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A Fermi Field Algebra as Crossed Product

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Abstract

On the example of Luttinger model and Schwinger model we consider the observable algebra of interacting fermi systems in two–dimensional space–time and construct field algebra related to it as a crossed product with some automorphism group. Fermi statistics results for conveniently chosen automorphisms. The extension of time evolution to the field algebra and its asymptotic behaviour are treated. For the Luttinger model time evolution is asymptotically anticommutative, while for the Schwinger model we find a reformulation of confinement.

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1 Introduction

The Bose–Fermi duality in one space dimension has been successfully used for solving various problems in (1+1)–dimensional field theories and in 1–dimensional models in solid state physics. Starting from the pioneering works by Jordan [J], Born [BN], Mattis–Lieb [ML], Klaiber [K], different aspects of this phenomenon and different approaches, both to its technical realization and to its physical meaning have been considered. However, consistent expressions exist so far only for fermion bilinears (directly connected to the observables of the theory) while the explicit reconstruction of fermions themselves back from the bosonic variables is more subtle. This problem traces back to a question of principal importance: whether and in which cases conclusions about the time evolution of charged fields can be drawn from the knowledge of the time evolution of the observables.

First rigorous results on the possible fermi behaviour of operators acting on a bose field can be found in [SW] on the example of the massless Skyrme model. An important contribution to the solution of the problem is done in [CR] where local fermi fields are constructed as strong limits on a dense set of states of specific bosonic models. In a series of papers [AMSa] further progress is achieved since these local fermi fields are constructed as ultrastrong limits of bosonic variables in all representations that are locally Fock with respect to the ground state of the massless scalar field. There, an appropriate framework for the construction of anticommuting variables out of commuting ones is found to be provided by the nonregular representations of (canonical extensions of) CCR algebras [AMSb]. However, the question about relation (if any) between charged and non–charged field evolution still remains open.

In the present paper we propose a solution to this problem that makes use of the construction of the field algebra as a crossed product of the observable one by the α–action of $\mathbf{Z}$, α being a not–inner automorphism of the latter.

The relevance of the crossed product C*–algebra extensions for the relation of the field algebra to the observable algebra is first pointed out in [DHR] where the problem of constructing field groups is reformulated as a problem of constructing extensions of the observable algebra by a group dual. Also, they are discussed in the context of C*– and W*–dynamical systems in [H]. Explicitly, crossed products of C*–algebras by semigroups of endomorphisms are introduced when proving the existence of a compact global gauge group in particle physics given only the local observables [DR]. The problem of extension of automorphisms from a unital C*–algebra to its crossed product by the action of a compact group dual becomes important in the structural analysis of symmetries in the algebraic setting of Quantum Field Theory [H], where in the case of a broken symmetry this allows for concrete conclusions about the vacuum degeneracy [BDLR].

We restrict ourselves to the more simple case of a crossed product generated by the action of a not inner automorphism of the observable algebra with a discrete group and identify the resulting object with a charged field algebra. The conditions under which space translations can be extended from the observable to the field algebra make the algebra extension essentially unique. This (noncanonical in the sense of [AMSb]) extension of the observable algebra yields an extension of its states whose properties are discussed.

The question about the fermion evolution finds here a natural answer, the crucial point being a compatibility relation between the automorphism used in the crossed product and the time evolution, i.e. a property taking place on the observable algebra. For its realization the structure of the energy spectrum of the model under consideration is essential. The gauge group and its action for a crossed product field algebra are also defined.

In our approach we stay state independent and do not consider strong limits. Thus we cannot get the CAR relations in the renormalized form $\{\Psi^\dagger(x), \Psi(y)\} = z\delta(x-y)$, where the renormalization constant z goes to zero in some limit. We rather take as characteristic of fermi fields their asymptotic anticommutativity.

Another advantage of envisaging the fermionization as a crossed product is the fact that the field algebra inherits in a natural way the net structure of the observable algebra. Therefore it is evident that global properties do not affect the construction of fermions in accordance with the observation in [AMSa].

On the other hand, the crossed product construction is not restrictive enough to guarantee a statistic theorem. On the contrary, an interesting feature of the algebra so obtained is the possibility, depending on the particular choice of functions that determine the automorphism α, to endow this algebra with a "zone" structure, where also fields with fractional statistics are present. The specific conditions under which such fields could be provided with a stable time evolution will be considered elsewhere.

2 The crossed product algebra

We start with the CCR (Weyl) algebra $\mathcal{A}(\mathcal{V}_0, \sigma)$ over the real symplectic space $\mathcal{V}_0$ with symplectic form σ, generated by unitaries $W(\Phi)$, with $\Phi := (f, g) \in \mathcal{V}_0$, which satisfy

$$\begin{aligned} W(\Phi_1)\, W(\Phi_2) &= e^{i\sigma(\Phi_1, \Phi_2)/2}\, W(\Phi_1 + \Phi_2) \\ W(\Phi)^* &= W(-\Phi) = W(\Phi)^{-1} \end{aligned} \tag{2.1}$$

The elements of $\mathcal{A}$ are of the form

$$A = \sum_i c^{(i)}\, W(\Phi_i) := \sum_i c^{(i)}\, W_i, \qquad c^{(i)} \in \mathbf{C}, \qquad \sum_i |c^{(i)}| < \infty. \tag{2.2}$$

We can also consider its closure as C*–algebra. This algebra can be enlarged to another CCR algebra by enlarging the space $\mathcal{V}_0$ to a space $\mathcal{V}$, in a way that σ in (2.1) appears to be the restriction on $\mathcal{V}_0$ of the symplectic form of $\mathcal{V}$. This view point was taken in [AMSa]. There, with appropriate ultrastrong limits it was shown that fermi field operators can be obtained, so that the fermi algebra belongs to the weak closure of $\mathrm{CCR}(\mathcal{V}, \sigma)$ in appropriately taken representations [AMSa, CR]. Instead of doing this we will construct a new algebra $\mathcal{F}$ such that

$$\mathrm{CCR}(\mathcal{V}_0) \subset \mathcal{F} \subset \mathrm{CCR}(\mathcal{V})$$

without referring to representations and then show that $\mathcal{F}$ can be considered as fermi field algebra.

For α a free [T] (so not inner) automorphism of $\mathrm{CCR}(\mathcal{V}_0, \sigma) = \mathcal{A}$ we can consider the crossed product

$$\mathcal{F} = \mathcal{A} \overset{\alpha}{\times} \mathbf{Z}.$$

This corresponds (compare [A]) to adding a unitary operator U with all its powers, so that one can formally write

$$\mathcal{F} = \sum_n \mathcal{A}\, U^n,$$

with U implementing the automorphism α in $\mathcal{A}$:

$$U\ A\ U^{-1} = \alpha\ A.$$

U should be thought of as charge creating operator and $\mathcal{F}$ is a minimal extension.

The multiplication law in $\mathcal{F}$ is

$$\sum_n A_n\ U^n \sum_k B_k\ U^k = \sum_{n,k} A_n\ \alpha^n\ B_k\ U^{n+k} \tag{2.3}$$

and we take α to be

$$\alpha\ W(\Phi) = e^{i\sigma(\bar{\Phi},\Phi)}\ W(\Phi), \qquad \alpha \equiv \alpha_{\bar{\Phi}}$$
$$\bar{\Phi} := (\bar{f}, \bar{g}) \in \mathcal{V} \setminus \mathcal{V}_0, \qquad \mathcal{V}_0 \subset \mathcal{V}.$$

Crossed products are unitarily equivalent, i.e.

$$\mathcal{A} \overset{\alpha}{\times} \mathbf{Z} \approx \mathcal{A} \overset{\widehat{\alpha}}{\times} \mathbf{Z}$$

if $\alpha \circ \widehat{\alpha}^{-1}$ is an inner automorphism of $\mathcal{A}$. Therefore our algebra $\mathcal{F}$ depends only on the equivalence class $\{\bar{\Phi}\} \in \mathcal{V}/\mathcal{V}_0$, though for the explicit calculations we will specify $\bar{\Phi} \in \mathcal{V} \setminus \mathcal{V}_0$. The automorphism α has to be free, so $\mathcal{F}$ has trivial center like $\mathcal{A}$. Since α is implemented by $W(\bar{\Phi})$ in $\mathrm{CCR}(\mathcal{V})$, $\mathcal{F}$ in a natural way (identifying $U = W(\bar{\Phi})$) is a subalgebra of $\mathrm{CCR}(\mathcal{V})$.

By writing an element $F \in \mathcal{F}$ as $F = \sum_n\ A_n\ U^n, A_n \in \mathcal{A}$, we see that it is convenient to consider $\mathcal{F}$ as (infinite) vector space with U^n as basic unit vectors and $A_n =: (F)_n$ the components of $F =: \{A_n\}$. The algebraic structure of $\mathcal{F}$ is such that multiplication is not componentwise but (2.3) says that

$$(F \cdot G)_m = \sum_n F_n\ \alpha^n\ G_{m-n}$$

The algebra $\mathcal{A}$ can be identified with the zero component in $\mathcal{F}$, $\mathcal{F}$ is actualy a left $\mathcal{A}$–module. In components we have

$$\begin{aligned} (U^k)_n &= \delta_{kn} \\ (U^*)_n &= \delta_{-1,n} \end{aligned}$$

so that the left action of U is the shift.

Further, we can write

$$F = \sum_n A_n \, U^n = \sum_{n,i} c_n^{(i)} W_i U^n := \bigoplus_n \sum_i c_n^{(i)} F_i^{(n)}. \tag{2.4}$$

The (non normalized) operator

$$F_i = \sum_n W_i \, U^n$$

defines the U–orbit through W_i, so the last of eqs. (2.4) gives an orbit decomposition of the elements of $\mathcal{F}$.

There are two questions that naturally arise. Given an automorphism on $\mathcal{A}$, can it be extended to $\mathcal{F}$ and how unique is this extension. In the physical applications we are especially concerned with space translations and time evolution. A similar question concerns the extension and its uniqueness for given states on $\mathcal{A}$.

We concentrate first on the extension $\tilde{\rho}$ of an automorphism ρ of $\mathcal{A}$. Since all elements can be written as sums and product of $A \in \mathcal{A}$ and U, i.e. $\{\delta_{1n}\}$ and the action of $\tilde{\rho}$ on A must coincide with ρ, we make the ansatz

$$\tilde{\rho}\{\delta_{1n}\} = \{V_{\rho_n}^{(1)}\},$$

$\tilde{\rho} \in \mathcal{F}$ requires $V_{\rho_n}^{(1)} \in \mathcal{A}$. Then $\{V_{\rho_n}^{(1)}\}$ will fix $\tilde{\rho}$. The consistency of ρ and $\tilde{\rho}$ on the subalgebra $\mathcal{A}$ of $\mathcal{F}$ requires

$$1 = U \cdot U^* = \tilde{\rho} U \cdot \tilde{\rho} U^* = \left\{ \sum_n V_{\rho_n}^{(1)} \alpha^n V_{\rho_{k-n}}^{-1)} \right\} = \{\delta_{0k}\}.$$

This equation is satisfied by

$$\{V_{\rho_n}^{(1)}\} = \{V_\rho^{(1)} \, \delta_{1n}\}, \qquad V_\rho^{(1)} \in \mathcal{A} \tag{2.5}$$

We refer to [HN] for a discussion on the uniqueness of this choice.

Further, for $W \in \mathcal{A}$, we have

$$\tilde{\rho}(U \cdot W) = \tilde{\rho} U \cdot \rho W = \tilde{\rho}(\alpha W \cdot U)$$

and from (2.3), (2.5) follows

$$\{V_\rho^{(1)} \delta_{1n}\}\{\rho W \delta_{0n}\} = \{\rho \alpha W \delta_{0n}\}\{V_\rho^{(1)} \delta_{1n}\}$$

so that

$$V_\rho^{(1)} \alpha \rho W = \rho \alpha (W) V_\rho^{(1)}$$

or, equivalently,

$$V_\rho^{(1)} \alpha \rho W V_\rho^{*(1)} =: \gamma_\rho \alpha \rho W = \rho \alpha W.$$

This can only be satisfied for some $V_\rho^{(1)} \in \mathcal{A}$, if the automorphism γ_ρ,

$$\gamma_\rho = \rho\alpha\rho^{-1}\alpha^{-1} \tag{2.6}$$

is an inner automorphism of $\mathcal{A}$. For example, α is easily seen to be extendible to an automorphism also of $\mathcal{F}$, since the corresponding γ_α is the identity transformation, so that $\tilde{\alpha}U = U$. Since the CCR($\mathcal{V}_0$) algebra $\mathcal{A}$ has trivial center, the unitary operator that implements an automorphism is unique up to a phase factor.

Apart from the condition that $\rho\alpha\rho^{-1}\alpha^{-1}$ is inner no other conditions have to be satisfied.

We should mention that the question if an automorphism of the observable algebra can be extended to the field algebra is also treated in [BDLR] in the context of the theory of [DR] where the field algebra is obtained as crossed product over a specially directed symmetric monoidal subcategory End $\mathcal{A}$ of unital endomorphisms of $\mathcal{A}$ as generalization of our automorphism group α. There two conditions enter, one is the appropriate replacement of our demand that $\rho\alpha\rho^{-1}\alpha^{-1}$ has to be inner, the other is a compatibility condition with the net structure. We do not have any counterpart to this condition. It will turn out that in our case the net structure of the field algebra is a consequence of the net structure of the observable algebra and of the compatibility relation for α.

We return to our explicitly chosen α. We consider ρ that are quasifree automorphisms on CCR($\mathcal{V}_0$), which means that they are of the form $\rho W(\Phi) = W(\Phi_\rho)$. The inverse of the map $\Phi \to \Phi_\rho$ we denote by $\Phi \to \Phi_{-\rho}$ and ρ has to preserve the symplectic structure in $\mathcal{V}_0$, so that $\sigma(\Psi, \Phi_{-\rho}) = \sigma(\Psi_\rho, \Phi)$. To start, this bijection is defined on $\mathcal{V}_0$. We have

$$\gamma_\rho W(\Phi) = e^{i\sigma(\bar{\Phi},\Phi_{-\rho}) - i\sigma(\bar{\Phi},\Phi)}\, W(\Phi).$$

Assume that

$$\sigma(\bar{\Phi}, \Phi_{-\rho}) - \sigma(\bar{\Phi}, \Phi) = \sigma(\Psi, \Phi)$$

for some $\Psi \in \mathcal{V}_0$. Then, on one hand, we have enlarged ρ to a quasifree automorphism on CCR($\mathcal{V}$) with $\bar{\Phi}_\rho = \bar{\Phi} + \Psi$, on the other hand, γ_ρ satisfies our requirement with

$$V_\rho = W(\Psi) = W(\bar{\Phi}_\rho - \bar{\Phi}) \in \mathcal{A}. \tag{2.7}$$

That the condition is satisfied for appropriately chosen $\bar{\Phi}$ if we consider space translation and with some restriction on time evolution will be discussed in Section 3. How this restriction is satisfied in physical models and what are the physical consequences will be discussed in Section 4.

The second principal question concerns construction of states over $\mathcal{F}$. Let $\omega(\cdot)$ be a state over the algebra $\mathcal{A}$ and π_ω the cyclic representation of $\mathcal{A}$ associated with it through the GNS construction

$$\omega(W(\Phi)) = \langle\omega|\pi_\omega(W(\Phi))|\omega\rangle = \langle\omega|\Phi\rangle_\omega,$$

where $|\omega\rangle$ denotes the vacuum. Then the vectors $|\Phi\rangle_\omega = \pi_\omega(W(\Phi))|\omega\rangle$ generate $\mathcal{H}_0$, the representation space of π_ω.

The representation π_ω itself is given by

$$\pi_\omega(W(\chi))|\Phi\rangle_\omega = e^{i\sigma(\chi,\Phi)/2}|\Phi+\chi\rangle_\omega$$

and the scalar product is

$$_\omega\langle\chi|\Phi\rangle_\omega = e^{-i\sigma(\chi,\Phi)/2}\omega(W(\Phi-\chi)).$$

The crossed product algebra acts in a larger Hilbert space $\mathcal{H}$ which may be considered as a direct sum of charge–n subspaces (the justification for this terminology will be given in Section 3), each of them being a representation space corresponding to the state $\omega\circ\alpha^{-n}$. We can imbed $\mathcal{H}_0$ into $\mathcal{H}$ and denote now the vacuum by $|\Omega\rangle$, thus expressing the fact that we consider it as a vector in $\mathcal{H}$. Then $U^k|\Omega\rangle$ can be denoted as

$$U^k|\Omega\rangle = |\Omega_k\rangle$$

and the vector space structure of $\mathcal{F}$ suggests that $\langle\Omega_k|\Omega_n\rangle = \delta_{kn}$ with the identification $|\Omega\rangle = |\Omega_0\rangle$.

Then

$$\begin{aligned}|F_i^{(k)}\rangle &= W(\Phi_i)|\Omega_k\rangle = U^k\alpha^{-k}W(\Phi_i)|\Omega\rangle = \\ &= e^{-ik\sigma(\bar\Phi,\Phi_i)}\,U^k|\Phi_i\rangle = e^{-ik\sigma(\bar\Phi,\Phi_i)}|\Phi_i^{(k)}\rangle\end{aligned} \tag{2.8}$$

so that $|F_i^{(k)}\rangle$, varying over Φ_i generate the charge–k space $\mathcal{H}^{(k)}$ and varying over k we get the complete Hilbert space $\mathcal{H}$.

Arbitrary linear functionals built by vectors in $\mathcal{H}$ considered as states over $\mathcal{A}$ read

$$\langle F_i^{(n)}|W(\chi)|F_i^{(k)}\rangle = \delta_{nk}e^{-i\sigma(\Phi_i,\chi)}\omega(\alpha^{-n}W(\chi)) = \delta_{nk}e^{-i\sigma(\Phi_i+n\bar\Phi,\chi)}\omega(W(\chi)). \tag{2.9}$$

On the other hand, given two states on $\mathcal{A}$, ω_1 and ω_2, a quantum mechanical superposition of them to a state on $\mathcal{F}$ is only possible if the same representation π is associated with both ω_1 and $\omega_2\circ\alpha^k$ for some k so that the new state is constructed with a vector

$$c_0|\Phi_1^{(0)}\rangle + c_k|\Phi_2^{(k)}\rangle, \qquad c_0, c_k \in \mathbf{C}$$

If we take into account that ω is irreducible, α free and $\omega\circ\alpha^n$ not normal with respect to ω we can conclude that the extension of the state over $\mathcal{A}$ to a state over $\mathcal{F}$ is uniquely given by the expectation value with $|\Omega_0\rangle$ in this representation. With

$$F = \sum_{i,n} c_n^{(i)}\,W(\Phi_i)U^n$$

we get

$$\omega(F^*F) = \sum_{i,j,n}\bar c_n^{(j)}c_n^{(i)}e^{i\sigma(\Phi_i,\Phi_j)/2}\,e^{-in\sigma(\bar\Phi,\Phi_i-\Phi_j)}\,\omega(W(\Phi_i-\Phi_j)).$$

Therefore, the states over $\mathcal{F}$ inherit in a natural way the whole structure and symmetry properties from the states over $\mathcal{A}$.

3 The crossed product as field algebra

The important result in [DR] is the theorem that the observable algebra $\mathcal{A}$ together with the set of particle states (that form a DR–category) can be enlarged to a field algebra on which a gauge group acts, that leaves the observable algebra elementwise invariant.

In the case of (free) fermions in one dimension the algebra is built by creation and annihilation operators $a(f)$, $a^\dagger(g)$, $f, g \in L^2(\mathbf{R})$. The observable algebra is built by monomials with the same number of creation and annihilation operators

$$\prod_{i=1}^{n} a^\dagger(f_i) \prod_{j=1}^{n} a(g_j). \tag{3.1}$$

They are invariant under the automorphism group

$$\gamma_\nu \, a(f) = e^{2\pi i \nu} \, a(f), \qquad \nu \in [0,1) = S_1. \tag{3.2}$$

The observable algebra contains the current algebra built by $a^\dagger(x)a(x)$, invariant under the local gauge group $a(f) = a(e^{i\nu(x)} f(x)$. One still has to check whether this algebra is well defined. ¿From e.g. [ML] we know that the current algebra leads to $\mathrm{CCR}(\mathcal{V}_0)$ in appropriate representations.

However, if we consider as observable algebra the C*–algebra obtained as a norm closure from (3.1) the passage to the CAR–algebra as crossed product is not possible as it was shown in [B]. Therefore the closure has to be taken with respect to some other topology. Consideration of $\mathrm{CCR}(\mathcal{V}_0)$ as a von Neumann algebra would solve the problem, but we do not favour it because we want to stay representation independent as much as possible. On the other hand, we cannot ignore the representation completely, because in the C*–norm $\mathrm{CCR}(\mathcal{V}_0)$ is not separable whereas the fermi observable algebra is. This is not really a problem: we are only interested in states that are locally normal with respect to the vacuum. Therefore we take the following view point: we consider $\mathrm{CCR}(\mathcal{V}_0)$ as a net of von Neumann algebras closed locally in some representations, so that there is no contradiction with [B]. To be more precise, we consider the local net

$$\mathcal{A}_\Lambda \subset \{\prod a^\dagger(f)a(g), \ \mathrm{supp}\ f, g \in \Lambda\}'',$$

$$\mathcal{A} = \overline{\bigvee_\Lambda \mathcal{A}_\Lambda}$$

The union is taken in norm and this algebra does coincide with

$$\mathcal{A} = \overline{\bigvee_\Lambda \mathrm{CCR}(\mathcal{V}_0, \Lambda)''}$$

where we have some freedom in choosing $\mathrm{CCR}(\mathcal{V}_0, \Lambda)$, e.g. $(f, g) \in \mathcal{V}_0 := (\mathcal{C}_0^\infty \times \mathcal{C}_0^\infty)$, supp $f, g \subset \Lambda$, where $\mathcal{C}_0^\infty$ is the space of test functions that are infinitely differentiable and with compact support.

Our first step in the identification of $\mathcal{F}$ with a Fermi type algebra

$$\overline{\bigvee_{\Lambda}\{b(f), b^{\dagger}(g),\ \mathrm{supp}\ f, g \in \Lambda\}''}$$

is to find the gauge automorphism, but this is trivial in the context of crossed products. Defining

$$\gamma_\nu\, U^n = e^{2\pi i\nu n}\, U^n \qquad \gamma_\nu\, W_i = W_i \tag{3.3}$$

and with (2.4) taken into account, we get

$$\gamma_\nu\, F_i = \sum_n e^{2\pi i\nu n}\, W_i U^n = \bigoplus_n e^{2\pi i\nu n}\, F_i^{(n)} \tag{3.4}$$

Evidently, $F_i^{(0)}$, the elements of $\mathcal{A}$ as subalgebra of $\mathcal{F}$, are invariant under the gauge automorphism (3.3)

$$\gamma_\nu\, F_i^{(0)} = \gamma_\nu\, \{W_i\, \delta_{0k}\} = \{W_i\, \delta_{0k}\} = F_i^{(0)}$$

Also, γ_ν commutes with the structural automorphism α, $\alpha \circ \gamma_\nu = \gamma_\nu \circ \alpha$.

For the representation π_Ω discussed in Section 2 we observe (see (2.8))

$$\gamma_\nu(W(\Phi_i)|\Omega_k\rangle) = e^{2\pi i\nu k}\{W(\Phi_i)\delta_{kn}\}|\Omega\rangle = e^{2\pi i\nu k}\, W(\Phi_i)|\Omega_k\rangle \tag{3.5}$$

so that we really can interpret vectors $|F_i^{(k)}\rangle$ as belonging to the charge–k subspace. Thus, the (gauge invariant) state over $\mathcal{A}$, $\omega(A)$, is extended to a gauge invariant state over $\mathcal{F}$

$$\gamma_\nu \circ \Omega\ (\{F_i^{(n)}\}) = \Omega\ (\gamma_\nu\{F_i^{(n)}\}) = \delta_{n0}\ \Omega\ (\{e^{2\pi i\nu n}\ F_i^{(n)}\}) = \Omega\ (F_i^{(n)}).$$

The next task is to reconstruct the net character of the field algebra. This means that we want to find subalgebras $\mathcal{F}_\Lambda$ for which the following relations take place

$$\begin{aligned} \mathcal{F}_\Lambda \subset \mathcal{F}_{\bar\Lambda}, &\qquad \text{if } \Lambda \subset \bar\Lambda \\ \sigma_x \mathcal{F}_\Lambda &= \mathcal{F}_{\Lambda+x} \\ \mathcal{F} &= \overline{\bigvee_\Lambda \mathcal{F}_\Lambda} \end{aligned}$$

To show that this is really the case, we shall make use of two important features of the crossed product algebras in question: first, the extendibility of space translations to automorphisms of the field algebra, and second, the unitary equivalence of crossed products with structural automorphisms which differ by an inner automorphism of the observable algebra $\mathcal{A}$.

Let us consider the observable algebra for a given region Λ and choose $\bar\Phi \in \mathcal{V} \setminus \mathcal{V}_0$, $\bar\Phi_x - \bar\Phi \in \mathcal{V}_0$ such that

$$\alpha_{\bar\Phi}|_{\mathcal{A}_{\widehat\Lambda}} = id, \qquad \widehat\Lambda \subset \Lambda^c,$$

where Λ^c is the causal complement of Λ. Then we define

$$\mathcal{F}_\Lambda := \mathcal{A}_\Lambda \overset{\alpha_{\bar\Phi}}{\times} \mathbf{Z} \subset \mathcal{F}.$$

Space translations act in $\mathcal{F}_\Lambda$ as

$$\sigma_x\{A_n\} = \{\sigma_x A_n \cdot U_x^{(n)}\},$$

with $U_x^{(n)}$ implementing the (inner) automorphism $\sigma_x \alpha_{\bar\Phi}^n \sigma_x^{-1} \alpha_{\bar\Phi}^{-n}$. We then get, in accordance with (2.3),

$$\{\sigma_x A_n U_x^{(n)}\}\{\sigma_x B_m U_x^{(m)}\} = \left\{\sum_k \sigma_x A_k \; \alpha_{\bar\Phi_x}^k \; B_{n-k} U_x^{(n)}\right\},$$

which is exactly the multiplication law for the crossed product algebra

$$\sigma_x \mathcal{A}_\Lambda \overset{\alpha_{\bar\Phi_x}}{\times} \mathbf{Z}$$

with

$$\begin{aligned} \alpha_{\bar\Phi_x} &= \sigma_x \alpha_{\bar\Phi} \sigma_{-x} \\ \alpha_{\bar\Phi_x}|_{A_{\widehat\Lambda}} &= id, \qquad \widehat\Lambda \subset (\Lambda + x)^c \end{aligned}$$

Therefore, we have

$$\sigma_x \mathcal{F}_\Lambda = \sigma_x \mathcal{A}_\Lambda \overset{\alpha_{\bar\Phi_x}}{\times} \mathbf{Z} = \mathcal{F}_{\Lambda+x}$$

The net structure of $\mathcal{F}$ appears as a consequence of the extendibility of space translations to an automorphism in $\mathcal{F}$ which requires for a choice of the structural automorphism $\alpha_{\bar\Phi}$ that is consistent with the net structure of $\mathcal{A}$.

Finally, we have to verify compatibility of anticommutation relations with the structure of the crossed product algebra $\mathcal{F}$ in order to ensure the existence of odd elements in it. This will complete the identification of $\mathcal{F}$ with a fermi field algebra corresponding to the observable algebra $\mathcal{A}$.

$\{\delta_{1n}\}$, the element in $\mathcal{F}$, that implements $\alpha_{\bar\Phi}$, is an odd element in $\mathcal{F}_\Lambda$ if

$$\sigma_x\{\delta_{1n}\} \cdot \{\delta_{1n}\} + \{\delta_{1n}\} \cdot \sigma_x\{\delta_{1n}\} = 0 \qquad \forall\, |x| > |\Lambda|. \tag{3.6}$$

With (2.3), (2.5) this means

$$\begin{aligned} &\{W(\bar\Phi_x - \bar\Phi)\delta_{1n}\} \cdot \{\delta_{1n}\} + \{\delta_{1n}\} \cdot \{W(\bar\Phi_x - \Phi)\delta_{1n}\} = \\ &= \{W(\bar\Phi_x - \bar\Phi)\delta_{2n}\} + \{\alpha_{\bar\Phi} W(\bar\Phi_x - \bar\Phi)\delta_{2n}\} = \\ &= \{(1 + e^{i\sigma(\bar\Phi, \bar\Phi_x - \bar\Phi)}) W(\bar\Phi_x - \bar\Phi)\delta_{2n}\} = 0 \end{aligned}$$

Thus, eq.(3.6) is satisfied if for $|x| > |\Lambda|$

$$e^{i\sigma(\bar\Phi, \bar\Phi_x - \bar\Phi)} = -1$$

that is if the following relation holds

$$\sigma(\bar{\Phi}, \bar{\Phi}_x) = (2k+1)\pi, \qquad k \text{ integer.} \tag{3.7}$$

This is exactly the normalization condition used in [AMSa] to choose an approrpiate $\bar{\Phi}$ for the construction of the fermion creation operator. It reads

$$\lim_{x\to\pm\infty} \sigma(\bar{\Phi}, \bar{\Phi}_x) = \pm(\bar{g}(\infty) - \bar{g}(-\infty)) \int \bar{f}(x)dx = (2k+1)\pi,$$

but the limit is already attained for $|x| > |\Lambda|$ because supp $f, g \subset \Lambda$.

Then, for the r–th class elements in $\mathcal{F}$ it follows:

$$\sigma_x\{\delta_{rn}\}\{\delta_{rn}\} = W_r(\bar{\Phi}_x - \bar{\Phi})\delta_{2r,k} = (-1)^{r^2}\{\delta_{rn}\}\sigma_x\{\delta_{rn}\} \tag{3.8}$$

so that for $r = 2n$ the elements commute and for $r = 2n+1$ they anticommute, thus providing a graded structure of $\mathcal{F}$.

Here, the sensitivity of the crossed product to the particular choice of $\bar{\Phi}$ shows up. If the pair $\bar{\Phi}$ is scaled to $\lambda\bar{\Phi}$, so that σ is scaled to $\lambda^2\sigma$, condition (3.6) in general fails. Instead, we get

$$\sigma_x\{\delta_{1n}\}\{\delta_{1n}\} = e^{-i\lambda^2(2k+1)\pi}\{\delta_{1n}\}\sigma_x\{\delta_{1n}\}, \tag{3.9}$$

which can be interpreted as a fractional statistics and therefore describes an essentially different physical system.

However, it might happen that elements obeying fractional statistics, are naturally present in the field algebra $\mathcal{F}$. In fact, this is exactly the situation, if the first odd element of $\mathcal{F}$ is not $\{\delta_{1n}\}$ but some $\{\delta_{\bar{n}n}\}$, i.e. if

$$\begin{aligned} &\sigma_x\{\delta_{\bar{n}n}\}\{\delta_{\bar{n}n}\} + \{\delta_{\bar{n}n}\}\sigma_x\{\delta_{\bar{n}n}\} \\ &= \{(W_{\bar{n}}(\bar{\Phi}_x - \bar{\Phi}) + \alpha^{\bar{n}} W_{\bar{n}}(\bar{\Phi}_x - \bar{\Phi}))\delta_{2\bar{n},k}\} = \\ &= \{(1 + e^{i\bar{n}^2\sigma(\bar{\Phi}, \bar{\Phi}_x - \bar{\Phi})})W_{\bar{n}}(\bar{\Phi}_x - \Phi)\delta_{2\bar{n},k}\} = 0 \end{aligned}$$

so, instead of (3.6) we have the relation

$$\sigma(\bar{\Phi}, \bar{\Phi}_x) = \frac{2k+1}{\bar{n}^2}\pi.$$

The graded structure is still present, with $2k\bar{n}$–classes being commuting and $(2k+1)\bar{n}$–classes anticommuting ones. The elements in the classes with numbers $m \in \mathbf{Z}/\mathbf{Z}_{\bar{n}}$ are characterized by fractional statistics, satisfying a relation in formal analogy to (3.9):

$$\sigma_x\{\delta_{mn}\}\{\delta_{mn}\} = e^{-i(m/\bar{n})^2(2k+1)\pi}\{\delta_{mn}\}\sigma_x\{\delta_{mn}\}.$$

This offers an alternative approach to construct models with fractional statistics.

Finally we note that $\mathcal{A}$ is a subalgebra of $\mathcal{F}$ for the gauge group $\mathcal{T} = [0, 1)$ while it is a subalgebra of CAR for the gauge group $\mathcal{T} \otimes \mathbf{R}$. Thus the crossed product algebra $\mathcal{F}$ being really a Fermi algebra, does not coincide with CAR but is only contained in it.

4 Examples

We are now going to discuss two typical examples which demonstrate the sensitivity of the construction described above to the physical content of the models. These two examples are Luttinger model [L] and Schwinger model [S]. It is not our aim here to give an overview on the enormous literature on these models or to enter in detail the far going conclusions drawn on their basis. What is important from the point of view of the crossed product algebra construction is the essential difference between the interactions they describe. The Luttinger model is an example of a one–dimensional interacting fermionic system which is nevertheless realistic enough (recently it has become even more popular in connection with the "Luttinger liquid" behaviour of normal metals [And]). The Schwinger model gives an example of confinement, being equivalent to a free massive scalar field theory in $(1+1)$–dimensional space–time.

The models in question are described by the following Lagrangians:

$$\mathcal{L}_S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi \tag{4.1a}$$

$$\mathcal{L}_L = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \int j^\mu(x)V(x-y)j_\mu(y)dy \tag{4.1b}$$

where $V(x-y)$ is an even smooth function, $\psi(x)$ is a two–component spinor, satisfying

$$\{\psi_i^\dagger(x), \psi_j(y)\} = \delta_{ij}\delta(x-y) \tag{4.2}$$

all other anticommutators vanishing, $A_\mu(x)$ is the vector potential,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mu,\nu = 0,1$$

and currents $j_\mu(x)$ are defined as

$$\begin{aligned} j_\mu(x) &= \bar{\psi}(x)\gamma_\mu\psi(x) \\ j_{5\mu}(x) &= \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) \\ j_{R,L}(x) &= \frac{1}{2}(j_0(x) \pm j_1(x)) \end{aligned} \tag{4.3}$$

with the two–dimensional γ–matrices

$$\gamma_0 = \sigma_1, \qquad \gamma_1 = -i\sigma_2, \qquad \gamma_5 = \gamma_0\gamma_1 = \sigma_3,$$

σ_k being the Pauli matrices.

Hamiltonian densities have the form

$$\mathcal{H}_S(x) = i\bar{\psi}(x)\gamma_1\partial_1\psi(x) - \frac{e^2}{2}\int (j_L(x)+j_R(x))|x-y|(j_L(y)+j_R(y))dy \tag{4.4a}$$

$$\mathcal{H}_L(x) = i\bar{\psi}(x)\gamma_1\partial_1\psi(x) + 4\int j_R(x)V(x-y)j_L(y)dy \tag{4.4b}$$

with $j_L(x)$ and $j_R(x)$ — left and right current respectively. Therefore, for a direct comparison of the results it is convenient to generalize (4.1b) (hence, (4.4b)) to the more realistic type of two–body interaction

$$\int (j_R(x)+j_L(x))V(x-y)(j_R(y)+j_L(y))dy \tag{4.5}$$

which does not affect solvability of the model but only causes minor changes in the spectrum of excitations.

Now, in the momentum space Hamiltonians are

$$\begin{aligned} H_S &= H_0 + \frac{e^2}{2\pi}\int_0^\infty [\rho_1(p)+\rho_2(p)]\frac{1}{p^2}[\rho_1(-p)+\rho_2(-p)]dp \\ H_L &= H_0 + \int_0^\infty \tilde{V}(p)[\rho_1(p)+\rho_2(p)][\rho_1(-p)+\rho_2(-p)]dp \end{aligned} \tag{4.6}$$

where H_0 is the free Hamiltonian and the following notation is used:

$$\begin{aligned} \rho_i(p) &= \int dk\ a_i^\dagger(k+p)a_i(k), \qquad p>0, \\ \rho_i(-p) &= \int dk\ a_i^\dagger(k)a_i(k+p), \qquad p>0 \end{aligned} \tag{4.7}$$

with $a_i(k)$, $a_i^\dagger(k)$ being the Fourier transformed of $\psi_i(x)$, $\psi_i^\dagger(x)$:

$$\psi_i(x) = \frac{1}{\sqrt{2\pi}}\int e^{ipx}\ a_i(p)\ dp$$

$$\{a_i^\dagger(q), a_j(p)\} = \delta_{ij}\ \delta(p-q).$$

The semiboundedness of the free Hamiltonian H_0 is achieved after a Bogolyubov transformation of a's and $a^\dagger$'s, which effectively describes the negative energy states filling (filling of the Dirac sea),

$$\begin{aligned} a_1(k) &= b(k)\theta(k) + c^\dagger(k)\theta(-k) \\ a_2(k) &= b(k)\theta(-k) + c^\dagger(k)\theta(k). \end{aligned} \tag{4.8}$$

The new creation and annihilation operators $b, b^\dagger, c, c^\dagger$ satisfy canonical anticommutation relations, but the vacuum is already defined as

$$b(k)|0\rangle = c(k)|0\rangle = 0. \tag{4.9}$$

This procedure results in the appearance of an anomalous term in the commutator of currents (4.7), which otherwise commuted

$$\begin{aligned} [\rho_1(p), \rho_1(p')] &= p\delta(p-p') \\ [\rho_2(p), \rho_2(p')] &= -p\delta(p-p') \end{aligned} \tag{4.10}$$

It is eqs. (4.10) that justify the so–called bosonization of the two–dimensional models with fermions. The calculation of the anomalies is done usually in this new vacuum, but this is not essential. In [GMR] the same result was also obtained in temperature states. This is not surprising. It is an algebraic relation, so it has to be state independent, provided the densities are well defined, i.e. smearing over p gives an (unbounded) operator. This only works in the new, Dirac vacuum and in all states that are locally normal with respect to it. It is one of the achievements of our approach that we can find the observable algebra as local net, so restricting the permitted states only on a local basis. From this local basis we come back to the field algebra and there is no need to check the anomalies in every state (that has not to be globally normal, i.e. permits temperature). In this sense we interpret the appearance of anomalies as a local effect.

For the corresponding spectra we get

$$\omega_L(p) = |p|(1-\widetilde{V}(p))^{1/2} \tag{4.11}$$

$$\omega_S(p) = |p|\left(1+\frac{m^2}{p^2}\right)^{1/2}, \qquad m=\frac{e}{\sqrt{\pi}}. \tag{4.12}$$

The CCR algebra $\mathcal{A}(\mathcal{V}_0,\sigma)$ in both cases is generated by the unitaries

$$W(\Phi) := W(f,g) = \exp\{i\int[f(x)\rho_A(x)+g(x)\rho_V(x)]dx\}$$

$$(f,g)\in\mathcal{V}_0=(\mathcal{C}_0^\infty\times\mathcal{C}_0^\infty),$$

$$\rho_A(x)=\rho_1(x)-\rho_2(x), \qquad \rho_V(x)=\rho_1(x)+\rho_2(x)$$

which satisfy

$$W(\Phi_1)\,W(\Phi_2)=e^{i\sigma(\Phi_1,\Phi_2)/2}\,W(\Phi_1+\Phi_2), \tag{4.13}$$

$$\sigma(\Phi_1,\Phi_2)\equiv\sigma((f_1g_1),(f_2g_2))=\int(f_1'g_2-f_2'g_1)dx.$$

The field algebra $\mathcal{F}$ may be constructed, following the procedure described in Sections 1 – 3, with the help of an automorphism $\alpha_{\bar{\Phi}}$

$$\alpha_{\bar{\Phi}}: W(\Phi)\to e^{i\sigma(\bar{\Phi},\Phi)}\,W(\Phi), \qquad \Phi\in\mathcal{V}_0,$$

$$\bar{\Phi} := (\bar{f},\bar{g})\in\mathcal{V}=(\partial^{-1}\mathcal{C}_o^\infty\times\partial^{-1}\mathcal{C}_0^\infty)$$

(a function $f(x)$ belongs to $\partial^{-1}\mathcal{C}_0^\infty$, $f(x)\in\partial^{-1}\mathcal{C}_o^\infty$ if $\partial f(x)\in\mathcal{C}_0^\infty$). Therefore, the functions f,g have bounded Fourier components at $p=0$

$$\int f(x)dx\sim\tilde{f}(0)<\infty \qquad \forall\, f(x)\in\mathcal{C}_o^\infty \tag{4.14}$$

while for functions $\bar{f},\bar{g}$, as well as for their space– and time–translated this components might diverge and only their boundary values at $\pm\infty$ are related through

$$\int\partial\bar{f}(x)dx=\bar{f}(\infty)-\bar{f}(-\infty)=M_{\bar{f}}<\infty. \tag{4.15}$$

The space translations as automorphism can be extended from the observable algebra $\mathcal{A}$ (4.13) to the field algebra $\mathcal{F}$

$$\mathcal{F} = \mathcal{A} \overset{\alpha_{\bar{\Phi}}}{\times} \mathbf{Z} \tag{4.16}$$

if, according to (2.7), condition (4.14) is satisfied for the functions

$$\bar{f}_x - \bar{f} = \widehat{f}_x, \qquad \bar{g}_x - \bar{g} = \widehat{g}_x$$

i.e. if $\widehat{\Phi}_x := (\widehat{f}_x, \widehat{g}_x) \in \mathcal{C}_0^\infty$ for finite x.

This is easily seen to be the case. Note, however, that due to (4.8) space translations are generated by $\hat{P} = exp\{i|p|x\}$ which restricts the Fourier transforms of all test functions to the subspace of even functions. Then the invariance of symplectic form σ under shifts in x determines the Fourier decompositions:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int e^{i|p|x}\, \widetilde{f}(p)\, dp \\ g(x) &= \frac{1}{\sqrt{2\pi}} \int e^{-i|p|x}\, \widetilde{g}(p)\, dp. \end{aligned}$$

This gives for the zero modes

$$\begin{aligned} \int_{-\infty}^{\infty} \widehat{f}_x(y)dy &= 2 \int_0^\infty e^{ipy} \widetilde{\bar{f}}(p)(e^{-ipx} - 1)dpdy \\ &= -ix\, (p\widetilde{\bar{f}}(p))\Big|_{p=0} = \\ &= -x \int \partial \bar{f}(y)dy = -xM_{\bar{f}} < \infty, \end{aligned} \tag{4.17}$$

according to (4.15), similarly for $\widehat{g}_x$. Eq. (4.17) also means that the singularity of the zero mode of $\bar{f}, \bar{g}$ is of the type $1/p$. For example, we can choose for $(\bar{f}, \bar{g})$ appropriately smeared θ–functions:

$$\begin{aligned} \bar{f}(x) &= \int F(x-y)\theta(y)dy = \int_0^\infty F(x-y)dy \\ \bar{g}(x) &= \int G(x-y)\theta(y)dy = \int_0^\infty G(x-y)dy \end{aligned}$$

with F, G being $\mathcal{C}_0^\infty$–functions, so that for the Fourier components we get

$$\widetilde{\bar{f}}(|p|) = \frac{1}{i} \frac{\widetilde{F}(p)}{p - i\varepsilon}, \qquad \widetilde{F}(0) \text{ finite.} \tag{4.18}$$

Since the space translations can be extended to the field algebra, we can discuss the asymptotic statistical behaviour of its elements. In particular, the existence of anticommuting variables, due to eq. (3.6), imposes some restrictions on the functions, defining the structural automorphism $\alpha_{\bar{\Phi}}$. For the algebra (4.16) this requirements reads

$$\sigma(\bar{\Phi}, \bar{\Phi}_x) = \pm[\bar{g}(\infty) - \bar{g}(-\infty)] \int \bar{f}'(y)dy = (2k+1)\pi, \qquad \text{for } |x| > \Lambda, \tag{4.19}$$

or

$$\pm M_{\bar{g}}\ M_{\bar{f}} = (2k+1)\pi, \qquad k \text{ integer.}$$

This is exactly the condition used in [AMSa] to choose the appropriate $\bar{\Phi}$ for the construction of the fermionic creation and annihilation operators, apart from the difference in the symplectic form, hence, in the choice of $\mathcal{V} = (\mathcal{C}_0^\infty \times \partial^{-1}\mathcal{C}_0^\infty)$ there. The finite quantities $M_{\bar{f}}, M_{\bar{g}}$ then may be given a meaning of charges.

In momentum representation condition (4.19) reads

$$\begin{aligned}
&\lim_{x\to\pm\infty} i \int |p|\widetilde{\bar{f}}(|p|)\widetilde{\bar{g}}(|p|)(e^{i|p|x} - e^{-i|p|x})dp = \\
&= \lim_{x\to\pm\infty} \int_{-\infty}^{\infty} |p|\ \frac{\widetilde{F}(p)\widetilde{G}(-p)}{(p+i\varepsilon)(p-i\varepsilon)}(e^{i|p|x} - e^{-i|p|x})dp \\
&= (2k+1)\pi = \pm 2\pi \widetilde{F}(0)\widetilde{G}(0). \qquad (4.20)
\end{aligned}$$

5 Extension of the time evolution to the field algebra

The success to consider the observable algebra as Weyl algebra stems from the fact that the time evolution of interacting fermi systems can be described as quasifree evolution of the Weyl algebra. To see how the extension procedure works we have to be more explicit:

$$\tau_t\, W(f,g) = W(f_t, g_t) = W(e^{-iht}f, e^{iht}g)$$

where h is the one–particle Hamiltonian. In our models e^{iht} maps $\mathcal{C}_0^\infty$ into $\mathcal{C}_0^\infty$. According to (2.7),

$$V_t = W(\bar{f}_t - \bar{f}, \bar{g}_t - \bar{g}),$$

and extendibility of time evolution asks that both $\bar{f}_t - \bar{f}$, $\bar{g}_t - \bar{g}$ must belong to the space $\mathcal{C}_0^\infty$. If time evolution commutes with space translation, we better work in momentum space, so that we have to consider

$$(e^{-i\omega(|p|)t} - 1)\widetilde{\bar{f}}(p), \qquad (e^{i\omega(|p|)t} - 1)\widetilde{\bar{g}}(p). \qquad (5.1)$$

According to (4.18) $\widetilde{\bar{f}}(p)$ does not stay bounded for $p \to 0$ but has a singularity of the type $1/(p - i\varepsilon)$.

Extendibility of the time evolution from the observable to the crossed product field algebra then depends on the structure of the spectrum $\omega(p)$. If relation

$$\lim_{p\to 0} \frac{\omega(|p|)}{p} = M_\omega < \infty \qquad (5.2)$$

holds, this is enough to cure the singularities of $\bar{f}, \bar{g}$, so that $V_t \in \mathcal{A}$.

Let us now look at the models described in Section 4. For reasonable potentials the spectrum of the Luttinger model (4.11) satisfies condition (5.2), while for the spectrum (4.12) this is not the case: there, an additional singularity is present, corresponding to the appearance of a massive scalar particle that does not allow extension of the time evolution as an automorphism on $\mathcal{F}$.

Since the functions $\bar{f}, \bar{g}$ are defined with some additional restrictions, following from the requirement for $\{\delta_{1n}\}$ to be an odd element of $\mathcal{F}$, a question arises about the importance of this additional condition (4.19) for the asymptotic statistical behaviour of the time evolution. This means, we are interested in the limit behaviour

$$\lim_{t\to\pm\infty} i \int |p|\widetilde{\bar{f}}(|p|)\widetilde{\bar{g}}(|p|)(e^{i\omega(|p|)t} - e^{-i\omega(|p|)t})dp$$

with $\omega(p)$ satisfying (5.2) so that we have to consider integrals of the type

$$\lim_{t\to\infty} i \int_0^\infty \frac{\widetilde{H}(p)}{p^2+\varepsilon^2} e^{i\omega(p)t} dp = \lim_{t\to\infty} i \int_0^\infty \frac{\widetilde{H}(\omega^{-1}(q))}{[\omega^{-1}(q)^2+\varepsilon^2]} \frac{e^{iqt}dq}{\omega'(\omega^{-1}(q))}$$

with $H(x)$ being an $\mathcal{C}_0^\infty$–function. These integrals have exactly the same singularities (due to (5.2)) as those in (4.20). Then, together with (4.19), an analogous relation takes place also for time–shifted $\bar{\Phi}$–pair

$$\lim_{t\to\pm\infty} \sigma(\bar{\Phi}_t, \bar{\Phi}) = (2k+1)\pi, \qquad k \text{ integer},$$

so that

$$\{\delta_{1n}\} \cdot \tau_t\{\delta_{1n}\} + \tau_t\{\delta_{1n}\} \cdot \{\delta_{1n}\} = 0.$$

Therefore, when an extension of the time evolution as automorphism from observable to the field algebra is possible, the asymptotic anticommutativity of space translations on the odd subalgebra of $\mathcal{F}$ provides asymptotic anti–Abelianess of the time evolution there (compare comments in [H], p.228).

We want to emphasize that it is by far not evident that asymptotic behaviour of time and space translations is the same. For example, in the XY–model new features appear [A, HN].

Finally, we wonder how time evolution can be interpreted if (5.2) is violated, e.g. in the Schwinger model. In [AMSb] the view point is taken that the algebra is enlarged even more to CCR($\mathcal{V}$) where $\mathcal{V} = \{\bigcup_t \bar{\Phi}_t \cup \mathcal{V}_0\}$. $\mathcal{V}$ is always a linear space and we have already observed that $\lambda\bar{\Phi}$, λ real, with varying λ leads to a larger algebra than the desired fermi field algebra (fractional statistics). We prefer to take the view point that we do not want to enlarge the algebra but are satisfied to have a well defined time evolution of states since we have found the possibility to extend any state on $\mathcal{A}$ to a gauge invariant state on $\mathcal{F}$ (Section 2). Accordingly, any gauge invariant state on $\mathcal{F}$ has a well defined time evolution. Especially, time invariant state on $\mathcal{A}$ induces a time invariant state on $\mathcal{F}$. Properties of such states also for finite temperature are discussed in [V].

We consider the states (2.10)

$$\langle F^{(k)}|\pi(W(\psi))|F^{(k)}\rangle = \langle\Phi|\pi(\alpha^{-k}W(\psi))|\Phi\rangle.$$

They evolve in the course of time to

$$\langle F_t^{(k)}|\pi(W(\psi))|F_t^{(k)}\rangle = \langle\Phi|\pi(\alpha^k\tau_t W(\psi))|\Phi\rangle = \langle\Phi|\pi\circ\tau_t\circ\tau_{-t}\alpha^k\tau_t W(\psi)|\Phi\rangle.$$

As we have already mentioned, two states, ω_1 and ω_2, over a gauge invariant algebra can be combined in a not gauge invariant state iff the representations π_1 and $\pi_2 \circ \alpha^k$ are equivalent for some k. In the course of time ω_1 and ω_2 evolve to states, corresponding to the representations $\pi_1 \circ \tau_t$ and $\pi_2 \circ \tau_t$. Since π_2 by assumption is equivalent to $\pi_1 \circ \alpha^{-k}$, we demand that

$$\begin{aligned} \pi_1 \circ \tau_t &\approx \pi_2 \circ \tau_t \circ \alpha^{-k} \\ \pi_1 &\approx \pi_2 \circ \alpha^{-k} \circ \alpha^k \tau_t \alpha^{-k} \tau_{-t} \end{aligned}$$

and this only holds if $\alpha^k \tau_t \alpha^{-k} \tau_{-t}$ is an automorphism of $(\mathcal{A})''$, which is not the case in the Schwinger model. Therefore states that are not gauge invariant have no well defined time evolution, so that they are not physically acceptable and we have screening of the charge (confinement).

To summarize, the time evolution on the fermionic field algebra can be obtained as a naturally extended automorphism from the time evolution of the observable fields only in cases when short range interactions determine the behaviour of the system. Existence of long range forces, typical example being the Schwinger model, appears to be an obstacle to this. Of course, the fermionic field algebra is still well defined but we can only consider the time evolution of gauge–invariant, hence, not charged fermionic structures (consisting of equal number fermions and antifermions). This nonextensibility of the time evolution may be viewed as another manifestation of the confinement, which takes place in the Schwinger model.

Concluding remarks

We have demonstrated on simple examples the possibility to construct fermionic field algebra as a crossed product of the observable algebra by a proper α–action of the group of integer numbers $\mathbf{Z}$. α has to be a free (not–inner) automorphism of the observable algebra $\mathcal{A}$ which is a simplification for dimension 2 as compared to the specially directed monoidal category of endomorphisms for 4 dimensions in [BDLR] but still provides an analysis of various models. The field algebra so obtained has a local net structure, the ingredients being von Neumann algebras. The extension of automorphisms from observable to the field algebra is shown to be possible under a compatibility condition between the automorphism in question and the structural one used in the crossed product. As a direct consequence of this compatibility relation for the special case of space translations and of the net structure of the observable algebra appears the net structure also of the field algebra so that no further restrictions on α have to be imposed to guarantee the latter. Also, the states are shown to be extendible to the field algebra, inheriting the structure and properties of the states over the algebra of observables.

In the two cases of automorphisms of particular interest — space translations and time evolution, we have the following situation: the conditions to be fulfilled in order to have space translations extended to the field algebra and to have anticommuting fields present in it, are enough to specify the automorphism $\alpha(\bar{\Phi})$ by fixing $\bar{\Phi}$. Then, time evolution appears to be extendible only in the case of short range interactions but then it is also asymptotically anti–Abelian for the anticommuting fields, so that space translations and time evolution have the same asymptotic statistical behaviour. In the cases when long range forces prevent consistent extension of time evolution from observable to the field algebra, a well defined time evolution is shown to exist for gauge invariant states on the latter. In this context, charge screening (or confinement) in the Schwinger model may be understood as an absence of well defined time evolution for charged states.

The crossed product field algebra allows also for fractional statistics in one space dimension. This interesting possibility as well as the description of vacuum degeneracy in gauge models in the crossed product scheme will be considered separately.

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References

[J] P. Jordan, Z. Phys. **93**, 464 (1935); **98**, 759 (1936); **99**, 109 (1936).

[BN] M. Born, N. Nagendra–Nath, Proc. Ind. Acad. Sci. **3**, 318 (1936).

[ML] D.C. Mattis and E. Lieb, J. Math. Phys. **6**, 304 (1965).

[K] B. Klaiber, Lectures in Theoretical Physics. Boulder Lectures, 1967 (Gordon and Breach, New York, 1968).

[SW] R. Streater, I. Wilde, Nucl. Phys. **B24**, 561 (1970).

[H] R. Haag, Local Quantum Physics, (Springer–Verlag, Berlin Heidelberg, 1993).

[CR] A.L. Carey, S.N.M. Ruijsenaars, Acta Appl. Math. **10**, 1 (1987).

[AMSa] F. Acerbi, G. Morchio, F. Strocchi, Lett. Math. Phys. **26**, 13 (1992).

[AMSb] F. Acerbi, G. Morchio, F. Strocchi, Lett. Math. Phys. **27**, 1 (1993).

[DHR] S. Doplicher, R. Haag, J. Roberts, Commun. Math. Phys. **15**, 173 (1969).

[DR] S. Doplicher, J.E. Roberts, Ann. Math. **130**, 75 (1989).

[BDLR] D. Buchholz, S. Doplicher, R. Longo, J.E. Roberts, Commun. Math. Phys. **155**, 123 (1993).

[T] M. Takesaki, Theory of Operator Algebras I (Springer–Verlag, New York, 1979).

[A] H. Araki, in *On Klauder's Path: A Field Trip*, Eds. G. Emch, G. Hegerfeldt, L. Streit (World Scientific, Singapore, 1994), p. 11.

[B] O. Bratelli, Duke Math. J. **46**, 1 (1979).

[L] J.M. Luttinger, J. Math. Phys. **4**, 1154 (1963).

[S] J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[And] P.W. Anderson, Phys. Rev. Lett. **64**, 1839 (1990).

[V] A. Verbeure, J. Math. Phys. **32**, 689 (1991).

[GMR] H. Grosse, W. Maderner, C. Reitberger, in *Proc. Int. Symposium 'Generalized Symmetries in Physics'*, Eds. H.–D. Döbner, V. Dobrev and A. Ushveridze (World Scientific, Singapore, 1994).

[HN] T. Hudetz, H. Narnhofer, in preparation.