Low-temperature phase diagrams of quantum lattice systems. I. Stability for quantum perturbations of classical systems with finitely-many ground states

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Abstract

Starting from classical lattice systems in $d \geq 2$ dimensions with a regular zero-temperature phase diagram, involving a finite number of periodic ground states, we prove that the addition of a small quantum perturbation and/or increasing the temperature produce only smooth deformations of their phase diagrams. The quantum perturbations can involve bosons or fermions and

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can be of infinite range but decaying exponentially fast with the size of the bonds. For fermions, the interactions must be given by monomials of even degree in creation and annihilation operators. Our methods can be applied to some anyonic systems as well. Our analysis is based on an extension of Pirogov-Sinai theory to contour expansions in $d+1$ dimensions obtained by iteration of the Duhamel formula.

Keywords: Phase diagrams; quantum lattice systems; Pirogov-Sinai theory; contour expansions; low-temperature expansions.

Contents

1 Introduction

The study of phase diagrams of quantum lattice systems is a much less developed subject than its classical counterpart. There has been extensive numerical work on quantum phase diagrams at zero temperature, but rigorous studies, which are often unexpectedly difficult and rich in surprises, have been very few in number. A systematic exploration of the different types of ground states has been carried out only for one-dimensional systems – chains – (see for instance [11, 20, 1] and references therein).

Our present understanding of phase diagrams at low, but nonzero temperature is very limited, as well, although many important quantum-spin lattice systems have been rigorously studied at low temperatures (see for instance [31, 14, 10, 12, 36, 37, 18, 19, 22, 24, 23, 3, 2]). These studies provide useful illustrations of some of the phenomena involved, but they focus on extracting detailed information for special models, rather than on developing a general formalism of wider applicability. In this paper, we take the opposite attitude: We present some general black-box type results which, although typically far from optimal for each specific model, allow us to understand the broad features of some regions in the phase diagrams of quantum lattice systems dominated by a classical interaction.

Usually, the study of low-temperature phase diagrams involves a two-step process: First, the zero-temperature phase diagram is drawn, and, second, one analyzes which of the zero-temperature phases survive at nonzero temperatures. For classical systems, there is a general theory to handle the second part of this process, namely Pirogov-Sinai theory [29, 30, 35, 39, 4]. The bottom line of this theory can be summarized as follows: If the zero-temperature phase diagram is a *regular* phase diagram (in the sense of satisfying the Gibbs phase rule) involving a *finite* number of *periodic* ground states, and if in addition, the excitations of these ground states have an energy proportional to the size of their *boundaries* (Peierls condition), then the phase diagrams for low enough temperatures are only small deformations of the zero-temperature phase diagrams. In other words, the theory says that for systems with finitely degenerate ground states and obeying the Peierls condition, the entropy contribution to the free energy, present at non-zero temperatures, is only a small correction to the internal energy at low temperatures.

In this paper we extend this theory to systems with small quantum perturbations, and we conclude that the addition of these perturbations only leads to small deformations of the phase diagram if the temperature is low (in particular equal to zero). It is somehow surprising that, except for a pioneer announcement [28] which was never followed by full proofs, this natural extension of Pirogov Sinai theory has never been considered previously[1]. One may argue that this is because the extension (apparently) refers to the least interesting regions of a quantum phase diagram, namely those where the quantum part does not trigger any new effect. This is, however, a poor reason on two accounts: First, a "no-go" result is needed and useful, because it allows people hunting for quantum effects to rule out large regions of the phase diagram, saving effort and misunderstandings. Second, and more importantly, some quantum effects can, in fact, be studied by using our results. Indeed, we shall show in a subsequent paper [8] how, by combining the present theory with a perturbation scheme, one can analyze degeneracy-breaking effects induced by quantum perturbations and the associated phase transitions.

A noteworthy difference between our approach and many of the previous ones (eg. [31, 14, 18]), is that, instead of using the Trotter formula, we resort to an imaginary-time Dyson expansion based on an iteration of Duhamel's formula. The resulting expansion roughly corresponds to performing part of the limit involved in the Trotter formula, so a sum over a large number of small subintervals is replaced by an integral. Thus, we work in subregions of $\mathbb{Z}^d \times [0, \beta]$; the last coordinate — the "time-direction" — being a continuous one (with periodic boundary conditions), and our contours are piecewise-cylindrical surfaces whose "time"-sections are ordinary Pirogov-Sinai classical contours. We think that this approach has several advantages. On one hand, contour considerations are based on the surfaces naturally associated to the expansion, without the additional projection introduced in approaches based on the Trotter formula. This allows for simpler and clearer geometrical and combinatorial arguments, a fact also exploited, for instance, in [1]. On the other hand, the effects of quantum perturbations have a nice visualization: they change the sections of contours. If the system were purely classical, the contours would be straight cylinders of constant section; the quantum terms produce deformations or the appearance of "vacuum fluctuations" in the form of contours that appear and/or disappear at intermediate values of the "time" coordinate.

These observations permit us to make rigorous the usual heuristics about quantum perturbations having an "entropy effect" comparable to temperature. Indeed, all our bounds are in terms of the maximum of the quantum-coupling parameter, λ, and a temperature-parameter of the form $e^{-\beta \tilde{J}}$, for some coupling $\widetilde{J} > 0$. In particular, by letting $\beta \to \infty$ our formalism yields information about ground states: the "classical-like" contours extending all the way along the interval $[0, \beta]$ disappear, and all that remains are vacuum fluctuations in a "sea of spins" configured as in a classical ground state. The fact that these expected features are exhibited in such a simple and immediate way is, we believe, a nice feature of our approach.

For the convenience of the reader we summarize the hypotheses and results in the following section, which can be read as a "recipe" section. Readers interested in

[1] We have recently been informed by C. Borgs and R. Kotecký that they have also constructed an extension of Pirogov-Sinai theory to systems of the type analyzed in this paper.

the method itself can continue with the proofs and technicalities of the remaining sections.

2 Hypotheses and results

2.1 A formalism for quantum lattice systems

We consider particles with a finite number, N, of internal degrees of freedom, on a d-dimensional lattice $\mathbb{Z}^d$. The Hilbert space associated with each site of the lattice is isomorphic to $\mathbb{C}^N$. The system is governed by a Hamiltonian of the form

$$\mathbf{H} \;=\; \mathbf{H}^{\rm cl} + \mathbf{V} \tag{2.1}$$

where $\mathbf{H}^{\rm cl}$ is interpreted as the "classical part", and $\mathbf{V}$ as the "quantum perturbation". The former consists of finite range interactions, and is assumed to have a finite number of periodic ground states. The interactions constituting $\mathbf{V}$ can be of infinite range, *provided* their strengths decay exponentially with the size of their supports. To make these assumptions precise, we need some standard definitions.

A quantum lattice model can be interpreted either as a *spin system* or as a *lattice gas*. In a quantum spin system, there is a particle at each site of the lattice having a finite number of internal degrees of freedom. In describing such a system there is no need to refer to the statistics of the particles. In contrast, the particles in a lattice gas are allowed to hop from site to site. Hence their statistics plays an important role, and it is necessary to introduce Fock spaces to describe them. The mathematical framework required to describe these systems has been introduced in [5, 15, 32]. We summarize the essential features below.

Quantum Spin Systems:

For a quantum spin system the Hilbert space, $\mathcal{H}_\Lambda$, associated with a finite subset Λ of the lattice is given by the tensor product

$$\mathcal{H}_\Lambda \;:=\; \bigotimes_{x\in\Lambda} \mathcal{H}_x \tag{2.2}$$

where each $\mathcal{H}_x$ is isomorphic to $\mathbb{C}^N$ [An infinite tensor product of Hilbert spaces is intentionally avoided, since it is not uniquely defined and is complicated to deal with. Infinite volume limits are considered only at the level of observable algebras and states].

Bases of $\mathcal{H}_\Lambda$ can be put in correspondence with configurations on Λ in the following way: We choose an orthonormal basis

$$\{e^x_\sigma\}_{\sigma\in I} \quad \text{with} \;\; I := \{1, \ldots, N\} \tag{2.3}$$

in $\mathcal{H}_x$. Let Ω_Λ be the set of configurations $\{\omega_\Lambda\}$ in Λ, defined as the set of all assignments $\{\sigma_x\}_{x\in\Lambda}$ of an element $\sigma_x \in I$ to each x. If $X \subset \Lambda$, then ω_X denotes the restriction of the configuration ω_Λ to the subset X. For each configuration $\omega_\Lambda = \{\sigma_x\}_{x\in\Lambda} \in \Omega_\Lambda$, let e_{ω_Λ} be the vector defined as

$$e_{\omega_\Lambda} := \otimes_{x\in\Lambda}\, e^x_{\sigma_x} \tag{2.4}$$

The set of vectors $\{e_{\omega_\Lambda}\}_{\omega_\Lambda\in\Omega_\Lambda}$ is an orthonormal basis of $\mathcal{H}_\Lambda$.

A *state* of a quantum lattice system is defined as a positive linear functional on a suitable C^*-algebra. To construct the latter, we start with the algebra $\mathcal{A}_\Lambda$ of all bounded operators (matrices), acting on $\mathcal{H}_\Lambda$, with the usual operator norm and with hermitian conjugation as the $*$-involution. The algebras $\mathcal{A}_\Lambda$ can be considered to be partially nested, i.e.,

$$\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2} \;\text{ if }\; \Lambda_1 \subset \Lambda_2, \tag{2.5}$$

by identifying each operator $A_1 \in \mathcal{A}_{\Lambda_1}$ with the operator $A_1 \otimes \mathbf{1}_{\Lambda_2\setminus\Lambda_1} \in \mathcal{A}_{\Lambda_2}$, where $\mathbf{1}$ denotes the identity operator. Moreover, the algebras $\mathcal{A}_\Lambda$ are *local* , i.e. if $A_1 \in \mathcal{A}_{\Lambda_1}$ and $A_2 \in \mathcal{A}_{\Lambda_2}$, and $\Lambda_1 \cap \Lambda_2 = \emptyset$, then

$$A_1\, A_2 = A_2\, A_1. \tag{2.6}$$

The norm closure of $\bigcup_{\Lambda\nearrow\mathbb{Z}^d} \mathcal{A}_\Lambda$ defines an algebra which we denote by $\mathcal{A}$. It is the quasilocal C^*-algebra of observables associated to the infinite lattice $\mathbb{Z}^d$. All local algebras $\mathcal{A}_\Lambda$ are subalgebras of $\mathcal{A}$.

The group $\mathbf{T}_{\mathbb{Z}^d}$ of space translations acts as a $*$-automorphism group $\{\tau_a : a \in \mathbb{Z}^d\}$ on $\mathcal{A}$, with

$$\mathcal{A}_{\Lambda+a} = \tau_a\, \mathcal{A}_\Lambda, \tag{2.7}$$

for $\Lambda \subset \mathbb{Z}^d$. (The definition of τ_a is obvious.)

An *interaction* of a quantum spin system is a function Φ from finite, nonempty subsets B of $\mathbb{Z}^d$ to selfadjoint observables $\Phi_B \in \mathcal{A}_B$. The set of interactions, $\{\Phi_B\}$, constitutes the Hamiltonian of the system. We shall assume that the interactions are translation invariant, i.e,

$$\tau_a\, \Phi_B = \Phi_{B+a}, \quad \text{for each } a \in \mathbb{Z}^d,\; B \subset \mathbb{Z}^d. \tag{2.8}$$

An interaction will be called *classical* if we can choose a basis $\{e^x_\sigma\}_{\sigma\in I}$ in $\mathcal{H}_x$ such that, for every finite $\Lambda \subset \mathbb{Z}^d$, the matrices Φ_B, $B \subset \Lambda$, are diagonal in the basis $\{e_{\omega_\Lambda}\}_{\{\omega_\Lambda\}\in\Omega_\Lambda}$, defined through (2.4). In this case the interactions are uniquely defined by the numbers

$$\begin{aligned} \phi_B(\omega) &:= \langle e_{\omega_B}|\Phi_B|e_{\omega_B}\rangle \\ &= \langle e_{\omega_\Lambda}|\Phi_B|e_{\omega_\Lambda}\rangle \quad , \quad \forall\Lambda \supset B\,. \end{aligned} \tag{2.9}$$

Quantum lattice gases:

In order to describe the itinerant particles of a quantum lattice gas, one starts with the one-particle Hilbert space

$$\mathcal{H}^{(1)} := \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \tag{2.10}$$

It represents a single particle which has N internal degrees of freedom and is confined to a lattice $\mathbb{Z}^d$. A basis of this space can be obtained from the bases $\{e^x_{\sigma_x}\}_{\sigma_x \in I}$ of $\mathcal{H}_x$, introduced above, (eqn.(2.3)). We shall also use $e^x_{\sigma_x}$ to denote the vector in $\mathcal{H}^{(1)}$ which has all other summands equal to zero.

To incorporate the statistics of the particles we construct Fock spaces

$$\mathcal{F}_P(\mathcal{H}^{(1)}) = P \bigoplus_{n \geq 0} \left(\mathcal{H}^{(1)}\right)^n \tag{2.11}$$

where

$$\left(\mathcal{H}^{(1)}\right)^n = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)} \otimes \cdots \otimes \mathcal{H}^{(1)} \tag{2.12}$$

denotes the n-fold tensor product of $\mathcal{H}^{(1)}$ with itself, $(\mathcal{H}^{(1)})^0 := \mathbb{C}$, and P is the orthogonal projection onto the subspace with the right symmetry properties. We shall consider bosons and fermions.

(i) For *bosons*, P is the projection onto the symmetric subspace, defined on each $\left(\mathcal{H}^{(1)}\right)^n$ by

$$P_{\text{Bose}}(e^{x_1}_{\sigma_1} \otimes \cdots \otimes e^{x_n}_{\sigma_n}) = \frac{1}{n!} \sum_{\pi} e^{x_{\pi 1}}_{\sigma_{\pi_1}} \otimes \cdots \otimes e^{x_{\pi n}}_{\sigma_{\pi n}}, \tag{2.13}$$

where $e^{x_i}_{\sigma_i} \in \mathcal{H}^{(1)}$, for all i, and π ranges over all permutations of the indices $(1, \ldots n)$.

(ii) For *fermions*, P is the projection onto the antisymmetric subspace

$$P_{\text{Fermi}}(e^{x_1}_{\sigma_1} \otimes \cdots \otimes e^{x_n}_{\sigma_n}) = \frac{1}{n!} \sum_{\pi} \operatorname{sgn}(\pi)\, e^{x_{\pi 1}}_{\sigma_{\pi_1}} \otimes \cdots \otimes e^{x_{\pi n}}_{\sigma_{\pi n}}, \tag{2.14}$$

where $\operatorname{sgn}(\pi)$ is $+1$ if the permutation π is even and -1 if it is odd. The RHS of (2.14) vanishes if any vector $e^{x_i}_{\sigma_i}$ appears more than once in the tensor product. This implies that it is impossible to create two fermions in the same state, in accordance with the *Pauli exclusion principle*.

Identical constructions can be made for finite volumes, i.e. when $\mathcal{H}^{(1)}$ is replaced by

$$\begin{aligned} \mathcal{H}^{(1)}_\Lambda &:= \ell^2(\Lambda) \otimes \mathbb{C}^N \\ &\simeq \bigoplus_{x \in \Lambda} \mathcal{H}_x \ , \end{aligned} \tag{2.15}$$

for finite $\Lambda \subset \mathbb{Z}^d$. The fermion Fock space for a finite volume Λ is given by

$$\mathcal{F}_{P_{\text{Fermi}}}(\mathcal{H}^{(1)}_\Lambda) = P_{\text{Fermi}} \bigoplus_{n \geq 0} \left(\mathcal{H}^{(1)}_\Lambda\right)^n \tag{2.16}$$

It follows from the Pauli principle that the direct sum in (2.16) terminates at $n = N|\Lambda|$.

The formulae (2.13) and (2.14) can be used to define some bases in the Fock spaces. One must take into account the fact that different vectors of $\left(\mathcal{H}^{(1)}\right)^n$, namely

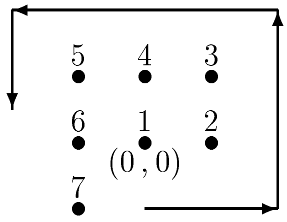

Figure 1: Spiral order in $\mathbb{Z}^2$

those which differ only in a permutation of the factors $e^{x_i}_{\sigma_i}$, are mapped by the projection operator P onto the *same* vector of the Fock space (up to a sign).

To avoid ambiguities, we choose a total ordering of the sites in $\mathbb{Z}^d$. For future convenience, we choose the *spiral order*, depicted in Figure 1 for $d = 2$. We shall say that (x_1, σ_1) is *earlier* than (x_2, σ_2) — and write $(x_1, \sigma_1) \preceq (x_2, \sigma_2)$ — if $x_1 \preceq x_2$ and, for $x_1 = x_2$, $\sigma_1 \leq \sigma_2$. The spiral order has the convenient property that

the set of sites earlier than those in a given, finite set B is also a finite *set $B^{(\preceq)}$.* (2.17)

This property will be useful in defining states corresponding to "classical" boundary conditions. [See discussion following (2.46)].

An orthonormal basis of $\mathcal{F}_P(\mathcal{H}^{(1)})$ is given by the vectors

$$|n_{x_1\sigma_1} \cdots n_{x_k\sigma_k}\rangle := \Big(\underbrace{e^{x_1}_{\sigma_1} \otimes \cdots \otimes e^{x_1}_{\sigma_1}}_{n_{x_1\sigma_1} \text{ times}} \otimes \ldots \otimes \underbrace{e^{x_k}_{\sigma_k} \otimes \cdots \otimes e^{x_k}_{\sigma_k}}_{n_{x_k\sigma_k} \text{ times}} \Big)_{\preceq} \tag{2.18}$$

where "$(\cdots)_{\preceq}$" indicates that the braced factors must be ordered such that each subscript $(x_i\sigma_i)$ is earlier than the ones to its right. The vectors (2.18) involve infinitely many occupation numbers $n_{x\sigma}$, but only finitely many, namely the indicated ones, are nonzero. By restricting the sites x_i to those in a finite region $\Lambda \subset \mathbb{Z}^d$, one obtains a basis of $\mathcal{F}_P(\mathcal{H}^{(1)}_\Lambda)$ in a similar manner. For fermions, each $n_{x\sigma}$ is either 0 or 1.

Having introduced the Fock spaces appropriate for the description of bosons and fermions, we proceed to define suitable C^**-algebras* of observables. The C^*-algebras are generated by the creation and annihilation operators on Fock space obeying the canonical anticommutation relations (CAR), for fermions, and the Weyl form of the canonical commutation relations (CCR), for bosons.

The annihilation and creation operators on $\mathcal{F}_P(\mathcal{H}^{(1)})$ are defined as

$$\mathbf{c}_{x\sigma} := P\,\mathbf{a}_{x\sigma}P \tag{2.19}$$

and

$$\mathbf{c}^*_{x\sigma} := P\,\mathbf{a}^*_{x\sigma}P, \tag{2.20}$$

with

$$\mathbf{a}_{x\sigma}(e^{x_1}_{\sigma_1}\otimes\cdots\otimes e^{x_n}_{\sigma_n}) := \sqrt{n}\,(e^x_\sigma, e^{x_1}_{\sigma_1})\ e^{x_2}_{\sigma_2}\otimes\cdots\otimes e^{x_n}_{\sigma_n}, \tag{2.21}$$

$$\mathbf{a}^*_{x\sigma}(e^{x_1}_{\sigma_1}\otimes\cdots\otimes e^{x_n}_{\sigma_n}) := \sqrt{n+1}\ e^x_\sigma\otimes e^{x_1}_{\sigma_1}\otimes\cdots\otimes e^{x_n}_{\sigma_n} \tag{2.22}$$

where $(e^x_\sigma, e^{x_1}_{\sigma_1})$ denotes the scalar product of the vectors e^x_σ and $e^{x_1}_{\sigma_1}$. Furthermore,

$$\mathbf{a}_{x\sigma}|0>:=0 \tag{2.23}$$

and

$$\mathbf{a}^*_{x\sigma}|0>:=e^x_\sigma \tag{2.24}$$

where

$$|0\rangle := (1,0,0,\cdots)\in\bigoplus_{n\geq 0}(\mathcal{H}^{(1)})^n \tag{2.25}$$

denotes the *vacuum*, i.e., the zero-particle state.

For bosons, the operators defined through eqs.(2.19) and (2.20) satisfy the canonical commutation relations (CCR):

$$\begin{aligned}[\mathbf{c}_{x_1\sigma_1},\mathbf{c}_{x_2\sigma_2}] &= [\mathbf{c}^*_{x_1\sigma_1},\mathbf{c}^*_{x_2\sigma_2}] = 0\\ [\mathbf{c}_{x_1\sigma_1},\mathbf{c}^*_{x_2\sigma_2}] &= (e^{x_1}_{\sigma_1},e^{x_2}_{\sigma_2})\mathbf{1},\end{aligned} \tag{2.26}$$

where $\mathbf{1}$ is the identity operator. For fermions, the corresponding operators satisfy the canonical anticommutation relations (CAR):

$$\begin{aligned}\{\mathbf{c}_{x_1\sigma_1},\mathbf{c}_{x_2\sigma_2}\} &= \{\mathbf{c}^*_{x_1\sigma_1},\mathbf{c}^*_{x_2\sigma_2}\} = 0\\ \{\mathbf{c}_{x_1\sigma_1},\mathbf{c}^*_{x_2\sigma_2}\} &= (e^{x_1}_{\sigma_1},e^{x_2}_{\sigma_2})\mathbf{1}\end{aligned} \tag{2.27}$$

It follows from the CAR that $\|\mathbf{c}_{x\sigma}\| = \|\mathbf{c}^*_{x\sigma}\| = 1$.

The basis vectors, $|n_{x_1\sigma_1}\cdots n_{x_k\sigma_k}\rangle$, defined by eqn.(2.18), can be alternatively expressed in terms of the action of the creation operators on the vacuum.

$$|n_{x_1\sigma_1}\cdots n_{x_k\sigma_k}\rangle = \frac{1}{\sqrt{\prod_{i=1}^k n_{x_i\sigma_i}!}}\Big((\mathbf{c}^*_{x_1\sigma_1})^{n_{x_1\sigma_1}}\cdots(\mathbf{c}^*_{x_k\sigma_k})^{n_{x_k\sigma_k}}\Big)_{\preceq}|0\rangle\ . \tag{2.28}$$

The labelling is consistent with the fact that these vectors are simultaneous eigenvectors of the *number operators*

$$\mathbf{n}_{x\sigma} := \mathbf{c}^*_{x\sigma}\mathbf{c}_{x\sigma} \tag{2.29}$$

with the eigenvalues $n_{x\sigma}$ taking values 0 or 1 for fermions, and 0 or any natural number for bosons. More generally

$$\mathbf{c}^*_{x_i\sigma_i}|n_{x_1\sigma_1}\cdots n_{x_k\sigma_k}\rangle = e^{i\alpha_\preceq}\sqrt{n_{x_i\sigma_i}+1}\ |n_{x_1\sigma_1}\cdots n_{x_i\sigma_i}+1\cdots n_{x_k\sigma_k}\rangle \tag{2.30}$$

and similarly

$$\mathbf{c}_{x_i\sigma_i}|n_{x_1\sigma_1}\cdots n_{x_k\sigma_k}\rangle = e^{i\alpha_\preceq}\sqrt{n_{x_i\sigma_i}}\ |n_{x_1\sigma_1}\cdots n_{x_i\sigma_i}-1\cdots n_{x_k\sigma_k}\rangle \tag{2.31}$$

where $\alpha_{\preceq}$ is a phase which depends on the $n_{x_j\sigma_j}$ with (x_j, σ_j) strictly earlier than (x_i, σ_i):

$$\alpha_{\preceq} = 0 \quad \text{for bosons,} \tag{2.32}$$
$$\alpha_{\preceq} = 0, \pi \quad \text{for fermions.} \tag{2.33}$$

Let $\mathcal{B}_\Lambda$ be the $*$-algebra generated by the identity and the fermionic annihilation operators, $\mathbf{c}_{x\sigma}$, with $x \in \Lambda$. It is referred to as the field algebra and is larger than the algebra of observables. The algebra $\mathcal{A}_\Lambda$ of local observables is the subalgebra of $\mathcal{B}_\Lambda$ consisting of all those operators which can be expressed as sums of monomials of even degree in the creation and annihilation operators associated with the lattice sites $x \in \Lambda$.[2] For $A_1 \in \mathcal{A}_{\Lambda_1}$ and $A_2 \in \mathcal{A}_{\Lambda_2}$

$$[A_1, A_2] = 0 \;\text{ if }\; \Lambda_1 \cap \Lambda_2 = \emptyset \tag{2.34}$$

and hence the algebras $\mathcal{A}_\Lambda$ are *local.* For bosons the creation and annihilation operators are not bounded. This is because there is no bound on the number of particles in the same one-particle state. The technical difficulties posed by this unboundedness can be avoided by considering bounded functions of these operators. One such choice yields the Weyl operators which are defined as

$$W_{x\sigma}(a,b) = \exp\left(ia\Phi_{x\sigma} + ib\Pi_{x\sigma}\right), \quad a, b \in \mathbb{R} \tag{2.35}$$

where

$$\Phi_{x\sigma} = \frac{\mathbf{c}_{x\sigma} + \mathbf{c}^*_{x\sigma}}{\sqrt{2}} \tag{2.36}$$

and

$$\Pi_{x\sigma} = \frac{\mathbf{c}_{x\sigma} - \mathbf{c}^*_{x\sigma}}{\sqrt{2}i} \tag{2.37}$$

The operators $\mathbf{c}_{x\sigma}$ and $\mathbf{c}^*_{x\sigma}$ are the bosonic annihilation and creation operators satisfying the CCR, (2.26). The Weyl operators satisfy the commutation relations

$$W_{x\sigma}(a,b)\, W_{x'\sigma'}(a',b') = \exp\Big(i(ab' - a'b)\delta_{xx'}\,\delta_{\sigma\sigma'}\Big) W_{x'\sigma'}(a',b')\, W_{x\sigma}(a,b) \tag{2.38}$$

[2]It is often reasonable to demand that observables are gauge invariant. A gauge transformation $\mathcal{G}_\phi$ is defined by its action on the annihilation operators

$$\mathcal{G}_\phi : \mathbf{c}_{x\sigma} \longmapsto e^{i\phi}\mathbf{c}_{x\sigma}; \;\; \phi \in \mathbb{R}$$

The algebra $\mathcal{A}_\Lambda$ of local observables could also be defined as the subalgebra of $\mathcal{B}_\Lambda$ consisting of all those elements which are invariant under the above transformation, i.e.,

$$A \in \mathcal{A}_\Lambda \;\text{ if }\; A \in \mathcal{B}_\Lambda \quad\text{and}\quad \mathcal{G}_\phi(A) = A.$$

Consequently a gauge invariant observable is given by a sum of terms each having an equal number of creation and annihilation operators.

which are called the Weyl form of the CCR. A quasilocal C^*-algebra suitable for the description of bosons can be generated from these Weyl operators.

In both cases, the fermionic and the bosonic case, the *local* algebras are nested w.r.t. inclusions of the localization regions, i.e.,

$$\mathcal{A}_{\Lambda_1} \subseteq \mathcal{A}_{\Lambda_2} \text{ if } \Lambda_1 \subseteq \Lambda_2 \tag{2.39}$$

and the quasilocal algebra of local observables is the norm closure of $\bigcup_{\Lambda \nearrow Z^\nu} \mathcal{A}_\Lambda$, in complete analogy with quantum spin systems. Furthermore, let $\mathcal{B}$ be the quasilocal C^*-algebra defined as

$$\mathcal{B} = \overline{\bigcup_{\Lambda \nearrow Z^\nu} \mathcal{B}_\Lambda}^{\text{norm}}, \tag{2.40}$$

where $\mathcal{B}_\Lambda$ is the $*$-algebra generated by the identity and annihilation operators for fermions, and by the Weyl operators for bosons.

As in a quantum spin system, an interaction in a quantum lattice gas is given by selfadjoint operators $\Phi_B \in \mathcal{A}_B$, for *finite* subsets B of the lattice. It is to be noted that it is implicitly assumed here that the Φ_B are bounded operators. This imposes severe restrictions on the allowed interactions in a bosonic lattice gas. For fermions, the interactions Φ_B are given by sums of monomials of even degree in creation and annihilation operators. We write this symbolically as

$$\Phi_B = \sum_{\underline{B}} \Phi_{\underline{B}} \tag{2.41}$$

where each $\Phi_{\underline{B}}$ is an even monomial and $\sum_{\underline{B}}$ denotes the sum over all such monomials (with support in B) comprising the interaction Φ_B. In our formulation of the low temperature expansion, it is necessary to express the fermionic interactions in terms of their constituent monomials, in order to arrive at a precise definition of quantum contours [see Sect. 3.3]. In order to have a unified treatment for bosons and fermions, we shall denote all quantum interactions in the sequel by $\Phi^{\mathrm{q}}_{\underline{B}}$, with the understanding that, for fermions, $\Phi^{\mathrm{q}}_{\underline{B}}$ is an even monomial (as mentioned above), whereas for bosons $\Phi^{\mathrm{q}}_{\underline{B}} = \Phi^{\mathrm{q}}_{B}$. Moreover, the notations

$$\underline{B} \ni 0 \quad ; \underline{B} \cap \Lambda \neq 0 \tag{2.42}$$

will be used to denote that the set B, corresponding to the monomials $\Phi^{\mathrm{q}}_{\underline{B}}$ (for fermions), and to the interaction Φ^{q}_{B} (for bosons), satisfies the above conditions, (2.42). For simplicity we shall assume that the interactions are translation invariant. However, our formalism can be easily extended to periodic interactions.

If an interaction, Φ_B, is diagonal in the basis formed by the vectors defined in (2.18), then it is called *classical* and is denoted by Φ^{cl}_{B}.

A *bond* is defined as a set $B \subset \mathbb{Z}^d$, for which $\Phi_B \neq 0$. In particular, a *quantum bond* is defined as follows:

Definition 2.1 *A* quantum bond *is a set B for which $\Phi^{\mathrm{q}}_{\underline{B}} \neq 0$. In the sequel, the symbol $\underline{B}$ will often be used to denote the support of $\Phi^q_{\underline{B}}$ and will also be referred to as a* quantum bond.

The *range* of the interactions is defined as the maximum of the diameters of the bonds (defined with any convenient translation invariant notion of distance on $\mathbb{Z}^d$). The classical part of the interactions will be assumed to be of finite range. The quantum perturbation can be of infinite range, but is assumed to satisfy a summability condition of the form

$$\sum_{\underline{B}\ni 0} \|\Phi^q_{\underline{B}}\| e^{\kappa\, \mathrm{diam}\, B} \;<\; \infty \tag{2.43}$$

for $\kappa > 0$ large enough.

If the occupancy of *every* site of the lattice $\mathbb{Z}^d$ is chosen to be one then the *lattice gas* reduces to a *spin system*.

We now introduce the notion of *boundary conditions* which is of crucial importance in the determination of phase diagrams. As mentioned before, we restrict our attention to finite subsets Λ of the lattice $\mathbb{Z}^d$. We choose some periodic configuration s on $\mathbb{Z}^d$, which is defined by the occupation numbers $n_{x\sigma} = s_{x\sigma}$ for $x \in \mathbb{Z}^d, 1 \le \sigma \le N$. The exterior configuration, s_{Λ^c}, with $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$, is called a *boundary condition*.

It is evident that our formalism must be generalized, because the configuration s does not correspond to a vector in the Fock space $\mathcal{F}_P(\mathcal{H}^{(1)})$. The vectors in Fock space are square-summable superpositions of vectors with *finite total occupation number*, whereas, for the vector $|s\rangle$, corresponding to a configuration s, this number is *infinite*, (unless $s_{x\sigma} = 0$, for all x, σ).

Thus, instead of considering the Fock space $\mathcal{F}_P(\mathcal{H}^{(1)})$, we consider a Hilbert space of states corresponding to *local alterations* of the configuration s. Technically, this means that we construct a space $\mathcal{F}^s_P(\mathcal{H}^{(1)})$ which has $|s\rangle$ as a *cyclic vector*. This construction is possible if we can prove that s defines a *state*, i.e., a positive, normalized linear functional on the quasilocal C^*-algebra $\mathcal{B}$. Since $\mathcal{B}$ is generated by the identity and the annihilation operators, this amounts to showing that we can define expectation values of the form

$$\langle s|\mathbf{b}_{x_1\sigma_1}\cdots\mathbf{b}_{x_n\sigma_n}|s\rangle\ , \tag{2.44}$$

where $\mathbf{b}_{x_i\sigma_i} = \mathbf{c}_{x_i\sigma_i}$, or $\mathbf{b}_{x_i\sigma_i} = \mathbf{c}^*_{x_i\sigma_i}$, for $1 \le i \le n$. We do this through the following limiting procedure: We consider the vectors $|s_\Lambda\rangle \in \mathcal{F}_P(\mathcal{H}^{(1)})$ defined by the occupation numbers

$$n_{x\sigma} \;=\; \begin{cases} s_{x\sigma} & \text{if } x \in \Lambda \\ 0 & \text{else}\ , \end{cases} \tag{2.45}$$

and we set

$$\langle s|\mathbf{b}_{x_1\sigma_1}\cdots\mathbf{b}_{x_n\sigma_n}|s\rangle \;:=\; \lim_{\Lambda\nearrow\mathbb{Z}^d} \langle s_\Lambda|\mathbf{b}_{x_1\sigma_1}\cdots\mathbf{b}_{x_n\sigma_n}|s_\Lambda\rangle\ . \tag{2.46}$$

This limit exists due to property (2.17) which implies that the phases of the matrix elements on the RHS of (2.46) stabilize once $\Lambda \supset \{x_1, \cdots, x_n\}_{(\preceq)}$. Our procedure defines the Hilbert spaces $\mathcal{F}_P^s(\mathcal{H}^{(1)})$ via the standard GNS construction [6, Section 2.3.3]. For $\Lambda \subset \mathbb{Z}^d$ and boundary condition s, we define the finite-volume *partial trace* for $\mathbf{A} \in \mathcal{B}_\Lambda$ as

$$\mathrm{Tr}\,_\Lambda^s \mathbf{A} \;:=\; \sum_{v_\Lambda} \langle v_\Lambda \otimes s_{\Lambda^c} | \mathbf{A} | v_\Lambda \otimes s_{\Lambda^c} \rangle \;, \tag{2.47}$$

where $\{|v_\Lambda\rangle\}$ is an orthonormal basis of $\mathcal{F}_P(\mathcal{H}_\Lambda^{(1)})$ of the form (2.28). (Note that $|v_\Lambda \otimes s_{\Lambda^c}\rangle \in \mathcal{F}_P^s(\mathcal{H}^{(1)})$.)

We define the Hamiltonian, $\mathbf{H}_\Lambda$, associated with a finite subset Λ of the lattice as follows:

$$\mathbf{H}_\Lambda \;:=\; \sum_{\underline{B} \cap \Lambda \neq \emptyset} \mathbf{P}_\Lambda^s \Phi_{\underline{B}} \mathbf{P}_\Lambda^s \;, \tag{2.48}$$

where $\Phi_{\underline{B}} \in \mathcal{A}_B$, and $\mathbf{P}_\Lambda^s$ is the orthogonal projection operator onto the subspace

$$\{\mathbf{A}|s\rangle : \mathbf{A} \in \mathcal{B}_\Lambda\}$$

of $\mathcal{F}_P^s(\mathcal{H}^{(1)})$. That is, we eliminate those matrix elements of the operators $\Phi_{\underline{B}}$, with B intersecting both Λ and its complement, that would lead to a change in the configuration outside Λ.

The *finite-volume free energy density* for a set of interactions $\{\Phi_{\underline{B}}\}$, boundary condition s, and inverse temperature β is given by the expression

$$f_s(\Lambda) \;:=\; \frac{-1}{\beta|\Lambda|} \ln \mathrm{Tr}\,_\Lambda^s e^{-\beta \mathbf{H}_\Lambda} \;. \tag{2.49}$$

Its infinite-volume limit

$$f \;:=\; \lim_{\Lambda \nearrow \mathbb{Z}^d} f_s(\Lambda) \;, \tag{2.50}$$

is the *free-energy density*. The limit may be taken in the sense of van Hove.

The *finite-volume Gibbs state* for a set of interactions $\{\Phi_{\underline{B}}\}$, boundary condition s and inverse temperature β is the linear functional on $\mathcal{B}_\Lambda$ defined by

$$\mathcal{B}_\Lambda \ni \mathbf{A} \;\mapsto\; \frac{\mathrm{Tr}\,_\Lambda^s \mathbf{A}\, e^{-\beta \mathbf{H}_\Lambda}}{\mathrm{Tr}\,_\Lambda^s e^{-\beta \mathbf{H}_\Lambda}} \;. \tag{2.51}$$

2.2 Assumptions on the interactions

We consider interactions of the form $\Phi_{\underline{\mu}\, B} = \Phi^{\mathrm{cl}}_{\underline{\mu}\, B} + \Phi^{\mathrm{q}}_{\underline{B}}$ which give rise to a class of Hamiltonians of the form

$$\begin{aligned} \mathbf{H}_\Lambda \;&=\; \mathbf{H}^{\mathrm{cl}}_{\underline{\mu}\, \Lambda} + \mathbf{V}_\Lambda \\ &=\; \Big[\sum_{B \cap \Lambda \neq \emptyset} \Phi^{\mathrm{cl}}_{\underline{\mu}\, B}\Big] + \Big[\sum_{\underline{B} \cap \Lambda \neq \emptyset} \Phi^{\mathrm{q}}_{\underline{B}}\Big] \;. \end{aligned} \tag{2.52}$$

Note: We remark that the *self-adjointness of the interactions* plays no essential role in this work. Hence we do not assume it. In particular, this means that the eigenvalues of the classical interactions, $\Phi^{\rm cl}_{\underline{\mu}\,B}$, are allowed to be complex.

We requires the following hypotheses:

(H1) $\{\Phi^{\rm cl}_{\underline{\mu}\,B}\}$ is a set of *classical, finite-range* interactions parametrized by $\underline{\mu} := (\mu_1, \ldots, \mu_{P-1})$. The "coordinate axes" of the phase diagram are labelled by μ_i, $1 \le i \le P-1$. The range of the interactions is assumed to be independent of $\underline{\mu}$. We shall assume translation invariance, but analogous results can be obtained for periodic interactions as well. The "classical" Hamiltonian $\mathbf{H}^{\rm cl}_{\underline{\mu}\,\Lambda}$ is assumed to satisfy the standard hypotheses of Pirogov-Sinai theory, namely:
There is a non-empty open set $\mathcal{O} \subset \mathbb{R}^{P-1}$ such that the following properties are satisfied:

(H1.1) *Existence of a common eigenbasis*. There is a basis of the form given in (2.18) in which all operators $\Phi^{\rm cl}_{\underline{\mu}\,B}$ are simultaneously diagonal, for all $\underline{\mu} \in \mathcal{O}$.

(H1.2) *Smoothness properties*. The functions $\mathcal{O} \ni \underline{\mu} \mapsto \Phi^{\rm cl}_{\underline{\mu}\,B}$ are *differentiable* in operator norm. The functions, as well as their derivatives, are uniformly bounded in norm. Typically, $\mathcal{O}$ is a bounded region. If the functions $\Phi^{\rm cl}_{\underline{\mu}\,B}$ have a *linear* dependence on $\underline{\mu}$, the parameters μ_i correspond to fields or chemical potentials.

(H1.3) *Finite degeneracy*. The set formed by *all periodic* ground states of $\{\Phi^{\rm cl}_{\underline{\mu}\,B}\}$, for all $\underline{\mu} \in \mathcal{O}$, is a finite family

$$\mathcal{K} \;=\; \{s_1, \ldots, s_P\}\;. \tag{2.53}$$

In the present situation, a periodic configuration s is a ground state for $\{\Phi^{\rm cl}_{\underline{\mu}\,B}\}$ if

$$\operatorname{Re} e_{\underline{\mu}}(s) \;=\; \min_{\tilde{s}\text{ periodic}} \operatorname{Re} e_{\underline{\mu}}(\tilde{s})\;, \tag{2.54}$$

where

$$e_{\underline{\mu}}(s) \;:=\; \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{B\cap\Lambda\neq\emptyset} \Phi^{\rm cl}_{\underline{\mu}\,B}(s)\;. \tag{2.55}$$

The symbol $|\Lambda|$ denotes the cardinality of the set Λ, and the limit is taken, for instance, via sequences of growing parallelepipeds. [*Note*: The definition of classical ground states is more complicated in the presence of infinite degeneracy or non-periodicity. See for example [38, Appendix B] and references therein.]

For *periodic* configurations the limit (2.55) exists, and the specific energy is equivalently given by the average energy contribution of each fundamental cell of the configuration, i.e.,

$$e_{\underline{\mu}}(s) \;=\; \frac{1}{|W|} \sum_{x\in W} e_{\underline{\mu}\,x}(s) \tag{2.56}$$

where $W \subset \mathbb{Z}^d$ is a choice of a fundamental cell of s, [i.e., a parallelepiped in which the length of each side is a multiple of the corresponding period of s], and

$$e_{\underline{\mu}\, x}(s) \ := \ \sum_{B \ni x} \frac{\Phi^{\rm cl}_{\underline{\mu}\, B}(s)}{\mid B \mid} \tag{2.57}$$

can be interpreted as the contribution of the site x to the energy. It will be referred to as the specific energy "at x" of the configuration s.

(H1.4) *Peierls condition.* For all $\underline{\mu} \in \mathcal{O}$ the Peierls condition is satisfied, for some *$\underline{\mu}$-independent* Peierls constant $\overline{J} > 0$. Roughly speaking, this means that the insertion of an excitation corresponding to a ground state configuration that is different from the one on the rest of the lattice costs an energy proportional to the surface area of the inserted droplet. The constant of proportionality is the Peierls constant. For a precise statement of this condition, see Definition 3.1 below.

(H1.5) *Regularity of the phase diagram.* The zero-temperature phase diagram for $\underline{\mu} \in \mathcal{O}$ is regular. We shall explain this notion below.

At zero temperature, the phase diagram is drawn using the set of ground states

$$\mathcal{Q}^{(\infty,0)}(\underline{\mu}) \ := \ \Big\{ s_p \in \mathcal{K} \ : \ \operatorname{Re} e_{\underline{\mu}}(s_p) = \min_{s_u \in \mathcal{K}} \operatorname{Re} e_{\underline{\mu}}(s_u) \Big\} \tag{2.58}$$

for each value of $\underline{\mu}$. The superscripts $(\infty, 0)$ correspond to the values of β (proportional to the inverse temperature) and the quantum perturbation parameter λ. The classical zero-temperature phase diagram is the family of manifolds

$$\mathcal{S}^{(\infty,0)}_{\{s_{p_1},\ldots,s_{p_k}\}} \ := \ \Big\{ \underline{\mu} \ : \ \mathcal{Q}^{(\infty,0)}(\underline{\mu}) = \{s_{p_1}, \ldots, s_{p_k}\} \Big\} \tag{2.59}$$

for $1 \le k \le P$, $s_{p_1}, \ldots, s_{p_k} \in \mathcal{K}$. These manifolds are called the *strata* of the phase diagram. The phase diagram defined by these strata is *regular* if the map

$$\underline{\mu} \ \longmapsto \ \Big(\operatorname{Re} e_{\underline{\mu}}(s_1) - \min_{1 \le i \le P} \operatorname{Re} e_{\underline{\mu}}(s_i) \, , \ \ldots \ , \ \operatorname{Re} e_{\underline{\mu}}(s_P) - \min_{1 \le i \le P} \operatorname{Re} e_{\underline{\mu}}(s_i) \Big) \tag{2.60}$$

is a homeomorphism of $\mathcal{O}$ into the boundary of the positive octant in the space $\mathbb{R}^P$. This means that the stratum of maximum coexistence, $\mathcal{S}_{\mathcal{K}}$, is a single point ($\cong$ the origin of $\mathbb{R}^P$), the strata with $P-1$ ground states are curves emanating from it ($\cong$ the coordinate semiaxes), and so on. The strata with $P-k$ ground states are k-dimensional manifolds bounded by the strata with $P-k+1$ ground states. (This geometry is also known as the *Gibbs phase rule*.)

(H2) The quantum perturbation $\{\Phi^{\rm q}_B\}$ is a translation invariant interaction satisfying *exponential decay*. The precise expression of this decay is based on a choice of *sampling plaquettes* $W_a(x) = \{y \in \mathbb{Z}^d \colon |x_i - y_i| \le a \text{ for } 1 \le i \le d\}$. [The constant a is chosen so as to have a one-to-one correspondence between configurations and classical contours, (see Section 3.1 below).] For a finite $B \subset \mathbb{Z}^d$, let

$$g(B) \ := \ \text{\textit{minimal number of plaquettes needed to cover B with a connected set.}} \tag{2.61}$$

Then the decay condition is given by

$$\|\Phi^{\mathrm{q}}_{\underline{B}}\| \;\leq\; c\,\lambda^{g(B)} \tag{2.62}$$

for some constant c and some $0 < \lambda < 1$.

2.3 Examples

In this section we give some simple examples of models to which our theory can be applied. For simplicity we consider models in $\mathbb{Z}^2$, but analogous results hold for models on $\mathbb{Z}^d$, $d > 2$.

Example 1: *Fisher antiferromagnet.* This is an example of a quantum-spin system. For simplicity we choose the spin at each site to be $1/2$. Hence $\mathcal{H}_x \simeq \mathbb{C}^2$. The system has a Hamiltonian given by

$$\begin{aligned}\mathbf{H}_\Lambda \;=\; & \Big[\sum_{<x,y>\,\pitchfork\,\Lambda} \sigma_x^{(3)}\sigma_y^{(3)} - K \sum_{\ll x,y\gg\,\pitchfork\,\Lambda} \sigma_x^{(3)}\sigma_y^{(3)} - h\sum_{x\in\Lambda}\sigma_x^{(3)} - h^{\mathrm{stagg}}\sum_{x\in\Lambda}(-1)^{|x|}\sigma_x^{(3)}\Big] \\ & + \Big[t \sum_{<x,y>\,\pitchfork\,\Lambda} \Big(\sigma_x^{(1)}\sigma_y^{(1)} + \sigma_x^{(2)}\sigma_y^{(2)}\Big) + \mathrm{h.c.}\Big]\,. \end{aligned} \tag{2.63}$$

$\sigma_x^{(i)}$, $i = 1, 2, 3$, are the spin operators (Pauli matrices); $<x, y>$ and $\ll x, y \gg$ denote nearest neighbour and next-nearest neighbour pairs, respectively, and the notation $B \pitchfork \Lambda$ is used to refer to the set:

$$\{B \subset \mathbb{Z}^d \,:\, B \cap \Lambda \neq \emptyset\}. \tag{2.64}$$

Finally, t is an exchange coupling constant.
Note: In this and the following examples we use square brackets to separate the classical and quantum parts of the Hamiltonian (as in (2.52)). Moreover, any perturbation satisfying hypothesis (H2) can be added to the quantum parts.

This model gives rise to phase diagrams of different degrees of complexity depending on which of the couplings are varied. Let us first consider $\underline{\mu} = (h, h^{\mathrm{stagg}})$. The parameter h^{stagg} modulates a *staggered field* whose sign changes as the parity of $|x| := |x_1| + \cdots + |x_d|$ changes. The ferromagnetic coupling K is assumed to have a *fixed* non-negative value. We use the symbols "+" and "−" to denote spin up and spin down respectively.

For $K > 0$, the set of ground states of the classical part is

$$\mathcal{K} \;=\; \{s_+, s_-, s_{+-}, s_{-+}\}\,, \tag{2.65}$$

where s_+ is the all-"+" configuration, s_- the all-"−", and s_{+-} and s_{-+} are the two Néel configurations with "+" spins in one sublattice and "−" spins in the other one.

Figure 2: Zero temperature phase diagram of the Fisher antiferromagnet (Example 1) for (a) $K > 0$ (b) $K = 0$ and $|h| < 2$.

The fundamental cell of these periodic configurations can be chosen to be a 2×2 square. Hence we write symbolically:

$$s_+ = \begin{pmatrix} + & + \\ + & + \end{pmatrix}\ , \ s_- = \begin{pmatrix} - & - \\ - & - \end{pmatrix}\ , \ s_{+-} = \begin{pmatrix} + & - \\ - & + \end{pmatrix}\ , \ s_{-+} = \begin{pmatrix} - & + \\ + & - \end{pmatrix}\ . \tag{2.66}$$

The corresponding zero-temperature phase diagram is depicted in Figure 2(a). The oblique lines for $h > 0$ are given by the equation

$$h = 2 + |h^{\rm stagg}|. \tag{2.67}$$

The corresponding lines for $h < 0$ are given by the equation

$$h = -2 - |h^{\rm stagg}|. \tag{2.68}$$

To construct the ground state phase diagram, it is convenient to rewrite the classical part of the Hamiltonian as a sum over terms corresponding to 2×2 blocks, M, i.e., $\sum_M \Phi_M^{cl}$ with

$$\begin{aligned} \Phi_M^{cl} = \frac{1}{2} \sum_{<x,y> \subset M} \sigma_x^{(3)} \sigma_y^{(3)} \ &- \ K \sum_{\ll x,y \gg \subset M} \sigma_x^{(3)} \sigma_y^{(3)} - \frac{h}{4} \sum_{x \in M} \sigma_x^{(3)} \\ &- \ \frac{h^{\rm stagg}}{4} \sum_{x \in M} (-1)^{|x|} \sigma_x^{(3)} \end{aligned} \tag{2.69}$$

and find the minimal energy configurations over any such block M. This is because the operators Φ_M^{cl} constitute an m-potential, [16].

This diagram is regular in the vicinity of the maximal-coexistence points P and Q. At zero temperature, this model exhibits a transition between ferromagnetic and antiferromagnetic order when any one of the oblique coexistence lines in the phase diagram is crossed. Our theory will show that this transition survives at nonzero temperatures and/or in the presence of small quantum perturbations, like the spin-flipping term added in the second line of (2.63). An alternative proof of this fact is presented in [2].

If we set $K = 0$, the oblique coexistence lines emanating from P and Q acquire infinitely many periodic ground states. For the upper lines these ground states result from periodic arrangements of the configurations:

$$\begin{pmatrix} + & - \\ + & + \end{pmatrix} \ \begin{pmatrix} + & + \\ - & + \end{pmatrix} \ \begin{pmatrix} + & + \\ + & - \end{pmatrix} \ \begin{pmatrix} - & + \\ + & + \end{pmatrix} \tag{2.70}$$

For the lower lines the configurations which contribute are the ones obtained from the above set by a spin-flip. These sectors of the phase diagram, therefore, lie outside the scope of our theory [violation of (H1.3)]. Nevertheless, we can still analyze the regions around the open vertical segment joining P with Q. This corresponds to fixing the parameter h at some value, such that $|h| < 2$, and considering the model to be parametrized by h^{stagg} alone. This yields the phase diagram of Figure 2 (b).

The fact that in each case the relevant classical part of the Hamiltonian satisfies the Peierls condition follows from a general theorem of Holsztynski and Slawny [16] and the stability of the Peierls condition under perturbation (Proposition 3.2 below).

Example 2: *Simple fermionic model.* We consider spinless fermions with interaction

$$\begin{aligned} \mathbf{H}_\Lambda &= \Big[\sum_{<x,y>\Cap\Lambda} \mathbf{n}_x\mathbf{n}_y - K\sum_{\ll x,y\gg\Cap\Lambda} \mathbf{n}_x\mathbf{n}_y - \mu\sum_{x\in\Lambda}\mathbf{n}_x - \mu^{\mathrm{stagg}}\sum_{x\in\Lambda}(-1)^{|x|}\mathbf{n}_x\Big] \\ &\quad + \Big[t\sum_{<x,y>\Cap\Lambda} \mathbf{c}_x^*\mathbf{c}_y + \mathrm{h.c.}\Big]\,. \end{aligned} \tag{2.71}$$

A lattice site can either be empty, or occupied by a single fermion. Hence $\mathcal{H}_x \simeq \mathbb{C}^2$. This model can be obtained from the one of the previous example by a transformation of spin variables to lattice gas variables. By suitably transcribing the results of Example 1, we obtain, for $K > 0$, the zero-temperature phase diagram shown in Figure 3 (a).

The latter involves the ground states

$$\mathcal{K} \;=\; \{s_\bullet, s_{\bullet\circ}, s_{\circ\bullet}\}\;, \tag{2.72}$$

where $s_\bullet$ is the configuration with exactly one fermion at each site, while $s_{\bullet\circ}$ and $s_{\circ\bullet}$ are the half-filled configurations having one fermion at each site of one of the sublattices and no particle in the other sublattice. Diagrammatically,

$$s_\bullet = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}\,,\; s_{\bullet\circ} = \begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}\,,\; s_{\circ\bullet} = \begin{pmatrix} \circ & \bullet \\ \bullet & \circ \end{pmatrix}\,. \tag{2.73}$$

The oblique lines in Figure 3 (a) are given by the equation

$$\mu = 4 - 2K + |\mu^{\mathrm{stagg}}|. \tag{2.74}$$

.

This diagram is regular in the vicinity of the maximal-coexistence point P. For $K = 0$ the oblique lines become lines of infinite degeneracy, with ground states having, in addition to the above configurations, (2.73), the following one:

$$\begin{pmatrix} \bullet & \bullet \\ \bullet & \circ \end{pmatrix},$$

and the three others obtained from it by rotations. Hence our theory can only be applied in the region around the vertical coexistence line up to, but *excluding*, the point P, i.e., to phase diagrams as in Figure 3 (b). The stability of these phase

Figure 3: Zero-temperature phase diagram of the simple fermionic model of Example 2 (a) for $K > 0$ and (b) for $K = 0$ and $0 \leq \mu < 4 - 2K$.

diagrams at low, but nonzero temperatures, and under quantum perturbations, has been studied in [22], in which the $K = 0$ model was introduced, and in [3], where the analogous region of the $K > 0$ model was analysed.

Example 3: *Simple model of fermions with spin.* By combining elements from the previous two examples, we can easily generate simple models involving itinerant particles with spin, which satisfy the hypotheses of Section 2.2. For instance, consider spin-1/2 fermions with Hamiltonians

$$\begin{aligned}
\mathbf{H}_\Lambda \;=\; & \Big[\sum_{<x,y> \Cap \Lambda} \Big(\frac{1 + \sigma_x^{(3)} \sigma_y^{(3)}}{2} \Big) \mathbf{n}_x \mathbf{n}_y - K \sum_{\ll x,y \gg \Cap \Lambda} \Big(\frac{1 + \sigma_x^{(3)} \sigma_y^{(3)}}{2} \Big) \mathbf{n}_x \mathbf{n}_y \\
& - \mu \sum_{x \in \Lambda} \mathbf{n}_x - \mu^{\rm stagg} \sum_{x \in \Lambda} (-1)^{|x|} \mathbf{n}_x - h \sum_{x \in \Lambda} \sigma_x^{(3)} \mathbf{n}_x - h^{\rm stagg} \sum_{x \in \Lambda} (-1)^{|x|} \sigma_x^{(3)} \mathbf{n}_x \Big] \\
& + \mathbf{V}_\Lambda \, . \qquad (2.75)
\end{aligned}$$

We have defined $\mathbf{n}_x := \sum_{\sigma=-1,1} \mathbf{n}_{x\sigma}$. In the most general case, namely when we choose some fixed $K > 0$ and consider the other four constants as parameters, the phase diagram is regular around the maximal-coexistence point

$$P \;=\; \Big(\mu = 2 - 2K \,,\; \mu^{\rm stagg} = 0 \,,\; h = 2 \,,\; h^{\rm stagg} = 0 \Big) \,, \qquad (2.76)$$

where there are five degenerate ground states:

$$\begin{aligned}
& s_{++} = \begin{pmatrix} + & + \\ + & + \end{pmatrix} \,,\; s_{+-} = \begin{pmatrix} + & - \\ - & + \end{pmatrix} \,,\; s_{-+} = \begin{pmatrix} - & + \\ + & - \end{pmatrix} \,, \\
& s_{+\circ} = \begin{pmatrix} + & \circ \\ \circ & + \end{pmatrix} \quad,\quad s_{\circ+} = \begin{pmatrix} \circ & + \\ + & \circ \end{pmatrix} \,. \qquad (2.77)
\end{aligned}$$

In Figure 4 we present the cross section of the $h > 0$ region of the phase diagram through the plane $h^{\rm stagg} = \mu^{\rm stagg} = 0$.

The validity of the Peierls condition is again a consequence of the results of Holsztynski and Slawny [16] and Proposition 3.2 given in Section 3.

In subsequent papers [8, 13] we consider a broader class of Hamiltonians whose classical part need not have a finite number of ground states (and hence may violate the Peierls condition). In [8] we develop a perturbation technique which, together with the contour expansion methods of this paper, permits us to study the degeneracy-breaking effects of a quantum perturbation on the classical part and to analyse the phase diagram of the Hamiltonian at low temperatures.

2.4 The main theorem

Our results show that, under the hypotheses listed in Section 2.2, the phase diagrams obtained at low temperatures, and for small quantum perturbations, are only small deformations of the zero-temperature diagram corresponding to the classical part $\{\Phi^{\rm cl}_{\underline{\mu}\,B}\}$. The precise statement of this result requires a notion of stability of phases.

Figure 4: Zero-temperature phase diagram of the model of fermionic spins of Example 3.

Definition 2.2 *We shall say that a configuration $s_p \in \mathcal{K}$ defines a stable phase (or that the s_p-phase is stable), for the interactions $\{\Phi_{\underline{\mu}\,B}\}$, if there exists a neighbourhood, $\mathcal{D}$, of the origin of $\mathbb{R}^2$, such that, for each pair $(e^{-\beta}, \lambda) \in \mathcal{D}$ and any local operator $\mathbf{A}$, the infinite-volume limit*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{\mathrm{Tr}_{\Lambda}^{s_p} \mathbf{A}\, e^{-\beta \mathbf{H}_\Lambda}}{\mathrm{Tr}_{\Lambda}^{s_p} e^{-\beta \mathbf{H}_\Lambda}} \;:=\; \langle \mathbf{A} \rangle_{\beta\,\lambda}^{s_p} \tag{2.78}$$

exists and satisfies

$$\lim_{\substack{\beta \to \infty \\ \lambda \to 0}} \langle \mathbf{A} \rangle_{\beta\,\lambda}^{s_p} \;=\; \langle s_p | \mathbf{A} | s_p \rangle\,. \tag{2.79}$$

In analogy to (2.58), we introduce the sets

$$\mathcal{Q}^{(\beta,\lambda)}(\underline{\mu}) \;:=\; \Big\{ s_p \;:\; \text{the } s_p\text{-phase is stable for } \{\Phi_{\underline{\mu}\,B}\} \Big\} \tag{2.80}$$

to define the strata

$$\mathcal{S}^{(\beta,\lambda)}_{\{s_{p_1},\ldots,s_{p_k}\}} \;:=\; \Big\{ \underline{\mu} \;:\; \mathcal{Q}^{(\beta,\lambda)}(\underline{\mu}) = \{s_{p_1},\ldots,s_{p_k}\} \Big\}\,. \tag{2.81}$$

Our paper presents a proof of the following theorem:

Theorem 2.3 *Under the hypotheses of Section 2.2, there are constants $\widetilde{J} > 0$ and $\varepsilon_0 > 0$ such that, for each β and λ in the region*

$$\max(e^{-\beta \tilde{J}}\,,\, \lambda) \;<\; \varepsilon_0, \tag{2.82}$$

there exists a non-empty open set $\mathcal{O}_{\beta\lambda} \in \mathbb{R}^{P-1}$ such that:

(i) The phase diagram defined by the strata $\mathcal{O}_{\beta\lambda} \cap \mathcal{S}^{(\beta,\lambda)}_{\{s_{p_1},\ldots,s_{p_k}\}}$ is regular [in the sense described below (2.60)] and these strata are differentiable manifolds.

(ii) As $\varepsilon_0 \to 0$, the strata $\mathcal{O}_{\beta\lambda} \cap \mathcal{S}^{(\beta,\lambda)}_{\{s_{p_1},\ldots,s_{p_k}\}}$ tend to the zero-temperature classical strata $\mathcal{O} \cap \mathcal{S}^{(\infty,0)}_{\{s_{p_1},\ldots,s_{p_k}\}}$, pointwise in $\underline{\mu}$. In particular, the distance between the maximal-coexistence manifolds $\mathcal{S}^{(\beta,\lambda)}_{\mathcal{K}}$ and $\mathcal{S}^{(\infty,0)}_{\mathcal{K}}$ is $\mathrm{O}(\varepsilon_0)$.

We shall say that a zero-temperature, classical phase diagram is *stable*, under temperature and quantum perturbations, if the conclusions (i) and (ii) of the theorem hold.

Using our contour expansion methods, we can further prove that, for a fixed value of $\underline{\mu}$ corresponding to a single–phase region of the phase diagram, the free energy density and the quantum expectations, defined in (2.78), are analytic functions of β and λ, provided $(\mathrm{Re}\,\beta)^{-1}$ and $|\lambda|$ are small enough.

As an illustration, let us describe the consequences of this theorem for the examples of Section 2.3. For the Fisher antiferromagnet, we conclude that, for $K > 0$ and

ε_0 small enough, the phase diagram around maximal-coexistence points looks like a smooth deformation of the diagram of Figure 2(a) in the vicinity of the points P and Q. Symmetry considerations imply that the coexistence line between Néel phases remains at $h^{\rm stagg} = 0$. These results have also been obtained in [2], using model-tailored dressing transformations. Likewise, our theory implies that for $K = 0$ and $|h| < 2$ the phase diagram of Figure 2(b) remains valid for small ε_0. In fact, the diagram remains *unchanged* because, by symmetry, the coexistence point stays at $h^{\rm stagg} = 0$.

Similar conclusions apply to the spinless fermion system of Example 2. In particular, we conclude that, for $K = 0$, the phase diagram of Figure 3 (b) remains unchanged when small kinetic terms (i.e., quantum perturbations) are added and the temperature is increased, as already proven in [22, 3]. Besides, we derive the stability of the phase diagram for $K > 0$ [Figure 3(a)] around the point P.

For the spin-1/2 fermion model of Example 3, we obtain the stability of the phase diagram around the maximal-coexistence point (2.76). By symmetry, the coexistence between Néel phases (defined by boundary conditions s_{+-} and s_{-+}) remains at $h^{\rm stagg} = 0$, and that of the half-filled phases (boundary conditions $s_{+\circ}$ and $s_{\circ+}$) at $\mu^{\rm stagg} = 0$. Hence we also obtain the stability of the (non-regular) phase diagram of Figure 4 around the point P.

The implications of Theorem 2.3 for more interesting, t-J-type, models will be the subject of a forthcoming paper [13].

3 Low-temperature expansion for quantum perturbations

The first step in the proof of our main result, Theorem 2.3, consists in constructing a suitable low-temperature expansion. This is the content of the present section. Our expansion is a type of polymer expansion in which the polymers are called quantum contours (and the consistency rules are more complicated than plain non-intersection). They are a generalization of the well-known *classical* contours of Pirogov-Sinai theory (see eg. [35, Chapter II]). We first recall the definition of these classical contours and of the associated Peierls condition.

In this and the following section we work with a *fixed* value of the parameters $\underline{\mu} \in \mathcal{O}$. Consequently, the parameters play no role and are hence not displayed. The definition of the contours depends only on the reference configurations $\mathcal{K} = \{s_1, \ldots, s_P\}$ and on the range r of the interactions. The parameters will be reintroduced in Section 7, where we will study the effect of varying them.

3.1 The classical contours. The Peierls condition

To define these contours we start with a set of periodic *reference configurations* $\mathcal{K} = \{s_1, \ldots, s_P\}$ and a number $r > 0$ which is an apriori bound on the range of the

classical interactions to be considered. We fix *sampling plaquettes* $W_a(x) = \{y \in \mathbb{Z}^d \colon |x_i - y_i| \le a$ for $1 \le i \le d\}$. The size a must be strictly larger than (i) the periods of the reference configurations $s_1, \ldots, s_P$, and (ii) the range r.
Condition (i) implies the following *extension property*:

$$\begin{array}{l} \textit{If } \omega \textit{ coincides with the configuration } s_p \textit{ on a plaquette } W_a(x) \\ \textit{and with } s_q \textit{ on } W_a(y) \textit{ with } \operatorname{dist}(x,y) \le 1, \textit{ then } s_p = s_q. \end{array} \tag{3.1}$$

To simplify the notation, we shall henceforth measure the cardinality of subsets A of $\mathbb{Z}^d$ in units of sampling plaquettes:

$$|A| \;:=\; \frac{\operatorname{card} A}{a^d}\;. \tag{3.2}$$

Two sets, A and B, in $\mathbb{Z}^d$ are said to be connected if $\operatorname{dist}(A, B) \le 1$ in lattice units. A subset M of a set $A \subset \mathbb{Z}^d$ is called a *component* of A if M is a maximal connected subset of A, i.e., M is connected and $M \subset M' \subset A$, $M \ne M'$ imply that M' cannot be connected.

The classical contours are constructed out of "incorrect" plaquettes. A site x is said to be p-correct, for a configuration ω, if the latter coincides with s_p on *every* sampling plaquette that contains x. The set of sites that are not p-correct for any p, $1 \le p \le P$, are referred to as "incorrect". The set of plaquettes for which at least one site is "incorrect" form the *defect set*, $\partial\omega$, of the configuration ω. Note that

$$\partial\omega \;=\; \bigcup_{x\in\mathbb{Z}^d} \Big\{W_a(x) \;:\; \omega_{W_a(x)} \ne (s_p)_{W_a(x)} \text{ for all } 1 \le p \le P\Big\}\;. \tag{3.3}$$

Typically we will consider configurations ω equal to some reference configuration $s \in \mathcal{K}$ almost everywhere, i.e., ω differs from s only on a finite set of lattice sites. In this situation $\partial\omega$ is a finite set. We shall refer to the plaquettes belonging to the defect set as *excited* plaquettes and the components of the defect set as *excitations.*

A *(classical) contour* of a configuration ω is a pair $\gamma = (M, \omega_M)$ where M is a component of the defect set $\partial\omega$. The set M is the *support* of γ, to be denoted by $\operatorname{supp}\gamma$. We shall often refer to the support of a contour γ again using the symbol γ and use the abbreviation

$$|\gamma| \;:=\; |\operatorname{supp}\gamma|\;. \tag{3.4}$$

According to our definition, the smallest contour is the one obtained when only a single site is "incorrect"; e.g. for a quantum spin system this results when one spin is misaligned, the corresponding contour being formed by all the plaquettes containing this spin. Hence the minimal nonzero value of $|\gamma|$ is given by

$$(2a-1)^d/a^d \;\ge\; 1\;. \tag{3.5}$$

Each configuration defines a unique family of contours from which it can be reconstructed, but *not* all families of contours correspond to admissible configurations.

The additional restrictions are that contours must not intersect and that configurations in the interiors and exteriors of nested contours must match. A family of contours which corresponds to an admissible configuration will be called *compatible*. Henceforth, we shall only consider *finite* contours (i.e., $|\operatorname{supp}\gamma| < \infty$). For each such contour, γ, the space $\mathbb{Z}^d \setminus \operatorname{supp}\gamma$ is divided into a finite number of components. Moreover, by the extension property (3.1), we can extend the configuration on a single plaquette in a component to a unique configuration of $\mathcal{K}$ in that component. In this way we can label each connected component of $\mathbb{Z}^d \setminus \operatorname{supp}\gamma$ by a particular reference configuration. Thus, we obtain the unique configuration ω^γ that has γ as its *only* contour. We shall refer to such a configuration as a one-contour configuration. The only infinite component of $\mathbb{Z}^d \setminus \gamma$ is called the *exterior* of the contour, $\operatorname{Ext}(\gamma)$, and the union of the other components constitute the *interior*, $\operatorname{Int}(\gamma)$. The union of components of $\operatorname{Int}(\gamma)$ labelled by a reference configuration s_q is called the q-interior, $\operatorname{Int}_q(\gamma)$. The contour is called a *p-contour* if its exterior is labelled by the configuration $s_p \in \mathcal{K}$.

A contour γ of a configuration ω is called an *exterior contour* of ω if its support is not contained in the interior of any other contour of ω, i.e., if $\gamma \subset \operatorname{Ext}(\gamma')$ holds, for any other contour γ' of ω.

Let $\{\phi_B^{\rm cl}\}$ be a set of classical interactions of range not exceeding r. The one-contour configurations can be used to compute energies of any allowed configuration, as we now explain.

Let ω^γ be a one-contour configuration which has the p-contour γ as its only contour. The *energy cost* of γ, relative to its exterior configuration s_p, is given by

$$H_\Lambda^{\rm cl}(\omega^\gamma|s_p) \;=\; \sum_{B\,\Cap[\operatorname{Int}(\gamma)\cup\operatorname{supp}\gamma]} \left[\Phi_B^{\rm cl}(\omega^\gamma) - \Phi_B^{\rm cl}(s_p)\right] , \tag{3.6}$$

where we have used the notation of (2.64). It is convenient to use the decomposition (see [39])

$$\begin{aligned}\sum_{B\,\Cap[\operatorname{Int}(\gamma)\cup\operatorname{supp}\gamma]} \Phi_B(\omega^\gamma) \;&=\; \sum_B \frac{|B\cap\operatorname{supp}\gamma|}{|B|}\,\Phi_B(\omega^\gamma) + \sum_B \frac{|B\cap\operatorname{Int}\gamma|}{|B|}\,\Phi_B(\omega^\gamma)\\ &\quad+ \sum_{B\,\Cap[\operatorname{Int}(\gamma)\cup\operatorname{supp}\gamma]} \frac{|B\cap\operatorname{Ext}\gamma|}{|B|}\,\Phi_B(\omega^\gamma)\,,\end{aligned} \tag{3.7}$$

to write (3.6) in the form

$$H^{\rm cl}(\omega^\gamma|s_p) \;=\; E(\gamma) + \sum_{u=1}^{P}\sum_{x\in\operatorname{Int}_u(\gamma)} [e_x(s_u) - e_x(s_p)] \tag{3.8}$$

where $e_x(s_u)$ is the specific energy "at x", (see (2.57)), of the configuration s_u, and

$$E(\gamma) \;=\; \sum_B \frac{|B\cap\operatorname{supp}\gamma|}{|B|}\left[\Phi_B^{\rm cl}(\omega^\gamma) - \Phi_B^{\rm cl}(s_p)\right] \tag{3.9}$$

is the *contour energy* of γ relative to the energy of its exterior configuration.
In obtaining (3.9) we have profited from having chosen the plaquette size *a larger* than the range r, so that

$$(\omega^\gamma)_B = (s_u)_B, \text{ if } B \cap \mathrm{Int}_u(\gamma) \neq \emptyset, \tag{3.10}$$

for any B with $\Phi_B^{\mathrm{cl}} \neq 0$, and, since γ is a p-contour,

$$(\omega^\gamma)_B = (s_p)_B \text{ if } B \cap \mathrm{Ext}(\gamma) \neq \emptyset. \tag{3.11}$$

Hence the latter bonds do not contribute to the contour energy $E(\gamma)$. In situations of maximal coexistence, all the reference configurations $s_p \in \mathcal{K}$ are groundstate configurations and have the same specific energy. In this case, it follows from (3.8) that the energy cost of a contour γ is simply given by $E(\gamma)$.

Consider a region Λ such that

$$(\omega^\gamma)_{\mathbb{Z}^d\setminus\Lambda} = (s_p)_{\mathbb{Z}^d\setminus\Lambda} \tag{3.12}$$

The total energy of the configuration ω^γ [needed for the partial trace of H_Λ in (2.47)] is given by

$$H^{\mathrm{cl}}_{\Lambda\, s_p}(\omega^\gamma) = H^{\mathrm{cl}}_\Lambda(\omega^\gamma|s_p) + \sum_{B \pitchfork \Lambda} \Phi_B^{\mathrm{cl}}(s_p). \tag{3.13}$$

We notice that a decomposition analogous to (3.7) yields

$$\sum_{B \pitchfork \Lambda} \Phi_B^{\mathrm{cl}}(s_p) \;=\; \sum_{x\in\Lambda} e_x(s_p) + \sum_{B \pitchfork \Lambda} \frac{|B\cap\Lambda^{\mathrm{c}}|}{|B|}\,\Phi_B^{\mathrm{cl}}(s_p) \tag{3.14}$$

However, the last term of (3.14) is a *boundary* term which does not contribute to the free energy density or to the expectation values of observables, (2.51), and is independent of the configuration ω. Hence we shall neglect it.

The energies of configurations with a *finite* number of contours (which are the only ones relevant in the sequel) can be reconstructed from energies of its contours. Let $\omega = \omega^\Gamma$ be a configuration corresponding to a compatible family of contours, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$, and coinciding at infinity with some reference configuration s_p. This implies that the exterior contours of the configuration ω are p-contours. Let $\omega^{\gamma_1}, \ldots, \omega^{\gamma_k}$ be the corresponding one-contour configurations. The energy cost

$$H^{\mathrm{cl}}(\omega^\Gamma|s_p) \;=\; \sum_{B \pitchfork [\mathrm{Int}(\gamma_j)\cup \mathrm{supp}\,\gamma_j]} [\Phi_B^{cl}(\omega^\Gamma) - \Phi_B^{cl}(s_p)] \tag{3.15}$$

of ω^Γ, relative to the exterior configuration s_p, is simply the sum of one-contour energy costs:

$$H^{\mathrm{cl}}(\omega^\Gamma|s_p) \;=\; \sum_{j=1}^{k} H^{\mathrm{cl}}(\omega^{\gamma_j}|s(\gamma_j))\ , \tag{3.16}$$

where $s(\gamma_j)$ is the reference configuration in the exterior of γ_j. It follows from eqs.(3.13),(3.14) and (3.16) that the total energy of ω^Γ is given by

$$H^{\rm cl}_\Lambda(\omega) \;=\; \sum_{x\in\Lambda} e_x(s_p) + E(\Gamma) + \sum_{u=1}^{P}\sum_{x\in\ell_u}\Big[e_x(s_u) - e_x(s_p)\Big]\;, \tag{3.17}$$

where

$$E(\Gamma) \;=\; \sum_{j}^{k} E(\gamma_j), \tag{3.18}$$

and ℓ_u is the set of sites in Λ that are either u-correct or belong to a u-contour. This expression is to be used for the partial trace $\mathrm{Tr}^{s_p}_\Lambda$ in (2.47). Hence the configuration ω^Γ must be such that

$$(\omega^\Gamma)_{\mathbb{Z}^d\setminus\Lambda} = (s_p)_{\mathbb{Z}^d\setminus\Lambda} \tag{3.19}$$

Remark: The contours γ_j may extend outside Λ. This happens if ω has some incorrect site *on* the boundary, $\partial\Lambda$. In this case all those plaquettes which contain this site, but extend outside Λ, also belong to a contour. Hence, in general, the contours are contained in the larger set formed by the plaquettes that touch Λ:

$$\widehat{\Lambda} \;:=\; \bigcup\{W_a(x)\;:\;W_a(x)\cap\Lambda\neq\emptyset\}\;. \tag{3.20}$$

This means that in $E(\gamma_j)$ one may be counting bonds $B\subset\Lambda^{\rm c}$ that are not counted in $H^{\rm cl}_{\Lambda\, s_p}$. However, the identity (3.17) remains valid, because these bonds do not contribute to the energy of a contour, [see sentence following (3.11)].

Note: We use the letter γ to denote individual contours and Γ to denote *families* of contours.

The Peierls condition can now be stated in terms of the contour energies defined in (3.9).

Definition 3.1 *An interaction $\Phi^{\rm cl}$ satisfies the Peierls condition with Peierls constant J if*

$$\mathrm{Re}\, E(\gamma) \;\geq\; J|\gamma|\;, \tag{3.21}$$

where $E(\gamma)$ is the contour energy defined through (3.9), and $|\gamma|$ is as in (3.4).

In general it is not simple to prove that the Peierls condition is satisfied for a particular model. One way of doing so is to show that the excess energy of each excited plaquette of the configuration is nonzero, *irrespective* of the particular configuration on the plaquettes surrounding it. However, this is true only in severely constrained systems or for highly symmetric situations (as in the Ising model [15, 9]). In most systems, it is often energetically favourable for a plaquette to have an "incorrect" configuration if the surrounding plaquettes are already excited. Hence, calculating the excess energy of a single excited plaquette is not sufficient for verifying the

Peierls condition. Instead, one may need to compute the energy balance of a possibly complex arrangement of plaquettes. However, one can avoid the complicated calculations that this involves by resorting to a theorem due to Holsztynski and Slawny [16], which states that the Peierls condition is satisfied if the interaction can be written as an *m-potential*, i.e., a potential admitting a finite number of ground states that minimize the contribution of each bond simultaneously. The only drawback of this important result is that its proof does not provide any estimate of the Peierls constant, a fact that in turn prevents one from explicitly estimating the range of temperatures for which our results concerning the phase diagram, are valid.

In this paper we have the additional complication of having to verify the Peierls condition simultaneously for a whole family of interactions, parametrized by $\underline{\mu}$ [Hypothesis (H1.4)]. However, the condition imposed on the size of the sampling plaquettes, namely $a > r$, simplifies the situation, since it allows us to make use of some perturbative results (discussed for instance in [38, pages 1126–1127]) which can be summarized in the following statement:

Proposition 3.2 *Consider a family of interactions $\{\Phi^{\rm cl}_{\underline{\mu}\,B}\}$ differentiable in $\underline{\mu}$. Assume that, for some value $\underline{\mu}_0$ of the parameters, the interaction $\{\Phi^{\rm cl}_{\underline{\mu}_0\,B}\}$ has a finite number of periodic ground states $\mathcal{K} = \{s_1, \ldots, s_P\}$ and that it satisfies the Peierls condition with Peierls constant J_0. Then, for $\delta > 0$ small enough, there exist open neighbourhoods $\mathcal{O}_\delta \ni \underline{\mu}_0$ such that all the interactions $\{\Phi^{\rm cl}_{\underline{\mu}\,B}\}$ with $\underline{\mu} \in \mathcal{O}_\delta$ satisfy the Peierls condition with Peierls constant $J_0 - \delta$ and for the same set of reference configurations $\mathcal{K}$.*

For instance, to verify the uniform-Peierls condition hypothesis [(H1.4) in Section 2.2] for the examples of Section 2.3, it is enough to check it at the points of maximal coexistence, which is an easy application of Holsztynski-Slawny theory.

3.2 The Duhamel expansion

We start by establishing a low-temperature expansion for the partition functions

$$\Xi_s(\Lambda) \;=\; \mathrm{Tr}^{\,s}_{\Lambda}\, e^{-\beta \mathbf{H}_\Lambda} \tag{3.22}$$

for finite regions $\Lambda \subset \mathbb{Z}^d$ with boundary condition $s = s_p \in \mathcal{K}$. To compute this trace we use the basis of $\mathcal{F}^s_P(\mathcal{H}^{(1)})$ spanned by the vectors $|v_\Lambda \otimes s_{\Lambda^c}\rangle$ corresponding to configurations $v_\Lambda s_p$ which coincide with s_p outside Λ. This basis is chosen because $H^{\rm cl}_\Lambda$ is diagonal in it [Hypothesis (H1.1)]. In turn, each of the configurations $v_\Lambda s_p$ is uniquely determined by a compatible family of contours $\Gamma^p = \Gamma^p(v_\Lambda s_p)$. The superscript, p, indicates that the *exterior contours*, i.e., the contours of Γ^p whose supports are not contained in the interior of any other contour of Γ^p, are p-contours. We can, therefore, relabel the basis in terms of these contours and write

$$\Xi_p(\Lambda) \;=\; \sum_{\Gamma^p} \langle \Gamma^p | e^{-\beta \mathbf{H}_\Lambda} | \Gamma^p \rangle, \tag{3.23}$$

(henceforth we shall denote $\Xi_{s_p} \equiv \Xi_p$). The presence of Λ in the above formula implies that all the contours involved must have supports contained in the larger set $\widehat{\Lambda}$ defined in (3.20).

Our starting point for the expansion is the (formal) series

$$
\begin{aligned}
e^{-\beta \mathbf{H}_\Lambda} \;&=\; e^{-\beta \mathbf{H}^{\rm cl}_\Lambda} + \sum_{n\ge 1} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \, \Theta\Big(\beta - \textstyle\sum_{i=1}^n \tau_i\Big) \\
&\qquad \times e^{-(\beta - \sum \tau_i)\mathbf{H}^{\rm cl}_\Lambda} \, (-\mathbf{V}_\Lambda)\, e^{-\tau_1 \mathbf{H}^{\rm cl}_\Lambda} \cdots (-\mathbf{V}_\Lambda)\, e^{-\tau_n \mathbf{H}^{\rm cl}_\Lambda}\;, \qquad (3.24)
\end{aligned}
$$

which is obtained by iterating Duhamel's formula

$$
e^{-\beta[\mathbf{H}^{\rm cl}_\Lambda + \mathbf{V}_\Lambda]} \;=\; e^{-\beta \mathbf{H}^{\rm cl}_\Lambda} - \int_0^\beta d\tau \, e^{-(\beta-\tau)\mathbf{H}^{\rm cl}_\Lambda} \, \mathbf{V}_\Lambda \, e^{-\tau[\mathbf{H}^{\rm cl}_\Lambda + \mathbf{V}_\Lambda]}\;. \qquad (3.25)
$$

We perform the following manipulations:

(a) Take $\mathrm{Tr}\,^{s}_{\Lambda}$ of (3.24) as in (3.23).

(b) Insert $\mathbf{1}_{\mathcal{F}^{s_p}_P(\mathcal{H}^{(1)}_\Lambda)} = \sum_{\Gamma^p} |\Gamma^p\rangle\langle\Gamma^p|$ around each operator $\mathbf{V}_\Lambda$ in (3.24).

(c) Use formula (3.17) to compute the (diagonal) matrix elements $\langle\Gamma^p| e^{-\tau \mathbf{H}^{\rm cl}_\Lambda} |\Gamma^p\rangle$.

(d) Expand each $\mathbf{V}_\Lambda$ as a sum of $\Phi^{\rm q}_{\underline{B}}$'s. In this way, at each time step, we obtain matrix elements involving only *one* quantum bond, or, for fermions, *one* creation-annihilation monomial.

The result is

$$
\begin{aligned}
\Xi_p(\Lambda) \;&=\; \exp\Big[-\beta \sum_{x\in\Lambda} e_x(s_p)\Big] \times \\
&\Big\{ 1 + \sum_{n\ge 1} \sum_{\substack{\Gamma^p_0, \Gamma^p_1, \ldots, \Gamma^p_n \\ \Gamma^p_i \text{ compatible family} \\ \Gamma^p_0 = \Gamma^p_n \text{ supported on } \Lambda}} \sum_{B_1,\ldots,B_n} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \, \Theta\Big(\beta - \textstyle\sum_{i=1}^n \tau_i\Big) \\
&\qquad \times \langle \Gamma^p_n | -\Phi^{\rm q}_{\underline{B}_n} | \Gamma^p_{n-1}\rangle \cdots \langle \Gamma^p_2 | -\Phi^{\rm q}_{\underline{B}_2} | \Gamma^p_1\rangle \langle \Gamma^p_1 | -\Phi^{\rm q}_{\underline{B}_1} | \Gamma^p_0\rangle \\
&\qquad \times \exp\Big\{ -\Big(\beta - \textstyle\sum_{i=1}^n \tau_i\Big) E(\Gamma^p_0) - \tau_1 E(\Gamma^p_1) \cdots - \tau_n E(\Gamma^p_n) \Big\} \\
&\qquad \times \prod_{u=1}^{P} \exp\Big\{ -\Big[\Big(\beta - \textstyle\sum_{i=1}^n \tau_i\Big) \displaystyle\sum_{x\in \ell_u(\Gamma^p_0)} + \tau_1 \sum_{x\in \ell_u(\Gamma^p_1)} + \cdots + \tau_n \sum_{x\in \ell_u(\Gamma^p_n)} \Big] \\
&\qquad\qquad \Big[e_x(s_u) - e_x(s_p) \Big] \Big\}\;,
\end{aligned}
$$

(3.26)

where Θ is the step function [i.e., $\Theta(t) = 1$ if $t > 0$ and 0 otherwise], and $\ell_u(\Gamma_i^p)$ refers to the set of sites $\{x\}$ in the subset

$$\bigcup_{\gamma\in\Gamma_i^p} \Big[\operatorname{supp}\gamma \cup \operatorname{Int}\gamma\Big]$$

of the lattice, which are either u-correct or belong to a u-contour.

Expression (3.26) can be interpreted as a "sum" of terms each of which is labelled by a sequence of the form

$$\Upsilon^p \;=\; (\Gamma_0^p, \underline{B}_1, \Gamma_1^p, \tau_1, \ldots, \underline{B}_n, \Gamma_0^p, \tau_n)\,, \tag{3.27}$$

where n is zero or a natural number. Each Γ_i^p is a compatible family of classical contours in $\widehat{\Lambda}$ having $s_p \in \mathcal{K}$ as its exterior configuration. The τ_i are real numbers in the interval $[0, \beta]$ with

$$\sum_{i=1}^{n} \tau_i \leq \beta, \tag{3.28}$$

and each $\underline{B}_i$ is a quantum bond. The sequence (3.27) can be visualized as a piecewise cylindrical surface in $d+1$ dimensions formed by cylindrical pieces of sections Γ_i^p and "flat" bridges corresponding to the quantum bonds $\underline{B}_i$, defined in Sect. 2.1.

We shall refer to $[0, \beta]$ as the "time" axis, and to $\mathbb{Z}^d \simeq \mathbb{Z}^d \times \{0\}$ as the "spatial" coordinates. In our construction, the boundary condition in the spatial direction is defined by one of the ground states. We always impose periodic boundary conditions in the "time" direction, i.e., throughout our analysis, the interval $[0, \beta]$ is endowed with the structure of a circle. This corresponds to taking the trace of the Boltzmann factor $e^{-\beta \mathbf{H}_\Lambda}$ as in (3.23).

Let V be a piecewise-cylindrical region in $d+1$ dimensions of the form

$$V = \Lambda \times [\tau_1, \tau_2], \quad \Lambda \subset \mathbb{Z}^d, \quad \tau_1, \tau_2 \in [0, \beta]. \tag{3.29}$$

In the following we shall use the symbol $\int_V$ to denote a summation over sites x in Λ (divided by a^d in accordance with our choice of sampling-plaquette units) and integration over the continuous variable τ, i.e.,

$$\int_V := \int_{\tau_1}^{\tau_2} d\tau \frac{1}{a^d} \sum_{x\in\Lambda} . \tag{3.30}$$

The surface Υ^p can be considered to be constructed in the following manner: Υ^p has a section Γ_0^p at "time" zero, which grows cylindrically during a "time" interval of length $(\beta - \sum_{i=1}^n \tau_i)$ at the end of which $\underline{B}_1$ is placed transversely, and the section changes suddenly to Γ_1^p. This results from the action of $\Phi^{\mathrm{q}}_{\underline{B}_1}$. The section Γ_1^p then grows cylindrically during a "time" interval τ_1 and so on. The action of the last quantum interaction $\Phi^{\mathrm{q}}_{\underline{B}_n}$ restores the section to Γ_0^p. This section propagates unchanged over a final "time" interval of length τ_n. This space-time picture motivates us to rewrite(3.26) in the following abbreviated form

$$\begin{aligned}\Xi_p(\Lambda) &= \exp\Big[-\int_{\Lambda\times[0,\beta]} e_x(s_p)\Big] \\ &\times \text{“}\sum_{\Upsilon^p}\text{”}\ w(\Upsilon^p) \prod_{u=1}^{P} \exp\Big[-\int_{L_u} [e_x(s_u)-e_x(s_p)]\Big]\ ,\end{aligned} \tag{3.31}$$

The space-time region L_u is the union of cylinders of bases $\ell_u(\Gamma_i^p)$ and heights τ_i. Its volume is given by

$$|L_u| = \sum_{i=0}^{n} \tau_i |l_u(\Gamma_i^p)| \tag{3.32}$$

where we have denoted $\tau_0 := \beta - \sum_{i=1}^n \tau_i$. Hence, using the notation of (3.30), we have that

$$\int_{L_u} = \sum_{i=0}^{n} \int_0^{\tau_i} d\tau \frac{1}{a^d} \sum_{x\in l_u(\Gamma_i^p)} \tag{3.33}$$

The definition of the weights $w(\Upsilon^p)$ can be readily inferred from (3.26). By convention, the case $n=0$ corresponds to $\Upsilon^p=\emptyset$, and we define $w(\emptyset)=1$ and $l_u(\emptyset)=\emptyset$. Following the analogy with classical contours, it would be natural to refer to the maximally connected components of the surface Υ^p as *quantum contours*. This is meaningful *only* if the "sum" (3.31) can be written as a "sum" over compatible families of such putative contours. This is possible if the integrals over τ_i's factorize, and if the weights $w(\Upsilon^p)$ can be written as a product of weights corresponding to individual, disjoint contours. In Section 4 we shall prove that these factorization properties are indeed satisfied.

3.3 Quantum Contours

In this section we give a precise definition of quantum contours, discuss their properties and introduce the quantum Peierls condition.

Definition 3.3 *A* p-quantum contour *for an interaction satisfying the hypothesis* **H2** *of Section 2.2 is a sequence of the form*

$$\zeta^p \ = \ (\Gamma_0^p, \underline{B}_1, \Gamma_1^p, \tau_1, \ldots, \underline{B}_n, \Gamma_0^p, \tau_n) \tag{3.34}$$

where n is a natural number (to be referred to as the number of transitions*). Each Γ_i^p is a compatible family of classical contours having $s_p \in \mathcal{K}$ as exterior configuration. Each τ_i is a non-negative real number such that $\sum_{i=1}^n \tau_i \le \beta$, and each $\underline{B}_i$ is a quantum bond. In addition we have the restrictions:*

(i) *Γ_i arises from Γ_{i-1} through the action of $\Phi^{\mathrm{q}}_{\underline{B}_i}$. This action can change the "spins" or the occupation numbers only in a subset of B_i (which can even be empty). Therefore*

$$0 \;\leq\; \Big||\Gamma_i| - |\Gamma_{i-1}|\Big| \;\leq\; |B_i|\,. \tag{3.35}$$

(ii) *The surface resulting from the toroidal boundary conditions in $[0,\beta]$ is connected [Figure 5(a)] or linked [Figures 6(b)]. [The condition of linking is relevant only in the case of anyons.]*

We shall omit the superscript indicating the exterior configuration whenever it plays no role in our discussion.

Due to the periodic boundary condition, the "time" axis has the topology of a circle. We make a distinction between contours that extend from "time" zero to "time" β and ones which do not. The former will be referred to as *long contours*, while the latter will be called *short contours*. Some examples of these have been illustrated in Figures 5 and 6.

As mentioned above, a contour ζ^p represents a surface in $d+1$ dimensions formed by successive cylinders of spatial sections Γ_i and time-height τ_i and flat pieces B_i, $i = 1,\ldots,n$, located at each transition. A quantum contour may have no connected section [Figures 5(a2), (a3) and 6 (b1), (b2)], but the different connected components cannot be very far away from each other, because they must become connected or linked through the actions of $\Phi^{\mathrm{q}}_{\underline{B}_1},\ldots,\Phi^{\mathrm{q}}_{\underline{B}_n}$. As a consequence

Sections of a quantum contour are such that no more than $|B_1| + \cdots + |B_n|$ additional plaquettes are needed to make them connected. (3.36)

(This statement is false in $d = 1$.) Because of this, we shall think of the quantum bonds B as "glue", and we shall refer to their cardinality $|B|$ as the "number of glue plaquettes". Note that observation (3.36) is also true for contours with no connected projections [Figure 6]. This is because if $|B_1|+\cdots+|B_n|$ glue plaquettes are needed for one component of a quantum contour to encircle another one, then an even smaller number of glue plaquettes is required to connect the two components.

A quantum contour, ζ, also has a well-defined notion of exterior and interior, with a unique configuration corresponding to each of its connected components. In analogy with the classical case, we shall use the notations $\mathrm{Ext}(\zeta)$ and $\mathrm{Int}_q(\zeta)$ to refer to them. We also define the support, $\mathrm{supp}\,\zeta$, of a quantum contour ζ as the union of the corresponding defect set (in $\mathbb{Z}^d \times [0,\beta]$) *and* the sites occupied by each of the quantum bonds B_i. Let

$$|\Gamma_i| := \sum_{\gamma\in\Gamma_i} |\gamma|. \tag{3.37}$$

Then the area $|\zeta| := |\,\mathrm{supp}\,\zeta|$ is computed by adding

$$|\zeta|_\perp \;:=\; |\Gamma_0|\Big(\beta - \sum_{i=1}^{n}\tau_i\Big) + |\Gamma_1|\tau_1 + \cdots + \tau_n|\Gamma_0|\,, \tag{3.38}$$

Figure 5: Examples of quantum contours: (a) Connected contours: (a1) Long contour (a2) Long contour with no connected section; (a3) Long contour (connectedness results from periodicity in the "time"-direction) (a4) Short contour.

Figure 6: Examples of quantum contours: (b1), (b2), (b3) Linked contours (b4) A linked contour whose projection is not connected; (c) A surface that is not a quantum contour, even though its spatial projection (= orthogonal projection onto $\mathbb{Z}^d \times \{0\}$) is connected.

which is the sum of the areas of the cylindrical portions, to the number of sites only contained in the glue plaquettes. This last number is bounded above by the total number of glue plaquettes:

$$|B(\zeta)| \;:=\; |B_1| + \cdots + |B_n|\,, \tag{3.39}$$

hence

$$|\zeta| \;:=\; |\mathrm{supp}\zeta| \;\leq\; |\zeta|_\perp + |B(\zeta)|\,. \tag{3.40}$$

A quantum contour ζ is said to be an *exterior* contour of a family of contours if its support is not contained in the interior of any other contour of the family. As in the classical case, we shall say that two contours are *compatible* if their supports do not intersect or form linked surfaces, *and* the labels of the configurations match, i.e., the exterior labels are the same if the contours are mutually exterior or, if they are nested, the exterior label of the internal contour coincides with the label of the component of the interior of the larger contour that contains it. A family of contours is compatible if its members are pairwise compatible. Such families can be associated to configurations on $\mathbb{Z}^d \times [0,\beta]$.

Note: The condition of non-linking of surfaces is not relevant for bosons or fermions. However, in view of applications of our theory to particles with other statistics, we shall include this condition of *non-linking* in the definition of compatibility.

The *weight* of a quantum contour ζ is given by

$$\begin{aligned} w(\zeta) \;=\; & \left[\prod_{i=1}^{n} \langle \Gamma_i | -\Phi^{q}_{\underline{B}_i} | \Gamma_{i-1}\rangle\right] \\ & \times \exp\{-\left[E(\Gamma_0)(\beta - \textstyle\sum_{i=1}^{n}\tau_i) + E(\Gamma_1)\tau_1 + \cdots + E(\Gamma_0)\tau_n\right]\}\,. \end{aligned} \tag{3.41}$$

The decay law (2.62) and the Peierls bound (3.21) [along with the linearity of $E(\Gamma)$, Eq. (3.18)] imply the bound

$$|w(\zeta)| \;\leq\; \lambda^{|B(\zeta)|}\,\exp[-J|\zeta|_\perp]\,. \tag{3.42}$$

This bound is the **quantum Peierls condition**.

4 Factorization properties

4.1 Factorization of the τ-integrals

This property follows from the fact that each quantum interaction $\Phi^q_{\underline{B}_i}$ affects a bond B_i in only *one* component of Υ^p. Hence, at the end of each "time" interval, τ_i, only the section of *one* of the components changes. The other components are not affected by the action of the quantum interaction at time τ_i, and hence their sections, and the corresponding exponential weights, remain unchanged. We shall explain this statement through a simple example: Let us assume that we are "summing" over a surface Υ^p consisting of two connected components, which we shall label by the symbols '$\sim$' and '$\frown$'. We also assume that only four bonds are affected by the quantum interactions, successive ones belonging to different components. This gives rise to sequences of the form:

$$\Big(\widehat{\Gamma}_0\widetilde{\Gamma}_0\,,\,\underline{\widehat{B}}_1\,,\,\widehat{\Gamma}_1\widetilde{\Gamma}_0\,,\,\tau_1\,,\,\underline{\widetilde{B}}_1\,,\,\widehat{\Gamma}_1\widetilde{\Gamma}_1\,,\,\tau_2\,,\,\underline{\widehat{B}}_2\,,\,\widehat{\Gamma}_2\widetilde{\Gamma}_1\,,\,\tau_3\,,\,\underline{\widetilde{B}}_2\,,\,\widehat{\Gamma}_2\widetilde{\Gamma}_2\,,\,\tau_4\Big) \tag{4.1}$$

[We have omitted the superscript p to simplify the notation]. We note that we do *not* assume that $\widehat{\Gamma}_2 = \widehat{\Gamma}_0$. We need to consider integrals of the form

$$\begin{aligned} I \;=\; & \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_4\, \Theta\Big(\beta - \textstyle\sum_{i=1}^4 \tau_i\Big) e^{-\left(\beta-\sum\tau_i\right)[f(\hat\Gamma_0)+f(\tilde\Gamma_0)]}\, e^{-\tau_1[f(\hat\Gamma_1)+f(\tilde\Gamma_0)]} \\ & \times e^{-\tau_2[f(\hat\Gamma_1)+f(\tilde\Gamma_1)]}\, e^{-\tau_3[f(\hat\Gamma_2)+f(\tilde\Gamma_1)]}\, e^{-\tau_4[f(\hat\Gamma_2)+f(\tilde\Gamma_2)]}\,, \end{aligned} \tag{4.2}$$

where f is some contour energy.
By regrouping the exponentials and performing the change of variables

$$\widehat{\tau}_1 = \tau_1 + \tau_2\ ;\ \widehat{\tau}_2 = \tau_3 + \tau_4\ ;\ \widetilde{\tau}_1 = \tau_2 + \tau_3\ ;\ \widetilde{\tau}_2 = \tau_4\ , \tag{4.3}$$

we find that I factorizes as follows:

$$I \;=\; \widehat{I}\,\widetilde{I} \tag{4.4}$$

with

$$\widehat{I} \;=\; \int_0^\beta d\tau_1 \int_0^\beta d\tau_2\, \Theta(\beta - \hat\tau_1 - \hat\tau_2)\, e^{-(\beta-\hat\tau_1-\hat\tau_2)f(\hat\Gamma_0)}\, e^{-\hat\tau_1 f(\hat\Gamma_1)}\, e^{-\hat\tau_2 f(\hat\Gamma_2)} \tag{4.5}$$

and $\widetilde{I}$ being given by a similar integral, but with the '$\frown$' replaced by '$\sim$'.

4.2 Factorization of the weights

For *bosons* it is easy to see that the weights $w(\Upsilon)$ can be written as a product of weights of pairwise disjoint, connected components. In this case the phase $\alpha_{\preceq}$ in (2.30) and (2.31) is zero. Hence we can absorb the matrix elements of the operators Φ^{q}_B in the states (2.28) into contour weights.

The situation for particles with other statistics is more complicated because the action of each $\Phi^{\mathrm{q}}_{B_i}$ gives rise to a phase that depends on the other contours present. However, in the case of *fermions*, the weights factorize because the interactions are assumed to be of the form

$$\Phi^{\mathrm{q}}_B = \sum_{\underline{B}} \Phi^{\mathrm{q}}_{\underline{B}}, \tag{4.6}$$

where each $\Phi^{\mathrm{q}}_{\underline{B}}$ is a monomial of even degree in the fermionic creation and annihilation operators.

Figure 7: A space-time surface, Υ^p, with two connected components, ζ_B and ζ_C. The sections of ζ_B and ζ_C at "time" zero are denoted by γ_B^0 and γ_C^0 respectively. The corresponding sections at "time" β are denoted by γ_B^β and γ_C^β.

We explain the factorization argument for a periodic space-time surface, Υ^p, consisting of two connected components. An example of such a surface is given in Figure 7.

The surface corresponds to the successive actions of a sequence of operators $\Phi^q_{\underline{D}_1} \cdots \Phi^q_{\underline{D}_{n+m}}$ belonging to the quasilocal algebra $\mathcal{A}$. Let ζ_B and ζ_C be the two connected components of the surface Υ^p. They correspond to the two families of operators $\mathcal{B} = \{\Phi^q_{\underline{B}_1}, \ldots, \Phi^q_{\underline{B}_n}\}$ and $\mathcal{C} = \{\Phi^q_{\underline{C}_1}, \ldots, \Phi^q_{\underline{C}_m}\}$. The sequence $(\underline{D}_1, \ldots, \underline{D}_{n+m})$ is a permutation of the sequence $(\underline{B}_1, \ldots, \underline{B}_n, \underline{C}_1, \ldots, \underline{C}_m)$ and is uniquely determined by the surface Υ^p. We assume that, for any subsequence of $\underline{D}_i$'s, the corresponding subsequences of $\underline{B}_i$'s and $\underline{C}_i$'s maintain an increasing order in i.

The surface Υ^p corresponds to a product,

$$\prod_{i=1}^{n+m} \langle \Gamma_i^p | \Phi_{\underline{D}_i}^q | \Gamma_{i-1}^p \rangle, \tag{4.7}$$

of matrix elements of the operators $\Phi_{\underline{D}_i}^q$, with $\Gamma_{n+m}^p = \Gamma_0^p$. In (4.7), Γ_i^p is a compatible family of contours describing the configuration that corresponds to the section of Υ^p at "time" τ, with $\tau_i < \tau < \tau_{i+1}$. Our definition of the operators $\Phi_{\underline{D}_i}^q$ guarantees that if $\langle \Gamma_i^p | \Phi_{\underline{D}_i}^q | \Gamma_{i-1}^p \rangle \neq 0$ then the family Γ_i^p of compatible contours is uniquely determined by the operator $\Phi_{\underline{D}_i}^q$ and by Γ_{i-1}^p and hence by $\Phi_{\underline{D}_i}^q, \cdots, \Phi_{\underline{D}_1}^q$ and by Γ_0^p, for all $i = 1, \ldots, n+m$. Moreover, since we are computing traces, $\Gamma_{n+m}^p = \Gamma_0^p$.

The periodicity and the disjointness of the components ζ_B and ζ_C implies properties (P1) and (P2) described below. To state them we introduce the following nomenclature:

We shall say that an operator $\mathbf{c}_{x\sigma}$ (resp. $\mathbf{c}_{x\sigma}^*$) *occurs* in $\Phi_{\underline{B}_i}^{\mathrm{q}}$, if the operator is part of the monomial defining $\Phi_{\underline{B}_i}^{\mathrm{q}}$. We shall also say that the pair (x, σ) occurs in $\underline{B}_i$ if $\mathbf{c}_{x\sigma}$ (resp. $\mathbf{c}_{x\sigma}^*$) *occurs* in $\Phi_{\underline{B}_i}^{\mathrm{q}}$. The phrase "occurrence of (x, σ) in $\mathcal{B}$" shall mean that the corresponding creation or annihilation operator *occurs* in some factor $\Phi_{\underline{D}_i}^q$, with $\underline{D}_i = \underline{B}_k$, for some $1 \leq k \leq n$. A site x is said to belong to a space-time surface ζ at a particular instant of time, if it is contained in the spatial section of ζ at that time.

The following properties are satisfied:

(P1) Given the vector $|\Gamma_0^p\rangle$, there exists some complex number $w_{\mathcal{B}} \neq 0$, depending on Γ_0^p such that

$$\Phi_{\underline{B}_n}^{\mathrm{q}} \cdots \Phi_{\underline{B}_1}^{\mathrm{q}} |\Gamma_0^p\rangle \;=\; w_{\mathcal{B}} |\Gamma_0^p\rangle \, . \tag{4.8}$$

This follows from the remarks after (4.7), in particular from the periodicity of the component ζ_B corresponding to the action of the $\Phi_{\underline{B}_i}^q$'s. We get a similar relation for the action of the $\Phi_{\underline{C}_i}^q$'s.

(P2) Between two subsequent occurrences of (x, σ) in $\mathcal{B}$ there is an *even* number of occurrences of (x, σ) in $\mathcal{C}$. This corresponds to the non-intersecting character of the components: If a site x belongs first to the components ζ_B, then to ζ_C and then once again to ζ_B, the operators $\Phi_{\underline{C}_j}^q$ must be such that, at the end of the intermediate period, they transform the configuration at the site x to what it was before they started acting.

With these properties we can prove the following result.

Proposition 4.1 *Consider two families $\mathcal{B}$ and $\mathcal{C}$ of operators, and an operator $\Phi_{\underline{\mathbf{D}}}^q := \Phi_{\underline{D}_{n+m}}^q \cdots \Phi_{\underline{D}_1}^q$, defined as above, where $(\underline{D}_{n+m} \cdots \underline{D}_1)$ is a permutation of the sequence $(\underline{B}_1, \ldots, \underline{B}_n, \underline{C}_1, \ldots, \underline{C}_m)$. Assume that properties (P1) and (P2) stated* *above are satisfied and that the operators $\Phi^{\mathrm{q}}_{\underline{B}_i}$ are monomials of even degree in the creation and annihilation operators, i.e.,*

$$[\mathbf{c}^*_{x\sigma}, \Phi^{\mathrm{q}}_{\underline{B}_i}] = 0 \quad \textit{and} \quad [\mathbf{c}_{x\sigma}, \Phi^{\mathrm{q}}_{\underline{B}_i}] = 0 \tag{4.9}$$

whenever (x,σ) does not occur in $\underline{B}_i$. Then, for any vector $|\Gamma^p_0\rangle$, there exists a complex number $w_{\mathcal{C}}$ such that

$$\langle\Gamma^p_0|\Phi^q_{\underline{D}_{n+m}} \cdots \Phi^q_{\underline{D}_1}|\Gamma^p_0\rangle \;=\; w_{\mathcal{B}}\, w_{\mathcal{C}}\ . \tag{4.10}$$

Proof. We first note that the result is obviously true if the operator $\Phi^q_{\underline{\mathbf{D}}} \equiv \prod_{i=1}^{n+m} \Phi^q_{\underline{D}_i}$ is identically zero. Hence we may assume that this is not the case. In particular, this implies that the creation and annihilation operators corresponding to a pair (x,σ) necessarily occur alternately in the sequence of operators constituting $\Phi^q_{\underline{\mathbf{D}}}$.

This observation allows us to restrict our attention to a situation in which no pair (x,σ) occurs in *both* sets, $\mathcal{B}$ and $\mathcal{C}$. Indeed, if (x,σ) occurs in $\mathcal{B}$ *and* $\mathcal{C}$, then by (P2) there is an even number of occurrences in the latter between two subsequent occurrences in the former. Therefore the occurrences of (x,σ) in $\mathcal{C}$ can be grouped into pairs not having any intermediate occurrence in $\mathcal{B}$. In this situation, if we use the anticommutation rules to move the first element of each pair to a position immediately preceding the second element, the commutation through intermediate operators in $\mathcal{B}$ does not produce any non-trivial phase because of (4.9). Hence we obtain a product of the form $\mathbf{c}_{x\sigma}\mathbf{c}^*_{x\sigma}$, (or $\mathbf{c}^*_{x\sigma}\mathbf{c}_{x\sigma}$), which we can replace by the identity operator $\mathbf{1}$ by using the anticommutation relation. This is because the opposite order of the operators would necessarily yield zero. In this way we obtain

$$\Phi^q_{\underline{\mathbf{D}}}|\Gamma^p_0\rangle \;=\; \epsilon^{x\sigma}_{\mathcal{C}}\, \widetilde{\Phi}^q_{\underline{\mathbf{D}}}|\Gamma^p_0\rangle\ , \tag{4.11}$$

where $\epsilon^{x\sigma}_{\mathcal{C}}$ is a phase and the symbol '$\sim$' has been used to indicate that $\mathcal{C}$ no longer consists of creation or annihilation operators corresponding to the pair (x,σ). We should point out that the operators constituting $\widetilde{\Phi}^q_{\underline{\mathbf{D}}}$ also satisfy properties (P1) and (P2) and the commutation relations (4.9). Note that the operators in $\mathcal{B}$ are *unchanged*. We can repeat this procedure for all the remaining pairs which occur in both sets, $\mathcal{B}$ and $\mathcal{C}$. At the end we obtain

$$\Phi^q_{\underline{\mathbf{D}}}|\Gamma^p_0\rangle = \epsilon_{\mathcal{C}}\, \widehat{\Phi}^q_{\underline{\mathbf{D}}}|\Gamma^p_0\rangle\ , \tag{4.12}$$

where $\epsilon_{\mathcal{C}}$ is the product of the phases, $\epsilon^{x\sigma}_{\mathcal{C}}$, for all pairs occurring in $\mathcal{B}$ *and* in $\mathcal{C}$, and the symbol '$\frown$' is used to indicate that the operators in $\mathcal{C}$ have been stripped of creation and annihilation operators corresponding to these pairs. Hence we no longer have any creation or annihilation operator in $\mathcal{C}$ corresponding to a pair (x,σ) occurring in $\mathcal{B}$, and we can move the remaining operators in $\mathcal{C}$ through the operators in $\mathcal{B}$, without producing any further phases. This results in an equation

$$\Phi^q_{\underline{D}_{n+m}} \cdots \Phi^q_{\underline{D}_1}|\Gamma^p_0\rangle \;=\; \epsilon_{\mathcal{C}}\, \widehat{\Phi}^q_{\underline{C}_m} \cdots \widehat{\Phi}^q_{\underline{C}_1}\, \Phi^{\mathrm{q}}_{\underline{B}_n} \cdots \Phi^{\mathrm{q}}_{\underline{B}_1}|\Gamma^p_0\rangle\ , \tag{4.13}$$

which, given (4.8), yields

$$\begin{aligned}\langle\Gamma_0^p|\Phi^q_{\underline{D}_{n+m}}\cdots\Phi^q_{\underline{D}_1}|\Gamma_0^p\rangle &= w_{\mathcal{B}}\,\epsilon_{\mathcal{C}}\,\langle\Gamma_0^p|\widehat{\Phi}^q_{\underline{C}_m}\cdots\widehat{\Phi}^q_{\underline{C}_1}|\Gamma_0^p\rangle \\ &:= w_{\mathcal{B}}\,w_{\mathcal{C}}\;. \;\blacksquare \end{aligned}\tag{4.14}$$

This proof can be extended to surfaces with more than two connected components. The situation is more complicated for anyons. In particular, for $d=2$, it is necessary to demand that the component formed by the $\Phi^q_{\underline{B}_i}$'s is not only disjoint but is also not linked with the one formed by the action of the $\Phi^q_{\underline{C}_i}$'s. For the sake of generality and in preparation for further studies, we shall consider this extra condition of non-linking as part of our definition of independent contours.

5 Contour expansions

5.1 Contour expansion for the partition functions

To construct the partition function corresponding to a boundary condition s_p, (defined in (3.26)), contours are added by summing over the "spatial" degrees of freedom and integrating over the "time" axis. We shall denote this sum-integral operation by a combined symbol: if g is a complex-valued function on quantum contours, then

$$\begin{aligned}\sum\!\!\!\!\!\!\int_{\zeta} g(\zeta) &:= 1+\sum_{n\geq 1}\sum_{(\underline{B}_1,\ldots,\underline{B}_n)}\sum_{(\Gamma_0,\ldots,\Gamma_n)}\mathrm{I}[(\mathrm{i}),(\mathrm{ii})] \\ &\qquad\times\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n\;\mathrm{I}\left[\beta\geq\textstyle\sum_{i=1}^n\tau_i\right]\,g(\zeta)\;.\end{aligned}\tag{5.1}$$

We sum each $\underline{B}_i$ over all quantum bonds, and each Γ_i^p over all possible families of compatible classical contours (with exterior p-contours). By $\mathrm{I}[E]$ we mean the indicator function of the event E; in particular, $\mathrm{I}[(\mathrm{i}),(\mathrm{ii})]$ in (5.1) vanishes unless the sections Γ_i satisfy the following conditions [see Definition 3.3]:

(i) $0\;\leq\;\Big||\Gamma_i|-|\Gamma_{i-1}|\Big|\;\leq\;|B_i|\;.$

(ii) $\Gamma_0^p=\Gamma_n^p$

We are interested in the "sums" corresponding to partition functions for piecewise-cylindrical finite regions V in $d+1$ dimensions. For such regions we define the volume, $|V|$, to be the sum of the volumes of the constituent cylindrical regions, where, however, in consistency with our choice of units for areas in $\mathbb{Z}^d$, the areas of the bases of the cylinders are measured in units of the sampling plaquette. Similarly, we obtain the area of the internal boundaries, ∂V, by adding the surface areas of these piecewise-cylindrical regions to the area of the bases. Again, we use sampling-plaquette units in $\mathbb{Z}^d$.

¿From now on, the symbol V will indicate a *piecewise-cylindrical* region of $\mathbb{Z}^d \times [0,\beta]$. The partition function for such a region V, with a spatial boundary condition s_p, is (formally) defined by the series

$$\begin{aligned}\Xi_p(V) \;&=\; \exp\Big[-\int_V e_x(s_p)\Big]\\ &\quad\times \sum_{\substack{\{\zeta_k\}\subset\hat V\\ \text{compatible}}} \Big[\prod_k w(\zeta_k)\Big]\Big[\prod_{u=1}^{P}\exp\Big\{-\int_{L_u}[e_x(s_u)-e_x(s_p)]\Big\}\Big],\end{aligned} \tag{5.2}$$

where the exterior contours of each compatible family are p-contours. The weights w are given by expression (3.41), and the region $\widehat{V}$ is obtained from V by adding to each spatial section the plaquettes touching V, i.e.,

$$\widehat{V} = \widehat{\Lambda}\times[0,\beta], \tag{5.3}$$

where $\widehat{\Lambda}$ is as defined in (3.20). The region $L_u = L_u(\{\zeta_k\})$ consists of the set of points in V that are either u-correct or that belong to a u-quantum contour. We shall call such a series a *contour expansion* of the partition function $\Xi_p(V)$. For V of the form $\Lambda\times[0,\beta]$ we recover expression (3.26). We shall use the letter V for space-time regions, Λ for spatial ones, and the abbreviation $\Xi_p(\Lambda) := \Xi_p(\Lambda\times[0,\beta])$.

We shall be interested in the quantity

$$f_p(V) \;:=\; \frac{-1}{|V|}\log\Xi_p(V)\;, \tag{5.4}$$

whenever the series (5.2) converges to a nonzero value, and in the limit

$$f \;:=\; \lim_{V\nearrow\mathbb{Z}^d\times[0,\beta]} f_p(V)\;, \tag{5.5}$$

whenever it exists.

Note that we have used the same symbols in the above definitions as those used in (2.49) and (2.50) to denote the free energy densities. This is in anticipation of the fact that, in the regimes analyzed in this paper, both definitions agree for regions of the form $V = \Lambda\times[0,\beta]$.

5.2 Contour expansion for the quantum expectations

An expansion for the expectations

$$\frac{\operatorname{Tr}_\Lambda^{s_p}\mathbf{A}\,e^{-\beta\mathbf{H}_\Lambda}}{\operatorname{Tr}_\Lambda^{s_p}e^{-\beta\mathbf{H}_\Lambda}} \;:=\; \frac{\Xi_p^{\mathbf{A}}(\Lambda)}{\Xi_p(\Lambda)} \tag{5.6}$$

can be constructed by expanding the numerator and the denominator. For the latter we have the previously developed expansion, (5.2). A similar expansion can be obtained for the numerator by proceeding as follows: We expand $\mathbf{A}e^{-\beta\mathbf{H}_\Lambda}$ with the help of the iterated-Duhamel formula (3.24) and perform steps (a)–(d) of Section 3.2. Let us assume, without loss of generality, that $\mathbf{A} \equiv \mathbf{A}_D \in \mathcal{A}_D$ for some finite $D \subset \mathbb{Z}^d$. Moreover, for *fermions*, we can also assume that $\mathbf{A}$ is an even monomial in creation and annihilation operators. Therefore we can view the operator $\mathbf{A}$ as giving rise to an extra "quantum bond" $\underline{D}$. We then obtain an expansion for $\Xi_p^{\mathbf{A}}(\Lambda)$ which is exactly of the same form (3.31), but where the terms of the sum are labelled by sequences of the form

$$\Upsilon_{\mathbf{A}}^p = (\Gamma_0^p, \underline{B}_1, \Gamma_1^p, \tau_1, \ldots, \underline{B}_n, \Gamma_n^p, \tau_n, \underline{D}, \Gamma_0^p) \tag{5.7}$$

with weights

$$\begin{aligned} w(\Upsilon_{\mathbf{A}}^p) &= \langle \Gamma_0^p | \mathbf{A} | \Gamma_n^p \rangle \langle \Gamma_n^p | - \Phi^{\mathrm{q}}_{\underline{B}_n} | \Gamma_{n-1}^p \rangle \cdots \langle \Gamma_1^p | - \Phi^{\mathrm{q}}_{\underline{B}_1} | \Gamma_0^p \rangle \\ &\quad \times e^{-\left(\beta - \sum \tau_i\right) E(\Gamma_0^p)} \, e^{-\tau_1 E(\Gamma_1^p)} \cdots e^{-\tau_n E(\Gamma_n^p)} \,. \end{aligned} \tag{5.8}$$

Note that the factor $\langle \Gamma_0^p | \mathbf{A} | \Gamma_n^p \rangle$ plays a role analogous to $\langle \Gamma_0^p | - \Phi^{\mathrm{q}}_{\underline{B}_{n+1}} | \Gamma_n^p \rangle$, with $\underline{B}_{n+1} = \underline{D}$.

To obtain a contour expansion, we must exhibit factorization of the expansion for $\Xi_p^{\mathbf{A}}(\Lambda)$ over the components of $\Upsilon_{\mathbf{A}}^p$. The "time" integrals involved are the same as for the partition function $\Xi_p(\Lambda)$, hence the factorization illustrated in Section 4.1 remains valid. For bosons the weights factorize as before. For fermions too, a factorization of the weights can be exhibited if we use the following manipulations: We group into a single entity all the components of $\Upsilon_{\mathbf{A}}^p$ whose sections at "time" β intersect the set $\underline{D}$. In this way, we obtain a special component $\zeta_{\mathbf{A}}$ corresponding to the action of $\mathbf{A}$. We shall refer to $\zeta_{\mathbf{A}}$ as the quantum contour *associated with* $\mathbf{A}$. It is defined by a sequence of the form (5.7) and has a weight given by (5.8). An example of such a contour is given in Figure 8, below.

All the other components of $\Upsilon_{\mathbf{A}}^p$ are quantum contours defined as in the previous section, i.e., their weights do not involve the operator $\mathbf{A}$. The factorization result of Proposition 4.1 applies. [In Figure 7, an example of a surface for which this proposition is valid is given. Note that the component ζ_C in this figure is a contour associated with a local observable.]. The weight $w(\Upsilon_{\mathbf{A}}^p)$ can hence be written as a product

$$w(\Upsilon_{\mathbf{A}}^p) = \Big[\prod_{\{\zeta_k\}} w(\zeta_k)\Big] \, w(\zeta_{\mathbf{A}}), \tag{5.9}$$

where

$$\Big(\bigcup_k \zeta_k\Big) \bigcup \zeta_{\mathbf{A}} = \Upsilon_{\mathbf{A}}^p. \tag{5.10}$$

Factorization implies that we can expand the numerator of the expectations (5.6) in the form

$$\Xi_p^{\mathbf{A}}(V) = \exp\Big[-\int_V e_x(s_p)\Big] \sum\!\!\!\!\!\!\int_{\zeta_{\mathbf{A}} \subset \hat{V}}^{*} w(\zeta_{\mathbf{A}})$$

Figure 8: A quantum contour, $\zeta_{\mathbf{A}}$, *associated with* a local observable $\mathbf{A} \in \mathcal{A}_D$ where D is a finite subset of the lattice.

$$
\times \sideset{}{^*}{\sum\!\!\!\!\!\!\int}_{\substack{\{\zeta_k\}\subset\hat{V} \\ \text{compatible}}} \left[\prod_k w(\zeta_k)\right] \left[\prod_{u=1}^{P} \exp\Big[-\int_{L_u} [e_x(s_u) - e_x(s_p)]\Big]\right]
$$

$$
\times \, \mathrm{I}\Big[\Big\{\zeta_{\mathbf{A}}, \{\zeta_k\}\Big\} \text{ compatible; exterior contours are } p\text{-contours}\Big] \, . \tag{5.11}
$$

The starred sum-integral refers to an expression of the form (5.1), but where in condition (ii) we demand

$$
\mathbf{A}|\Gamma^p_n > \propto \ |\Gamma^p_{n+1} >, \quad \text{with} \quad \Gamma^p_{n+1} = \Gamma^p_0. \tag{5.12}
$$

The weights $w(\zeta_{\mathbf{A}})$ satisfy a Peierls bound:

$$
|w(\zeta_{\mathbf{A}})| \ \leq \ \lambda^{|B(\zeta_{\mathbf{A}})|} \exp[-J|\zeta_{\mathbf{A}}|_{\perp}] \, \frac{||A||}{\lambda^{|D|}} \, , \tag{5.13}
$$

where

$$|\zeta_{\mathbf{A}}|_{\perp} := |\Gamma_0|\Big(\beta - \sum_{i=1}^{n} \tau_i\Big) + |\Gamma_1|\tau_1 + \cdots + \tau_n|\Gamma_n| \;, \tag{5.14}$$

$$|B(\zeta_{\mathbf{A}})| := |B_1| + \cdots + |B_n| + |D| \tag{5.15}$$

and $\|\cdot\|$ denotes the usual operator norm.

6 Cluster expansion for the symmetric or the single-phase regime

6.1 The result

Having formulated a convenient low-temperature expansion in terms of contours, we must address the task of proving its convergence for some open set of values of β and λ. The main mathematical complications arise from the requirement of compatibility of the contours in expression (5.2). Compatibility is a highly non-local condition (two arbitrarily far away nested contours can be rendered incompatible by a mismatch between the labels of their exterior and interior configurations).

In this section we analyze the simpler case in which compatibility reduces to non-linking (or non-intersection), i.e., when the labels of the configurations outside the support of the contours become irrelevant. The results of this section form the basis for the full-fledged theory which is to be discussed in Section 7. We are concerned with an expansion of the form

$$\Xi_p(V) \;=\; \sum\!\!\!\!\!\!\int_{\substack{\{\zeta_k^p\}\subset\widehat{V} \\ \mathrm{non-linked}}} \prod_k w(\zeta_k^p) \;, \tag{6.1}$$

with weights as in (3.41) satisfying the quantum Peierls condition (3.42). This type of expansion is obtained when there is a single ground state for an open set of parameters $\underline{\mu}$, or, more generally, for values of $\underline{\mu}$ for which the ground states are related by some symmetry operation. In the latter case, the removal of any contour of a compatible family leads to another compatible family of contours. These have been among the first situations treated by contour arguments [9, 15]. Among our examples of Section 2.3, this symmetric situation occurs at the coexistence point $h^{\mathrm{stagg}} = 0$ for the Fisher antiferromagnet, with $K = 0$ and $|h| < 2$ [Figure 2 (b)], and at the coexistence point $\mu_{\mathrm{stagg}} = 0$ for its lattice-gas transcription, [i.e., Example 2], with $K = 0$ and $\mu < 2$ [Figure 3 (b)].

There is a well established technology to analyze (the log of) "volume-exclusion" expansions like (6.1), namely the methods of cluster-expansion [32, Chapter 4]. The method involves some standard combinatorics (reviewed in Section 6.2 below) and a bound on the sum of the weights of contours containing a fixed point.

Let

$$f_p \;=\; -\lim_{V\nearrow\mathbb{Z}^d\times[0,\beta]} \frac{1}{|V|}\log\Xi_p(V)\;, \tag{6.2}$$

In Sections 6.4 and 6.5 we shall prove the following key result.

Theorem 6.1 *There exist strictly positive constants $\widetilde{J}$ and ε_0 such that, for each β and λ in the region*

$$\max(e^{-\beta\tilde{J}}\,,\,\lambda) \;<\; \varepsilon_0, \tag{6.3}$$

the cluster expansion (6.1) converges absolutely for all piecewise-cylindrical regions V. The free-energy density f_p exists in this region and is jointly analytic in $e^{-\beta\tilde{J}}$ and λ. Moreover it satisfies

$$f_p \;=\; \mathrm{O}(\varepsilon_0)\;, \tag{6.4}$$

and

$$\Big||V|f_p + \log\Xi_p(V)\Big| \;\le\; |\partial V|\,\mathrm{O}(\varepsilon_0)\;. \tag{6.5}$$

Corollary 6.2 *There exists a positive constant $\widetilde{\varepsilon}_0$ such that, in the region* $\max(e^{-\beta\tilde{J}}\,,\,\lambda) < \widetilde{\varepsilon}_0$*, the limit*

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\Xi_p^{\mathbf{A}}(\Lambda)}{\Xi_p(\Lambda)} \;=:\; \langle\mathbf{A}\rangle_{\beta\,\lambda} \tag{6.6}$$

exists and is jointly analytic as a function of $e^{-\beta}$ and λ, for any local observable $\mathbf{A}$. Moreover

$$\lim_{\substack{\beta\to\infty\\ \lambda\to 0}}\langle\mathbf{A}\rangle_{\beta\,\lambda} \;=\; \langle s_p|\mathbf{A}|s_p\rangle\;. \tag{6.7}$$

This proves the stability of all the phases occurring in the present situation. In particular, this proves that the phase diagrams of Figures 2 (b) and 3 (b) remain *undeformed* at low temperatures and under small quantum perturbations.

Remark: In the arguments that follow we shall impose conditions of the form "J sufficiently large" and "λ sufficiently small". As long as the temperature is sufficiently low and λ is sufficiently small, we can choose J large enough simply by a rescaling. This is because the Hamiltonian $\mathbf{H}_\Lambda$, which depends on the parameters J and λ, appears in the partition function in the form $\beta\,\mathbf{H}_\Lambda$ which we can write as

$$\beta(\mathbf{H}_\Lambda^{\mathrm{cl}} + \mathbf{V}_\Lambda) \;=\; \frac{\beta}{\beta'}\,(\beta'\mathbf{H}_\Lambda^{\mathrm{cl}} + \beta'\mathbf{V}_\Lambda)\;. \tag{6.8}$$

Upon rescaling, $\beta' J \to J$, we see that a large β' leads to a large J. The product $\beta'\lambda$, which is the rescaled perturbation parameter, is small if λ is sufficiently small. Once β' is fixed we adjust β such that β/β' is sufficiently large.

6.2 Combinatorics of the cluster expansion

We summarize some results on cluster expansions, adapted to the case where contours are discrete in all dimensions, except one. We can extend the conventional proofs for the convergence of cluster expansions (applicable when the compatibility relation corresponds to non-intersection) to the more general situation when the contours are either disjoint *or* non-linked. For the usual proofs the reader may consult for instance [33], [7] or [27].

We use the notation $\zeta \imath \widetilde{\zeta}$, see [21], to denote the condition that $\operatorname{supp} \zeta$ either intersects or is linked with $\operatorname{supp} \widetilde{\zeta}$. In the following, we shall use the term linking to refer to both intersection and linking. A *cluster* is a *finite* family $\{\zeta_1, \cdots, \zeta_n\}$ of contours that cannot be decomposed into two non-linked subfamilies. To simplify our notation we have omitted the superscripts of the contours ζ referring to their exterior configurations.

The following theorem gives the condition for the convergence of a cluster expansion.

Theorem 6.3 *If*

$$C \;:=\; \sup_{\widetilde{\zeta}} \frac{1}{|\widetilde{\zeta}|} \not\!\!\int\!\!\!\!\!\!\sum_{\zeta \imath \widetilde{\zeta}} |w(\zeta)|\, e^{|\zeta|} \;<\; 1\,, \tag{6.9}$$

then the expansion (6.1) is absolutely convergent and $\log \Xi(V)$ *has an absolutely convergent expansion of the form*

$$\log \Xi(V) \;=\; \sum_{N \geq 1} \frac{1}{N!} \not\!\!\int\!\!\!\!\!\!\sum_{\zeta_1 \subset \widehat{V}} \cdots \not\!\!\int\!\!\!\!\!\!\sum_{\zeta_N \subset \widehat{V}} w^{\mathrm{T}}(\{\zeta_1, \ldots, \zeta_N\}) \tag{6.10}$$

(the cluster expansion*), where* w^{T} *is a function on families of contours with the properties that*

$$w^{\mathrm{T}}(\{\zeta_1, \cdots, \zeta_N\}) \;=\; 0 \quad \text{if } \{\zeta_1, \cdots, \zeta_N\} \text{ is not a cluster}\,, \tag{6.11}$$

and satisfying the bound

$$\sup_{\vec{x},t} \not\!\!\int\!\!\!\!\!\!\sum_{\{\zeta_1,\ldots,\zeta_N\} \ni (\vec{x},t)} \frac{|w^{\mathrm{T}}(\zeta_1, \ldots, \zeta_N)|}{|\zeta_1 \cup \cdots \cup \zeta_N|} \;\leq\; \alpha_N \tag{6.12}$$

where α_N *is a constant of the order of*

$$\sup \prod_{i=1}^{N} |w(\zeta_i)|,$$

the supremum being taken over all sets of N *contours (not necessarily compatible).*

[The notation $\{\zeta_1, \ldots, \zeta_N\} \ni (\vec{x}, t)$ means that $(\vec{x}, t)$ belongs to the union of the supports of the quantum contours $\zeta_1, \ldots, \zeta_N$.] ¿From this and the periodicity of the contour ensemble one obtains an expression for the quantity f defined in (5.5).

Corollary 6.4 *If*

$$C \ := \ \sup_{\widetilde{\zeta}} \frac{1}{|\widetilde{\zeta}|} \mathop{\sum\!\!\!\!\!\!\int}_{\zeta \,\imath\, \widetilde{\zeta}} |w(\zeta)|\, e^{|\zeta|} \ < \ 1 \,, \tag{6.13}$$

then the quantity f, defined in (5.5), exists, and is given by

$$\begin{aligned} f \ &= \ -\sum_{N \geq 1} \frac{1}{|W|} \sum_{(\vec{x},0) \in W} \mathop{\sum\!\!\!\!\!\!\int}_{\{\zeta_1, \ldots, \zeta_N\} \ni (\vec{x},0)} \frac{w^{\mathrm{T}}(\zeta_1, \ldots, \zeta_N)}{|\zeta_1 \cup \cdots \cup \zeta_N|} \\ &= \ \alpha', \end{aligned} \tag{6.14}$$

where W is a fundamental cell of the configuration and α' is a constant of order C. Moreover,

$$\Big| |V| f + \log \Xi(V) \Big| \ \leq \ |\partial V| \, \mathrm{O}(C) \,. \tag{6.15}$$

The contour ensembles we are considering in this paper are all *periodic*, rather than just translation-invariant. That is why in (6.12) we have taken a supremum over sites and in (6.14) we have summed over a fundamental cell W of the configuration. Nevertheless, the relevant estimates will be done majorizing the contour weights via the Peierls condition (3.42). This majorizing ensemble is then *translation-invariant*, and hence this supremun over sites is superfluous in the key estimate that follows.

The limit f can be interpreted as a free energy of the ensemble of contours (6.1). The coefficient on the RHS of (6.15) can be interpreted as a surface tension representing a finite-volume correction.

6.3 The key estimate

In view of Corollary 6.4 and the freedom of rescaling discussed at the end of Section 6.1, we see that the following lemma is the key step needed to prove Theorem 6.1.

Lemma 6.5 *There exist $\lambda_0 > 0$ and $\beta_0 < \infty$ such that, for $\lambda \leq \lambda_0$ and $\beta \geq \beta_0$,*

$$\mathop{\sum\!\!\!\!\!\!\int}_{\operatorname{supp} \zeta \ni (\vec{0},0)} \lambda^{|B(\zeta)|} \exp[-J|\zeta|_\perp] \ \leq \ \mathrm{O}(e^{-\beta J/2}) + \mathrm{O}(\lambda) \,. \tag{6.16}$$

Proof. We use the fact that the integrand on the LHS of (6.16) depends on the sections Γ_i and the quantum bonds $\underline{B}_i$ *only* through their sizes, to write it as a sum of contributions of "entropy" and "energy" factors. We first decompose the LHS as

$$\mathop{\sum\!\!\!\!\!\!\int}_{\operatorname{supp} \zeta \ni (\vec{0},0)} \lambda^{|B(\zeta)|} \exp[-J|\zeta|_\perp] \ = \ S^0 + S^{>0} \,, \tag{6.17}$$

where S^0 is the contribution due to contours without transition (perfectly cylindrical contours), and $S^{>0}$ is the rest. The bound on S^0 is exactly as in the usual Peierls argument:

$$\begin{aligned} S^0 &\leq \sum_{l\geq 1} \mathrm{card}\{\Gamma \,:\, |\Gamma| = l, \mathrm{supp}\,\Gamma \ni \vec{0}\}\; e^{-\beta J l} \\ &= \mathrm{O}(e^{-\beta J})\;. \end{aligned} \tag{6.18}$$

Regarding $S^{>0}$, we have:

$$\begin{aligned} S^{>0} \;\leq\; & \sum_{n\geq 1} \sum_{(j_1,\ldots,j_n)} \lambda^{j_1+\cdots+j_n} \sum_{\substack{(l_1,\ldots,l_n) \\ |l_i - l_{i-1}| \leq j_i \;(l_0 \equiv l_n)}} N(j_1, l_1, \ldots, j_n, l_n) \\ & \times \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \; \mathrm{I}\,[\beta \geq \textstyle\sum_{i=1}^n \tau_i] \\ & \quad \exp\{-J\,[l_0(\beta - \textstyle\sum_{i=1}^n \tau_i) + l_1\tau_1 + \cdots + l_n\tau_n]\}\;, \end{aligned} \tag{6.19}$$

where

$$N(j_1, l_1, \ldots, j_n, l_n) \;:=\; \mathrm{card}\Big\{(\Gamma_0, \ldots, \Gamma_n) \,:\, \text{(a) and (b) below}\Big\} \;: \tag{6.20}$$

(a) $|\Gamma_i| = l_i$; $\Gamma_0 = \Gamma_n$.

(b) There exists a sequence $(\underline{B}_1, \ldots, \underline{B}_n)$ of quantum bonds with $|\underline{B}_i| = j_i$, such that there is a contour ζ formed by the sections Γ_i and the bonds $\underline{B}_i$, with $\mathrm{supp}\,\zeta \ni (\vec{0}, 0)$. Conditions necessary for this property to be satisfied are:

  (b.1) Γ_i is obtained from Γ_{i-1} by acting with some $\Phi^{\mathrm{q}}_{\underline{B}_i}$, with $|\underline{B}_i| = j_i$, on $|\Gamma_{i-1}\rangle$.

  (b.2) There exists one section Γ_i or one quantum bond $\underline{B}_i$ such that $\vec{0} \in \Gamma_i$ or $\vec{0} \in \underline{B}_i$.

We can distinguish a contribution due to "entropy" [the factor $N(j_1, \ldots, l_n)$], and another due to "energy" (the exponential and the powers of λ). To prove (6.16), we must show that "energy" overwhelms "entropy".

The two contributions on the RHS of (6.16) arise from two different types of quantum contours, namely the long contours and the short contours defined in Section 3.3. A *long* contour extends all the way from "time" zero to β, i.e., *none of the sections* Γ_i *are empty*. We shall denote the set of such contours by Q_l. A short contour has a "time"-height strictly smaller than β. It appears and disappears under the action of the quantum interactions $\Phi^{\mathrm{q}}_{\underline{B}_1}, \ldots, \Phi^{\mathrm{q}}_{\underline{B}_n}$. Hence, under the action of *one* of these n interactions, the section size of the contour reduces to zero, i.e., there is one (and only one) value of i, $[1 \leq i \leq n]$, for which

$$\Phi^{\mathrm{q}}_{\underline{B}_i} : \Gamma_{i-1} \longmapsto \Gamma_i, \quad \text{with } |\Gamma_{i-1}| \neq 0 \text{ and } |\Gamma_i| = 0.$$

The set of short contours will be denoted by Q_s.

Bound for long contours.

We start with the *entropy bound*, that is the bound on $N(j_1, l_1, \ldots, j_n, l_n)$. By condition (b.2) above,

$$N(l_0, \ldots, l_n) \;\le\; (l_{\max} + j_{\max})\, \widetilde{N}(j_1, l_1, \ldots, j_n, l_n)\;, \tag{6.21}$$

where $l_{\max} = \max_i l_i$, $j_{\max} = \max_i j_i$, and $\widetilde{N}(j_1, l_1, \ldots, j_n, l_n)$ is the number of "pinned" contours, that is, contours with the given section and quantum bond sizes for which $(\vec{0}, 0)$ is the first point (e.g. in lexicographic order) of its support $[\vec{0} \in \Gamma_i,\ or\, \vec{0} \in B_i]$. To evaluate $\widetilde{N}$ we imagine that we "construct" the quantum contour by starting from a section with minimal size l_{min}:

$$\widetilde{N}(j_1, l_1, \ldots, j_n, l_n) \;\le\; \sum_{\Gamma \in \mathrm{CC}(l_{\min}, j_1, \ldots, j_n)} \mathcal{N}_{\Gamma \to \Gamma}(j_{i_{\min}+1}, l_{i_{\min}+1}, \ldots, j_{i_{\min}-1} l_{i_{\min}-1}, j_{i_{\min}})\;. \tag{6.22}$$

Here, $i_{\min}$ satisfies $l_{i_{\min}} = \min_i l_i := l_{\min}$,

$$\mathrm{CC}(l, j_1, \ldots, j_n) \;:=\; \{\Gamma : |\Gamma| = l, \text{ and } \Gamma \text{ is a section of a quantum contour with } n \text{ transitions given by the actions of } \Phi^{\mathrm{q}}_{\underline{B}_1}, \ldots, \Phi^{\mathrm{q}}_{\underline{B}_n}\}$$

and

$\mathcal{N}_{\Gamma_0 \to \Gamma_n}(j_1, l_1, \ldots, j_{n-1}, l_{n-1}, j_n) \;:=\;$ number of ways of choosing sections $\Gamma_1, \ldots, \Gamma_{n-1}$ of areas $l_1, \ldots, l_{n-1}$ and quantum bonds $\underline{B}_1, \ldots, \underline{B}_n$ of areas $j_1, \ldots, j_n$ such that two consecutive sections Γ_i and Γ_{i-1} of the sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_n$ are obtained from each other by acting with $\Phi^{\mathrm{q}}_{\underline{B}_i}$.

By induction one can see that

$$\mathcal{N}_{\Gamma_0 \to \Gamma_n}(j_1, l_1, \ldots, j_{n-1}, l_{n-1}, j_n) \;\le\; (a^{2d})^n \prod_{i=1}^{n} (l_i + j_i)\, j_i\, {c_d}^{j_i} \tag{6.23}$$

where c_d is a dimension-dependent constant (the one for the Königsberg bridge lemma). The proof of this fact is presented at the end of this subsection. Thus

$$\widetilde{N}(j_1, l_1, \ldots, j_n, l_n) \;\le\; \mathrm{card}\Big(\mathrm{CC}(l_{\min}, j_1, \ldots, j_n)\Big)(a^{2d})^n \prod_{i=1}^{n} (l_i + j_i)\, j_i\, c_d^{j_i}\;. \tag{6.24}$$

The bound on $\mathrm{card}\Big(\mathrm{CC}(l_{\min}, j_1, \ldots, j_n)\Big)$ must take care of the fact that the section of area $l_{\min}$ need not be connected [see discussion following (3.36)], but its components cannot be too dispersed . More precisely, property (3.36) implies that for each $l_{\min}$ there is a *connected* set formed by a number of plaquettes ranging from $l_{\min}$ to $l_{\min} + j_1 + \cdots + j_n$. Therefore (by the Königsberg bridge lemma), there exists a constant $c_d \geq 1$, depending only on the spatial dimension, such that

$$\begin{aligned} \operatorname{card}\Big(\mathrm{CC}(l_{\min}, j_1, \ldots, j_n)\Big) &\leq c_d^{l_{\min}} + \cdots + c_d^{l_{\min}+j_1+\cdots+j_n} \\ &\leq (j_1 + \cdots + j_n + 1)\, c_d^{l_{\min}+j_1+\cdots+j_n} \;. \end{aligned} \tag{6.25}$$

Substituting (6.21), (6.24) and (6.25) in (6.19) we obtain the bound

$$\begin{aligned} S_{Q_l}^{>0} \;\leq\; & \sum_{n\geq 1} (a^{2d})^n \sum_{(j_1,\ldots,j_n)\,:\,j_i\geq 1} (j_1 + \cdots + j_n + 1)\, \lambda^{j_1+\cdots+j_n} \\ & \sum_{\substack{(l_1,\ldots,l_n)\,:\,l_i\geq 1 \\ |l_i - l_{i-1}|\leq j_i\ (l_0\equiv l_n)}} (l_{\max} + j_{\max})\, c_d^{l_{\min}} \Big[\prod_{i=1}^n (l_i + j_i)\, j_i\, (c_d)^{2j_i}\Big]\, R(l_1, \ldots, l_n) \;, \end{aligned} \tag{6.26}$$

with

$$\begin{aligned} R(l_1, \ldots, l_n) \;:=\; & \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \;\mathrm{I}\,[\beta \geq \textstyle\sum_{i=1}^n \tau_i] \\ & \times \exp -J\,[l_0(\beta - \textstyle\sum_{i=1}^n \tau_i) + l_1\tau_1 + \cdots + l_n\tau_n] \;. \end{aligned} \tag{6.27}$$

In order to obtain a bound on this last integral, the *energy bound*, we proceed as follows: We use the bound

$$e^{-J\alpha\, l_i} \;\leq\; e^{-J\alpha\, l_{\min}/2} e^{-J\alpha\, l_i/2} \;, \tag{6.28}$$

for $\alpha = \tau_1, \ldots, \tau_n, (\beta - \sum_{i=1}^n \tau_i)$, to extract an overall factor $e^{-\beta J l_{\min}/2}$ outside the integral on the RHS. of (6.27). The remaining integral is the same as the original one, but with J replaced by $J/2$. By neglecting the indicator function and the term proportional to l_0 in the exponent, and extending the limits of integration to infinity, one obtains

$$R(l_1, \ldots, l_n) \;\leq\; \left(\frac{2}{Jl_1}\right) \cdots \left(\frac{2}{Jl_n}\right)\, e^{-\beta J l_{\min}/2} \;, \tag{6.29}$$

which, by (6.26), implies that

$$\begin{aligned} S_{Q_l}^{>0} \;\leq\; & \sum_{n\geq 1} \left(\frac{2a^{2d}}{J}\right)^n \sum_{(j_1,\ldots,j_n)\,:\,j_i\geq 1} (j_1 + \cdots + j_n + 1)\, \lambda^{j_1+\cdots+j_n} \\ & \sum_{\substack{(l_1,\ldots,l_n)\ l_i\geq 1 \\ |l_i - l_{i-1}|\leq j_i\ (l_0\equiv l_n)}} (l_{\max} + j_{\max})\, \left(c_d\, e^{-\beta J/2}\right)^{l_{\min}} \Big[\prod_{i=1}^n (1 + j_i)\, j_i\, (c_d)^{2j_i}\Big] \;. \end{aligned} \tag{6.30}$$

The sum over the l_i, $1 \le i \le n$, can be written purely in terms of $l_{\rm min}$ (≥ 1) and $j_1, \ldots, j_n$. Indeed, for each l_i, there are only $2j_i + 1$ possible values for l_{i+1}. Hence, once $l_{\rm min}$ is given, the sum over the remaining l_i's yields an extra factor $\prod_i (2j_i + 1)$. One now notes that the maximum size, $l_{\rm max}$, of a section of the contour satisfies the bound

$$l_{\rm max} \;\le\; l_{\rm min} + j_1 + \cdots + j_n . \tag{6.31}$$

This is because the section of "area" $l_{\rm max}$ is obtained from the section of "area" $l_{\rm min}$ by the action of *at most* n quantum interactions, $\Phi^{\rm q}_{\underline{B}_1}, \ldots, \Phi^{\rm q}_{\underline{B}_n}$, the latter corresponding to quantum bonds $\underline{B}_1, \ldots, \underline{B}_n$ of sizes $j_1, \cdots, j_n$. Also,

$$j_{\rm max} \;\le\; j_1 + \cdots + j_n \;. \tag{6.32}$$

Thus

$$\begin{aligned} S^{>0}_{Q_l} \;&\le\; \sum_{n\ge 1} \left(\frac{2a^{2d}}{J}\right)^n \sum_{(j_1,\ldots,j_n)\,:\,j_i\ge 1} (j_1 + \cdots + j_n + 1)\Big[\prod_{i=1}^n (1+2j_i)^2\, j_i\, (c_d)^{2j_i}\Big]\lambda^{j_1+\cdots+j_n} \\ &\quad \times \sum_{l_{\rm min}\ge 1} (l_{\rm min} + 2j_1 + \cdots + 2j_n)\, \Big(c_d\, e^{-\beta J/2}\Big)^{l_{\rm min}} \\ &= \; {\rm O}(e^{-\beta J/2}) \;. \end{aligned} \tag{6.33}$$

Bound for the short contours.

There are two types of short contours. There is a "collapsed" type, corresponding to actions of all the $\Phi^{\rm q}_{\underline{B}}$ that do not alter the configuration. By the exponential bound (2.62), the contribution of these fluctuations to the expansion (6.16) is simply given by

$$\begin{aligned} c \sum_{B\ni\vec{0}} \lambda^{|B|} \;&\le\; c \sum_{j\ge 1} (c_d\lambda)^j \\ &\le\; {\rm O}(\lambda) \end{aligned} \tag{6.34}$$

The remaining short contours have a finite extension in the "time"-direction. We shall assume, without loss of generality, that the quantum interaction $\Phi^{\rm q}_{\underline{B}_1}$ reduces the section size of the contour to zero, i.e., we assume that $\Gamma_1 = \emptyset$ and hence $l_1 = 0$.

The entropy bound can be obtained in a way similar to that for long contours, but with the following modifications:

- $l_{\rm min}$ is the minimum of the sizes of the *non-empty* sections, $l_2, \ldots, l_n (= l_0)$.
- the "area" l_1 on the RHS of (6.24) is zero.
- The entire contour (which includes the sections of areas $l_{\rm min}$ and $l_{\rm max}$), corresponds to the actions of $\Phi^{\rm q}_{\underline{B}_1}, \ldots, \Phi^{\rm q}_{\underline{B}_n}$.

Hence

$$l_{max}, j_{max} \leq j_1 + \ldots + j_n, \tag{6.35}$$

which implies that

$$l_{max} + j_{max} \leq 2(j_1 + \ldots + j_n). \tag{6.36}$$

Moreover, in place of the bound (6.25), we have that

$$\mathrm{card}\Big(\mathrm{CC}(l_{min}, j_1, \ldots, j_n)\Big) \leq (j_1 + \ldots + j_n) c^{(j_1+\ldots+j_n)} \tag{6.37}$$

With the above modifications, we obtain that

$$\begin{aligned} & N(j_1, 0, j_2, l_2, \ldots, j_n, l_n) \\ & \leq \; 2(j_1 + \ldots + j_n)^2 \, c_d^{j_1+\cdots+j_n} \, (a^{2d})^n \prod_{i=1}^{n} (l_i + j_i) \, j_i \, c_d^{j_i} \; . \end{aligned} \tag{6.38}$$

In calculating the energy bound, we make use of the inequality $e^{-J\alpha l_i} \leq e^{-J/2} \, e^{-J\alpha l_i/2}$ for $i \neq 1$. Proceeding as in the case of long contours, we obtain the bound

$$\begin{aligned} R(0, l_2, \ldots, l_n) \; & \leq \; \left(\frac{2}{Jl_2}\right) \cdots \left(\frac{2}{Jl_n}\right) \int_0^\beta d\tau_1 \, e^{-(\beta-\tau_1)J/2} \\ & \leq \; \left(\frac{2}{Jl_2}\right) \cdots \left(\frac{2}{Jl_n}\right) \left(\frac{2}{J}\right) \; , \end{aligned} \tag{6.39}$$

where the last inequality results from extending the limit of integration to infinity. Hence

$$\begin{aligned} S_{Q_s}^{>0} \; & \leq \; \mathrm{O}(\lambda) + \sum_{n\geq 2} n \left(\frac{2a^{2d}}{J}\right)^n \sum_{(j_1,\ldots,j_n)\,:\, j_i \geq 1} 2(j_1 + \cdots + j_n)^2 \\ & \qquad \times \Big[\prod_{i=1}^{n} (1 + 2j_i)^2 \, j_i \, (c_d)^{2j_i}\Big] \lambda^{j_1+\cdots+j_n} \\ & \leq \; \mathrm{O}(\lambda) + \mathrm{O}(\lambda^2) \; . \; \blacksquare \end{aligned} \tag{6.40}$$

Proof of the claim (6.23). The proof follows by induction in n. The induction step is carried out as follows:

$$\begin{aligned} & \mathcal{N}_{\Gamma_0 \to \Gamma_n}(j_1, l_1, \ldots, j_{n-1}, l_{n-1}, j_n) \; = \\ & \sum_{\Gamma_{n-1} \in \mathrm{CC}_{l_{n-1}, j_n}(\Gamma_n)} \mathcal{N}_{\Gamma_0 \to \Gamma_{n-1}}(j_1, l_1, \ldots, j_{n-2}, l_{n-2}, j_{n-1}) \; , \end{aligned} \tag{6.41}$$

where

$\mathrm{CC}_{l_{n-1}, j_n}(\Gamma_n) \; := \; \{\Gamma \, : \, |\Gamma| = l_{n-1}$, and Γ differs from Γ_n through the action of some $\Phi^{\mathrm{q}}_{\underline{B}_n}$ with $|\underline{B}_n| = j_n\}$.

By the inductive hypothesis

$$\mathcal{N}_{\Gamma_0\to\Gamma_n}(j_1 l_1,\ldots,j_{n-1},l_{n-1}) \;\le\; \mathrm{card}\Big(\mathrm{CC}_{l_{n-1},j_n}(\Gamma_n)\Big)\,(a^{2d})^{n-1}\prod_{i=1}^{n-1}(l_i+j_i)\,j_i\,c_d^{j_i}\;. \tag{6.42}$$

We have to consider two cases:

(i) $l_{n-1}\ge l_n$. In this case, the bond B_n must intersect $\mathrm{supp}\,\Gamma_{n-1}$. The number of such possibilities is bounded by the product of the number of sites in Γ_{n-1} ($= l_{n-1}a^d$), the number of sites in B_{n-1} ($= j_{i-1}a^d$) and the number of bonds $\underline{B}_n$ with $|B_n| = j_n$. The latter is less than or equal to $c_d^{j_n}$, for some constant c_d depending on the dimension d.

(ii) $l_{n-1} < l_n$. In this case, B_n must intersect $\mathrm{supp}\,\Gamma_n$. Hence we can use the preceeding argument, with Γ_{n-1} replaced by Γ_n.

Therefore, in both cases,

$$\begin{aligned}\mathrm{card}\Big(\mathrm{CC}_{l_{n-1},j_n}(\Gamma_n)\Big) &\le (a^d)^2\,\max(l_{n-1},l_n)\,j_n\,c_d^{j_n}\\ &\le a^{2d}\,(l_n+j_n)j_n\,c_d^{j_n}\;,\end{aligned}\tag{6.43}$$

where we have used that $l_{n-1}\le l_n+j_n$ because of (3.35). ∎

6.4 Convergence of the expansion for the free energy

Estimate (6.16) proves (6.13) and hence Theorem 6.1 for *non-intersecting* (as opposed to non-linked) contours. Expression (6.14) shows that the free energy is analytic in $e^{-\beta\tilde J}$, $\tilde\lambda$ on a region of $\mathbb{C}^2$ of the form

$$\max(|e^{-\beta\tilde J}|,|\tilde\lambda|) \;<\; \varepsilon_0. \tag{6.44}$$

We use tildes because J and λ must be rescaled as in (6.8), in order to absorb the extra factor $e^{|\zeta|}$ in (6.13). A small additional argument is needed to establish a comparable bound when contours are also formed by components whose supports are *linked*. The argument goes as follows: If a contour ζ winds around $\tilde\zeta$, so as to yield linking, then there must exist points $(\vec x,t)\in\zeta$ and $(\vec y,t)\in\tilde\zeta$ such that $|B(\zeta)|\ge 2\,\mathrm{dist}(\vec x,\vec y)$. Moreover, $|B_1|+\cdots+|B_n|$ must exceed the number of plaquettes needed to turn around a single plaquette, which is a large dimension-dependent number. A safe lower bound is for instance $j_1+\cdots+j_n\ge 16$ (for the more demanding case $d=2$). Hence

$$\begin{aligned}\frac{1}{|\tilde\zeta|}\sum\!\!\!\!\!\!\not\;\;\;_{\zeta\,\imath\,\tilde\zeta}|w(\zeta)|\,e^{|\zeta|} &\le \sum\!\!\!\!\!\!\not\;\;\;_{\mathrm{supp}\,\zeta\ni(\vec 0,0)}|w(\zeta)|\,e^{|\zeta|}\\ &\quad+\sum_{\vec x\in\mathbb{Z}^d}\;\sum\!\!\!\!\!\!\not\;\;\;_{\substack{\mathrm{supp}\,\zeta\ni(\vec x,0)\\ |B(\zeta)|\ge\max\{2\,\mathrm{dist}(\vec x,\vec 0)\,,\,16\}}}|w(\zeta)|\,e^{|\zeta|}\\ &:= C+C_1\;.\end{aligned}\tag{6.45}$$

The arguments used in the proof of the key estimate also prove that

$$\sum\!\!\!\!\!\!\!\int_{\substack{\mathrm{supp}\,\zeta\ni(\vec{x},0)\\ |B(\zeta)|\geq\max\{2\,\mathrm{dist}(\vec{x},\vec{0})\,,\,16\}}} |w(\zeta)|\,e^{|\zeta|} \;\leq\; \mathrm{O}(\lambda^{\max\{\mathrm{dist}(\vec{x},\vec{0})\,,\,16\}})\;, \tag{6.46}$$

and hence

$$C_1 \;\leq\; \sum_{\vec{x}} \mathrm{O}(\lambda^{\max\{\mathrm{dist}(\vec{x},\vec{0})\,,\,16\}}) \tag{6.47}$$

which is only a negligible correction to C.

In some cases we shall restrict our attention to a smaller family, $\mathcal{F}$, of contours and consider the partition function

$$\Xi^{\mathcal{F}}(V) \;:=\; \sum\!\!\!\!\!\!\!\int_{\substack{\{\zeta_k\}\text{ pairwise non-linked}\\ \zeta_k\subset V\\ \zeta_k\in\mathcal{F}}} \prod_k w(\zeta_k)\;. \tag{6.48}$$

Repeating the arguments of Sections 6.2 and 6.3, we obtain an analogue of Theorem 6.1, but where the convergence parameter is

$$C^{\mathcal{F}} \;=\; \mathrm{O}(L_l^{\mathcal{F}}) + \mathrm{O}(L_s^{\mathcal{F}}) \tag{6.49}$$

where $L_l^{\mathcal{F}}$ [resp. $L_s^{\mathcal{F}}$] is the lowest-order term in the sum (6.33) [resp. (6.40)] for contours of the family $\mathcal{F}$. We already invoked this fact in (6.46). Another case, used in Section 7 below, is when $\mathcal{F}$ is the set of ("large") contours with $l_{\min}+j_1+\cdots+j_n > m$. In this case we can replace the constant ε_0 in (6.4) and (6.5) by $(\varepsilon_0)^m$.

6.5 Stability of phases in the symmetric regime

Here we prove Corollary 6.2. Our proof is based on the original Peierls argument, but uses the cluster expansion technique to get around the fact that weights may be negative and imaginary. We choose an observable $\mathbf{A} \in \mathcal{A}_D$. ¿From the discussion of Section 5.2 we have that

$$\frac{\Xi_p^{\mathbf{A}}(\Lambda)}{\Xi_p(\Lambda)} \;=\; \sum\!\!\!\!\!\!\!\int^{*}_{\zeta_{\mathbf{A}}\subset\widehat{V}} w(\zeta_{\mathbf{A}})\,U_V(\zeta_{\mathbf{A}}), \tag{6.50}$$

where

$$U_V(\zeta_{\mathbf{A}}) \;:=\; \frac{\sum\!\!\!\!\!\!\int_{\substack{\{\zeta_k\}\subset\widehat{V}\\ \text{non-linked}}} \Big[\prod_k w(\zeta_k)\Big]\;\mathrm{I}\Big[\zeta_k \text{ non-linked to } \zeta_{\mathbf{A}}\Big]}{\sum\!\!\!\!\!\!\int_{\{\zeta_k\}\subset\widehat{V}} \Big[\prod_k w(\zeta_k)\Big]}, \tag{6.51}$$

for $V = \Lambda\times[0,\beta]$. The quotient on the RHS of (6.51) is amenable to the cluster-expansion technology: The logarithm of the numerator produces clusters for which no contour is linked with $\zeta_{\mathbf{A}}$, while the logarithm of the denominator produces all clusters, with no restrictions. The quotient corresponds to the clusters with at least one contour linked with $\zeta_{\mathbf{A}}$. Hence,

$$\log U_V(\zeta_{\mathbf{A}}) \;=\; \sum_{N\geq 1} \; \sum_{\substack{\{\zeta_1,\ldots,\zeta_N\} \wr \zeta_{\mathbf{A}} \\ \zeta_j \subset \widehat{V}}} \!\int \;\; w^{\mathrm{T}}(\zeta_1,\ldots,\zeta_N)\;, \tag{6.52}$$

and we have the bound

$$|U_{\zeta_{\mathbf{A}}}(V)| \;\leq\; \exp\Big[|\zeta_{\mathbf{A}}|\mathrm{O}(\varepsilon_0)\Big] \tag{6.53}$$

uniformly in Λ, and, due to the Peierls bound (5.13),

$$|w(\zeta_{\mathbf{A}})U_V(\zeta_{\mathbf{A}})| \;\leq\; \frac{\|\mathbf{A}\|}{\lambda^{|D|}}\,\widetilde{\lambda}^{|B(\zeta_{\mathbf{A}})|}\,\exp[-\widetilde{J}|\zeta_{\mathbf{A}}|_{\perp}], \tag{6.54}$$

again uniformly in Λ. We use the notation $\widetilde{\lambda} = \lambda e^{\mathrm{O}(\varepsilon_0)}$ and $\widetilde{J} = J - \mathrm{O}(\varepsilon_0)$.

As a consequence, expression (6.50) can be treated by cluster-expansion methods, as the key estimate (6.16) is valid. The "dominant" contribution in (6.50) arises from terms for which $\zeta_{\mathbf{A}}$ corresponds to $\langle s_p|\mathbf{A}|s_p\rangle$ and has a support

$$\operatorname{supp}\zeta_{\mathbf{A}} = D.$$

We obtain that

$$\frac{\Xi_p^{\mathbf{A}}(\Lambda)}{\Xi_p(\Lambda)} \;=\; \langle s_p|\mathbf{A}|s_p\rangle \;+ S_1(\beta,\lambda) \;+\; S_2(\beta,\lambda)\;, \tag{6.55}$$

where $S_1(\beta,\lambda)$ is the contribution of short contours which intersect the set D at the "time" β, whereas $S_2(\beta,\lambda)$ is the contribution of long contours intersecting D at "time" β. The key estimate given in Lemma 6.5, and the fact that all the components of $\zeta_{\mathbf{A}}$ must intersect some site in D, imply that $S_1(\beta,\lambda)$ and $S_2(\beta,\lambda)$ are bounded by

$$|D|\frac{\|\mathbf{A}\|}{\lambda^{|D|}}\,\mathrm{O}(\varepsilon_0)\;.$$

The expression (6.55) and the above bound together prove Corollary 6.2. They prove that the quotients $\Xi_p^{\mathbf{A}}(\Lambda)/\Xi_p(\Lambda)$ are analytic functions of $(e^{-\beta\tilde{J}},\widetilde{\lambda})$ in a region

$$\max(|e^{-\beta\tilde{J}}|,|\widetilde{\lambda}|) \;<\; \widetilde{\varepsilon} \tag{6.56}$$

where $\widetilde{\varepsilon}$ is a constant independent of V. Each term of these series converges to the corresponding term of the series (6.50) *without* the condition $\zeta_{\mathbf{A}} \subset \widehat{V}$. In addition, the finite-volume series and this infinite-volume limit are majorized by the same absolutely convergent series. By dominated convergence, the limit $\Lambda \nearrow \mathbb{Z}^d$ of the series is the series of the limits, throughout the region (6.56), which, therefore, is also the region of analyticity of the limit. We conclude that the quantum expectations satisfy

$$\begin{aligned}\langle \mathbf{A}\rangle_{\beta\,\lambda} &= \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\Xi_p^{\mathbf{A}}(\Lambda)}{\Xi_p(\Lambda)} \\ &= \langle s_p|\mathbf{A}|s_p\rangle \;+\; |D|\frac{\|\mathbf{A}\|}{\lambda^{|D|}}\,\mathrm{O}(\varepsilon_0)\,. \end{aligned} \tag{6.57}$$

This proves equation (6.7).

6.6 Differentiability of the expansion

Let us reintroduce the parameters $\underline{\mu}$ in the interaction and hence in the weights w, and discuss the consequences of the smoothness hypothesis (H1.2) (Section 2.2). This hypothesis implies that the derivatives of the weights satisfy a Peierls condition analogous to the one obeyed by the weights themselves, except for a factor proportional to $|\zeta|$. This factor can be absorbed in a rescaling of J and λ; so we can assume that

$$\left|\frac{\partial w_{\underline{\mu}}(\zeta)}{\partial\mu_i}\right| \;\le\; \widetilde{\lambda}^{|B(\zeta)|}\, e^{-\beta\tilde{J}}. \tag{6.58}$$

Therefore, from the preceding results, we conclude that the series formed by the derivatives of the weights converge uniformly in a small interval around each $\underline{\mu}\in\mathcal{O}$ (and absolutely in $e^{-\beta\tilde{J}}$ and λ). This implies that the series for the partition function can be differentiated term by term, and thus it and its logarithm are differentiable functions of $\underline{\mu}$. Using the cluster expansion we obtain

$$\left|\frac{\partial}{\partial\mu_i}\log\Xi_p(V)\right| \;\le\; |V|\,\mathrm{O}(\varepsilon_0) \tag{6.59}$$

and, writing $\Xi_p = \exp(\log\Xi_p)$,

$$\left|\frac{\partial}{\partial\mu_i}\Xi_p(V)\right| \;\le\; |V|\,|\Xi_p(V)|\,\mathrm{O}(\varepsilon_0)\,. \tag{6.60}$$

These observations will be useful in Section 7.3 below.

7 Pirogov-Sinai theory for quantum perturbations

7.1 Overview. The initial trick

We now turn to the proof of the main Theorem 2.3 in the general (non-symmetric) situation. This involves dealing with the contour expansions (5.2) [which we repeat in (7.1) below for the reader's convenience] with a non-local compatibility condition among contours (the matching of the labels of nested interiors and exteriors). Following the standard approach, going back to the original work of Pirogov and Sinai [29, 30], the theory is constructed in two parts: First (Section 7.2 below), a criterion is established to determine the stable phases for a *fixed* interaction; second (Section 7.3), the stability of the phase diagram as a whole is determined. The parameters $\underline{\mu}$ play a role only in the second part (and hence will not be displayed in the first part).

In our treatment, we closely follow the excellent presentation of Borgs and Imbrie [4]. Our proofs are basically a transcription of those in [4], except for some small adaptations and simplifications. The starting point of our proof is the formal expression (5.2) for the partition functions $\Xi_p(V)$ and, more generally, the expressions (5.11) for $\Xi_p^{\mathbf{A}}(V)$. For the convenience of the reader we repeat (5.2) here:

$$\begin{aligned} \Xi_p(V) \;&=\; \exp\Big[-\int_V e_x(s_p)\Big] \\ &\times \sum\!\!\!\!\!\!\!\!\not\quad_{\substack{\{\zeta_k\}\subset\widehat{V}\\ \text{compatible}}} \Big[\prod_k w(\zeta_k)\Big]\Big[\prod_{u=1}^{P}\exp\Big\{-\int_{L_u}\left[e_x(s_u)-e_x(s_p)\right]\Big\}\Big], \end{aligned} \tag{7.1}$$

where the exterior contours of each compatible family are p-contours. Both quantities, the energies $e_x(s_u)$ and the weights $w(\zeta)$, are complex-valued, and the latter satisfy the quantum Peierls bound (3.42).

We follow the procedure, introduced by Minlos and Sinai [25, 26], to eliminate the inconvenient compatibility condition in (7.1). We first resum (7.1) (formally!) over the contours in the interior of the exterior contours,

$$\begin{aligned} \Xi_p(V) \;&=\; \exp\Big[-\int_V e_x(s_p)\Big] \\ &\times \sum\!\!\!\!\!\!\!\!\not\quad_{\substack{\{\zeta_k^p\}\subset\widehat{V}\\ \text{exterior}}} \Big[\prod_k w(\zeta_k^p)\prod_{u=1}^{P}\Xi_u\Big(\mathrm{Int}_u(\zeta_k^p)\Big)\exp\Big\{\int_{\mathrm{Int}_u(\zeta_k^p)} e_x(s_p)\Big\}\Big], \end{aligned} \tag{7.2}$$

and then multiply and divide the RHS by $\Xi_p\Big(\mathrm{Int}_u(\zeta_k^p)\Big)$ to obtain

$$\begin{aligned} &\Xi_p(V)\,\exp\Big[\int_V e_x(s_p)\Big] \;= \\ &\sum\!\!\!\!\!\!\!\!\not\quad_{\substack{\{\zeta_k^p\}\subset\widehat{V}\\ \text{exterior}}} \Big[\prod_k W(\zeta_k^p)\prod_{u=1}^{P}\Xi_p\Big(\mathrm{Int}_u(\zeta_k^p)\Big)\exp\Big\{\int_{\mathrm{Int}_u(\zeta_k^p)} e_x(s_p)\Big\}\Big], \end{aligned} \tag{7.3}$$

with the new weights

$$W(\zeta^p) \;:=\; w(\zeta^p) \prod_{u=1}^{P} \frac{\Xi_u\Big(\mathrm{Int}_u(\zeta^p)\Big)}{\Xi_p\Big(\mathrm{Int}_u(\zeta^p)\Big)}\,. \tag{7.4}$$

One can now repeat the same procedure for each factor,

$$\Xi_p\Big(\mathrm{Int}_u(\zeta_k^p)\Big)\exp\Big\{\int_{\mathrm{Int}_u(\zeta_k^p)} e_x(s_p)\Big\}, \tag{7.5}$$

in (7.3). This iteration finally yields the expression

$$\begin{aligned}
\Xi_p(V) \;&=\; \exp\Big\{-\int_V e_x(s_p)\Big\} \sum\!\!\!\!\!\!\!\!\not\;\;\;\;_{\substack{\{\zeta_k^p\}\subset\widehat{V}\\ \mathrm{non-linked}}} \prod_k W(\zeta_k^p)\\
:=\;& \exp\Big\{-\int_V ; e_x(s_p)\Big\}\, \widehat{\Xi}_p(V)
\end{aligned} \tag{7.6}$$

Identity (7.6) yields an alternative formal expression for the partition function Ξ_p defined in (7.1). If either one of these two expansions converges uniformly, then so does the other. In (7.6), however, the contours are only required to be non-linked, rather than compatible in the sense used in (7.1). Hence, at least from a combinatorial point of view, (7.6) is simpler to deal with. Moreover, the factor $\widehat{\Xi}_p(V)$ is of the form (6.1) and hence is amenable to the cluster-expansion techniques of Section 3.

However, one is confronted with the following problem: Whereas the original weights $w(\zeta^p)$ satisfied the quantum Peierls condition (3.42), there is no a priori bound on the new weights $W(\zeta^p)$, defined in (7.4). Hence, in order to prove the convergence of the expansions in (7.1) and (7.6), we have to devise a method to control the new weights. This is done in the following section. The same considerations apply to $\Xi_p^{\mathbf{A}}$.

7.2 Criterion for the stability of phases

A sufficient condition for the stability of the s_p-phase is the absolute convergence of the expansions for $\log\Xi_p(V)$, (with $\Xi_p(V)$ as in (7.6)) and of the analogous expansions for $\log\Xi_p^{\mathbf{A}}$. ¿From the discussions in Sections 3 and 6 and the similarity of the expansion (7.6) of $\widehat{\Xi}_p(V)$ to that of $\Xi_p(V)$ in (6.1), we conclude that it suffices to check that the new weights $W(\zeta^p)$ satisfy a quantum Peierls condition (3.42), provided J is large enough and λ is small enough. Definition (7.4) of these weights implies in turn that a sufficient condition for the Peierls condition to be valid is that there exist some constant z such that

$$\left|\frac{\Xi_u\Big(\mathrm{Int}_u(\zeta^p)\Big)}{\Xi_p\Big(\mathrm{Int}_u(\zeta^p)\Big)}\right| \;\le\; \exp[z|\partial\,\mathrm{Int}_u(\zeta^p)|], \tag{7.7}$$

for all u and all contours ζ^p. Hence, by choosing a sufficiently large J and a small λ, we may attempt to ensure that the new weights $W(\zeta^p)$ are exponentially damped. The constant z will be fixed once and for all. Inspired by [4] we choose $z = 4\alpha$, where

$$\alpha \;=\; 1 + a^d \max_{1\le u\le P} |e(s_u)| \;. \tag{7.8}$$

(If the configurations s_u all have period 1, i.e., are constant, $\alpha = 1$ suffices.) In the expansion of $\Xi_p^{\mathbf{A}}$ we also have to consider the special surface $\zeta_{\mathbf{A}}^p$ which is a quantum contour *associated with* $\mathbf{A}$ (defined in Section 3.6). To cover all cases, we prove a bound of the form

$$\left|\frac{\Xi_u(V)}{\Xi_p(V)}\right| \;\le\; \exp[4\alpha|\partial V|] \tag{7.9}$$

if s_p is stable, for all regions $V \subset \mathbb{Z}^d \times [0,\beta]$. We interpret (7.9) as the condition for the stability of a space-time region V. In particular, the bound (7.7) is used to define stable contours. Following [39], we require

Definition 7.1

(i) A region $V \subset \mathbb{Z}^d \times [0,\beta]$ is p-stable if $\Xi_p(V) \neq 0$ and (7.9) is satisfied for all u.

(ii) A p-contour ζ^p is stable if each $\mathrm{Int}_u(\zeta^p)$ is p-stable, for $1 \le u \le P$.

It is evident that the weights $W(\zeta)$ of *stable* contours satisfy the quantum Peierls condition. Hence, if we restrict the sums in (7.6) to stable p-contours then we can apply the cluster expansion technology of Section 3. This observation motivates us to define *truncated contour partition functions*, as in [39]:

$$\begin{aligned}\Xi'_p(V) \;&:=\; \exp\Bigl\{-\int_V e_x(s_p)\Bigr\} \sum_{\substack{\{\zeta_k^p\}\subset\widehat{V} \\ \text{stable} \\ \text{non-linked}}} \prod_k W(\zeta_k^p) \\ &:=\; \exp\Bigl\{-\int_V e_x(s_p)\Bigr\}\, \widehat{\Xi}'_p(V)\;,\end{aligned} \tag{7.10}$$

The cluster expansion for the truncated partition function $\Xi'_p(V)$ converges absolutely. In particular, by Theorem 6.1 we have that, for β large and λ small, the *truncated contour free energies*

$$f'_p \;:=\; -\lim_{V\nearrow\mathbb{Z}^d\times[0,\beta]} \frac{1}{|V|} \log \Xi'_p(V) \tag{7.11}$$

exist, and are of the form

$$f'_p \;=\; e(s_p) + \widehat{f}'_p, \tag{7.12}$$

where $e(s_p)$ is the specific energy (2.56), and the remainder $\widehat{f}'_p$ is given by the cluster expansion (6.14), with w replaced by the new weights W. Note that

$$\widehat{f}'_p \;=\; \mathrm{O}(\varepsilon_0)\;, \tag{7.13}$$

by (6.4), where ε_0 is the constant appearing in Theorem 6.1, and thus

$$\begin{aligned}\Big||V|f'_p + \log \Xi'_p(V)\Big| \;&\leq\; \Big|-\int_V e_x(s_p) + e(s_p)|V|\Big| + \mathrm{O}(\varepsilon_0)|\partial V| \\ &\leq\; \alpha|\partial V|\;.\end{aligned} \tag{7.14}$$

This bound is precisely the reason for our choice of α. We also observe that

$$\begin{aligned}\Big|\int_V e_x(s_p) - f'_p|V|\Big| \;&\leq\; \Big|-\int_V e_x(s_p) + e(s_p)|V|\Big| + |\widehat{f}'_p|\,|V| \\ &\leq\; \alpha|V|\;.\end{aligned} \tag{7.15}$$

We see that if, for a given boundary condition s_p, *all* contours are stable then $f_p = f'_p$. More generally, if *all* regions V are p-stable we have that $(\Xi_p^{\mathbf{A}})' = \Xi_p^{\mathbf{A}}$ and the primed quantum expectations equal the unprimed ones. (The prime indicates that the summation is over stable contours only.) The key observation of Pirogov-Sinai theory, in the formulation due to Zahradník [39], is that all regions V are p-stable if and only if the truncated free energy corresponding to the boundary condition s_p is *minimal*. Let

$$a_p \;:=\; \mathrm{Re}\, f'_p - \min_{s_u\in\mathcal{K}} \mathrm{Re}\, f'_u\;. \tag{7.16}$$

Then the stability criterion can be stated as follows:

Theorem 7.2 *If $a_p = 0$ the s_p phase is stable. Moreover, there is a region of $\mathbb{C}^2$ of the form $\max(|e^{-\beta\tilde{J}}|, |\lambda|) < \varepsilon_0$ where the free-energy density and all the quantum expectations are analytic functions of $e^{-\beta}$ and λ.*

The key lemma needed in the proof of this theorem is the following:

Lemma 7.3 *The following statements are equivalent:*

(i) $a_p = 0$.

(ii) All regions V are p-stable.

Proof. We first prove that (ii)$\Longrightarrow$(i), assuming that (i)$\Longrightarrow$(ii) holds. For this purpose, we consider a boundary condition s_v for which $a_v = 0$. For each V, we have that $\Xi_p(V) = \Xi'_p(V)$, by assumption, and $\Xi_v(V) = \Xi'_v(V)$ holds because (i)$\Longrightarrow$(ii). Therefore

$$\exp\Big[4\,\alpha\mathrm{O}(|\partial V|)\Big] \;\geq\; \left|\frac{\Xi_v(V)}{\Xi_p(V)}\right| \;=\; \exp\Big[a_p\,|V| + \gamma\,|\partial V|\Big]\;, \tag{7.17}$$

where γ is a constant of order ε_0. The leftmost inequality expresses p-stability of the region V, while the equality on the right is a consequence of (7.14). If $a_p > 0$ the equation in (7.17) leads to a contradiction for regions V with diverging volume-to-surface-area ratio. Thus we conclude that $a_p = 0$.

Proof of (i)$\Longrightarrow$(ii). In order to understand the steps and definitions that follow, it is useful to inspect the ratio of partition functions corresponding to different boundary conditions. From (7.14) we have that, for any $s_v, s_q \in \mathcal{K}$,

$$\left|\frac{\Xi'_v(V)}{\Xi'_q(V)}\right| \;=\; \exp\Bigl[-(a_v - a_q)\,|V| + \delta\,|\partial V|\Bigr]\;, \tag{7.18}$$

where δ is bounded by the constant ε_0. From (7.18) we conclude that if $a_q = 0$ then

$$\left|\frac{\Xi'_v(V)}{\Xi'_q(V)}\right| \;\le\; \exp(|\partial V|), \tag{7.19}$$

for large β and small λ. Hence, in this case, the proof would be complete if the truncated partition functions in (7.19) could be replaced by the untruncated ones and

$$\left|\frac{\Xi_v(V)}{\Xi_q(V)}\right| \;\le\; \exp(\mathrm{const}|\partial V|). \tag{7.20}$$

More generally, for regions V for which

$$a_q\,|V| \;\le\; |\partial V|\;, \tag{7.21}$$

we have from (7.18) that, for large β and small λ,

$$\left|\frac{\Xi'_v(V)}{\Xi'_q(V)}\right| \;\le\; \exp(2\,|\partial V|). \tag{7.22}$$

As a first step, we would like to show that the primes in (7.19) and (7.22) can be removed for regions satisfying (7.21). If condition (7.21) were inherited by subregions of V then we could prove inductively, from (7.22), that $\Xi_p(V) = \Xi'_p(V)$. However, it is not true that the bound (7.21) remains valid for arbitrary subregions of V. Therefore it is convenient to resort to a sufficient condition that has this hereditary feature. For this purpose, we introduce the notion of *small regions* and *small contours*, adopting the definitions of [4]. For a piecewise-cylindrical region V of spatial sections $V_1, \ldots, V_n$, we define the spatial diameter of V as follows:

$$\mathrm{spdiam}\, V \;=\; \max_i \mathrm{diam}\, V_i\;. \tag{7.23}$$

Definition 7.4

(i) A region V is q-small if

$$a_q\, \mathrm{spdiam}\, \widehat{V} \;\le\; 1\;. \tag{7.24}$$

(ii) A contour ζ is q-small if

$$a_q \,\mathrm{spdiam}\, \zeta \;\leq\; 1 \; ; \tag{7.25}$$

*otherwise the contour is called q-*large*,*

[where $\widehat{V}$ is as defined in (5.3)].

It is clear that smallness is inherited by subregions. Moreover, the bound (7.21) is valid for q-small regions, because

$$\begin{aligned} a_q\, |V| \;&\leq\; a_q \,\mathrm{spdiam}\, V \, |\partial V| \\ &\leq\; |\partial V| \; . \end{aligned} \tag{7.26}$$

In particular, all contours inside a q-small region are q-small contours. [It is for the sake of this property that we used $\widehat{V}$ in (7.24)]. The hypothesis that $a_p = 0$ implies that *all* regions are p-small. As a consequence, the proof of the implication (i)$\Longrightarrow$(ii) is completed by proving the following lemma [39]:

Lemma 7.5 *For all q, q-small regions are q-stable. As a consequence, all q-contours contained in q-small regions are stable.*

We prove this lemma by induction in the spatial diameters of the regions.
Let us assume that, *for all u*, $a_u \,\mathrm{spdiam}\, V \leq 1$ implies that $\Xi_u(\widetilde{V}) \neq 0$ and

$$\left| \frac{\Xi_v(\widetilde{V})}{\Xi_u(\widetilde{V})} \right| \;\leq\; \exp[4\alpha |\partial V|], \tag{7.27}$$

for all v and for all regions $\widetilde{V}$, contained in V, with spatial diameter less than or equal to m. We pick some $s_q \in \mathcal{K}$ and some q-small region $\widehat{V}$ of spatial diameter $m+1$, and prove the bound (7.27), with $u = q$. All contours ζ^q in this region are q-small, hence their interiors are q-small and of spatial diameter strictly smaller than $m+1$. By the inductive hypothesis such interiors satisfy (7.27), and hence these contours are stable, yielding

$$\Xi'_q(V) \;=\; \Xi_q(V) \; . \tag{7.28}$$

We remark that if $a_v = 0$ then the proof is complete. This is because if $a_v = 0$ *all* regions are v-small, and, consequently, all v-contours in $\widehat{V}$ are stable. This implies that $\Xi'_v(V) \;=\; \Xi_v(V)$, which, along with (7.28) and (7.22), implies that (7.9) is true.

Let us now consider a boundary condition s_v for which $a_v \neq 0$. To estimate $\Xi_v(V)/\Xi_q(V)$ we resort to maneuvres that are justified, a-posteriori, by the proof of uniform convergence. We start with expression (7.2) for the partition function of an ensemble of mutually exterior contours with exterior configuration s_v, in a space-time volume $\widehat{V}$, and resum the contribution of *v-small exterior contours*. This yields

$$\frac{\Xi_v(V)}{\Xi_q(V)} \;=\; \sum_{\substack{\{\zeta_k^v\}\subset\widehat{V}\\ v-\text{large}\\ \text{exterior}}}\!\!\!\!\!\!/\;\; \frac{\Xi_v^{\text{small}}(\text{Ext})}{\Xi_q(V)} \prod_k w(\zeta_k^v)\,\exp\Bigl\{-\int_{\text{supp}(\zeta_k^v)} e_x(s_v)\Bigr\}\,\Xi\Bigl(\text{Int}(\zeta_k^v)\Bigr)\;. \tag{7.29}$$

Here "Ext" is the region outside the v-large exterior contours $\{\zeta_k^v\}$, the label "small" indicates a restriction to configurations where all the exterior contours are v-small, and $\Xi(\text{Int}(\zeta^v)) := \prod_{\tilde{v}} \Xi_{\tilde{v}}(\text{Int}_{\tilde{v}}(\zeta^v))$. If we multiply and divide the RHS of (7.29) by

$$\Xi_q(\text{Int}) \;:=\; \prod_k \prod_{\tilde{v}} \Xi_q(\text{Int}_{\tilde{v}}(\zeta_k^v))\;, \tag{7.30}$$

we obtain

$$\frac{\Xi_v(V)}{\Xi_q(V)} \;=\; \sum_{\substack{\{\zeta_k^v\}\subset\widehat{V}\\ v-\text{large}\\ \text{exterior}}}\!\!\!\!\!\!/\;\; \frac{\Xi_v^{\text{small}}(\text{Ext})\,\Xi_q(\text{Int})\;\exp\Bigl\{-\sum_k \int_{\text{supp}(\zeta_k^v)} e_x(s_v)\Bigr\}}{\Xi_q(V)} \prod_k Y(\zeta_k^v)\;, \tag{7.31}$$

with

$$Y(\zeta_k^v) \;:=\; w(\zeta_k^v)\prod_{\tilde{v}} \frac{\Xi_{\tilde{v}}(\text{Int}_{\tilde{v}}(\zeta_k^v))}{\Xi_q(\text{Int}_{\tilde{v}}(\zeta_k^v))}\;. \tag{7.32}$$

We observe that, by the inductive hypothesis,

$$\Xi_v'^{\;\text{small}}(\text{Ext}) \;=\; \Xi_v^{\text{small}}(\text{Ext}) \tag{7.33}$$

Identities (7.28) and (7.33) allow us to apply the finite-volume bound (7.14) to all the factors in (7.31), except $\prod_k Y(\zeta_k^v)$. We then obtain

$$\begin{aligned} &\left|\frac{\Xi_v^{\text{small}}(\text{Ext})\,\Xi_q(\text{Int})\;\exp\Bigl\{-\sum_k \int_{\text{supp}(\zeta_k^v)} e_x(s_v)\Bigr\}}{\Xi_q(V)}\right| \\ &\quad\le\; \exp\Bigl[-\text{Re}\,(f_v'^{\text{small}} - f_q')\,|V\setminus\text{Int}\,| \;+\; 2\alpha\,|\partial V|\Bigr] \prod_k e^{(2d+1)\alpha|\zeta_k^v|}\;. \end{aligned} \tag{7.34}$$

We have used (7.15) and the geometrical bound $|\partial\,\text{Ext}\,|+|\partial\,\text{Int}\,| \le |\partial V|+2d\sum_k|\zeta_k^v|$. We now use the q-smallness, inequality (7.26), of V to bound

$$\begin{aligned} -\text{Re}\,(f_v'^{\text{small}} - f_q')\,|V\setminus\text{Int}\,| \;&=\; (-a_v^{\text{small}} + a_q)\,|V\setminus\text{Int}\,| \\ &\le\; -a_v^{\text{small}}|V\setminus\text{Int}\,| + |\partial V|\;. \end{aligned} \tag{7.35}$$

Furthermore, the quantum Peierls condition (3.42) and the inductive hypothesis (7.27) for $u = q$ (combined with the bound $|\partial\,\text{Int}(\zeta_k^v)| \le 2d|\zeta_k^v|$) imply that

$$|Y(\zeta_k^v)| \;\le\; \lambda^{|B(\zeta_k^v)|}\,e^{-J|\zeta_k^v|_\perp}\,e^{8d\alpha|\zeta_k^v|}\;. \tag{7.36}$$

Substituting (7.34), (7.35) and (7.36) in (7.31), we get the bound

$$\begin{aligned}
\left|\frac{\Xi_v(V)}{\Xi_q(V)}\right| &\le e^{3\alpha|\partial V|} \sum_{\substack{\{\zeta_k^v\}\subset \widehat{V} \\ v-\text{large} \\ \text{exterior}}} e^{-a_v^{\text{small}}|V\setminus \text{Int}\,|} \prod_k \lambda^{|B(\zeta_k^v)|}\, e^{-J|\zeta_k^v|_\perp}\, e^{(10d+1)\alpha|\zeta_k^v|} \\
&:= e^{3\alpha|\partial V|} \sum_{\substack{\{\zeta_k^v\}\subset \widehat{V} \\ v-\text{large} \\ \text{exterior}}} e^{-a_v^{\text{small}}|V\setminus \text{Int}\,|} \prod_k w^*(\zeta_k^v)\,. \qquad (7.37)
\end{aligned}$$

To show that $e^{4\alpha|\partial V|}$ is an upper bound for (7.37), and hence complete the proof of the lemma, it is convenient to follow [39] and consider the quantity

$$\widetilde{\Xi}_v^{\text{large}}(V) := \sum_{\substack{\{\zeta_k\}\subset \widehat{V} \\ non-linked}} \prod_k w^*(\zeta_k)\, e^{2d|\zeta_k|}, \qquad (7.38)$$

where the label "large" indicates restriction to configurations where all the exterior contours are v-large. This quantity can be interpreted as the partition function of an ensemble of contours having weights

$$\widetilde{w}(\zeta) \;:=\; w^*(\zeta)\, e^{2d|\zeta|} \qquad (7.39)$$

and confined to a space-time volume $\widehat{V}$. It is evident that, for ε_0 small enough, the contour weights $\widetilde{w}(\zeta)$ satisfy the quantum Peierls condition, and hence the cluster expansion converges. Moreover, if $\widetilde{f}_v^{\text{large}}$ is the corresponding free energy density (defined as in (6.2)) then it follows from Theorem 4.1 that

$$[\widetilde{\Xi}_v^{\text{large}}(V)]^{-1} \;\le\; e^{\widetilde{f}_v^{\text{large}}|V|}\, e^{\mathrm{O}(\varepsilon_0)|\partial V|}\;, \qquad (7.40)$$

We claim that

$$a_v^{\text{small}} \;\ge\; -\widetilde{f}_v^{\text{large}}\;. \qquad (7.41)$$

Indeed, by (7.12) and Theorem 6.1,

$$a_v^{\text{small}} \;=\; a_v + \mathrm{O}(C^{\text{large}})\;. \qquad (7.42)$$

[see discussion after (6.49)]. Moreover, for every v-large contour

$$\begin{aligned}
a_v\,(l_{\min} + j_1 + \cdots + j_n) &\;\ge\; a_v\ \text{spdiam}\,\zeta \\
&\;>\; 1\,, \qquad (7.43)
\end{aligned}$$

where the first inequality follows from (the important) property (3.36), and the second one is just the definition of largeness. Therefore, by the final comment in Section 6.4,

$$C^{\text{large}} \;=\; \mathrm{O}\Big((\varepsilon_0)^{1/a_v}\Big)\;. \qquad (7.44)$$

By the same argument,

$$\widetilde{f}_v^{\text{large}} \;=\; \mathrm{O}\Big((\varepsilon_0)^{1/a_v}\Big)\;. \tag{7.45}$$

Hence

$$a_v^{\text{small}} + \widetilde{f}_v^{\text{large}} \;\geq\; a_v + \mathrm{O}\Big((\varepsilon_0)^{1/a_v}\Big) \tag{7.46}$$

which is non-negative for β large and λ small, proving (7.41).

For future purposes, we summarize the rest of the argument in the following lemma which shows that (7.41) causes the sum-integral in (7.37) to yield at most a contribution exponential in the boundary. By substituting the bound (7.47), shown below, into the RHS of (7.37), we obtain the bound (7.27). This completes the inductive proof. ∎

Lemma 7.6 *Consider weights $w^*(\zeta)$ satisfying a Peierls bound (3.42), and let $\widetilde{f}$ denote the contour free energy for the weights $\widetilde{w}(\zeta) = w^*(\zeta)\, e^{2d|\zeta|}$ (well defined if ε_0 is small enough). Then, for $g \geq -\widetilde{f}$,*

$$\sum\!\!\!\!\!\!\!\int_{\substack{\{\zeta_k\}\subset V\\ \text{exterior}}} e^{-g|V\setminus\mathrm{Int}\,|} \prod_k w^*(\zeta_k) \;\leq\; \exp\Big[\mathrm{O}(\varepsilon_0)|\partial V|\Big]\;. \tag{7.47}$$

Proof. (This is Lemma 3.2 of [4]. The proof given there applies verbatim; we transcribe it for the sake of completeness.) Multiply and divide the LHS of (7.47) by $\widetilde{\Xi}_v^{\text{large}}(\mathrm{Int})$. Using the analogue of (7.40) for the region $\mathrm{Int} := \bigcup_{\tilde{v},k} \mathrm{Int}_{\tilde{v}}(\zeta_k)$, and the bound $|\partial\,\mathrm{Int}\,| \leq 2d\sum_k |\zeta_k|$ we obtain

$$[\widetilde{\Xi}(\mathrm{Int})]^{-1} \;\leq\; e^{\widetilde{f}|\,\mathrm{Int}\,|} \prod_k e^{2d\sum_k|\zeta_k|} \tag{7.48}$$

for ε_0 small enough. Thus the LHS of (7.47) satisfies

$$\mathrm{LHS} \;\leq\; \sum\!\!\!\!\!\!\!\int_{\substack{\{\zeta_k\}\subset \widehat{V}\\ \text{exterior}}} e^{-g|V\setminus\mathrm{Int}\,|}\, e^{\widetilde{f}|\,\mathrm{Int}\,|}\, \widetilde{\Xi}^{\text{large}}(\mathrm{Int}) \prod_k \widetilde{w}(\zeta_k)\;, \tag{7.49}$$

and, since $-g \leq \widetilde{f}$, we have that

$$\begin{aligned} \mathrm{LHS} \;&\leq\; e^{\tilde{f}|V|} \sum\!\!\!\!\!\!\!\int_{\substack{\{\zeta_k\}\subset \widehat{V}\\ \text{exterior}}} \widetilde{\Xi}^{\text{large}}(\mathrm{Int}) \prod_k \widetilde{w}(\zeta_k) \\ &\leq\; e^{\tilde{f}|V|}\, \widetilde{\Xi}^{\text{large}}(V)\;. \end{aligned} \tag{7.50}$$

Hence, by the analogue of (7.48) for the region V,

$$\mathrm{LHS} \;\leq\; e^{\mathrm{O}(\varepsilon_0)|\partial V|}\;. \;∎ \tag{7.51}$$

7.3 Stability of phase diagrams

Finally we are in a condition to prove Theorem 2.3. The proof of this theorem involves two steps:

Step 1: Prove that the exponential damping of the original weights w and their derivatives implies an analogous damping of the *new* weights W (for small contours) and of their derivatives.

Step 2: Prove that these differentiability properties of the weights W imply that at low temperature and small λ, the contour free energies f'_p are so close to the energy densities $e(s_p)$ that the manifolds defined by

$$\operatorname{Re} f'_{p_1} = \cdots = \operatorname{Re} f'_{p_k} < \operatorname{Re} f'_{p_{k+1}}, \cdots, \operatorname{Re} f'_{p_n} \;, \tag{7.52}$$

[i.e., $\mathcal{S}^{(\beta,\lambda)}_{\{s_{p_1},\ldots,s_{p_k}\}}$ defined in (7.52)] are close to those defined in terms of the energies $e(s_p)$ [i.e., $\mathcal{S}^{(\infty,0)}_{\{s_{p_1},\ldots,s_{p_k}\}}$], and have differentiability properties similar to those of the weights W.

The proof of step 2, given step 1, is, *in principle*, an exercise in implicit-function theorem technology. However, it is somewhat subtle in cases, as the one we are interested in here, where the weights may fail to be positive or (even) real. As pointed out in [4], there may appear zeroes of the partition functions that destroy the continuity of the excess free energies a_v.

In the sequel we shall only prove step 1; the proof of step 2 is a straightforward adaptation of the argument given in [4, Section 6] (replacing "diam" by "spdiam").

Theorem 7.7 *Assume that there is a non-empty open set $\mathcal{O} \subset \mathrm{I\!R}^{P-1}$ such that, for $\underline{\mu} \in \mathcal{O}$, the quantities $e_{\underline{\mu}\,x}(s_v)$ and $w_{\underline{\mu}}$ are continuously differentiable functions of $\underline{\mu}$ and, moreover,*

$$\left|w_{\underline{\mu}}(\zeta)\right| \;,\; \left|\frac{\partial w_{\underline{\mu}}(\zeta)}{\partial \mu_i}\right| \;\le\; \lambda^{|B(\zeta)|}\, e^{-J|\zeta|_\perp}, \tag{7.53}$$

for all contours ζ, and

$$\left|e_{\underline{\mu}\,x}(s_v)\right| \;\le\; \alpha - 1 \quad ; \quad \left|\frac{\partial e_{\underline{\mu}\,x}(s_v)}{\partial \mu_i}\right| \;\le\; 1 \tag{7.54}$$

*for all $s_v \in \mathcal{K}$, $1 \le i \le P-1$, for some $\alpha < \infty$. Then there exists a constant $\widetilde{J} > 0$ such that if $\varepsilon_0 = \max(e^{-\beta\tilde{J}}, \lambda)$ is sufficiently small, we have that, for all $\underline{\mu} \in \mathcal{O}$ and all q-*small *contours ζ^q,*

$$\left|W_{\underline{\mu}}(\zeta^q)\right| \;,\; \left|\frac{\partial W_{\underline{\mu}}(\zeta^q)}{\partial \mu_i}\right| \;\le\; \widehat{\lambda}^{|B(\zeta^q)|}\, e^{-\hat{J}|\zeta^q|_\perp} \tag{7.55}$$

with $\widehat{\lambda} = \lambda e^{15d\alpha}$ and $\widehat{J} = J - 15d\alpha$.

Remark: Conditions (7.53) and (7.54) express the uniform boundedness requirement of Hypothesis (H1.2). We can always rescale J, λ and the parameters μ_i such that there are no further constants in these bounds.

Proof. Pick a small q-contour ζ^q. The bound on $W(\zeta^q)$ is immediate because of the stability of small q-contours (Lemma 7.5). For the derivatives we would like to use the Leibnitz formula,

$$\begin{aligned}\frac{\partial W(\zeta^q)}{\partial \mu_i} &= \frac{\partial w(\zeta^q)}{\partial \mu_i} \prod_{v=1}^{P} \frac{\Xi_v\Big(\mathrm{Int}_v(\zeta^q)\Big)}{\Xi_q\Big(\mathrm{Int}_v(\zeta^q)\Big)} \\ &\quad + w(\zeta^q) \sum_{v:v\neq u} \frac{\partial}{\partial \mu_i}\left(\frac{\Xi_v\Big(\mathrm{Int}_v(\zeta^q)\Big)}{\Xi_q\Big(\mathrm{Int}_v(\zeta^q)\Big)}\right) \prod_{\tilde v:\tilde v\neq v} \frac{\Xi_{\tilde v}\Big(\mathrm{Int}_{\tilde v}(\zeta^q)\Big)}{\Xi_q\Big(\mathrm{Int}_{\tilde v}(\zeta^q)\Big)}\,,\end{aligned} \tag{7.56}$$

and find suitable bounds for each term. This approach requires two arguments: First, one must show that each of the factors is differentiable and, second, one must exhibit bounds on the derivatives.

The stability of q-small contours — already proven above — implies the correct bounds for all factors $|\Xi_v(\mathrm{Int}_v(\zeta^q))/\Xi_q(\mathrm{Int}_v(\zeta^q))|$, and the hypotheses take care of the differentiability of $w(\zeta^q)$ and of the bounds on these quantities and on $|\partial w(\zeta^q)/\partial\mu_i|$. What remains is to prove the differentiability of the ratios

$$\frac{\Xi_v\Big(\mathrm{Int}_v(\zeta^q)\Big)}{\Xi_q\Big(\mathrm{Int}_v(\zeta^q)\Big)} \tag{7.57}$$

and to find a bound on their derivatives. It is easy to treat Ξ_q, because it only contains q-small, and hence stable, contours, and we can apply the results of Section 6.6. For the numerators, however, we need to take into account v-large, and hence possibly unstable, contours. It is imperative, at this point, to work with the quotient Ξ_v/Ξ_q. In fact, proceeding very much like in the proof of Lemma 7.5, one shows the following:

Claim. If V is q-small then, for any v, the quotient $\Xi_v(V)/\Xi_q(V)$ is differentiable, and

$$\left|\frac{\partial}{\partial\mu_i}\left(\frac{\Xi_v(V)}{\Xi_q(V)}\right)\right| \;\le\; 5\,|V|\,e^{4\alpha|\partial V|}\,. \tag{7.58}$$

It is clear that this claim implies the proposed inequality (7.55). Indeed, inserting the hypotheses (7.53)–(7.54), the stability condition (7.27) and the claimed inequality (7.58) — for $V = \mathrm{Int}_v(\zeta^q)$ — into (7.56), we obtain

$$\begin{aligned}\left|\frac{\partial W(\zeta^q)}{\partial\mu_i}\right| &\le \lambda^{|B(\zeta^q)|}\, e^{-J|\zeta^q|_\perp}\, e^{4\alpha|\partial\,\mathrm{Int}(\zeta^q)|}\Big[1+5|\,\mathrm{Int}(\zeta^q)|\Big] \\ &\le \lambda^{|B(\zeta^q)|}\, e^{-J|\zeta^q|_\perp}\, e^{8d\alpha|\,\mathrm{Int}(\zeta^q)|}\, e^{7d|\zeta^q|} \\ &\le \lambda^{|B(\zeta^q)|}\, e^{-J|\zeta^q|_\perp}\, e^{15d\alpha|\zeta^q|}\,,\end{aligned} \tag{7.59}$$

where we have used the bounds

$$|\partial\,\mathrm{Int}(\zeta^q)| \leq 2d|\zeta^q| \quad ; \quad |\,\mathrm{Int}(\zeta^q)| \leq |\partial\,\mathrm{Int}(\zeta^q)|^2$$

and $1+5x^2 \leq \exp[7x/2]$. As $|\zeta| \leq |B(\zeta)| + |\zeta|_\perp$, this last bound implies (7.55).

To prove the claim we proceed, once again, by induction in the spatial diameter of V. We assume that (7.58) is true for all u and for all u-small regions $\widetilde{V}$ with spatial diameter m or less. We shall now prove (7.58) for some q and some q-small region V of diameter $m+1$. We start with the resummed expression (7.31) which we repeat for the reader's convenience:

$$\frac{\Xi_v(V)}{\Xi_q(V)} = \sum\!\!\!\!\!\!\int_{\substack{\{\zeta_k^v\}\subset\widehat{V} \\ v-\text{large} \\ \text{exterior}}} \frac{\Xi_v^{\text{small}}(\text{Ext})\,\Xi_q(\text{Int})\ \exp\Bigl\{-\sum_k \int_{\text{supp}(\zeta_k^v)} e_x(s_v)\Bigr\}}{\Xi_q(V)} \prod_k Y(\zeta_k^v)\,. \tag{7.60}$$

We shall use the product rule to calculate the μ_i-derivative of the RHS. For the partition functions on the RHS we can use the cluster-expansion technology, because only stable contours are involved (Lemma 7.5 for Ξ^{small}, and the inductive hypothesis for Ξ_q). Thus, we conclude differentiability and a bound analogous to (6.60):

$$\begin{aligned}
\left|\frac{\partial}{\partial\mu_i}\Xi_v^{\text{small}}(\text{Ext})\right| &\leq \left|\Bigl[\exp\Bigl\{-\int_{\text{Ext}} e_x(s_v)\Bigr\}\Bigr]\frac{\partial}{\partial\mu_i}\widehat{\Xi}_v^{\text{small}}(\text{Ext})\right| \\
&\qquad + \left|\frac{\partial}{\partial\mu_i}\Bigl[-\exp\Bigl\{\int_{\text{Ext}} e_x(s_v)\Bigr\}\Bigr]\,\widehat{\Xi}_v^{\text{small}}(\text{Ext})\right| \\
&\leq \left|\Xi_v^{\text{small}}(\text{Ext})\right| |\,\text{Ext}\,| + \int_{\text{Ext}}\left|\frac{\partial e_x(s_v)}{\partial\mu_i}\right|\,\left|\Xi_v^{\text{small}}(\text{Ext})\right| \\
&\leq 2\left|\Xi_v^{\text{small}}(\text{Ext})\right|\,|\,\text{Ext}\,|\,.
\end{aligned} \tag{7.61}$$

The first inequality makes use of (7.6), the second one is due to hypothesis (7.54), and the third one follows from (6.60). Similarly,

$$\left|\frac{\partial}{\partial\mu_i}\left(\frac{\Xi_q(\text{Int})}{\Xi_q(V)}\right)\right| \leq 2\left|\frac{\Xi_q(\text{Int})}{\Xi_q(V)}\right|\,|V\setminus\text{Int}\,|\,. \tag{7.62}$$

Moreover, from the differentiability of the energy-densities and the bound (7.54),

$$\begin{aligned}
&\left|\frac{\partial}{\partial\mu_i}\exp\Bigl\{-\textstyle\sum_k \int_{\text{supp}(\zeta_k^v)} e_x(s_v)\Bigr\}\right| \\
&\qquad \leq \sum_k |\zeta_k^v|\,\left|\exp\Bigl\{-\textstyle\sum_k \int_{\text{supp}(\zeta_k^v)} e_x(s_v)\Bigr\}\right|\,.
\end{aligned} \tag{7.63}$$

Finally, for the weights

$$Y(\zeta_k^v) \;=\; w(\zeta_k^v) \prod_{u=1}^{P} \frac{\Xi_u\Big(\mathrm{Int}_u(\zeta_k^v)\Big)}{\Xi_q\Big(\mathrm{Int}_u(\zeta_k^v)\Big)} \tag{7.64}$$

we use the hypotheses made on the original weights w and the inductive hypothesis. They imply that each factor is differentiable, and we can use the bound:

$$\left|\frac{\partial Y(\zeta_k^v)}{\partial \mu_i}\right| \;\le\; \left|\frac{\partial w(\zeta_k^v)}{\partial \mu_i}\right| \prod_{u=1}^{P} \left|\frac{\Xi_u\Big(\mathrm{Int}_u(\zeta_k^v)\Big)}{\Xi_q\Big(\mathrm{Int}_u(\zeta_k^v)\Big)}\right| \;+\; w(\zeta_k^v) \sum_{u=1}^{P} \left|\frac{\partial}{\partial \mu_i}\left(\frac{\Xi_u\Big(\mathrm{Int}_u(\zeta_k^v)\Big)}{\Xi_q\Big(\mathrm{Int}_u(\zeta_k^v)\Big)}\right)\right| \,. \tag{7.65}$$

Using the q-stability of the interior regions and the inductive bound we obtain the upper bound

$$\begin{aligned}\left|\frac{\partial Y(\zeta_k^v)}{\partial \mu_i}\right| \;&\le\; \lambda^{|B(\zeta_k^v)|} e^{-J|\zeta_k^v|_\perp} \Big[e^{4\alpha|\partial\,\mathrm{Int}\,\zeta_k^v|} + 5|\,\mathrm{Int}\,\zeta_k^v| e^{4\alpha|\partial\,\mathrm{Int}\,\zeta_k^v|}\Big] \\ &\le\; \widehat{\lambda}^{|B(\zeta_k^v)|}\, e^{-\hat{J}|\zeta_k^v|_\perp} \;=:\; W^*(\zeta_k^v)\,. \end{aligned} \tag{7.66}$$

[To obtain the last line we proceeded as in (7.59).] With the bounds (7.61)–(7.66) and the already known bounds (7.34)–(7.35), expression (7.60) yields

$$\begin{aligned}\left|\frac{\partial}{\partial \mu_i} \frac{\Xi_v(V)}{\Xi_q(V)}\right| \;\le\; e^{2\alpha|\partial V|} & \sum\!\!\!\!\!\!\not\;\;\;_{\substack{\{\zeta_k^v\}\subset \widehat{V} \\ v-\text{large} \\ \text{exterior}}} \exp\Big[-\mathrm{Re}\,(f_v'^{\,\text{small}} - f_q')\,|V\setminus \mathrm{Int}\,|\Big] \\ \times \prod_k W^*(\zeta_k^v)\, e^{(2d+1)\alpha|\zeta_k^v|} & \Big[2(|\,\mathrm{Ext}\,| + |V\setminus \mathrm{Int}\,|) \;+\; \sum_k (|\zeta_k^v| + 1)\Big]\,. \end{aligned} \tag{7.67}$$

The square bracket is bounded by

$$4|V\setminus \mathrm{Int}| + \sum_k 1 \;\le\; 5|V|\,, \tag{7.68}$$

and the remaining sum is bounded by $e^{\alpha|\partial V|}$, by Lemma 7.6. ■

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