A MULTIFRACTAL ANALYSIS OF GIBBS MEASURES FOR CONFORMAL EXPANDING MAPS AND MARKOV MORAN GEOMETRIC CONSTRUCTIONS

YAKOV PESIN AND HOWARD WEISS

The Pennsylvania State University

April 25, 1997 10:43pm

ABSTRACT. We establish the complete multifractal formalism for Gibbs measures for conformal expanding maps and Markov Moran geometric constructions. Examples include Markov maps of an interval, hyperbolic Julia sets, and conformal toral endomorphisms.

I: Introduction

This paper describes the multifractal analysis of measures invariant under dynamical systems. The concept of a multifractal analysis was suggested by several physicists in the seminal paper [HJKPS] and became a popular interdisciplinary subject of study. A search of several electronic databases showed that there are now hundreds of related papers in the physical and mathematical literature.

The first rigorous multifractal analysis was carried out in [CLP] for a special class of measures invariant under some one-dimensional Markov maps, and in [Ra] for Gibbs measures for Cookie-Cutter maps. Lopes [Lo] studied the measure of maximal entropy for a hyperbolic Julia set. Recently, Simpelaere [Si] effected a complete multifractal analysis for Gibbs measures of Axiom A surface diffeomorphisms.

The two major components of the multifractal analysis are the Hentschel-Procaccia (HP) spectrum for dimensions and the $f(\alpha)$ spectrum for dimensions. We will provide motivation for introducing these spectra by considering the BRS (Bowen-Ruelle-Sinai) measures on a hyperbolic attractors.

Let $g\colon M \to M$ be a diffeomorphism of a smooth Riemannian manifold M and $\Lambda \subset M$ a compact hyperbolic attractor for g. For simplicity, we assume that g is topologically mixing on Λ. In [B], Bowen showed that the evolution of the Lebesgue measure in a

Key words and phrases. Hausdorff dimension, pointwise dimension, multifractal analysis, dimension spectrum, HP spectrum, expanding map, Markov partition.

The work of the first author was partially supported by a National Science Foundation grant #DMS91-02887. The work of the second author was partially supported by a National Science Foundation grant #DMS-9403724.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

basin of Λ converges to the BRS-measure. From the physical point of view, this is the natural measure on the attractor since it describes the orbit distribution of points in the basin which are typical with respect to the Lebesgue measure. This distribution is not uniform, and as computer pictures show, there exist spots of high and low density of visits sometimes called *hot* and *cold* spots. This phenomenon also has been observed for a more general class of attractors (hyperbolic attractors with singularities), which includes the Lorenz attractor, the Lozi attractor, and the Belich attractor. The attempts to analyze this measure in computer simulations are based on partitioning the basin into a very fine grid and estimating the measure of each box by the frequency with which a typical orbit visits it. This leads to an enormous amount of data.

A more "practical" approach involves the study of correlations of the distributions of $q-$tuples along a typical orbit for $q = 2, 3, \dots$ More precisely, let $g\colon X \to X$ be a map on a metric space (X, ρ) preserving a Borel probability measure ν. We set

$$C(x, q, r, n) = \frac{1}{n^q}\text{card}\{(i_1 \dots i_q) : \rho(g^{i_j}x, g^{i_k}x) \leq r \text{ for all } 0 \leq i_j \leq i_k < n\}.$$

We define the *correlation dimension* of order q by

$$C_q(x) = \frac{1}{1-q} \lim_{r\to 0} \lim_{n\to\infty} \frac{\log C(x, q, r, n)}{\log r}$$

(*provided the limits exist*). If ν is ergodic, it was shown in [Pe1] (see also [PT]) that for ν almost every x

$$\lim_{n\to\infty} C(x, q, r, n) = \int_X \nu(B(y, r))^{q-1}\, d\nu(y),$$

where $B(y, r)$ denotes the ball of radius r centered at the point y. Thus, for $q = 2, 3, \dots$

$$C_q(x) = \frac{1}{1-q} \lim_{r\to 0} \left(\frac{\log \int_X \nu(B(y, r))^{q-1}\, d\nu(y)}{\log r}\right)$$

provided the limit exists. Let us emphasize that, in general, one does not expect this limit to exist. In [PT], the authors constructed an example of a continuous map on an interval that preserves a measure absolutely continuous with respect to the Lebesgue measure, for which the above limit does not exist for almost every x in a large interval. Combining this with the results in [K] one can construct a diffeomorphism of the two-torus preserving an ergodic measure that is absolutely continuous with respect to the Lebesgue measure, having positive topological entropy, and for which the above limit does not exist for almost every x in some large set. In this paper, we show that this limit exists for a broad class of measures including Gibbs measures for conformal repellers. This unifies and extends almost all cases in the literature.

The natural extension of the correlation dimension of order $q = 2, 3, \dots$ to $q \in \mathbb{R}$, $q \neq 1$ was introduced by Hentschel and Procaccia in [HP]. Let ν be a Borel probability measure on a metric space (X, ρ). For $q > 0$, $q \neq 1$ we define the **HP spectrum for dimensions** by

$$HP_\nu(q) = \frac{1}{1-q} \lim_{r\to 0} \frac{\log \int \nu(B(y, r))^{q-1}\, d\nu(y)}{\log r}$$

provided the limit exists. A precursor to this definition was suggested by Rényi and is known as the Rényi spectrum for dimensions. It can be obtained by replacing coverings by partitions.

We define a new class of measures that incorporate the metric structure of the underlying metric space. Namely, we call a measure ν **isotropic** if for each $A > 1$ there exists $K > 0$ such that for every x and any sufficiently small $r > 0$ we have

$$\nu(B(x, Ar)) \leq K\nu(B(x, r)). \tag{1}$$

It is shown in [Pe2] that if ν is isotropic then for any $q \neq 1$

$$HP_\nu(q) = \frac{1}{1-q} \lim_{r\to 0} \frac{\log \inf_{\mathfrak{V}_r} \sum_{B\in\mathfrak{V}_r} \nu(B)^q}{\log r},$$

where the infinum is taken over all covers $\mathfrak{V}_r$ of X by balls of radius r, *provided the limit exists.* We will use this definition of HP spectrum for dimensions in our proofs.

In [Pe2], Pesin showed that the Rényi spectrum coincides with the $HP-$spectrum for isotropic measures.

In general, even *good* measures may not be isotropic. One can construct a smooth ergodic measure for a diffeomorphism of a compact manifold which is not isotropic [Pe2].

In this paper, we consider continuous expanding maps on compact metric spaces and prove that any Gibbs measure is isotropic. Our main tool is a construction of a Markov partition for continuous expanding maps. This construction is geometrically natural and simplier than other constructions for smooth expanding maps that we are aware of. This construction is specially adapted to a given point (or any finite collection of points) such that the partition element containing this point also contains a "large" ball centered at the point. The same approach can be used to construct special Markov partitions for Axiom A diffeomorphisms and their continuos version. Hence, Gibbs measures for Axiom A diffeomorphisms are isotropic.

We turn to the second ingredient in our multifractal analysis and define the $f(\alpha)$ spectrum for dimensions. Given $x \in X$ we consider the *upper* and *lower pointwise dimensions of* ν at x,

$$\overline{d}_\nu(x) = \limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \quad \text{and} \quad \underline{d}_\nu(x) = \liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}.$$

If $\underline{d}_\nu(x) = \overline{d}_\nu(x)$ we call the common value the **pointwise dimension** at x and denote it by $d_\nu(x)$. We call ν *exact dimensional* if

$$\overline{d}_\nu(x) = \underline{d}_\nu(x) = d_\nu(x) = d$$

for $\nu-$almost every x where d is a non-negative constant. If ν is exact dimensional then Young [Y] showed that the Hausdorff dimension of ν coincides with the box dimension of ν. In general one does not expect the pointwise dimension of ν to exist at a typical point even for *nice* measures which are invariant under dynamical systems [LM, PW]. Even when the pointwise dimension of ν does exist it is not necessarily exact dimensional [C, PW].

Nevertheless, measures which are invariant under smooth dynamical systems with hyperbolic behavior often turn out to be exact dimensional. Eckmann and Ruelle have conjectured that *hyperbolic measures* (i.e., ergodic measures with non-zero Lyapunov exponents almost everywhere) are exact dimensional. This has been established for hyperbolic measures in the two dimensional case in [Y] and for hyperbolic BRS-measures and Gibbs measures for Axiom A diffeomorphisms in [Le, PY].

The multifractal analysis is a description of the fine-scale geometry of the set X whose constituent components are the sets

$$L_\alpha = \{x \in X \mid \underline{d}_\nu(x) = \overline{d}_\nu(x) = \alpha\},$$

for $\alpha \in \mathbb{R}$. The $f(\alpha)$ **spectrum for dimensions** is defined by

$$f(\alpha) = \dim_H L_\alpha,$$

where $= \dim_H L_\alpha$ denotes the Hausdorff dimension of the set L_α. Note that if ν is exact dimensional then $\nu(L_d) = 1$. It is important to emphasize that even in this case the union of the sets L_α may not be all of X. Shereshevsky [Sh] showed that for a large class of C^2 Axiom A surface diffeomorphisms, the Hausdorff dimension of the set $X - \cup_\alpha L_\alpha$ is positive for any Gibbs measure ν.

In [HJKPS], the authors presented a heuristic argument showing that the HP spectrum for dimensions (multiplied by $1-q$) and the $f(\alpha)$ spectrum for dimensions form a Legendre transform pair. For this to make sense one must first establish that the two spectra are smooth and strictly convex on some interval. A priori this seems quite amazing since in general one expects the functions $f(\alpha)$ and $HP_\nu(q)$ to be only measurable. Furthermore, it is not at all clear whether, even in the exact dimensional case, the pointwise dimension attains *any* values besides d. Once the Legendre transform relation between the two dimension spectrums is established, one can compute the delicate and seemingly intractable $f(\alpha)$ spectrum through the $HP-$spectrum, which is completely determined by the statistics of a typical trajectory.

Our concept of a complete multifractal analysis includes establishing the coincidence of the Hentschel-Procaccia spectrum and the Rényi spectrum for dimensions, and the Legrendre transform relation between them and the $f(\alpha)$ spectrum for dimensions. We believe that a meaningful class of measures to analyze should include isotropic measures for which the Hentschel-Procaccia spectrum and Rényi spectrum coincide. We conjecture that any Gibbs measure on a locally maximal hyperbolic set is isotropic. In this paper we verify this conjecture for Gibbs measures for expanding maps (see Appendix 1). We also conjecture that Gibbs measures on a locally maximal hyperbolic set admit a multifractal analysis. This was recently established in [Si] for Gibbs measures for Axiom A surface diffeomorphisms. Simpelaere's analysis uses the theory of large deviations in an essential way. Furthermore, his proof exploits the smoothness of stable and unstable foliations in the two-dimensional case (which is in general false in higher dimensions) and is based upon the fact that the restriction of the diffeomorphism to unstable leaves is a one-dimensional expanding map.

In this paper we effect a thorough multifractal analysis of Gibbs measures for smooth conformal expanding maps. Examples include Markov maps of an interval, hyperbolic

Julia sets, and conformal toral endomorphisms. We prove that the functions $f(\alpha)$ and $(1-q)HP_\nu(q)$ are analytic and strictly convex on an interval and form a Legrendre transform pair, provided the measure is not the measure of maximal entropy (see Theorem 2). In particular this implies that the pointwise dimension attains a continuum of values. Our results generalize and extend all known results related to the multifractal analysis of smooth conformal expanding maps.

Another class of examples that we consider are Moran symbolic geometric constructions where the basic sets comply with a given symbolic dynamical system. Moran first studied these constructions in the case of the full shift and computed the Hausdorff dimension of the limit set. In [PW], we studied the general symbolic case, and in particular computed the Hausdorff and box dimensions of the limit sets.

In this paper we also undertake a complete multifractal analysis of Gibbs measures for a large class of Moran geometric constructions (see Theorem 1). Our analysis is based upon the dynamical properties of the map on the limit set induced by the shift map on symbolic space. This induced map need not be expanding even when the Moran construction is modeled by the full shift. Whether the induced map is expanding strongly depends on the symbolic dynamics and its embedding into Euclidean space, i.e., the gaps between the basic sets. Our multifractal analysis of these "expanding" geometric constructions is intimately related to our analysis for smooth conformal expanding maps.

II: Multifractal Analysis of Gibbs Measures on Limit Sets of Moran Geometric Constructions

In [Mo], Moran introduced geometric constructions in $\mathbb{R}^q$ given by p non-overlapping balls $\Delta_{i_1\dots i_n}$ satisfying $\mathrm{diam}(\Delta_{i_1\dots i_n j}) = \lambda_j \mathrm{diam}(\Delta_{i_1\dots i_n})$, where $0 < \lambda_j < 1$ for $j = 1, \dots, p$ are the *ratio coefficients*. Moran discovered the formula for the Hausdorff dimension of the limit set F of the construction: $s = \dim_H F$, where s is the unique root of the equation

$$\sum_{i=1}^{p} \lambda_i^t = 1.$$

He also showed that the $s-$dimensional Hausdorff measure of the limit set is finite and strictly positive.

Moran's great insight was to realize that the spacing of the balls $\Delta_{i_1\cdots i_n}$ is not important in the calculation of the Hausdorff dimension of the limit set: the dimension depends only on the ratio coefficients. Moran proved this using the (uniform) mass distribution principle applied to the $s-$dimensional Hausdorff measure. Our basic construction, which we call a *Moran geometric construction*, defines a Cantor-like set of $\mathbb{R}^q$ by using a symbolic description in the space of all one-sided infinite sequences $\omega = (i_1 i_2 \cdots)$ on p symbols. We denote this space by Σ_p^+ and endow it with its usual topology.

A *Moran geometric construction* is defined by

a) a compact set $Q \subset \Sigma_p^+$ invariant under the shift σ (i.e., $\sigma(Q) = Q$) such that $\sigma \mid Q$ is topologically transitive,

b) a collection of numbers λ_k, $k = 1, \dots, p$ called the *ratio coefficients* such that $0 < \lambda_k < 1$,

c) a family of balls, called *basic sets*, $\Delta_{i_1 \cdots i_n} \subset \mathbb{R}^q$ for $i_j = 1, 2, \ldots, p$ and $n \in \mathbb{N}$ where the n-tuples $(i_1 \cdots i_n)$ are admissible with respect to Q (i.e., there exists $\omega = (i'_1, i'_2, \cdots) \in Q$ such that $i'_1 = i_1, i'_2 = i_2, \cdots, i'_n = i_n$) and these sets satisfy

$$\operatorname{diam}(\Delta_{i_1 \cdots i_n}) = \prod_{k=0}^{n} \lambda_{i_k}. \tag{2}$$

For any admissible sequence $(i_1 \cdots i_{n+1})$ $in\{1, \cdots, p\}^{n+1}$, we require that

$$\Delta_{i_1 \cdots i_{n+1}} \subset \Delta_{i_1 \cdots i_n}$$

and that the sets $\Delta_{i_1 \cdots i_n}$ be disjoint, i.e.,

$$\Delta_{i_1 \cdots i_n} \cap \Delta_{i'_1 \cdots i'_n} = \emptyset \quad \text{if } (i_1 \cdots i_n) \neq (i'_1 \cdots i'_n).$$

The *limit set* F for this construction is defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \Delta_{i_1 \cdots i_n}.$$

The set F is a Cantor-like set, i.e., it is a perfect, nowhere dense, and totally disconnected set (see Figure 1). We emphasize that the placement of the balls $\{\Delta_{i_1 \cdots i_n}\}$ is completely arbitrary.

FIGURE 1. MORAN GEOMETRIC CONSTRUCTION

Given $x \in F$ and $n > 0$, there exists a unique set $\Delta_{i_1\cdots i_n}(x)$ that contains x and hence $x = \bigcap_{n=1}^{\infty} \Delta_{i_1\cdots i_n}(x)$. This gives a unique one-sided infinite sequence $\omega = (i_1 i_2 \cdots)$ such that the mapping $\chi\colon Q \to F$ defined by $\chi(\omega) = x$ is a homeomorphism from Q onto F. The map χ is Hölder continuous. To see this let $\omega_1 = (i_1 i_2 \dots i_n j \dots)$ and $\omega_2 = (i_1 i_2 \dots i_n k \dots)$, $j \neq k$ be two points in Q. We have

$$\rho(\chi(\omega_1), \chi(\omega_2)) \leq \prod_{j=1}^{n} \lambda_{i_j} \leq \lambda_{\max}^n \leq C\rho(\omega_1, \omega_2)^{\alpha},$$

where $C > 0$, $0 < \alpha < 1$ are constants and $\rho(\cdot, \cdot)$ denotes the Euclidean metric on F. Therefore, any Hölder continuous function on F pulls back to a Hölder continuous function on Q. However, since the spacing between the basic sets at each step of the construction (the "gaps" between them) is arbitrary, the map χ^{-1} is, in general, not Hölder continuous. Therefore, the pushforward of any Hölder continuous function may not be Hölder continuous on F.

The shift map σ on Q induces a continuous map $G\colon F \to F$ defined by $G(x) = \chi \circ \sigma \circ \chi^{-1}(x)$. Since the spacing between the basic sets is arbitrary, the map G need not be expanding. See Remarks 1 and a the end of Section II.

Given a Hölder continuous function ϕ on F there exists the equilibrium measure ν_ϕ on F corresponding to ϕ, i.e., the measure that satisfies

$$P_G(\phi) = \sup \left(h_\nu(G) + \int_F \phi \, d\nu \right) = h_{\nu_\phi}(G) + \int_F \phi \, d\nu_\phi$$

where the supremum is taken over all $G-$invariant measures on F, $h_\nu(G)$ is the metric or Kolmogorov-Sinai entropy of G with respect to ν, and $P_G(\phi)$ is the topological pressure (see Appendix 2).

In this paper, we effect a multifractal analysis of equilibrium measures corresponding to Hölder continuous functions on F. We begin with the following statement that shows how to compute the dimension of the limit set for a Moran geometric construction.

Given s, $0 \leq s \leq 1$ consider the function $\omega = (i_1 i_2 \cdots) \mapsto s \log \lambda_{i_1}$ which is clearly a Hölder continuous function on Q. It is well-known [B] that there exists the unique root of *Bowen's equation* $P_Q(s \log \lambda_{i_1}) = 0$ that we denote by d. Let η be the Gibbs measure corresponding to the Hölder continuous function $d \log \lambda_{i_1}$ and let m be the pullforward of η under the coding map χ.

Theorem A [PW]**.** *Let F be the limit set for a Moran geometric construction. Then*

(1) *the Hausdorff dimension of F, $\dim_H F$, and the lower and upper box dimensions of F, $\underline{\dim}_B F$ and $\overline{\dim}_B F$, satisfy $\dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = d$,*
(2) *the $d-$Hausdorff measure $m_H(d, \cdot)$ is equivalent to m.*
(3) *$\underline{d}_m(x) = \overline{d}_m(x) = d_m(x)$ for every $x \in F$.*

Let ξ be a Hölder continuous function on F and ν the corresponding equilibrium measure. Denote by ϕ the pull back of ξ under the coding map and by μ the pull back of the measure ν. We need the following statement.

Lemma 1. *There exists a positive Hölder continuous function ψ such that $P(\log\psi) = 0$ and μ is a Gibbs measure for $\log\psi$.*

Proof. Choose ψ such that $\log\psi = \phi - P(\phi)$. $\square$

Define the one-parameter family of functions ϕ_q, $q \in (-\infty, \infty)$ on Σ_A^+ by $\phi_q(\omega) = T(q)\log\lambda_{i_1} + q\log\psi(\omega)$ where $T(q)$ is chosen such that $P(\phi_q) = 0$. It is obvious that ϕ_q are Hölder continuous.

Henceforth, we will assume that the symbolic system (Q, σ) is a transitive subshift of finite type (Σ_A^+, σ).

The following theorem is our main result on the multifractal analysis of measures on the limit sets of Moran geometric constructions.

Theorem 1. *Let ν be the equilibrium measure on F corresponding to a Hölder continuous function ξ. Let μ be the pull back of ν to Σ_A^+. Then*

(1) *the pointwise dimension $d_\nu(x)$ exists for $\nu-$almost every $x \in F$. Moreover*

$$d_\nu(x) = \frac{\int_{\Sigma_A^+} \log(\psi(\omega))\, d\mu}{\int_{\Sigma_A^+} \log\lambda_{i_1}(\omega)\, d\mu} = \frac{h_\mu(\sigma)}{\int_{\Sigma_A^+} \log\lambda_{i_1}(\omega)\, d\mu}$$

where ψ is chosen as in Lemma 1, and $h_\mu(\sigma)$ denotes the Kolmogorov-Sinai entropy of the shift with respect to the measure ν.

(2) *if ν is not the measure of maximal entropy, then the multifractal spectrum $f_\nu(\alpha)$ is real analytic on some open interval and $f_\nu(\alpha(q)) = T(q) + q\alpha(q)$ where $\alpha(q) = -T'(q)$ is positive. The range of $f_\nu(\alpha)$ contains the number $d = T(0)$.*

(3) *if ν is not the measure of maximal entropy, then the functions $f_\nu(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair (see Appendix 2).*

(4) *the $\nu-$measure of any ball centered at points in F is positive, and if ν is isotropic (see (1)) then for any $q \in \mathbb{R}$ we have*

$$T(q) = \lim_{r\to 0} \frac{\log \inf_{\mathfrak{V}_r} \sum_{B\in\mathfrak{V}_r} \nu(B)^q}{\log r},$$

where the infimum is taken over all covers $\mathfrak{V}_r$ of X by balls B of radius r.

Proof. Let μ_q denote the Gibbs measure corresponding to ϕ_q. We also denote the push-forward of μ_q to F by ν_q. Clearly $T(0) = \dim_H F$.

To prove Statement 1 we need the following lemma.

Lemma 2. *There exist constants $C_1 = C_1(q, d), C_2 = C_2(q, d) > 0$ such that for all basic sets $\Delta_{i_1\dots i_n}$,*

$$C_1 \le \frac{\nu_q(\Delta_{i_1\dots i_n})}{m(\Delta_{i_1\dots i_n})^{\frac{T(q)}{d}}\nu(\Delta_{i_1\dots i_n})^q} \le C_2. \tag{3}$$

Proof. This follows immediately from the fact that η, μ, and μ_q are Gibbs measures corresponding to the Hölder continuous functions $d\log\lambda_{i_1}$, $\log\psi$, and $T(q)\log\lambda_{i_1} + q\log\psi$

respectively, see Appendix 2 (recall that m, ν, and ν_q are the pushforward measures of η, μ, and μ_q). $\square$

Given $0 < r < 1$ and a vector of numbers $(\lambda_1, \cdots, \lambda_p)$, $0 < \lambda_i < 1$, $i = 1, \ldots, p$, we define, following [M, PW], a special disjoint cover $\mathfrak{U}_r = \mathfrak{U}_r(\lambda)$ of the limit set F by basic sets with radius approximately equal to r. For any $x \in F$, let $n(x)$ denote the unique positive integer such that $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(x)}} > r$ and $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(x)+1}} \leq r$ where $\chi^{-1}(x) = (i_1 i_2 \cdots)$. It is easy to see that $n(x) \to \infty$ as $r \to 0$ uniformly in x. Fix $x \in F$ and consider the set $\Delta_{i_1\cdots i_{n(x)}}$. We have $x \in \Delta_{i_1\cdots i_{n(x)}}$, and if $y \in \Delta_{i_1\cdots i_{n(x)}}$ and $n(y) \geq n(x)$, then

$$\Delta_{i_1\cdots i_{n(y)}} \subset \Delta_{i_1\cdots i_{n(x)}}.$$

Let $\Delta(x)$ be the largest set containing x with the property that $\Delta(x) = \Delta_{i_1\cdots i_{n(z)}}$ for some $z \in \Delta(x)$ and $\Delta_{i_1\cdots i_{n(y)}} \subset \Delta(x)$ for any $y \in \Delta(x)$. The sets $\Delta(x)$ corresponding to different $x \in F$ either coincide or are disjoint. We denote these sets by Δ_r^j, $j = 1, \cdots, N_r$. There exist points x_j such that $\Delta_r^j = \Delta_{i_1\cdots i_{n(x_j)}}$. These sets form a disjoint cover of F.

Consider the open Euclidean ball $B(x, r)$ of radius r centered at a point x. Let $N(x, r)$ denote the number of sets $\Delta_r^j \in \mathfrak{U}_r$ that have a non-empty intersection with $B(x, r)$. Since our basic sets are balls, it immediately follows that $N(x, r) \leq L$, uniformly in x and r for some finite L which we call the *Moran number*.

Since η is the Gibbs measure for the function $d \log \lambda_{i_1}$ it follows that for any basic set $\Delta_{i_1\ldots i_n}$ we have

$$D_1 \operatorname{diam}(\Delta_{i_1\ldots i_n})^d \leq m(\Delta_{i_1\ldots i_n}) \leq D_2 \operatorname{diam}(\Delta_{i_1\ldots i_n})^d, \tag{4}$$

where D_1 and D_2 are positive constants. It follows from the choice of our special cover and (4) that there exist positive numbers C_3, C_4 such that for every $\Delta_r^j \in \mathfrak{U}_r$

$$C_3 r^d \leq m(\Delta_r^j) \leq C_4 r^d. \tag{5}$$

Since $\mathfrak{U}_r$ is a disjoint cover of F, we have

$$\sum_{\Delta_r^j \in \mathfrak{U}_r} \nu_q(\Delta_r^j) = 1.$$

Hence, summing (3) over the cover $\mathfrak{U}_r$ we obtain that there exist positive constants C_5, C_6 such that

$$C_5 \leq r^{T(q)} \sum_{\Delta_r^j \in \mathfrak{U}_r} \nu(\Delta_r^j)^q \leq C_6.$$

Taking logs and dividing by $\log r$ yields

$$\lim_{r \to 0} \frac{\log \sum_{\Delta_r^j \in \mathfrak{U}_r} \nu(\Delta_r^j)^q}{\log r} = T(q). \tag{6}$$

We now prove Statement 1 of the theorem. Let ν be an arbitrary measure on F. In [PW], we found a method to compute its pointwise dimension which is better adapted to the structure of a Moran geometric construction. The idea is to replace balls containing a point x with the basic set containing x. Let

$$\overline{d}_{\nu,C}(x) \equiv \limsup_{n\to\infty} \frac{\log \nu(\Delta_{i_1\dots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1\dots i_n}(x))} \quad \text{and} \quad \underline{d}_{\nu,C}(x) \equiv \liminf_{n\to\infty} \frac{\log \nu(\Delta_{i_1\dots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1\dots i_n}(x))}.$$

If $\overline{d}_{\nu,C}(x) = \underline{d}_{\nu,C}(x)$ we denote the common value by $d_{\nu,C}(x)$. We need the following result from [PW] that describes the relations between $\overline{d}_{\nu,C}(x)$, $\underline{d}_{\nu,C}(x)$, and the lower and upper pointwise dimensions at x.

Theorem B [PW]**.**

(1) $\underline{d}_{\nu,C}(x) \le \underline{d}_{\nu}(x) \le \overline{d}_{\nu}(x) \le \overline{d}_{\nu,C}(x)$ *for all* $x \in F$.

(2) *If* $\underline{d}_{\nu,C}(x) = \overline{d}_{\nu,C}(x)$ *for* $\nu-$*almost every* $x \in F$, *then*

$$\underline{d}_{\nu}(x) = \overline{d}_{\nu}(x) = d(x)$$

for $\nu-$*almost every* $x \in F$.

Hence if $d_{\nu,C}(x)$ exists for some point $x \in F$ then $d_{\nu,C}(x)$ coincides with the pointwise dimension at x.

For all $q, r \in \mathbb{R}$ we define $\phi_{q,r}(\omega) = \log \lambda_{i_1} + q \log \psi(\omega)$ and set

$$\alpha(q) \equiv \frac{\frac{\partial P(\phi_{q,r})}{\partial q}}{\frac{\partial P(\phi_{q,r})}{\partial r}} = \frac{\int_{\Sigma_A^+} \log(\psi(\omega))\, d\mu_q}{\int_{\Sigma_A^+} \log \lambda_{i_1}(\omega)\, d\mu_q},$$

where the partial derivatives are taken at the point $(q, T(q))$. By definition of ψ we have $\int_{\Sigma_A^+} \log(\psi(\omega))\, d\mu_q = -h_{\mu_q}(\sigma) < 0$. Since $0 < \lambda_k < 1$ for all k, it follows that $\int_{\Sigma_A^+} \log \lambda_{i_1}(\omega)\, d\mu_q < 0$. This implies that the function $\alpha(q) > 0$ for all q.

We first compute the pointwise dimension of the measures μ_q. Define the set

$$K_\beta = \left\{ x \in F \ : \ \lim_{n\to\infty} \frac{\log \prod_{k=1}^{n(x)} \psi(\sigma^k(\chi^{-1}(x)))}{\log \prod_{k=1}^{n(x)} \lambda_{i_k}(\chi^{-1}(x))} = \beta \right\}.$$

Proposition 1. *The pointwise dimension* $d_{\nu_q}(x) = T(q) + q\alpha(q)$ *for all* $x \in K_{\alpha(q)}$.

Proof. Consider the functions $\omega \mapsto \log(\psi(\omega))$ and $\omega \mapsto \log(\lambda_{i_1})$ where $\omega = (i_1 i_2 \dots)$. Since μ_q is ergodic, the Birkhoff ergodic theorem yields that

$$\lim_{n\to\infty} \frac{\frac{1}{n} \log \prod_{k=1}^{n} \psi(\sigma^k \omega)}{\frac{1}{n} \log \prod_{k=1}^{n} \lambda_{\omega_k}(\omega)} = \alpha(q)$$

for μ_q-almost every $\omega \in \Sigma_A^+$. Since μ is a Gibbs measure for μ_q-almost every $\omega = \chi^{-1}(x)$ we have

$$\frac{\log C_1 \prod_{k=1}^{n} \psi(\sigma^k \omega)}{\log \prod_{k=1}^{n} \lambda_{\omega_k}} \le \frac{\log \nu(\Delta_{i_1\dots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1\dots i_n}(x))} \le \frac{\log C_2 \prod_{k=1}^{n} \psi(\sigma^k \omega)}{\log \prod_{k=1}^{n} \lambda_{\omega_k}}.$$

Hence,

$$d_{\nu,C}(x) = \lim_{n\to\infty} \frac{\log \prod_{k=1}^n \psi(\sigma^k\omega)}{\log \prod_{k=1}^n \lambda_{\omega_k}} = \alpha(q)$$

for ν_q-almost every $x \in F$. It follows that $d_\nu(x) = \alpha(q)$ for ν_q-almost every $x \in F$ and hence, $\nu_q(K_{\alpha(q)}) = 1$.

Again, using the fact that μ_q is a Gibbs measure we obtain for $x \in F$, $\omega = \chi^{-1}(x)$,

$$\frac{\log C_1 \prod_{k=1}^n \lambda_{\omega_k}^{T(q)} \psi(\sigma^k\omega)^q}{\log \prod_{k=1}^n \lambda_{\omega_k}} \le \frac{\log \nu_q(\Delta_{i_1\dots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1\dots i_n}(x))} \le \frac{\log C_2 \prod_{k=1}^n \lambda_{\omega_k}^{T(q)} \psi(\sigma^k\omega)^q}{\log \prod_{k=1}^n \lambda_{\omega_k}}$$

Hence, for every $x \in K_{\alpha(q)}$,

$$d_{\nu_{q,C}}(x) = \lim_{n\to\infty} \frac{\log \prod_{k=1}^n \lambda_{\omega_k}^{T(q)} \psi(\sigma^k\omega)^q}{\log \prod_{k=1}^n \lambda_{\omega_k}} = \lim_{n\to\infty} \frac{T(q)\log \prod_{k=1}^n \lambda_{\omega_k} + q \log \prod_{k=1}^n \psi(\sigma^k\omega)}{\log \prod_{k=1}^n \lambda_{\omega_k}}$$

$$= T(q) + q d_\nu(x) = T(q) + q\alpha(q).$$

By Theorem B, $d_{\nu_q}(x) = d_{\nu_{q,C}}(x) = T(q) + q\alpha(q)$ for all $x \in K_{\alpha(q)}$. □

It immediately follows from (6) that $T(1) = 0$ and thus $\mu = \mu_1$. The first statement of Theorem 1 now follows from Proposition 1.

We now prove Statement 2 of the theorem. We first show that $\dim_H K_{\alpha(q)} = T(q) + q\alpha(q)$. This is an easy consequence of Proposition 1 and the following general statement.

Proposition 2. *Let X be a metric space of finite topological dimension and ν a Borel probability measure on X. If $K = \{x \in X \mid d_\nu(x) = \beta\}$ then $\dim_H K = \beta$.*

Proof. It follows from the famous result by Young [Y] that the Hausdorff dimension of the measure ν, $\dim_H \nu = \beta$. This immediately implies that $\dim_H K \ge \dim_H \nu \ge \beta$.

Fix $\gamma > 0$. It follows from the definition of the pointwise dimension that for any $x \in K$ there exists $\varepsilon(x) > 0$ such that

$$\nu(B(x,\varepsilon)) \ge \varepsilon^{\beta+\gamma} \tag{7}$$

for any $\varepsilon \le \varepsilon(x)$.

Define $K_r = \{x \in K \mid \varepsilon(x) = r^{-1}\}$. Clearly, $K = \cup_{r=1}^\infty K_r$. It is enough to show that $\dim_H K_r \le \beta$ for each r. Since the topological dimension of X is finite for any $\varepsilon \le r^{-1}$ there exists a cover $\mathcal{Y}$ of K_r by balls of radius ε of finite multiplicity $L > 0$. Applying (7) yields

$$L \ge \sum_{B\in\mathcal{Y}} \nu(B) \ge \sum_{B\in\mathcal{Y}} \operatorname{diam}(B)^{\beta+\gamma}.$$

It follows that $\dim_H K \ge \beta + \gamma$. Since this holds for all $\gamma > 0$ the proposition follows. □

To complete the proof of Statement 2 we need the following lemma.

Lemma 3. *The function $T(q)$ is real analytic.*

Proof. Consider the function $c\colon \mathbb{R}^2 \to C^\alpha(\Sigma_A^+, \mathbb{R})$ defined by $c(p,q) = p \log \lambda_{i_1} + q \log \psi$. This function is clearly real analytic. Since the pressure P is real analytic the desired lemma follows immediately from the Implicit Function Theorem once we verify the non-degeneracy hypothesis. The latter is that

$$\frac{\partial P(c(p,q))}{\partial p}\bigg|_{(p_0,q_0)} = \int_{\Sigma_A^+} \log \lambda_{i_1}\, d\mu_{p_0,q_0} \neq 0$$

where μ_{p_0,q_0} is the Gibbs measure for $c(p_0,q_0)$. □

Lemma 4. *For all q we have $\alpha(q) = -T'(q)$.*

Proof. Since $P(\phi_q) = 0$ for all q we have $\frac{d}{dq}P(\phi_q) = 0$. Applying the chain rule and the well known formula for the derivative of Pressure (see Appendix 2) we obtain for all q_0

$$0 = \frac{dP(\log(\lambda_{i_1}^{T(q)}\psi(\omega)^q))}{dq}\bigg|_{q_0} = \int_{\Sigma_A^+} (\log \psi(\omega) + T'(q) \log \lambda_{i_1})\, d\mu_{q_0}.$$

The lemma follows. □

It follows from Lemma 3 and Lemma 4 that the function $\alpha(q)$ is analytic and $\alpha'(q) = -T''(q) < 0$. Hence the range of $\alpha(q)$ contains an interval.

We now prove Statement 3 of the theorem.

Lemma 5. *If P is strictly convex then $T(q)$ is strictly convex.*

Proof. The function

$$\frac{d^2T(q)}{dq^2}\bigg|_q = -\frac{(\frac{\partial P(\phi_{q,r})}{\partial r})^2(\frac{\partial^2 P(\phi_{q,r})}{\partial q^2}) + (\frac{\partial P(\phi_{q,r})}{\partial q})^2(\frac{\partial^2 P(\phi_{q,r})}{\partial r^2})}{(\frac{\partial P(\phi_{q,r})}{\partial r})^3}$$

is clearly positive if $\frac{\partial^2 P(\phi_{q,r})}{\partial q^2} > 0$, where the partial derivatives are evaluated at the point $(q, T(q))$. □

We remark that $\frac{\partial^2 P(\phi_{q,r})}{\partial q^2} > 0$ at the point $(q, T(q))$ if and only if the measure μ is not the measure of maximal entropy. Statement 3 of the theorem follows from Statement 2, Lemma 4, and Lemma 5.

We now prove the final statement of the theorem. We remind the reader that the definition of $HP-$spectrum requires us to consider covers by balls of radius precisely r.

Since $r \leq \operatorname{diam}\Delta_r^j \leq \frac{r}{\min_k \lambda_k}$, it follows that there exists $A > 1$ such that $B(x_j, r) \subset \Delta_r^j \subset B(x_j, Ar)$. For each $\Delta_r^j \in \mathfrak{U}_r$ choose a point $z_j \in \Delta_r^j \cap F$. Then $\Delta_r^j \subset B(z_j, 2Ar)$ and $B(z_j, r) \subset B(x_j, 2Ar)$. Since ν is isotropic it follows from (1) that there exists $K > 0$ such that $\nu(B(x_j, 2Ar)) < K\nu(B(x_j, r))$. Hence

$$\nu(B(z_j, r)) \leq \nu(B(x_j, 2Ar)) \leq K\nu(B(x_j, r)) \leq K\nu(\Delta_r^j).$$

Consider the cover $\mathfrak{C}_{2Ar}$ by balls of radius $2Ar$ about the points z_j. We have

$$\inf_{\substack{F\subset\cup_i B_i\\ \mathrm{diam}(B_i)=2Ar}} \sum_i \nu(B_i)^q \le \sum_i \nu(B_i)^q \le K \sum_{\Delta_r^j\in\mathfrak{U}_r} \nu(\Delta_r^j)^q. \tag{8}$$

Let us choose an arbitrary cover $\mathfrak{Y}_r$ of the limit set F by balls $B(y_i, r)$. For each $k \in \mathbb{N}$, there exists $B(y_{j_k}, r) \in \mathfrak{Y}_r$ such that $B(y_{j_k}, r) \cap \Delta_r^k \neq \emptyset$. Recall that $\Delta_r^k \subset B(y_{j_k}, 2Ar)$. Consider the new cover of F by the balls $B(y_{j_k}, 2Ar)$. Clearly each basic set $\Delta_r^j \in \mathfrak{U}_r$ is contained in at least one element of the new cover.

Define an equivalence relation on the basic sets Δ_r^j by saying that two basic sets are equivalent if they are both contained in the same element of the new cover. Since the basic sets Δ_j^r and the elements of the new cover are all balls of radius at least r, each equivalence class contains at most L elements, where L is the Moran number. For each equivalence class ξ_k determined by some ball $B(y_{j_k}, 2Ar)$, we have

$$\sum_{\Delta_r^j\in\xi_k} \nu(\Delta_r^j)^q \le L^q \nu(B(y_{j_k}, 2Ar))^q.$$

Using Lemma 3 it follows that

$$\sum_{\mathfrak{U}_r} \nu(\Delta_r^j)^q \le L^q \sum_i \nu(B(y_{j_k}, 2Ar))^q \le L^q K' \sum_i \nu(B(y_{j_k}, r))^q = L^q K' \sum_{B\in\mathfrak{Y}_r} \nu(B)^q, \tag{9}$$

where K' is a positive constant. Statement 4 of Theorem 1 follows immediately from (6), (8), and (9). □

Remarks.

(1) To show that the $f(\alpha)$ spectrum and $HP-$spectrum form a Legendre transform pair, we exploit the fact that the measure ν is isotropic. It easily follows from (5) that the measure m is isotropic. In general, one would not expect an arbitrary equilibrium measure ν to be isotropic. This is intimately related to expanding properties of the induced map $G = \chi \circ \sigma \circ \chi^{-1}$ on the limit set F.

We remind the reader that a continuous map $G\colon X \to X$ on a compact metric space X is **expanding** if G is a local homeomorphism and (10) holds.

Since we consider Moran geometric constructions which are modeled by a subshift of finite type (Q, σ), the induced map G on the limit set is a local homeomorphism. Moreover, if one builds a Moran geometric construction modeled by an arbitrary symbolic system (Q, σ) with the induced map on the limit set being expanding, then Q must be a subshift of finite type. This follows from a result of Parry [Pa]. This is one of the reasons why we restrict our study to Moran geometric constructions modeled by subshifts of finite type.

The following result establishes the isotropic property of Gibbs measures on the limit sets for Moran geometric constructions such that the induced map G is expanding. This result is an immediate consequence of Theorem 4 in Appendix 1.

Corollary 1. *If the induced map G on the limit set F for a Moran geometric construction is expanding then any Gibbs measure on F is isotropic and hence admits a multifractal analysis established by Theorem 1.*

(2) If ν is the measure of maximal entropy (i.e., the corresponding function $\xi = 0$) the function ψ is constant, and hence, the functions $\phi_q(\omega)$ are homological to the function $T(q)\log\lambda_{i_1}$ for every q. This implies that $\nu_q = m$ for every q. Therefore, $T(q) = d = \dim_H F$ (see Theorem A). It also follows that $\alpha(q) = d$ and, hence, $f_\nu(\alpha(q)) = T(q) = d$ identically in q. Let us notice that if the ratio coefficients λ_k all coincide, the measure m is the measure of maximal entropy. Hence $f_m(\alpha(q)) = T(q) = d$ for every q.

(3) Consider geometric constructions generated by *contraction maps* (see [PW]). This means that the basic sets $\Delta_{i_1\cdots i_n}$ are given by

$$\Delta_{i_1\cdots i_n} = h_{i_1} \circ \cdots \circ h_{i_n}(\Delta)$$

where $h_1, \dots, h_p \colon \Delta \to \Delta$ are contraction maps, i.e., $\rho(h_i(x), h_i(y)) \leq L_i \rho(x, y)$ with $L_i < 1$ and $x, y \in \Delta$ (a ball in $\mathbb{R}^q$). Most of the results in the literature in dimension theory require that the process be described in this way. We stress that this is a very special case of Moran geometric construction that we consider in the present paper. One can easily see that the map G on the limit set F is expanding if the maps h_i are all invertible and their inverse are Hölder continuous. This is certainly the case if one assumes that the maps h_i are *affine* (in this case the maps h_i are called *similarity maps* and the corresponding geometric construction is *self-similar*). Thus Theorem 1 applies and yields the following statement.

Corollary 2. *Let ν be a Gibbs measure on the limit set for a geometric construction generated by contraction maps h_i which are invertible with Hölder continuous inverse. Then ν admits the multifractal analysis established by Theorem 1.*

This is a new result although a part of it (related to $f(\alpha)$ spectrum for dimensions) has been studied by several authors (see for example [CM, Ri]).

(4) In the definition of Moran geometric construction we assume that the basic sets on each step are disjoint. In fact, it readily follows from the above arguments that Theorem 1 is valid under the weaker assumption that the basic sets do not overlap (their boundaries may intersect).

(5) The assumption that the geometric construction is Moran, i.e., the basic sets are balls, is crucial. In [PW], we provide an example of a geometric construction with rectangles for which the measure m is not exact dimensional and thus the multifractal analysis can not be effected (at least for this measure which is the measure of maximal entropy).

(6) Theorem 1 is valid for any Moran geometric construction modeled by a symbolic system on which the pressure function is smooth. One of the reasons why we require the symbolic model (Q, σ) to be a subshift of finite type is that the smoothness of the pressure function is essentially known only in this case.

III: Conformal Repellers

In this section we undertake a multifractal analysis for smooth expanding maps. Let M be a smooth Riemannian manifold and $g\colon M \to M$ a smooth map. Let J be a compact subset of M and V an open neighborhood of J. We say that g is *smoothly expanding* on J and that J is a *repeller* if

(1) $J = \{x \in V \,:\, g^n x \in V \text{ for all } n \geq 0\}$,
(2) there exist $C > 0$ and $\alpha > 1$ such that $\|dg^n_x u\| \geq C\alpha^n \|u\|$ for all $x \in J$, $u \in T_x M$, and $n \geq 1$ (for some Riemannian metric on M),
(3) g is topologically transitive on J.

Note that a smoothly expanding map is expanding (see Appendix 1). It follows from the definition that $g(J) = J$.

We recall some facts about expanding maps. For simplicity we assume that the map F on J is topologically mixing (see [B]). In [B, Ru1], Bowen and Ruelle showed that for any Hölder continuous function ξ on J there exists a unique equilibrium measure $\nu = \nu_\xi$ on J. It is well known that expanding maps have Markov partitions [Ru1, Ru2] consisting of partition elements, called *rectangles*, $\{R_1, \dots, R_M\}$ of (arbitrarily small) diameter δ such that

(1) each rectangle R is the closure of its interior $\overset{\circ}{R}$,
(2) $J = \cup_i R_i$,
(3) $\overset{\circ}{R}_i \cap \overset{\circ}{R}_j = \emptyset$ unless $i = j$,
(4) each $g(R_i)$ is a union of rectangles R_j.

The Markov partition allows us to set up a complete analogy between Moran geometric constructions and repellers for expanding maps. Define the *basic sets*

$$\Delta_{i_1 \dots i_n} = R_{i_1} \cap g^{-1} R_{i_2} \cap \dots \cap g^{-n} R_{i_n}.$$

Although the basic sets of Moran geometric constructions are balls, these sets, are typically Cantor-like sets and may intersect along their boundaries (see Remark 4 at the end of the previous section). By the Markov property, every basic set $\Delta_{i_1 \dots i_n} = g^{-n}(R_{i_n}) \cap R_{i_1}$ for some branch of g^{-n}.

Let $\mathcal{R} = \{R_1, \dots, R_M\}$ be a Markov partition for (g, J). It is well-known that the Markov partition generates a symbolic model of the repeller by a subshift of finite type (Σ_A^+, σ) where A is the incidence matrix of the Markov partition. This gives a coding map $\chi\colon \Sigma_A^+ \to J$ which is Hölder continuous and injective on the set of points whose trajectories never hit the boundary of any element of the Markov partition. The pullback by χ of any Hölder continuous function on J is Hölder continuous on Σ_A^+. We need the following well-known estimate about expanding maps (see [M]).

Lemma 6. *There exist positive constants C_1, C_2 such that for $x, y \in R_k$, $k = 1, \dots, M$ and any branch of g^{-n}*

$$C_1 \leq \frac{|Jac\, g^{-n}(x)|}{|Jac\, g^{-n}(y)|} \leq C_2,$$

where $Jac\, g$ denotes the Jacobian of g.

We remark that any equilibrium measure on the repeller corresponding to a Hölder continuous function is isotropic. This follows from Theorem 4 in Appendix 1.

A smooth map g is called *conformal* if $dg_x = a(x)Isom_x$ where $Isom_x$ denotes an isometry of T_xM. A smooth conformal map g is called an expanding map if $|a(x)| > 1$ for all points x. The repeller J for a conformal expanding map g is called a *conformal repeller*. Examples of conformal repellers include one-dimensional Markov maps and hyperbolic Julia sets (see below). Ruelle [Ru2] showed that the Hausdorff dimension d of a conformal repeller J is given by *Bowen's formula* $P(-d\log|a|) = 0$ and that the $d-$Hausdorff measure is equivalent to the Gibbs measure m corresponding to $-d\log|a|$.

Let ξ be a Hölder continuous function on J and ν the corresponding equilibrium measure for g. Denote by ϕ the pull back of ξ under the coding map and by μ the pull back of the measure ν.

Define the one parameter family of functions ϕ_q, $q \in (-\infty,\infty)$ on Σ_A^+ by $\phi_q(\omega) = T(q)\log|a(\chi(\omega))| + q\log\psi(\omega)$ where $T(q)$ is chosen such that $P(\phi_q) = 0$ and the function ψ is defined in Lemma 1. It is obvious that the functions ϕ_q are Hölder continuous.

We now state our main theorem for conformal expanding maps.

Theorem 2. *Let ν be the equilibrium measure on J corresponding to a Hölder continuous function ξ. Let μ be the pull back of ν to Σ_A^+. Then*

(1) *the $\nu-$measure of any ball centered at points in J is positive and for any $q \in \mathbb{R}$ we have*

$$T(q) = \lim_{r\to 0} \frac{\log\inf_{\mathfrak{U}_r}\sum_{B\in\mathfrak{U}_r}\nu(B)^q}{\log r},$$

where the infimum is taken over all covers $\mathfrak{V}_r$ of X by balls of radius r.

(2) *the pointwise dimension $d_\nu(x)$ exists for $\nu-$almost every $x \in J$ and*

$$d_\nu(x) = \frac{\int_{\Sigma_A^+}\log(\psi(\omega))\,d\mu}{\int_{\Sigma_A^+}\log|(\chi^*a(\omega)|)\,d\mu(\omega)}$$

where $(\chi^*a)(\omega) = a(\chi(\omega))$

(3) *if ν is not the measure of maximal entropy, then the multifractal spectrum $f_\nu(\alpha)$ is real analytic on some open interval and $f_\nu(\alpha(q)) = T(q) + q\alpha(q)$ where $\alpha(q) = -T'(q)$ is positive. The range of $\alpha(q)$ contains the number $d = T(0)$.*

(4) *if ν is not the measure of maximal entropy, then the functions $f_\nu(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair.*

Proof. The proof will be completely analogous to the proof of Theorem 1. We first establish the analog of (2) for repellers.

Lemma 7. *There exist positive constants C_3 and C_4 such that for every $x \in J$*

$$C_3 \le \frac{diam(\Delta_{i_1\ldots i_n}(x))}{\prod_{k=0}^{n} 1/|a(g^kx)|} \le C_4,$$

where $\Delta_{i_1\ldots i_n}(x)$ denotes a basic set containing x.

Proof. Since g is conformal and expanding on J we have

$$\|dg_x^n\| = \prod_{k=0}^{n}|a(g^kx)| = Jac\,g^n(x).$$

This fact and Lemma 6 imply

$$\operatorname{diam}(\Delta_{i_1\dots i_n}(x)) \le \operatorname{diam}(R_{i_n}) \max_{y\in R_{i_n}} \|dg^{-n}(y)\| = \operatorname{diam}(R_{i_n}) \max_{y\in R_{i_n}} |Jac\, g^{-n}(y)|$$

$$= \operatorname{diam}(R_{i_n}) \left(\frac{\max_{y\in R_{i_n}} |Jac\, g^{-n}(y)|}{|Jac\, g^{-n}(g^n(x))|}\right) |Jac\, g^{-n}(g^n(x))|$$

$$\le \delta C_2 \prod_{k=0}^{n} |a(g^k(x))|^{-1}.$$

Since each R_j is the closure of its interior, we have

$$\operatorname{diam}(\Delta_{i_1\dots i_n}(x)) \ge \operatorname{diam}(R_{i_n}) \min_{y\in R_{i_n}} \|dg^{-n}(y)\| = \operatorname{diam}(R_{i_n}) \min_{y\in R_{i_n}} |Jac\, g^{-n}(y)|$$

$$= \operatorname{diam}(R_{i_n}) \left(\frac{\min_{y\in R_{i_n}} |Jac\, g^{-n}(y)|}{|Jac\, g^{-n}(g^n(x))|}\right) |Jac\, g^{-n}(g^n(x))|$$

$$\ge \delta C_1 \prod_{k=0}^{n} |a(g^k(x))|^{-1}.$$

This completes the proof of Lemma 7. □

The following proposition is the key point in our analysis. It establishes the analog of the estimates in (4) and thus allows us to imitate the proof of Theorem 1.

Proposition 3. *There exist positive constants C_5, C_6 such that*

$$C_5 \le \frac{m(\Delta_{i_1\dots i_n}(x))}{diam(\Delta_{i_1\dots i_n}(x))^d} \le C_6.$$

Proof. Since ν is a Gibbs measure, there exists positive constants C_7, C_8 such that

$$C_7 \le \frac{m(\Delta_{i_1\dots i_n}(x))}{\prod_{k=0}^{n} |a(g^k x)|^{-d}} \le C_8. \tag{11}$$

Proposition 3 immediately follows from (11), Lemma 7, and Theorem 4 in Appendix 1. □

The rest of the proof of Theorem 2 uses the fact that the measure ν is isotropic and is completely analogous to the proof of Theorem 1. □

IV: Examples

Theorem 2 allows us to effect a multifractal analysis of Gibbs measures for hyperbolic rational maps, one-dimensional Markov maps, and conformal toral endomorphims. We first consider rational maps.

Let $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere. The map R, being holomorphic, is clearly conformal. The *Julia set* J_R of R is the closure of the set of repelling periodic points of R (recall that a periodic point p of period m is repelling if $|(R^m)'(p)|) > 1$). One says that R is *hyperbolic* (or that the Julia set is *hyperbolic*) if the map R is expanding on J_R, i.e., if it satisfies conditions (1)–(3) in the definition of smooth expanding map with respect to the spherical metric on $\hat{\mathbb{C}}$ [CG]). It is known that the map $z \to z^2 + c$ is hyperbolic provided $|c| < \frac{1}{4}$. It is conjectured that a dense set of rational maps are hyperbolic. Since the Julia set of a hyperbolic rational map is a conformal repeller, Theorem 2 immediately implies the following statement.

Corollary 3. *If ν is a Gibbs measure for a hyperbolic rational map then Statements 1–3 of Theorem 2 hold.*

We now consider one-dimensional Markov maps. Let g be a *Markov map* of the interval $I = [0,1]$. This means that there exists a finite family $I_1, I_2, \dots I_M \subset I$ of disjoint open intervals such that

(1) for every j, there is a subset $K = K(j)$ of indices with $g(I_j) = \cup_{k \in K} I_k \mod 0$,
(2) for every $x \in \cup_j I_j$, the derivative of g exists and satisfies $|\,g'(x)\,| \geq \alpha$ for some fixed $\alpha > 0$,
(3) there exists $\lambda > 1$ and $n_0 > 0$ such that if $g^m(x) \in \cup_j I_j$, for all $0 \leq m \leq n_0 - 1$ then $|(g^{n_0})'(x)| \geq \lambda$.

Let $J = \{x \in I \,|\, g^n(x) \in \cup_{k=1}^M I_j \text{ for all } j \in \mathbb{N}\}$. The set J is a repeller for the map g. It is conformal because the domain of g is one-dimensional. Hence Theorem 2 immediately implies the following statement.

Corollary 4. *If ν is a Gibbs measure for a Markov map then Statements 1–3 of Theorem 2 hold.*

In [Ra], Rand carried out a multifractal analysis of Gibbs measures for a *Cookie-Cutter map.* This is a special case of a Markov map where one has only two subintervals which get mapped onto I under g.

Another example of a conformal expanding map is a conformal toral endomorphism defined by a diagonal matrix $(m, \dots, m)$ where m is an integer and $|m| > 1$.

Corollary 5. *If ν is a Gibbs measure for a conformal toral endomorphism, then Statements 1–3 of Theorem 2 hold.*

V: Appendix 1

The Isotropic Property of Equilibrium Measures for Continuous Expanding Maps

Let X be a compact metric space with metric ρ. We say that a continuous map $g\colon X \to X$ is **expanding** if g is a local homeomorphism and there exist constants $B \geq A > 1$ and $r_0 > 0$ such that

$$B(g(x), A\,r) \subset g(B(x,r)) \subset B(g(x), B\,r)$$

for every $x \in X$ and $0 < r < r_0$.

Without loss of generality we may assume that for any $x \in X$, the map g restricted to the ball $B(x, r_0)$ is a homeomorphism.

We recall that a *Markov partition* for an expanding map $g\colon X \to X$ is a finite cover of X by elements, called *rectangles*, $\{R_1, \dots, R_M\}$ such that:

(1) each rectangle R is the closure of its interior $\overset{\circ}{R}$,
(2) $\overset{\circ}{R}_i \cap \overset{\circ}{R}_j = \emptyset$ unless $i = j$,
(3) each $g(R_i)$ is a union of rectangles R_j.

We construct a special Markov partition for an expanding map such that the rectangle containing a given point in X is *almost* a ball. Let $R(x)$ denote the rectangle in $\mathcal{R}$ that contains the point x.

Theorem 3. *There are positive constants C_1, C_2, and a positive integer k such that for any $0 < r \leq r_0$ and any $x \in X$, there exists a Markov partition $\mathcal{R}_x = \{R_1, \dots, R_M\}$ for the map g^k such that $diam(R_i) \leq C_2 r$ for all $i = 1, \dots, M$ and $B(x, C_1 r) \subset R(x)$.*

Proof. Let $k > 1$ be an integer which we will specify later. Fix a point $x \in X$ and choose r such that $10\, r < \frac{1}{B^k} r_0$. Let us also choose a finite cover $\mathcal{B}^0$ of X by balls $B_i^0 = B(x_i, r)$ such that $x_1 = x$ and

$$\bigcup_{i \geq 2} B_i^0 \cap B(x, \frac{3r}{4}) = \emptyset.$$

Given i, consider a cover $\mathcal{C}_i^0$ of the set $g^k(B_i^0)$ by balls $B_j^0 \in \mathcal{B}^0$. Let

$$B_i^1 = \bigcup_{B_j^0 \in \mathcal{C}_i^0} g^{-k}(B_j^0).$$

Lemma 1. *We have $B_i^1 \subset B(x_i, r + 2\, A^{-k}\, r)$.*

Proof of Lemma 1. Consider a ball $B_j^0 \in \mathcal{C}_i^0$ and a point $y \in B_j^0 \setminus g^k(B_i^0)$. Choose $z \in B_j^0 \cap g^k(B_i^0)$. Clearly the distance $\rho(z, y) \leq 2r$. By (10), $\rho(g^{-k}y, g^{-k}z) \leq A^{-k} 2r$. The lemma follows since $g^{-k}z \in B_i^0$. □

Consider the cover $\mathcal{B}^1$ of X by sets $\{B_i^1\}$. Given i, we have

$$g^k(B_i^1) = \bigcup_{B_j^0 \in \mathcal{C}_i^0} B_j^0. \tag{12}$$

Let $\mathcal{C}_i^1$ be the cover of the set $g^k(B_i^1)$ by sets $B_j^1 \in \mathcal{B}^1$ with $B_j^0 \in \mathcal{C}_i^0$. Set

$$B_i^2 = \bigcup_{B_j^1 \in \mathcal{C}_i^1} g^{-k}(B_j^1).$$

Lemma 2. *We have* $B_i^2 \subset B(x_i, r + 2A^{-k}r + 2A^{-2k}r)$.

Proof of Lemma 2. Consider a set $B_j^1 \in \mathcal{C}_i^1$ and a point $y \in B_j^1 \setminus g^k(B_i^1)$. Choose $z \in B_j^1 \cap g^k(B_i^1)$. Clearly the distance $\rho(z, y) \le 2r$. By (12) and Lemma 1, we have $\rho(y, z) \le 2A^{-k}r$. The lemma follows. $\square$

By induction we construct covers $\mathcal{B}^n = \{B_i^n\}$, $n = 2, 3, \dots$ with the following properties:

(1) $g^k(B_i^n) = \bigcup_{B_j^0 \in \mathcal{C}_i^0} B_j^{n-1}$
(2) $B_i^n \subset B(x_i, r + 2r \sum_{l=1}^n A^{-lk})$.

We consider the cover $\mathcal{B}^\infty$ which consists of the sets

$$B_i^\infty = \bigcup_{n=0}^{\infty} B_i^n.$$

Lemma 3. *For k sufficiently large we have*

(1) $g^k(B_i^\infty) = \bigcup_{B_j^0 \in \mathcal{C}_i^0} B_j^\infty$
(2) $B_i^\infty \subset B(x_i, r(1 + \frac{2A^{-k}}{1-A^{-k}}))$
(3) *For sufficiently large k the set* $B_i^\infty \subset B(x_i, \frac{5}{4}r)$
(4) $\bigcup_{i \ge 2} B_i^\infty \cap B(x, \frac{r}{4}) = \emptyset$.

Proof of Lemma 3. The first two statements are an immediate consequence of the above properties (1) and (2) of covers $\mathcal{B}^n$. Statements (3) and (4) follow directly from them. $\square$

The first statement of Lemma 3 means that the cover $\mathcal{B}^\infty$ is a *Markov cover*, i.e., its elements satisfy properties (1) and (3) in the definition of Markov partition. We will cut elements of this Markov cover to obtain the desired Markov partition.

Given $y \in X$ let $s(y) = (i_1 \dots i_n)$ be the set of integers such that $y \in B_{i_j}^\infty$. Set

$$R(y) = \bigcap_{i_j \in s(y)} B_{i_j}^\infty.$$

Lemma 4.

(1) *For every $y \in X$ the set $R(y)$ is open.*
(2) *If $z \in R(y)$ then $R(z) \subset R(y)$.*
(3) *If $z \notin R(y)$ then $R(z) \cap R(y) = \emptyset$.*
(4) *For every $z \in X$ we have $R(g^k(z)) \subset g^k(R(z))$.*

Proof of Lemma 4. The first statement is obvious since the sets B_i^∞ are open. Now assume that $z \in R(y)$. Then $z \in B_{i_j}^\infty$ for every $i_j \in s(y)$ and $s(y) \subset s(z)$. Hence

$$R(z) = \bigcap_{i_j \in s(z)} B_{i_j}^\infty \subset \bigcap_{i_j \in s(y)} B_{i_j}^\infty.$$

Now assume that $z \notin R(y)$. If there exists $w \in R(z) \cap R(y)$ then by Statement 2 we have $R(z) \subset R(y)$. Thus $z \in R(y)$ and we obtain a contradiction.

To prove the last statement, consider a point $z \in R(z) = \bigcap_{i_j \in s(z)} B_{i_j}^{\infty}$. Then $g^k(z) \in \bigcap_{i_j \in s(z)} g^k(B_{i_j}^{\infty})$. By Statement 1 of Lemma 3, $g^k(z) \in \bigcap_{i_l \in s(g^k(z))} B_{i_l}^{\infty}$ and hence

$$R(g^k(z)) \subset \bigcap_{i_l \in s(g^k(z))} B_{i_l}^{\infty} \subset \bigcap_{i_j \in s(z)} g^k(B_{i_j}^{\infty}).$$

This completes the proof of Lemma 4. □

Lemma 4 implies that there exists a cover $\mathcal{R}_x$ of X by closed sets $\{R_1, \dots, R_M\}$ and an integer $1 \le k \le M$ satisfying

(1) For every $1 \le j \le k$ and every $z \in \overset{\circ}{R}_j$ we have $R_j = \overline{R(z)}$.
(2) For every $k+1 \le j \le N$ there exist finitely many points $y_{j_l} \in X$ such that for every $z \in \overset{\circ}{R}_j$, we have $R_j = \overline{R(z)} \setminus \cup_l R(y_{j_l})$.

We claim that the cover $\mathcal{R}_x$ is a Markov partition for g^k. We need only check Property 3 in the definition of Markov partition since the other properties follow from Lemmas 3 and 4.

Given a set R_i and a point $z \in \overset{\circ}{R}_i$ assume that $g^k(z) \in \overset{\circ}{R}_j$.

If $1 \le i \le k$ then $R_i = R(z)$. By Statement 4 of Lemma 4 we have that $R(g^k(z)) \subset g^k(R_i)$. Since $R_j \subset R(g^k(z)$, this implies the Markov property.

If $k+1 \le i \le N$ then $R_i = \overline{R(z)} \setminus \bigcup_l R(y_{i_l})$. By Statement 4 of Lemma 4 we have that $R(g^k(z)) \subset g^k(R(z))$. Since $R_j \subset R(g^k(z))$, this implies that $R_j \subset g^k(R(z))$. Applying an appropriate branch of the inverse map g^{-k} we have that $g^{-k}R_j \subset R(z)$. Assume that there is a point $w \in g^{-k}R_j$ which does not belong to R_i. Then $w \in R(y_{i_l})$ for some $y_{i_l} \in X$. This implies that $g^k(w) \in R_j$ and hence $R_j \subset R(g^k(w))$. By Statement 4 of Lemma 4 we have

$$g^{-k}(R_j) \subset g^{-k}(R(g^k(w))) \subset R(w) \subset R(y_{i_l}).$$

This is impossible since $z \in g^{-k}(R_j)$ and the Markov property has been verified.

It follows directly from Statement 4 of Lemma 3 that the Markov partition $\mathcal{R}_x$ has the desired property with respect to the given point x. □

We use the special Markov partition constructed in Theorem 3 to prove the following statement.

Theorem 4. *Let ϕ be a Hölder continuous function on X. Then any equilibrium measure for ϕ with respect to g is isotropic.*

Proof. Given $x \in X$ and a number $r > 0$ consider the Markov partition $\mathcal{R}_x$ for the map g^k constructed in Theorem 3 with respect to the point x and number $A\,r/C_1$. We have that

$$B(x, Ar) \subset R(x) \subset B(x, AC_2 r/C_1). \tag{13}$$

Define $\omega = (i_1 i_2 \dots)$ such that $\chi(\omega) = x$ where χ is the coding map. Since the diameter of every rectangle does not exceed $AC_2 r/C_1$, it follows from (12) and (13) that there exists a number n independent of r and x such that

$$\Delta_{i_1 \dots i_n}(x) \subset B(x, r).$$

This implies that

$$\Delta_{i_1\ldots i_n}(x) \subset B(x,r) \subset B(x,Ar) \subset R(x).$$

Let $\tilde{\phi}$ be the pull back of the function ϕ by the coding map χ and let μ be the pull back of the equilibrium measure ν corresponding to the function ϕ. It follows from the definition of Gibbs measures (see Appendix 2 (5)) that

$$\nu(B(x,Ar)) \le \nu(R(x)) \le D_1 \exp(-P(\tilde{\phi}) + \phi(x))$$

$$\le D_1 D_2 \frac{\exp(-P(\tilde{\phi}) + \phi(x))}{\exp(-nP(\tilde{\phi}) + \sum_{k=0}^{n-1} \tilde{\phi}(\sigma^k \omega))} \nu(\Delta_{i_1\ldots i_n}(x)) \le K\nu(B(x,r)),$$

where K is a positive constant. $\square$

VI: Appendix 2

Facts About Pressure (see [Ru1]).

(1) Consider the *pressure function* $P\colon C(Q) \to \mathbb{R}$ defined by

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\substack{(i_1\cdots i_n) \\ \text{Q-admissible}}} \inf_{x\in\Delta_{i_1\cdots i_n}} \exp(S_n\phi(x)) \right),$$

where $S_n\phi(x) = \sum_{i=0}^{n-1} \phi(\sigma^i x)$. The value $P(\phi)$ is called the *topological pressure* of ϕ.

(2) *Variational Principle:* Let $\phi \in C(Q)$. Then

$$P(\phi) = \sup_{\mu\in\mathfrak{M}(Q)} \left(h_\mu(\sigma) + \int_Q \phi\, d\mu \right),$$

where $\mathfrak{M}(Q)$ denotes the set of shift-invariant probability measures on Q. A Borel probability measure $\mu = \mu_\phi$ on Q is called an *equilibrium measure* for the potential ϕ if

$$P(\phi) = h_\mu(\sigma) + \int_{\Sigma_A^+} \phi\, d\mu.$$

(3) The pressure function $P\colon C^\alpha(\Sigma_A^+, \mathbb{R}) \to \mathbb{R}$ is real analytic. We remark that this result may not be true if Σ_A^+ is replaced by an arbitrary symbolic system.

(4) Let $\phi \in C^\alpha(\Sigma_A^+, \mathbb{R})$. The map $\mathbb{R} \to \mathbb{R}$ defined by $t \to P(t\phi)$ is convex. It is strictly convex unless ϕ is cohomologous to a constant, i.e., there exists $C > 0$ and $g \in C^\alpha(\Sigma_A^+, \mathbb{R})$ such that $\phi(x) = g(\sigma x) - g(x) + C$.

(5) Let $\phi \in C(Q)$. A Borel probability measure $\mu = \mu_\phi$ on Q is called a *Gibbs measure* for the potential ϕ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \le \frac{\mu\{y : y_i = x_i,\ i = 0, \cdots, n-1\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k x))} \le D_2$$

for all $x = (x_1 x_2 \cdots) \in \Sigma_A^+$ and $n \geq 0$. For subshifts of finite type, Gibbs measures exist for any Holdër continuous potential ϕ, are unique, and coincide with the equilibrium measure for ϕ.

(6) For $f, g \in C^\alpha(\Sigma_A^+, \mathbb{R})$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} P(f + \varepsilon g) = \int_{\Sigma_A^+} g \, d\mu_f$$

where μ_f denotes the Gibbs measure for the potential g.

Facts About The Legendre Transform (see [A]).

Let f be a C^2 strictly convex map on an interval I, hence $f''(x) > 0$ for all $x \in I$. The *Legendre transform* of f is the function f of a new variable p defined by

$$g(p) = \max_{x \in I}(p\,x - f(x)).$$

It is easy to show that g is strictly convex and that the Legendre transform is involutive. One can also show that strictly convex functions f and g form a Legendre transform pair if and only if $g(\alpha) = f(q) + q\alpha$, where $\alpha(q) = -f'(q)$ and $q = g'(\alpha)$. See [A] for more details.

References

[A] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Verlag, 1978.

[B] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, SLN # 470, Springer Verlag, 1975.

[CM] R. Cawley and R. D. Mauldin, *Multifractal Decompositions of Moran Fractals*, Advances in Math. **92** (1992), 196–236.

[C] C. D. Cutler, *Connecting Ergodicity and Dimension in Dynamical Systems*, Ergod. Th. and Dynam. Systems **10** (1990), 451–462.

[CG] L. Carleson and T. Gamelin, *Complex Dynamics*, Springer Verlag, 1993.

[CLP] P. Collet, J. L. Lebowitz, and A. Porzio, *The Dimension Spectrum of Some Dynamical Systems*, J. Stat. Physics **47** (1987), 609–644.

[HP] H. G. E. Hentschel and I. Procaccia, *The Infinite Number of Generalized Dimensions of Fractals and Strange Attractors*, Physica 8D (1983), 435–444.

[HJKPS] Thomas C. Halsey, M. Jensen, L. Kadanoff, I. Procaccia, B. Shraiman, *Fractal Measures and Their Singularities: The Characterization of Strange Sets*, Physical Review A, 33, N2 (1986), 1141-1151.

[K] A. Katok, *Bernoulli diffeomorphisms on surfaces.*, Ann. of Math. **110** (1979).

[Le] F. Ledrappier, *Proceedings of the Conference on Ergodic Theory and Related Topics*, vol. 11, Teubner, 1986, pp. 116–124.

[Lo] A. Lopes, *The Dimension Spectrum of the Maximal Measure*, SIAM J. Math. Analysis **20** (1989), 1243–1254.

[LM] F. Ledrappier and M. Misiurewicz, *Dimension of Invariant Measures for Maps with Exponent Zero*, Ergod. Th. and Dynam. Systems **5** (1985), 595–610.

[M] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer Verlag, 1987.

[Mo] P. Moran, *Additive Functions of Intervals and Hausdorff Dimension*, Proceedings of the Cambridge Philosophical Society **42** (1946), 15–23.

[Pa] W. Parry, *Symbolic Dynamics and Transformations of the Unit Interval*, Trans. AMS **122** (1966), 368–378.

[PP] W. Parry and M. Pollicott, *Zeta Functions and the Periodic Orbit Structures of Hyperbolic Dynamics*, Astérisque **187–188** (1990).

[Pe1] Y. Pesin, *On rigorous mathematical definition of correlation dimension and generalized spectrum for dimensions*, Journal of Stat. Phys. **7** (1993).

[Pe2] Y. Pesin, *Dimension Theory in Dynamical Systems. Rigorous Results and Applications*, to be published by the Chicago University Press, 1995.

[PT] Y. Pesin and A. Tempelman, *Correlation Dimension of Measures Invariant Under Group Actions*, to be published in Random and Computational Dynamics (1995).

[PW] Y. Pesin and H. Weiss, *On the Dimension of Deterministic and Random Cantor-like Sets*, Math. Research Letters **1** (1994), 519–529.

[PY] Y. Pesin and C. B. Yue, *Hausdorff Dimension of Measures with Non-zero Lyapunov Exponents and Local Product Structure*, PSU preprint (1993).

[Ra] A. Rand, *The Singularity Spectrum for Cookie-Cutters*, ETDS **9** (1989), 527–541.

[Ri] R. Riedi, *An improved multifractal formalism and self-similar measures*, ETH Preprint (1994).

[Ru1] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, 1978.

[Ru2] D. Ruelle, *Repellers for Real Analytic Maps*, ETDS **2**, 99–107.

[Sh] M. Shereshevsky, *A Complement to Young's Theorem on Measure Dimension: The Difference Between Lower and Upper Pointwise Dimension*, Nonlinearity **4** (1991), 15–25.

[Si] D. Simpelaere, *Dimension Spectrum of Axiom A Diffeomorphisms. II. Gibbs Measures*, Journal of Stat. Phys. **76** (1994), 1359–1375.

[Y] L. S. Young, *Dimension, Entropy, and Lyapunov Exponents*, Ergod. Th. and Dynam. Systems **2** (1982), 109–124.

Yakov Pesin
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802
U.S.A.
Email: pesin@math.psu.edu

Howard Weiss
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802
U.S.A.
Email:weiss@math.psu.edu