n\}} \Big[\prod_{k=1}^{n} K_q(Y_k)\Big] \phi_c(\mathcal{Y}_\Psi, Y_1, \cdots, Y_n). \tag{6.12}$$

Here $\phi_c(\mathcal{Y}_\Psi, Y_1, \cdots, Y_n)$ is a combinatoric factor defined in terms of the connectivity properties of the graph $G(\mathcal{Y}_0, Y_1, \cdots, Y_n)$, introduced as the graph on the vertex set $\{0, 1, \cdots, n\}$ which has an edge between two vertices $i \geq 1$ and $j \geq 1$, $i \neq j$, whenever $\operatorname{supp} Y_i \cap \operatorname{supp} Y_j \neq \emptyset$, and an edge between the vertex 0 and a vertex $i \neq 0$ whenever $V(Y_i) \cap \operatorname{supp} \mathcal{Y}_\Psi \neq \emptyset$. (see for example [Sei82] or [Dob94] for a new simple and very lucid proof). The combinatoric factor $\phi_c(\mathcal{Y}_\Psi, Y_1, \cdots, Y_n)$ is zero if $G(Y_\Psi, Y_1, \cdots, Y_n)$ has more than one component.

To prove the convergence of (6.12) one has to show that the weights $K_{q,\Psi}$ and K_q decay sufficiently fast with the size of $\mathcal{Y}_\Psi$ and Y, respectively. To this end it is useful to introduce truncated models. For a contour Y with $V(Y) \cap \mathcal{S}(\Psi) = \emptyset$, we define $K'_q(Y)$ as before, see (5.9), while for the collection $\mathcal{Y}_\Psi$ we define

$$K'_{q,\Psi}(\mathcal{Y}_\Psi) = \rho_\Psi(\mathcal{Y}_\Psi) e^{\tilde{\beta} e_q |\text{supp}\, \mathcal{Y}_\Psi|} \prod_{m=1}^{r} \frac{Z_m(\text{Int}\,_m \mathcal{Y}_\Psi)}{Z'_q(\text{Int}\,_m \mathcal{Y}_\Psi)} \prod_{Y \in \mathcal{Y}_\Psi} \chi'_q(Y) \,, \tag{6.13}$$

with $\chi'_q(Y)$ as in (5.10). Given this definition, we introduce

$$Z'^{\,\Psi}_q(V) = e^{-\tilde{\beta} e_q |V|} \sum_{\mathcal{Y}_\Psi} K'_{q,\Psi}(\mathcal{Y}_\Psi) \sum_{\{Y_1,\dots,Y_n\}} \prod_{k=1}^{n} K'_q(Y_k) \tag{6.14}$$

and

$$\langle \Psi \rangle'_{q,V} = \frac{Z'^{\,\Psi}_q(V)}{Z'_q(V)} \,, \tag{6.15}$$

which can again be expanded as

$$\langle \Psi \rangle'_{q,V} = \sum_{\mathcal{Y}_\Psi} K'_{q,\Psi}(\mathcal{Y}_\Psi) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{Y_1,\dots,Y_n\}} \left[\prod_{k=1}^{n} K'_q(Y_k) \right] \phi_c(\mathcal{Y}_\Psi, Y_1, \cdots, Y_n) \,. \tag{6.16}$$

The following Lemma gives the absolute convergence of the expansion (6.16), which in turn will yield Theorem 2.1 ii), iii) and iv).

Lemma 6.1. *Let ϵ, ϵ_0 and $\overline{\alpha}$ be as in Lemma 5.1, and assume that $\epsilon < \epsilon_0$ and $\overline{\alpha} \geq 1$. Then:*

i) For every collection $\mathcal{Y}_\Psi$ one has

$$\left| K'_{q,\Psi}(\mathcal{Y}_\Psi) \right| \leq \|\Psi\| e^{(\alpha+2)|\text{supp}\, \Psi|} \epsilon^{|\text{supp}\, \mathcal{Y}_\Psi \setminus \mathcal{S}(\Psi)|} \,. \tag{6.17}$$

ii) If

$$a_q \max_{Y \in \mathcal{Y}_\Psi} \delta(Y) \leq \overline{\alpha} \,, \tag{6.18}$$

then $K'_{q,\Psi}(\mathcal{Y}_\Psi) = K_{q,\Psi}(\mathcal{Y}_\Psi)$.

iii) The cluster expansion (6.16) is absolutely convergent, and

$$\begin{aligned} \left| \langle \Psi \rangle'_{q,V} \right| &\leq \sum_{\mathcal{Y}_\Psi} \left| K'_{q,\Psi}(\mathcal{Y}_\Psi) \right| \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{Y_1,\dots,Y_n\}} \left[\prod_{k=1}^{n} \left| K'_q(Y_k) \right| \right] \left| \phi_c(\mathcal{Y}_\Psi, Y_1, \cdots, Y_n) \right| \\ &\leq \|\Psi\| e^{(\alpha+2+O(\epsilon))|\text{supp}\, \Psi|} \,. \end{aligned} \tag{6.19}$$

iv) If $a_q \delta(\mathbb{T}_\Lambda) \leq \bar{\alpha}$, then $\langle \Psi \rangle_{q,\Lambda} = \langle \Psi \rangle'_{q,\Lambda}$.

v) For an arbitrary volume $V \subset \mathbb{T}$,

$$|Z^\Psi_q(V)| \leq \|\Psi\| e^{(\gamma+1)|\text{supp}\, \Psi|} e^{-\tilde{\beta} f_0 |V| + O(\epsilon)|\partial V|} \,. \tag{6.20}$$

vi) There is a constant $K = K(d) > 0$ such that for $V = \mathbb{T}_\Lambda$,

$$|Z_q^\Psi(V)| \le \|\Psi\| e^{(\gamma+1)|\operatorname{supp}\Psi|} e^{-\tilde\beta f_0|V| + O(\epsilon)|\partial V|} \max\left(e^{-\frac{a_q}{4}|V|}, e^{-K\gamma|\partial V|}\right). \quad (6.21)$$

Proof. Observing that $\delta(W) \le \max_{Y\in\mathcal{Y}_\Psi} \delta(Y)$ for all components W of $\operatorname{Int}\mathcal{Y}_\Psi$, the statement ii) of Lemma 6.1 immediately follows from Lemma 5.1.

In order to prove i), we first note that $\rho_\Psi(Y_\Psi) \le \|\Psi\|\tilde\rho(Y_\Psi)$, where $\tilde\rho(Y_\Psi)$ satisfies an analog of the bound (4.4) (however, Y_Ψ is not necessarily excited on $\mathcal{S}(\Psi)$), namely

$$|\tilde\rho(Y_\Psi)| \le e^{-\tilde\beta e_0|\operatorname{supp} Y_\Psi|} e^{-\gamma|\operatorname{supp} Y_\Psi\setminus\mathcal{S}(\Psi)|}(1+e^{-\gamma})^{|\mathcal{S}(\Psi)|}. \quad (6.22)$$

Hence

$$\rho_\Psi(\mathcal{Y}_\Psi) e^{\tilde\beta e_q|\operatorname{supp}\mathcal{Y}_\Psi|} \le \|\Psi\| e^{-\tilde\beta(e_0-e_q)|\operatorname{supp}\mathcal{Y}_\Psi|} e^{-\gamma|\operatorname{supp}\mathcal{Y}_\Psi\setminus\mathcal{S}(\Psi)|}(1+e^{-\gamma})^{|\mathcal{S}(\Psi)|}. \quad (6.23)$$

Using Lemma 5.1 we get

$$\prod_{m=1}^{r} \left|\frac{Z_m(\operatorname{Int}\mathcal{Y}_\Psi)}{Z_q'(\operatorname{Int}\mathcal{Y}_\Psi)}\right| \le e^{a_q|\operatorname{Int}\mathcal{Y}_\Psi| + O(\epsilon)|\partial\operatorname{Int}\mathcal{Y}_\Psi|}. \quad (6.24)$$

Combined with the bound $\tilde\beta(e_q - e_0) \le a_q + O(\epsilon)$ we get

$$\left|K'_{q,\Psi}(\mathcal{Y}_\Psi)\right| \le \|\Psi\| e^{a_q|\operatorname{supp}\mathcal{Y}_\Psi\cup\operatorname{Int}\mathcal{Y}_\Psi| + O(\epsilon)|\partial\operatorname{Int}\mathcal{Y}_\Psi|} e^{-\gamma|\operatorname{supp}\mathcal{Y}_\Psi\setminus\mathcal{S}(\Psi)|} e^{O(\epsilon)|\operatorname{supp}\mathcal{Y}_\Psi|}. \quad (6.25)$$

Observing that $\prod_{Y\in\mathcal{Y}_\Psi}\chi_q(Y) \ne 0$ implies that $a_q|V(Y)| \le (\alpha+1+O(\epsilon))|Y|$ for each $Y\in\mathcal{Y}_\Psi$, and noting that $|\mathcal{S}(\Psi)| = |\operatorname{supp}\Psi|$, we finally get

$$\left|K'_{q,\Psi}(\mathcal{Y}_\Psi)\right| \le \|\Psi\| e^{-(\gamma-\alpha-1-O(\epsilon))|\operatorname{supp}\mathcal{Y}_\Psi\setminus\mathcal{S}(\Psi)|} e^{(\alpha+1+O(\epsilon))|\operatorname{supp}\Psi|}. \quad (6.26)$$

Statement iii) follows from i) and Lemma 5.1, i), while statement iv) follows from ii) and Lemma 5.1, ii) and iii).

To get the statement v) and vi), we first write $Z_q^\Psi(V)$ as

$$Z_q^\Psi(V) = \sum_{\mathcal{Y}_\Psi} \rho_\Psi(\mathcal{Y}_\Psi) Z_q(\operatorname{Ext}\mathcal{Y}_\Psi) \prod_m Z_m(\operatorname{Int}\mathcal{Y}_\Psi). \quad (6.27)$$

Using now (6.23) and the bound (5.18) in its strengthened form (A.43), we get

$$\left|Z_q^\Psi(V)\right| \le \|\Psi\| e^{\gamma|\mathcal{S}(\Psi)|} e^{-\tilde\beta f_0|V|} e^{O(\epsilon)|\partial V|} \sum_{\mathcal{Y}_\Psi} e^{-\gamma|\operatorname{supp}\mathcal{Y}_\Psi|} e^{O(\epsilon)|\operatorname{supp}\mathcal{Y}_\Psi|} \times$$
$$\times \max_{U\subset\operatorname{Ext}\mathcal{Y}_\Psi} \left(e^{-\frac{a_q}{4}|\operatorname{Ext}\mathcal{Y}_\Psi\setminus U|} e^{-K(d)\gamma|\partial U|}\right). \quad (6.28)$$

Extracting a factor

$$\max_{\mathcal{Y}_\Psi} e^{-\frac{\gamma}{2}|\operatorname{supp}\mathcal{Y}_\Psi|} \max_{U\subset \operatorname{Ext}\mathcal{Y}_\Psi} \big(e^{-\frac{a_q}{4}|\operatorname{Ext}\mathcal{Y}_\Psi\setminus U|} e^{-K(d)\gamma|\partial U|}\big) \le$$
$$\le \max_{\mathcal{Y}_\Psi} e^{-K(d)\gamma|\partial V(\mathcal{Y}_\Psi)|} \max_{U\subset V\setminus V(\mathcal{Y}_\Psi)} \big(e^{-\frac{a_q}{4}|(V\setminus V(\mathcal{Y}_\Psi))\setminus U|} e^{-K(d)\gamma|\partial U|}\big) \le$$
$$\le \max_{\tilde U\subset V}\big(e^{-\frac{a_q}{4}|V\setminus\tilde U|} e^{-K(d)\gamma|\partial\tilde U|}\big), \quad (6.29)$$

where $V(\mathcal{Y}_\Psi) = V\setminus \operatorname{Ext}\mathcal{Y}_\Psi$, and bounding

$$\sum_{\mathcal{Y}_\Psi} e^{-\frac{\gamma}{2}|\operatorname{supp}\mathcal{Y}_\Psi|} \le (1+O(e^{-\frac{\gamma}{2}}))^{|\operatorname{supp}\Psi|}, \quad (6.30)$$

we get

$$\big|Z_q^\Psi(V)\big| \le \|\Psi\| e^{(\gamma+1)|\operatorname{supp}\Psi|} e^{-\tilde\beta f_0|V|} e^{O(\epsilon)|\partial V|} \max_{\tilde U\subset V}\big(e^{-\frac{a_q}{4}|V\setminus\tilde U|} e^{-K(d)\gamma|\partial\tilde U|}\big). \quad (6.31)$$

Bounding the last factor by one, we get statement v); continuing as in the proof of Lemma A.3 in the appendix, we get statement vi). □

7. PROOF OF THEOREMS 2.1 AND 2.2

Proof of Theorem 2.1. In order to prove the theorem, we will use Lemmas 5.1 and Lemma 6.1, with $\alpha = \gamma/2$. We therefore need that $\gamma > 4 - 2\log\epsilon_0$, which in turn requires $\tilde\beta$ and γ_Q are sufficiently large, see (4.5). Choosing appropriate β_0 and γ_Q, we take $\tilde\beta \in (\frac12\beta_0, \beta_0]$ and choose λ_0 according to (2.13). Then (4.3) and (4.21) are satisfied once $|\lambda| \le \lambda_0$. For every $\beta \ge \beta_0$ we now choose M such that $\tilde\beta = \frac{\beta}{M} \in (\frac12\beta_0, \beta_0]$. Whenever $a_q = 0$ we can then use the claims ii) from Lemma 5.1 and ii) from Lemma 6.1 to conclude that $K_q(Y) = K'_q(Y)$ for all Y and $K'_{q,\Psi}(\mathcal{Y}_\Psi) = K_{q,\Psi}(\mathcal{Y}_\Psi)$ for all $\mathcal{Y}_\Psi$. As a result the bounds (5.5), (5.17), and (6.17) are valid for $K_q(Y)$ and $K_{q,\Psi}(\mathcal{Y}_\Psi)$. This allows us to use cluster expansions for evaluation of $Z_q(V)$ as well as $Z_q^\Psi(V)$.

Hence (2.25) follows directly from (5.2), (5.13) and Lemma 5.1 iii). Similarly, we get (2.26) (with an explicit formula for $\langle\Psi\rangle_q$) from (6.12). Finally, given the absolute convergence of the cluster expansion for the expectation values of local observables and the exponential decay of the contour activities, the bound (2.27) is standard.

To evaluate the expectation value of the projection operator (2.28), we apply the expansion (6.10) for the particular observable $\Psi = P^{(m)}_{U(x)} \equiv |g^{(m)}_{U(x)}\rangle\langle g^{(m)}_{U(x)}|$. Using the factor $\epsilon^{|\operatorname{supp}\mathcal{Y}_\Psi\setminus\mathcal{S}(\Psi)|}$ in (6.17), one can show that the sum of all terms with $\operatorname{supp}\mathcal{Y}_\Psi \ne \mathcal{S}(\Psi)$ is of the order const ϵ, a term that can be made small by taking ϵ small. We are thus left with contributions coming from the term $K_{q,\Psi}(\mathcal{Y}_\Psi)$ with

$$\operatorname{supp}\mathcal{Y}_\Psi = \mathcal{S}(\Psi) \equiv \bigcup_{x\in U(X)} C(x,1). \quad (7.1)$$

This means that necessarily $\mathcal{Y}_\Psi = \{Y_\Psi\}$ with $\operatorname{supp} Y_\Psi = \mathcal{S}(\Psi)$ and $\alpha(\partial\mathcal{S}(\Psi)) = q$. The only configurations σ yielding this Y_Ψ are those for which $\sigma = \{\sigma_{y,t}\}$ agrees with $g^{(q)}$, except possibly for the point $(y,t) = (x,1)$. For the activity

$$K_{q,\Psi}(\mathcal{Y}_\Psi) = \rho_\Psi(\{Y_\Psi\}) e^{\tilde\beta e_q |U(x)|}\,, \tag{7.2}$$

this gives a contribution $O(\epsilon)$ if $\sigma_{x,1} \neq g_x^{(q)}$, and a contribution

$$e^{\tilde\beta e_q |\Lambda| M} \operatorname{Tr}_{\mathcal{H}_\Lambda} \Big(\Psi\, |g_\Lambda^{(q)}\rangle\langle g_\Lambda^{(q)}| \prod_{t=1}^{M} e^{-\tilde\beta \sum_{y\in\Lambda} \Phi_y(g^{(q)}_{U(y)})} \Big) =$$

$$= \operatorname{Tr}_{\mathcal{H}_\Lambda} \big(\Psi\, |g_\Lambda^{(q)}\rangle\langle g_\Lambda^{(q)}| \big) = \delta_{m,q}. \tag{7.3}$$

if $\sigma_{x,1} = g_x^{(q)}$. Putting everything together, we obtain the bound

$$\langle P^{(m)}_{U(x)} \rangle_q = \delta_{m,q} + O(\epsilon)\,, \tag{7.4}$$

which proves iv).

Finally, v) is a standard claim in the Pirogov-Sinai theory. Namely, given the bounds (5.5), (5.17), and the fact that f_q can be analysed by a convergent cluster expansion, we get $|f_q - e_q| \leq O(\epsilon)$ and a similar bound for the derivatives of f_q. Statement v) then follows from the corresponding assumptions (2.4) and (2.5) on the functions $e_q(\mu)$. $\square$

Proof of Theorem 2.2.

The partition functions

$$Z_{\mathrm{per},\Lambda(L)} = \operatorname{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H_{\mathrm{per},\Lambda(L)}} \tag{7.5}$$

and

$$Z^\Psi_{\mathrm{per},\Lambda(L)} = \operatorname{Tr}_{\mathcal{H}_\Lambda} \Psi e^{-\beta H_{\mathrm{per},\Lambda(L)}} \tag{7.6}$$

have representations $Z(\mathbb{T}_{\Lambda_{\mathrm{per}}(L)})$ and $Z^\Psi(\mathbb{T}_{\Lambda_{\mathrm{per}}(L)})$ similar to (5.1) and (6.5). The proof of Theorem 2.2 follows from this representation in a standard manner [BI 89, BK90]. One only has to notice that contours contributing to these partition functions may be wrapped around the torus $\mathbb{T}_{\Lambda_{\mathrm{per}}(L)}$ in time as well as space directions. Nevertheless, whenever a contour Y satisfies the condition $\delta(Y) \leq \frac{L}{3}$, one can define $\operatorname{Ext} Y$ as the largest component of $\mathbb{T}_{\Lambda_{\mathrm{per}}} \setminus \operatorname{supp} Y$. For every configuration containing only such contours, all external contours have clearly the same external label. Splitting now $Z(\mathbb{T}_{\Lambda_{\mathrm{per}}(L)})$ (resp. $Z^\Psi(\mathbb{T}_{\Lambda_{\mathrm{per}}(L)})$) into contributions containing at least one contour such that $\delta(Y) > \frac{L}{3}$ and those where all contributing contours are such that $\delta(Y) \leq \frac{L}{3}$, we get

$$Z(\mathbb{T}_{\Lambda_{\mathrm{per}}}) = Z^{\mathrm{big}}(\mathbb{T}_{\Lambda_{\mathrm{per}}}) + \sum_{m=1}^{r} Z_m(\mathbb{T}_{\Lambda_{\mathrm{per}}}). \tag{7.7}$$

Here $Z_m(\mathbb{T}_{\Lambda_{\text{per}}})$ is given as a sum over all configurations containing only those contours for which $\delta(Y) \le \frac{L}{3}$ and such that the common external label of external contours is m. Taking into account the fact that term $Z^{\text{big}}(\mathbb{T}_{\Lambda_{\text{per}}})$ is exponentially suppressed in L (one can use verbatim the proof from [BI89]), we get

$$\Big| Z(\mathbb{T}_{\Lambda_{\text{per}}}) - \sum_{m=1}^{r} Z_m(\mathbb{T}_{\Lambda_{\text{per}}}) \Big| \le e^{-\tilde\beta f_0 M(2L+1)^d} e^{-\text{const}\,\gamma L}, \tag{7.8}$$

and similarly

$$\Big| Z^{\Psi}(\mathbb{T}_{\Lambda_{\text{per}}}) - \sum_{m=1}^{r} Z^{\Psi}_m(\mathbb{T}_{\Lambda_{\text{per}}}) \Big| \le \|\Psi\| e^{(\gamma+1)|\operatorname{supp}\Psi|} e^{-\tilde\beta f_0 M(2L+1)^d} e^{-\text{const}\,\gamma L}. \tag{7.9}$$

Moreover, whenever m is stable, $m \in Q(\mu)$, we have $Z_m(\mathbb{T}_{\Lambda_{\text{per}}}) = Z'_m(\mathbb{T}_{\Lambda_{\text{per}}})$ and we can use the cluster expansion of $\log Z'_m(\mathbb{T}_{\Lambda_{\text{per}}})$ and f_m to show that

$$\big| Z_m(\mathbb{T}_{\Lambda_{\text{per}}}) - e^{-\tilde\beta f_m M(2L+1)^d} \big| \le e^{-\tilde\beta f_0 M(2L+1)^d} M(2L+1)^d e^{-\text{const}\,\gamma L}. \tag{7.10}$$

Namely, we just observe that the first terms in which these two cluster expansions differ are of the order $e^{-\text{const}\,\gamma L}$ (clusters wrapped around $\mathbb{T}_{\Lambda_{\text{per}}}$ in spatial directions).

For $m \notin Q(\mu)$, on the other hand, we proceed as in the proof of Lemma A.3 to get

$$\begin{aligned} \Big| Z_m(\mathbb{T}_{\Lambda_{\text{per}}}) \Big| \le \exp(M(2L+1)^d e^{-\text{const}\,\gamma L})\, e^{-\tilde\beta f_0 M(2L+1)^d} \times \\ \times \max\Big\{ e^{-\frac{a_q}{4} M(2L+1)^d},\, e^{-K(d)\gamma M(2L+1)^{d-1}} \Big\}. \end{aligned} \tag{7.11}$$

For $m \notin Q(\mu)$ and L sufficiently large, we therefore get

$$\Big| Z_m(\mathbb{T}_{\Lambda_{\text{per}}}) \Big| \le 2 e^{-\tilde\beta f_0 M(2L+1)^d} e^{-K(d)\gamma M(2L+1)^{d-1}}. \tag{7.12}$$

In a similar way, proceeding now as in the proof of Lemma 6.1, we get

$$\Big| Z^{\Psi}_m(\mathbb{T}_{\Lambda_{\text{per}}}) \Big| \le 2\|\Psi\| e^{(\gamma+1)|\operatorname{supp}\Psi|} e^{-\tilde\beta f_0 M(2L+1)^d} e^{-K(d)\gamma M(2L+1)^{d-1}}, \tag{7.13}$$

provided $m \notin Q(\mu)$ and L is sufficiently large.

Combining the bounds (7.8), (7.10) and (7.12), and using the fact that $f_m = \operatorname{Re} f_m$ if the couling constant λ is real, which in turn implies $f_m = f_0$ if $m \in Q(\mu)$, we get

$$\Big| Z(\mathbb{T}_{\Lambda_{\text{per}}}) - |Q(\mu)| e^{-\tilde\beta f_0 M(2L+1)^d} \Big| \le e^{-\tilde\beta f_0 M(2L+1)^d} O(e^{-\text{const}\gamma L}) \tag{7.14}$$

provided L is sufficiently large.

Introducing now the truncated expectation values

$$\langle\Psi\rangle'_{q,\mathbb{T}_{\Lambda_{\rm per}}} = \frac{{Z'_q}^{\Psi}(\mathbb{T}_{\Lambda_{\rm per}})}{Z'_q(\mathbb{T}_{\Lambda_{\rm per}})} \tag{7.15}$$

(cf. (5.6) and (6.14)), we get

$$\begin{aligned}\Big|Z^{\Psi}(\mathbb{T}_{\Lambda_{\rm per}}) - \sum_{m\in Q(\mu)} e^{-\tilde\beta \operatorname{Re} f_m M(2L+1)^d}\langle\Psi\rangle'_{m,\mathbb{T}_{\Lambda_{\rm per}}}\Big| \le \\ \le \sum_{m\in Q(\mu)}\Big|\langle\Psi\rangle'_{m,\mathbb{T}_{\Lambda_{\rm per}}}\big(Z'_m(\mathbb{T}_{\Lambda_{\rm per}}) - e^{-\tilde\beta \operatorname{Re} f_m M(2L+1)^d}\big)\Big| + \\ + \sum_{m\notin Q(\mu)}\Big|Z^{\Psi}_m(\mathbb{T}_{\Lambda_{\rm per}})\Big| + \|\Psi\| e^{(\gamma+1)|\operatorname{supp}\Psi|} e^{-\tilde\beta f_0 M(2L+1)^d} e^{-\mathrm{const}\,\gamma L}.\end{aligned} \tag{7.16}$$

Next, we observe that $\langle\Psi\rangle'_{m,\mathbb{T}_{\Lambda_{\rm per}}}$ can be bounded in the same way as $\langle\Psi\rangle'_{m,\mathbb{T}_{\Lambda}}$, namely

$$\big|\langle\Psi\rangle'_{m,\mathbb{T}_{\Lambda_{\rm per}}}\big| \le \|\Psi\| e^{(\alpha+2+O(\epsilon))|\operatorname{supp}\Psi|} \le \|\Psi\| e^{(\gamma+1)|\operatorname{supp}\Psi|}\,. \tag{7.17}$$

Inserting the bounds (7.10), (7.17) and (7.13) into (7.16), and dividing both sides of the resulting bound by $|Q(\mu)|e^{-\tilde\beta f_0 M(2L+1)^d}$, we get

$$\Big|\langle\Psi\rangle_{\,{\rm per},\Lambda(L)} - \sum_{m\in Q(\mu)}\langle\Psi\rangle_{m,\mathbb{T}_{\Lambda_{\rm per}}}\Big| \le e^{-\mathrm{const}\,\gamma L}\|\Psi\| e^{(\gamma+1)|\operatorname{supp}\Psi|}\,, \tag{7.18}$$

provided L is sufficiently large. In the limit $L\to\infty$, this yields the claim of Theorem 2.2. $\square$

Appendix. Pirogov-Sinai theory in thin slabs — proof of Lemma 5.1

In this appendix, we prove Lemma 5.1. Actually, it is a direct consequence of the Lemma A.1 below. In order to state the lemma, we recall the definition of $f_q^{(n)}$ as the free energy of an auxilliary contour model with activities

$$K^{(n)}(Y) = \begin{cases} K'(Y) & \text{if } \delta(Y) \le n, \\ 0 & \text{otherwise,} \end{cases} \tag{A.1}$$

and define

$$f_0^{(n)} = \min_q \operatorname{Re} f_q^{(n)}, \tag{A.2}$$

$$a_q^{(n)} = \tilde\beta(\operatorname{Re} f_q^{(n)} - f_0^{(n)}). \tag{A.3}$$

Observing that $f_0 = \lim_{n\to\infty} f_0^{(n)}$ and $a_q = \lim_{n\to\infty} a_q^{(n)}$, Lemma 5.1 follows directly from the following.

Lemma A.1. *Assume that $\rho(\cdot)$ obeys the conditions* (4.4) *and* (4.22) *and let*

$$\epsilon = e^{-\gamma+\alpha+2} \qquad \text{and} \qquad \bar\alpha = \alpha - 2. \tag{A.4}$$

Then there is a constant ϵ_0, depending only on d and r, such that the following statements are true once (γ is such that) $\epsilon < \epsilon_0$ and $\alpha \ge 3$. For all $n \ge 0$ and Y and V such that $\delta(Y) \le n$, $\delta(V) \le n$, one has:

i) $|K_q'(Y)| \le \epsilon^{|Y|}$.

ii) $\left|\frac{\partial}{\partial\mu_i} K_q'(Y)\right| \le (3r\tilde\beta C_0 + 2)|V(Y)|\epsilon^{|Y|}$.

iii) *If* $a_q^{(n)}\delta(Y) \le \bar\alpha$ *then* $\chi_q'(Y) = 1$.

iv) *If* $a_q^{(n)}\delta(Y) \le \bar\alpha$ *then* $K_q'(Y) = K_q(Y)$.

v) *If* $a_q^{(n)}\delta(V) \le \bar\alpha$ *then* $Z_q'(V) = Z_q(V)$.

vi) $|Z_q(V)| \le e^{-\tilde\beta f_0^{(n)}|V|} e^{O(\epsilon)|\partial V|}$.

vii) $\left|\frac{\partial}{\partial\mu_i} Z_q(V)\right| \le (2\tilde\beta C_0 + 1)|V| e^{-\tilde\beta f_0^{(n)}|V|} e^{O(\epsilon)|\partial V|}$.

Proof. We proceed by induction on n.

I. The case $n = 0$.

There are no contours with $\delta(Y) = 0$. This makes i) – iv) trivial statements and implies that $f_q^{(0)} = e_q$. On the other hand, $\delta(V) = 0$ implies $|V| = 0$ and $Z_q(V) = Z_q'(V) = 1$, which makes v) – vii) trivial statements.

II. Induction step $n - 1 \to n$.

Proof of i) for $\delta(Y) = n$. Clearly, $\delta(\operatorname{Int} Y) < n$, and all contours $\tilde Y$ contributing to $Z_q'(\operatorname{Int}_m Y)$ obey the condition $\delta(\tilde Y) < n$. This implies that $|K_q'(\tilde Y)| \le \epsilon^{|\tilde Y|}$ by the inductive assumption i). As a consequence, the logarithm of $Z'_q(\mathrm{Int}_m Y)$ can be analyzed by a convergent expansion, and

$$\left|\log Z'_q(\mathrm{Int}_m Y) + \tilde{\beta} f_q^{(n-1)} |\mathrm{Int}_m Y|\right| \le O(\epsilon)|\partial \mathrm{Int}_m Y|. \tag{A.5}$$

Combining (A.5) with the induction assumption vi), we get

$$\begin{aligned}\prod_m \left|\frac{Z_m(\mathrm{Int}_m Y)}{Z'_q(\mathrm{Int}_m Y)}\right| &\le e^{a_q^{(n-1)}|\mathrm{Int}\, Y|} e^{O(\epsilon)\sum_m |\partial \mathrm{Int}_m Y|} \\ &\le e^{a_q^{(n-1)}|\mathrm{Int}\, Y|} e^{O(\epsilon)|Y|}.\end{aligned} \tag{A.6}$$

Observing that

$$|\tilde{\beta} e_m - \tilde{\beta} f_m^{(n-1)}| \le O(\epsilon), \tag{A.7}$$

which implies the bound

$$|\tilde{\beta}(e_q - e_0) - a_q^{(n-1)}| \le O(\epsilon), \tag{A.8}$$

we use the bound (4.4) to evaluate

$$|\rho(Y) e^{\tilde{\beta} e_q |Y|}| \le e^{-\gamma|Y|} e^{\tilde{\beta}(e_q - e_0)|Y|} \le e^{-(\gamma - O(\epsilon))|Y|} e^{a_q^{(n-1)}|Y|}. \tag{A.9}$$

Applying now the bounds (A.9) and (A.6) to the definition (5.9) and using the equation $|V(Y)| = |\mathrm{Int}\, Y| + |Y|$, we obtain

$$|K'_q(Y)| \le \chi'_q(Y) e^{a_q^{(n-1)}|V(Y)|} e^{-(\gamma - O(\epsilon))|Y|}. \tag{A.10}$$

Without loss of generality, we may assume that $\chi'_q(Y) > 0$ (otherwise $K'_q(Y) = 0$ and the statement i) is trivial). Let us notice that

$$|V(Y)| \le \delta(Y)|Y|. \tag{A.11}$$

Indeed, considering a disjoint union T of "rows" consisting of elementary cubes $C(x,t)$ with fixed cordinates $x_2, \dots, x_d$ and t, one notices that the set $V(Y)$ intersects at most $\delta(Y)$ of elementary cubes in each such row and there is at most $|Y|$ such rows that have a nonempty intersection with $V(Y)$. By the definition of $\chi'_q(Y)$ and (A.11), we get

$$\tilde{\beta}|f_q^{(n-1)} - f_m^{(n-1)}|\ |V(Y)| \le (1+\alpha)|Y|$$

for all $m \ne q$. As a consequence,

$$a_q^{(n-1)}|V(Y)| \le (1+\alpha)|Y|, \tag{A.12}$$

provided $\chi'_q(Y) \ne 0$. Combined with (A.10) and the fact that $\chi'_q(Y) \le 1$, this implies that

$$|K'_q(Y)| \le e^{-[\gamma - 1 - \alpha - O(\epsilon)]|Y|}, \tag{A.13}$$

which yields the desired bound i) for $\delta(Y) = n$.

Proof of ii) for $\delta(Y) = n$. Using (4.22), (2.7), (4.4), and (A.8), we get

$$\Big|\frac{\partial}{\partial \mu_i}\big(\rho(Y) e^{\tilde\beta e_q |Y|}\big)\Big| \le (2\tilde\beta C_0 + 1)|Y| e^{-(\tilde\beta e_0 + \gamma)|Y|} e^{\tilde\beta e_q |Y|} \le$$
$$\le (2\tilde\beta C_0 + 1)|Y| e^{-(\gamma - O(\epsilon))|Y|} e^{a_q^{(n-1)}|Y|}. \tag{A.14}$$

Using inductive assumption i) and ii) for contours contributing to $Z'_q(\operatorname{Int}_m Y)$, we can apply the cluster expansion to get the bounds

$$\Big|\frac{\partial}{\partial \mu_i}\log Z'_q(\operatorname{Int}_m Y)\Big| \le \big[\tilde\beta C_0 + (1+\tilde\beta C_0) O(\epsilon)\big] |\operatorname{Int}_m Y| \tag{A.15}$$

and

$$\tilde\beta\Big|\frac{\partial}{\partial \mu_i} f_m^{(n-1)}\Big| \le \big[\tilde\beta C_0 + (1+\tilde\beta C_0) O(\epsilon)\big]\,. \tag{A.16}$$

Using further the inductive assumption vi) and vii), as well as the bounds (A.5) and (A.6), we get

$$\Big|\frac{\partial}{\partial \mu_i}\prod_m \frac{Z_m(\operatorname{Int}_m Y)}{Z'_q(\operatorname{Int}_m Y)}\Big| \le \big[(3\tilde\beta C_0 + 1) + (1+\tilde\beta C_0) O(\epsilon)\big] |\operatorname{Int} Y| e^{a_q^{(n-1)}|\operatorname{Int} Y|} e^{O(\epsilon)|Y|}$$
$$\le \big[(2r\tilde\beta C_0 + 1) + (1+\tilde\beta C_0) O(\epsilon)\big] |\operatorname{Int} Y| e^{a_q^{(n-1)}|\operatorname{Int} Y|} e^{O(\epsilon)|Y|}. \tag{A.17}$$

With the help of (A.16) and (5.11) we get

$$\Big|\frac{\partial}{\partial \mu_i}\chi'_q(Y)\Big| \le 2(r-1)[\tilde\beta C_0 + (1+\tilde\beta C_0) O(\epsilon)]\delta(Y)$$
$$\le 2(r-1)[\tilde\beta C_0 + (1+\tilde\beta C_0) O(\epsilon)]|Y|. \tag{A.18}$$

Combining now (A.14), (A.17), and (A.18) with (A.6), (A.9) and the observation that $|V(Y)| = |Y| + |\operatorname{Int} Y|$, we get

$$\Big|\frac{\partial}{\partial \mu_i} K'_q(Y)\Big| \le \big\{2r\tilde\beta C_0 + 1 + (1+\tilde\beta C_0) O(\epsilon)\big\} |V(Y)| e^{a_q^{(n-1)}|V(Y)|} e^{-(\gamma - O(\epsilon))|Y|}\,. \tag{A.19}$$

Using now again the fact that $K'_q(Y)$ (and its derivatives) vanishes unless (A.12) is fulfiled, we get the desired bound, provided ϵ is sufficiently small.

Proof of iii) for $k = \delta(Y) \le n$ *and* $a_q^{(n)}\delta(Y) \le \bar\alpha$. We just have proved that i) is true for all contours Y with $\delta(Y) \le n$. As a consequence, both $f_m^{(k-1)}$ and $f_m^{(n)}$ may be analyzed by a convergent cluster expansion. Using the definition of $f_m^{(n)}$ and the obvious fact that $|Y| \ge \delta(Y)$ (again, $d \ge 2$), one concludes that all contours Y

contributing to the cluster expansion of the difference $f_m^{(k-1)} - f_m^{(n)}$ obey the bound $|Y| \geq k$. As a consequence,

$$\tilde{\beta}|f_m^{(k-1)} - f_m^{(n)}| \leq (K\epsilon)^k \tag{A.20}$$

and

$$\tilde{\beta}|f_m^{(k-1)} - f_m^{(n)}|\delta(Y) \leq (K\epsilon)^k\delta(Y) = k(K\epsilon)^k \leq O(\epsilon). \tag{A.21}$$

where K is a constant depending only on the dimension d and the number of phases r. Combining (A.21) with the assumption $a_q^{(n)}\delta(Y) \leq \bar{\alpha}$, we obtain the lower bound

$$\alpha - \tilde{\beta}[f_q^{(k-1)} - f_m^{(k-1)}]\delta(Y) \geq \alpha - a_q^{(n)\delta(Y)} - O(\epsilon) \geq \alpha - \bar{\alpha} - O(\epsilon) = 2 - O(\epsilon),$$

with $\bar{\alpha}$ defined by (A.4). Combining this with (5.12) we get the equality $\chi_q'(Y) = 1$.

Proof of iv) and v). The statement follows from the just proven fact that $\chi_q'(Y) = 1$ for all contours Y with $\delta(Y)a_q^{(n)} \leq \bar{\alpha}$, the definiton (5.9) of $K_q'(Y)$ and the relations (5.6) and (5.4). We proceed by a second induction on the diameters of Y and V. For $\delta(Y) = 0$ or $\delta(V) = 0$ the statement is trivial. Assume now that $K_q'(Y) = K_q(Y)$ for all Y with $\delta(Y) \leq k < n$ and $\delta(Y)a_q^{(n)} \leq \bar{\alpha}$. Comparing (5.4) and (5.6), we conclude that $Z_q'(V) = Z_q(V)$ for all V with $\delta(V) \leq k < n$ and $\delta(V)a_q^{(n)} \leq \bar{\alpha}$. Let Y be a contour with $\delta(Y) = k+1$ and $\delta(Y)a_q^{(n)} \leq \bar{\alpha}$. Then $Z_q'(\mathrm{Int}_m Y) = Z_q(\mathrm{Int}_m Y)$ since $\delta(\mathrm{Int}_m Y) \leq \delta(Y) - 1 = k$. Combined with the fact that $\chi_q'(Y) = 1$ by iii), we conclude that $K_q'(Y) = K_q(Y)$. This completes the induction on k.

Proof of vi) for $\delta(V) = n$. We say that a contour Y is *small* if $a_q^{(n)}\delta(Y) \leq \bar{\alpha}$ while it is *large* if $a_q^{(n)}\delta(Y) > \bar{\alpha}$. We then rewrite the partition function $Z_q(V)$ given in the form (5.1) by splitting the set of external contours into small and large contours and, for a fixed collection of large external contours $\{X_1, \cdots X_k\}_{\mathrm{ext}}$, we resum over (mutually external) small q-contours in Ext $= V \setminus \bigcup_{i=1}^{k} V(X_i)$. As a result we get

$$Z_q(V) = \sum_{\{X_1,\cdots,X_k\}_{\mathrm{ext}}} Z_q^{\mathrm{small}}(\mathrm{Ext}) \prod_{i=1}^{k} \Big[\rho(X_i) \prod_m Z_m(\mathrm{Int}_m X_i)\Big] \tag{A.22}$$

with the sum going over sets of mutually external large contours in V. The partition function $Z_q^{\mathrm{small}}(\mathrm{Ext})$ is obtained from $Z_q(\mathrm{Ext})$ by dropping all large external q-contours.

Due to the inductive assumption iv), $K_q(Y) = K_q'(Y)$ if Y is small. Since $|K_q'(Y)| \leq \epsilon^{|Y|}$ by i), $Z_q^{\mathrm{small}}(\mathrm{Ext})$ can be evaluated by a convergent cluster expansion, and

$$\big|Z_q^{\mathrm{small}}(\mathrm{Ext})\big| \leq e^{-\tilde{\beta}\mathrm{Re}\, f_q^{\mathrm{small}}|\mathrm{Ext}|} e^{O(\epsilon)|\partial\mathrm{Ext}|}. \tag{A.23}$$

Here f_q^{small} is the free energy of the contour model with activities

$$K_q^{\mathrm{small}}(Y) = \begin{cases} K'_q(Y) & \text{if } \delta(Y) \le n \text{ and } Y \text{ is small,} \\ 0 & \text{otherwise.} \end{cases} \tag{A.24}$$

On the other hand,

$$\prod_m |Z_m(\mathrm{Int}_m X_i)| \le e^{-\tilde\beta f_0^{(n-1)}|\mathrm{Int}\, X_i|} e^{O(\epsilon)|\partial \mathrm{Int}\, X_i|}$$

by the induction assumption vi). Observing that the smallest contours contributing to the difference of $f_m^{(n)}$ and $f_m^{(n-1)}$ obey the bound $|Y| \ge n$, while

$$|V(X_i)| \le \delta(X_i)\,|X_i| \le n|X_i|\,,$$

we may continue as in the proof of (A.21) to bound

$$\tilde\beta|f_0^{(n-1)} - f_0^{(n)}||\mathrm{Int}\, X_i| \le \tilde\beta|f_0^{(n-1)} - f_0^{(n)}||V(X_i)| \le n(K\epsilon)^n|X_i| \le O(\epsilon)|X_i|\,.$$

Thus

$$\prod_m |Z_m(\mathrm{Int}\, X_i)| \le e^{-\tilde\beta f_0^{(n)}|\mathrm{Int}\, X_i|} e^{O(\epsilon)|X_i|}\,. \tag{A.25}$$

Combining (A.23) and (A.25) with the bounds

$$|\rho(X_i)| \le e^{-\gamma|X_i| - \tilde\beta e_0|X_i|} \le e^{-(\gamma - O(\epsilon))|X_i|} e^{-\tilde\beta f_0^{(n)}|X_i|} \tag{A.26}$$

and

$$|\partial \mathrm{Ext}\,| \le |\partial V| + \sum_{i=1}^k |\partial V(X_i)| \le |\partial V| + 2(d+1)\sum_{i=1}^k |X_i|, \tag{A.27}$$

we conclude that

$$|Z_q(V)| \le e^{O(\epsilon)|\partial V|} e^{-\tilde\beta f_0^{(n)}|V|} \sum_{\{X_1,\cdots,X_k\}_{\mathrm{ext}}} e^{-\tilde\beta[\mathrm{Re}\, f_q^{\mathrm{small}} - \mathrm{Re}\, f_q^{(n)}]|\mathrm{Ext}\,|} \prod_{i=1}^k e^{-(\gamma - O(\epsilon))|X_i|}. \tag{A.28}$$

Next, we bound the difference $f_q^{\mathrm{small}} - f_q^{(n)}$. First, for all large contours X, we have

$$|X| \ge \delta(X) \ge \ell_0 := \frac{\bar\alpha}{a_q^{(n)}}. \tag{A.29}$$

Next, we observe that

$$\tilde\beta|f_q^{(n)} - f_q^{\mathrm{small}}| \le (K\epsilon)^{\ell_0} \le \frac{1}{-\ell_0 \log(K\epsilon)}, \tag{A.30}$$

where K is a constant depending only on d and N. Recalling the condition $\bar{\alpha} \geq 1$, we get

$$\tilde{\beta}|f_q^{(n)} - f_q^{\text{small}}| \leq \frac{a_q^{(n)}}{-\log(K\epsilon)} \leq \frac{1}{2} a_q^{(n)}, \tag{A.31}$$

provided ϵ is chosen small enough. Combining (A.28) with (A.31), we finally obtain

$$|Z_q(V)| \leq e^{O(\epsilon)|\partial V|} e^{-\tilde{\beta} f_0^{(n)}|V|} \sum_{\{X_1,\cdots,X_k\}_{\text{ext}}} e^{-\frac{a_q^{(n)}}{2}|\text{Ext}|} \prod_{i=1}^{k} e^{-\tilde{\gamma}|X_i|} \tag{A.32}$$

with

$$\tilde{\gamma} = \gamma - 1. \tag{A.33}$$

At this point we need the following Lemma A.2, which is a variant of a lemma first proven in [Zah84] (see also [BI89] and [BK94] for the proof of this lemma exactly in the following formulation).

Lemma A.2. *Consider an arbitrary contour activity $\tilde{K}_q(Y) \geq 0$, and let $\tilde{Z}_q$ be the partition function*

$$\tilde{Z}_q(V) = \sum_{\{Y_1,\cdots,Y_n\}} \prod_{i=1}^{n} (\tilde{K}_q(Y_i) e^{|Y_i|}). \tag{A.34}$$

Let $\tilde{s}_q$ be the corresponding free energy, and assume that $\tilde{K}_q(Y) \leq \tilde{\epsilon}^{|Y|}$, where $\tilde{\epsilon}$ is small (depending on r and d). Then for any $\tilde{a} \geq -\tilde{s}_q$ the following bound is true

$$\sum_{\{Y_1,\cdots,Y_k\}_{\text{ext}}} e^{\tilde{a}|\text{Ext}|} \prod_i \tilde{K}_q(Y_i) \leq e^{O(\tilde{\epsilon})|\partial V|}, \tag{A.35}$$

with the sum running over all sets of mutually external q-contours in V.

In order to apply the lemma, we define $\tilde{K}_q(Y) = e^{-\tilde{\tau}|Y|}$ if Y is a large q-contour, and $\tilde{K}_q(Y) = 0$ otherwise. With this choice,

$$0 \leq -\tilde{s}_q \leq (K\epsilon)^{\ell_0} \leq \frac{1}{-\ell_0 \log(K\epsilon)}. \tag{A.36}$$

As a consequence,

$$-\tilde{s}_q \leq \tilde{a} := \frac{a_q^{(n)}}{2} \tag{A.37}$$

provided ϵ is small enough. Applying Lemma A.2 to the right hand side of (A.32), and observing that $\tilde{\epsilon} := e^{-\tilde{\tau}} \leq \epsilon$, we finally obtain the desired inequality

$$|Z_q(V)| \leq e^{O(\epsilon)|\partial V|} e^{-\tilde{\beta} f_0^{(n)}|V|}. \tag{A.38}$$

Proof of vii) for $\delta(V) = n$.

Beginning from the formula (A.22) above, we first notice that

$$\Big|\frac{\partial}{\partial\mu_i}\log Z_q^{\rm small}(\mathrm{Ext}\,Y)\Big| \le (\tilde\beta C_0 + (1+\tilde\beta C_0)O(\epsilon))|\mathrm{Ext}\,Y| \tag{A.39}$$

(cf. (A.15)) since we can use ii) for small Y. Using further the bound (A.23) we can conclude that

$$\Big|\frac{\partial}{\partial\mu_i}Z_q^{\rm small}(\mathrm{Ext}\,)\Big| \le (\tilde\beta C_0 + (1+\tilde\beta C_0)O(\epsilon))|\mathrm{Ext}\,Y| e^{-\tilde\beta \mathrm{Re}\, f_q^{\rm small}|\mathrm{Ext}\,|} e^{O(\epsilon)|\partial\mathrm{Ext}\,|}. \tag{A.40}$$

Combining this with the bounds (4.22) as well as (A.23) for $Z_q^{\rm small}(\mathrm{Ext}\,)$ and the inductive assumptions (vii) and (vi) for $Z_m(\mathrm{Int}\,_m X_i)$, we get

$$\begin{aligned}\Big|\frac{\partial}{\partial\mu_i}Z_q(V)\Big| &\le e^{O(\epsilon)|\partial V|}e^{-\tilde\beta f_0^{(n)}|V|}\\ &\sum_{\{X_1,\cdots,X_k\}_{\rm ext}} \big|(\tilde\beta C_0 + (1+\tilde\beta C_0)O(\epsilon))|\mathrm{Ext}\,Y| + (2\tilde\beta C_0+1)|\mathrm{Int}\,Y| + (\tilde\beta C_0+1)|Y|\big|\\ &\qquad e^{-\tilde\beta[\mathrm{Re}\, f_q^{\rm small} - \mathrm{Re}\, f_q^{(n)}]|\mathrm{Ext}\,|}\prod_{i=1}^k e^{-(\gamma - O(\epsilon))|X_i|}.\end{aligned} \tag{A.41}$$

Proceding now as in the proof of vi) from (A.28), we finally get vii).

This concludes the inductive proof of Lemma A.1. □

For $V = \mathbb{T}_\Lambda$, the bound from Lemma A.1 vi) can be actually strengthened:

Lemma A.3. *Under the assumptions of Lemma A.1 we have*

$$|Z_q(\mathbb{T}_\Lambda)| \le e^{-\tilde\beta f_0|\mathbb{T}_\Lambda|}e^{O(\epsilon)|\partial\mathbb{T}_\Lambda|}\max\big\{e^{-\frac{a_q}{4}|\mathbb{T}_\Lambda|},\, e^{-K(d)\gamma|\partial\mathbb{T}_\Lambda|}\big\} \tag{A.42}$$

with a constant $K(d)$ that depends only on the dimension d.

Proof. Extracting from (A.32) the factor $\max_{\{X_1,\cdots,X_k\}_{\rm ext}} e^{-\frac{a_q^{(n)}}{4}|\mathrm{Ext}\,|}\prod_{i=1}^k e^{-\frac12\tilde\gamma|X_i|}$, we still get the same bound. This factor can be, in the limit $n\to\infty$, clearly bounded by $\max_{U\subset V}\big(e^{-\frac{a_q}{4}|V\setminus U|}e^{-K(d)\gamma|\partial U|}\big)$. As a consequence,

$$|Z_q(V)| \le e^{-\tilde\beta f_0|V|}e^{O(\epsilon)|\partial V|}\max_{U\subset V}\big(e^{-\frac{a_q}{4}|V\setminus U|}e^{-K(d)\gamma|\partial U|}\big), \tag{A.43}$$

which is still true for arbitrary $V\subset\mathbb{T}$. We now decompose U and V into timeslices, $U = \cup_t U_t$ and $V = \sum_t V_t$, and observe that $|V\setminus U| = \sum_t\big(|V_t| - |U_t|\big)$, while $|\partial U| \ge$

$\sum_t |\partial\pi(U_t)|$, where $\pi(\cdot)$ denotes the projection onto $\mathbb{R}^d$. Using the isoperimetric inequality on $\mathbb{R}^d$, we now bound

$$\begin{aligned}\max_{U\subset V}\Big(e^{-\frac{aq}{4}|V\setminus U|}e^{-2dK(d)\gamma|U|^{\frac{d-1}{d}}}\Big) &\leq \prod_{t=1}^{M}\max_{U_t\subset V_t}\Big(e^{-\frac{aq}{4}(|V_t|-|U_t|)}e^{-2dK(d)\gamma|U_t|^{\frac{d-1}{d}}}\Big)\\ &\leq \prod_{t=1}^{M}\max\Big\{e^{-\frac{aq}{4}|V_t|}, e^{-2dK(d)\gamma|V_t|^{\frac{d-1}{d}}}\Big\}\,.\end{aligned}\tag{A.44}$$

Restricting ourselves to $V=\mathbb{T}_\Lambda$, we observe that $|V_t|=|V|/M$ is independent of t in this case. As consequence,

$$\begin{aligned}&\max_{U\subset V}\Big(e^{-\frac{aq}{4}|V\setminus U|}e^{-2dK(d)\gamma|U|^{\frac{d-1}{d}}}\Big) \leq \Big(\max\Big\{e^{-\frac{aq}{4}|V|/M}, e^{-2dK(d)\gamma(|V|/M)^{\frac{d-1}{d}}}\Big\}\Big)^M\\ &=\max\Big\{e^{-\frac{aq}{4}|V|}, e^{-K(d)\gamma|\partial V|}\Big\}\,,\end{aligned}\tag{A.45}$$

where we used that $|\partial V| = 2dM(|V|/M)^{\frac{d-1}{d}}$ in the last step. $\square$

References

[Bor88] C. Borgs, *Confinement, Deconfinement and Freezing in Lattice Yang-Mills theories with Continuous Time*, Commun. Math. Phys. **116** (1988), 309–342.

[BI89] C. Borgs, J. Imbrie, *A Unified Approach to Phase Diagrams in Field Theory and Statistical Mechanics*, Commun. Math. Phys. **123** (1989), 305–328.

[BK90] C. Borgs, R. Kotecký, *A Rigorous Theory of Finite Size Scaling at First Order Phase Transitions*, Journ. Stat. Phys. **61** (1990), 79–119.

[BK94] C. Borgs, R. Kotecký, *Surface induced finite size effects for first order phase transitions*, Journ. Stat. Phys. (1995), in print.

[BK95] C. Borgs, R. Kotecký, *Staggered charge-order in the narrow band extended Hubbard model* (1995), in preparation.

[DFF95] N. Datta, R. Fernandez, J. Fröhlich, *private communication* (1995).

[Dob65] Dobrushin, R. L., *Existence of a phase transition in the two-dimensional and three-dimensional Ising models*, Soviet Physics Doklady **10** (1965), 111-113.

[Dob94] R. L. Dobrushin, *Estimates of Semiinvariants for the Ising Model at Low Temperatures* (1994), preprint ESI 125, available from FTP.ESI.AC.AT.

[GKK] K. Gawędzki, R. Kotecký, A. Kupiainen, *Coarse-Graining Approach to First-Order Phase Transitions*, Journ. Stat. Phys. **47** (1988), 701–724.

[Gin69] Ginibre, J., *Existence of phase transitions for quantum lattice systems*, Commun. Math. Phys. **14** (1969), 205.

[Gri64] Griffiths, R. B., *Peierls proof of spontaneous magnetization in a two-dimensional Ising ferromagnet*, Physical Review **A 136** (1964), 437–439.

[Ken85] T. Kennedy, *Long Range Order in the Anisotropic Quantum Ferromagnetic Heisenberg Model*, Commun. Math. Phys. **100** (1985), 447–462.

[KP84] R. Kotecký, D. Preiss, *An Inductive Approach to the Pirogov-Sinai Theory*, Suppl. ai Rendiconti del Circolo Matem. di Palermo, Ser. II **3** (1984), 161–164.

[Pei36] Peierls, R., *On the Ising model of ferromagnetism*, Proceedings of the Cambridge Philosophical Society **32** (1936), 477–481.

[PS75] Pirogov, S. and Sinai, Ya. G., *Phase diagrams of classical lattice systems*, Theoretical and Mathematical Physics **25** (1975), 1185–1192; **26** (1976), 39–49.

[Sin82] Ya. G. Sinai, *Theory of phase transitions: Rigorous results*, Pergamon Press, Oxford–New York–etc, 1982.

[SS76] E. Seiler and B. Simon, *Nelson's symmetry and all that in Yukawa and* $(\phi^4)_3$ *theories.*, Ann. Phys. **97** (1976), 470.

[Zah84] M Zahradník, *An Alternate Version of Pirogov-Sinai Theory*, Commun. Math. Phys. **93** (1984), 559–581.

CHRISTIAN BORGS
FREIE UNIVERSITÄT BERLIN, INSTITUT FÜR THEORETISCHE PHYSIK
ARNIMALLEE 14, 14195 BERLIN
E-mail address: borgs@dirac.physik.fu-berlin.de

ROMAN KOTECKÝ
CENTER FOR THEORETICAL STUDY, CHARLES UNIVERSITY,
JILSKÁ 1, 110 00 PRAHA 1, CZECH REPUBLIC
AND
DEPARTMENT OF THEORETICAL PHYSICS, CHARLES UNIVERSITY,
V HOLEŠOVIČKÁCH 2, 180 00 PRAHA 8, CZECH REPUBLIC
E-mail address: kotecky@aci.cvut.cz

DANIEL UELTSCHI
INSTITUT DE PHYSIQUE THÉORIQUE, EPFL,
CH-1015 LAUSANNE, SWITZERLAND
E-mail address: ueltschi@eldpa.epfl.ch