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Bounds on the mass gap of the ferromagnetic XXZ chain

Bruno Nachtergaele
Department of Physics
Princeton University
Princeton, NJ 08544-0708, USA
E-mail: bxn@math.princeton.edu

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Abstract

We prove rigorous lower and upper bounds for the mass gap of the ferromagnetic spin 1/2 XXZ chain. The two bounds coincide asymptotically in the Ising limit $\Delta \to \infty$. Near the isotropic point, $\Delta = 1$, the estimates are good enough to determine the critical behaviour of the mass gap unambiguously. The derivation does not rely on exact solutions.

Keywords: quantum spin chains, Heisenberg model, XXZ chain, mass gap

1 Introduction

The main purpose of this paper is to establish rigorous upper and lower bounds on the spectral gap of the one-dimensional ferromagnetic spin 1/2 XXZ Heisenberg chain. See below in Theorem 1.1 for the explicit formulae. The upper bound is generally assumed to be the exact result. The lower bound has essentially the same behaviour and is reasonably close to the upper bound. It is good enough to determine unambiguously the critical exponent α of the model. We hope to make clear that our method relies only on certain properties of the ferromagnetic XXZ chain and not on the exact solution. We expect that the method will be useful in the study of other models that share the same general properties without being exactly soluble. In fact, similar ideas were already applied to a class of quantum spin chains with discrete symmetry breaking in [1]. We refer the reader to Section 3 for a discussion of the method and further references.

The ferromagnetic regime of the XXZ chain has not been studied as extensively as the antiferromagnetic and the critical regimes. One could think that this is because it is less interesting or less challenging. We think this is only partly correct. In fact, the literature makes it very clear that the model is not so well understood as one might deduce from the fact that it is "Bethe Ansatz soluble". The more careful practitioners of the art do not neglect to point out that the validity of the Bethe Ansatz solution relies on the so-called string hypothesis, which remains unproven till now. In fact, it is known that the string hypothesis cannot universally hold, i.e., it is violated for some finite chains [2].

Proofs of the completeness of the Bethe Ansatz based on combinatorial arguments (counting the number of solutions of the Bethe equations) always assume the string hypothesis [3]. The proof of Yang and Yang [4] of the validity of the Bethe Ansatz for the ground state of the XXZ chain only covers the range $-\infty < \Delta < -1$, which is the complement of the regime studied in the present work. For the case $\Delta = 1$ a form of completeness in the thermodynamic limit was shown in [5].

In view of the general uniqueness theorem [6] for the Gibbs state of one-dimensional quantum spin models with translation invariant finite range interactions, there should be no surprises in the finite temperature behaviour of the XXZ chain. Yet, there are still unresolved questions about the low temperature behaviour of the specific heat of the model in the ferromagnetic region (See [7] and [8, Chapter 6] for a discussion.)

These unresolved questions clearly demonstrate the need for rigorous arguments. The arguments presented in this paper do not address all of them, but they provide unambiguous information on the behaviour of the mass gap.

The XXZ-Hamiltonian for a finite chain of L sites, including the special boundary conditions that we consider, is

$$H_L^{XXZ} = A(\Delta)(S_L^3 - S_1^3) - \sum_{x=1}^{L-1} \frac{1}{\Delta}(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2) + S_x^3 S_{x+1}^3 \tag{1.1}$$

where $S_x^\alpha, \alpha = 1, 2, 3$, are the usual 2×2 spin matrices (with eigenvalues $\pm 1/2$) acting on the site x, and $A(\Delta) = \pm\frac{1}{2}\sqrt{1 - 1/\Delta^2}$. The two Hamiltonians corresponding to the positive and negative choice of $A(\Delta)$ are obviously unitarily equivalent by left-right symmetry. Unless explicitly mentioned we will always refer to the Hamiltonian with the positive choice for $A(\Delta)$. The boundary conditions and normalization of (1.1) are natural for the following reasons. First of all they make the ground state degeneracy equal to $L + 1$ *for all* $\Delta \geq 1$. This property can be explained in terms of a quantum group symmetry that the Hamiltonian, with these particular boundary terms included, possesses [9]. The normalization is such that one can consider the limit $\Delta \to \infty$ without difficulty. In this limit the model becomes the ferromagnetic Ising chain with a boundary term that allows for ground states with a kink, i.e., for any site x in the finite chain the configuration with all spins to the left of x *up* ($\uparrow$) and all spins to the right of x *down* ($\downarrow$), is a ground state. Obviously there are $L - 1$ of such kink states. Together with the two translation invariant configurations this yields $L+1$ ground states. As we will see in Section 4 the boundary terms also make the computation of the GNS Hamiltonians of the infinite chain immediate.

In the isotropic limit ($\Delta = 1$) the $L + 1$-fold degeneracy is the dimension of the spin $L/2$ representation of SU(2). Note that the boundary terms vanish for $\Delta = 1$. In the thermodynamic limit ($L \to \infty$) all translation invariant ground states are states of perfectly aligned spins. No non translation invariant ground states are known.

If $\Delta > 1$, there are four different classes of known ground states of the model on the infinite chain, which could be called *up, down, kink*, and *antikink*. They consist, respectively, of the state with all spins $\uparrow$, the state with all spins $\downarrow$, an infinite number of states in which the spins are $\uparrow$ at $-\infty$ and $\downarrow$ at $+\infty$, and an infinite number of states in which the spins are $\downarrow$ at $-\infty$ and $\uparrow$ at $+\infty$. The infinite degeneracy of the ground state in the latter two sectors corresponds to the possible choices for the location of the kink or antikink, i.e., the location on the chain where the spins turn over form up to down in the first and down to up in the second case. The kinks are strictly speaking located at a single bond only in the Ising limit ($\Delta \to \infty$). For $1 < \Delta < \infty$ the ground states are not described by a single configuration because of the quantum fluctuations, but the kinks, respectively antikinks, are quasilocalized for all $1 < \Delta < \infty$. We refer to Section 2 for more explicit properties of the ground states.

Our main results are the upper and lower bounds on the gap in the spectrum of the Hamiltonian in the thermodynamic limit in the aforementioned four superselection sectors. Each of these sectors corresponds to a different representation of the observable algebra of the system. In these representations the Heisenberg dynamics of the model is generated by a densely defined self-adjoint, non-negative definite operator H. Theorem 1.1 below refers to the gap above zero in the spectrum of this operator. Alternatively, we can define the gap of the model with respect to an infinite volume ground state $\langle\,\cdot\,\rangle$ as the largest constant $\gamma \geq 0$ such that for all *local* observables X

$$\langle X^* H^3 X\rangle = \langle X^*[H^3, X]\rangle \geq \gamma\langle X^*[H^2, X]\rangle = \gamma\langle X^* H^2 X\rangle \tag{1.2}$$

By a local observable X we simply mean here a polynomial of a finite number of spin matrices. All local observables obtained from spin matrices S_x^α with site index x in a given finite subset

Λ of the chain form an algebra which is denoted by $\mathcal{A}_\Lambda$. The inequality (1.2) expresses the property that the Hamiltonian is larger than γ on its range, i.e., on the othogonal complement of the space of ground states. If the ground state is non-degenerate in the representation under consideration, the following equivalent inequality is more customary:

$$\langle X^*[H,X]\rangle \geq \gamma(\langle X^*X\rangle - |\langle X\rangle|^2) \tag{1.3}$$

In both (1.2) and (1.3) the commutator has to be interpreted as the limit of successive commutators

$$[H^p, X] = \lim_{\Lambda_p\to\infty} \cdots \lim_{\Lambda_1\to\infty} [H_{\Lambda_p}, [H_{\Lambda_{p-1}}, \cdots [H_{\Lambda_1}, X]\cdots]] \tag{1.4}$$

which, due to the fact that the interaction is of finite range, is a local observable for all local X.

Theorem 1.1 *In each of the sectors described above as* up, down, kink*, and* antikink*, the infinite volume gap γ satisfies*

$$\Gamma_{\text{low}}(\Delta) \leq \gamma \leq \Gamma_{\text{upp}}(\Delta) \tag{1.5}$$

for $\Delta > 1$ and where

$$\Gamma_{\text{upp}}(\Delta) \;=\; 1 - \Delta^{-1} \tag{1.6}$$

$$\Gamma_{\text{low}}(\Delta) \;=\; \left(1 - \sqrt{1 - \sqrt{1 - \frac{1}{\Delta^2}}}\right)^2 \tag{1.7}$$

For $\Delta = 1$, the difference γ_L between the lowest and second-lowest eigenvalue of the Hamiltonian H_L of a finite chain of length L, $L \geq 2$, satisfies

$$\frac{1}{4L^2} \leq (1 - \sqrt{1 - \frac{1}{L-1}})^2 \leq \gamma_L \leq 4\pi^2(\frac{1}{L^2} + \frac{4}{L^3}) \tag{1.8}$$

Two other parametrizations common in the literature are given by $\Delta = \cosh\lambda = (\mu + \mu^{-1})/2$. The range $1 \leq \Delta \leq +\infty$ corresponds to $0 \leq \lambda \leq +\infty$ and $1 \geq \mu \geq 0$. The parameter μ is often denoted by q. The most common Hamiltonian is Δ times H^{XXZ} defined in (1.1) (up to the boundary terms). In terms of the parameter λ the bounds of Theorem 1.1 are

$$\Delta\Gamma_{\text{upp}} \;=\; \cosh\lambda - 1 \tag{1.9}$$

$$\Delta\Gamma_{\text{low}} \;=\; \cosh\lambda \left(1 - \sqrt{1 - \tanh\lambda}\right)^2 \tag{1.10}$$

Near the isotropic ferromagnet ($\Delta = 1$) the upper and lower bound both behave linearly, with a slope 2 and 1 respectively. This fixes the critical exponent α which governs the behavior of the gap (as well as the low-temperature behaviour of the specific heat) near $\Delta = 1$ to be equal to 1, which is in agreement with the exact result of Johnson, Krinsky and McCoy [10]. Near $\Delta = +\infty$ the lower bound behaves as $1 - \sqrt{2}/\Delta$.

Figure 1 gives an idea of the difference between the bounds $\Gamma_{\rm low}$ and $\Gamma_{\rm upp}$. $\Gamma_{\rm upp}$, here derived by a variational argument in Section 4, is the exact solution given by the Bethe Ansatz [11, 12, 13]. Also for the finite volume estimates at $\Delta = 1$, there are exact expressions for the coefficient of $1/L^2$ [14].

The advantages of the approach in this paper are: 1) The method can be applied also to Hamiltonians that cannot be explicitly diagonalized; 2) As the the treatment is short, transparent, and completely rigorous, it should also deliver a better insight in the determining properties of the gap; 3) It is possible to obtain useful bounds down to the critical point (here $\Delta = 1$), whereas this is usually not possible with other rigorous methods such as, e.g., the polymer expansion technique of [15].

2 The ground states of the XXZ chain

Only these aspects of the ground states of the XXZ chain that have direct relevance to our estimates and understanding of the spectral gap of the model will be presented here. A more detailed analysis can be found in [16], and various aspect of the ground states have been discussed in the literature (see e.g. [17] and the references therein). It should be mentioned that a full analysis of the ground state problem for the infinite chain has not been achieved yet. Below we give a clear description of what is believed to be the complete set of ground states for the infinite chain. I am not aware of a rigorous proof that this is indeed the case. Loosely speaking one would obtain a description of the complete set of ground states by studying the thermodynamic limit with arbitrary boundary conditions. The difficulty is that a simple description of a sufficiently large class of boundary conditions is not available. Fortunately the statements in this article do not depend on the completeness of the set of ground states considered.

For the study of the finite chains in this section we shall employ the special boundary conditions introduced in (1.1). This choice of boundary conditions simplifies the study of the thermodynamic limit. It is also convenient to add a constant to the Hamiltonian to make the ground state energy vanish. This way, using the parameter μ, we arrive at the Hamiltonian,

$$H_L^\mu = H_L^{XXZ} + (L-1)/4 = \sum_{x=1}^{L-1} h_{x,x+1}^\mu \tag{2.1}$$

where $h_{x,x+1}^\mu$ is the orthogonal projection on the vector

$$\xi_\mu = \frac{1}{1+\mu^2}(\mu\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{2.2}$$

In terms of the spin matrices $h^{\mu}_{1,2}$ is

$$h^{\mu}_{1,2} = -\Delta^{-1}(S^1_1 S^1_2 + S^2_1 S^2_2) - S^3_1 S^3_2 + \frac{1}{4} + A(\Delta)(S^3_2 - S^3_1) \tag{2.3}$$

with A defined following (1.1). From the definition of ξ_μ it is obvious that $h^{\mu}_{x,x+1}\,|\uparrow\cdots\uparrow\rangle = 0$ for all $x = 1,\ldots,L-1$. As H^{μ}_L is the sum of the $h^{\mu}_{x,x+1}$, which are positive, this implies that the ground state energy of H^{μ}_L is zero and that $|\uparrow\cdots\uparrow\rangle$ is a ground state. For all $0 \le \mu \le 1$, the ground state space ($\equiv \ker H^{\mu}_L$) is $L+1$-dimensional. An explicit description of the $L+1$ ground states can be given in several ways.

For all $\Delta \ge 1$ the uniform states $|\uparrow\cdots\uparrow\rangle$ and $|\downarrow\cdots\downarrow\rangle$ are ground states of the XXZ chain. If $\Delta = 1$ the $L+1$-dimensional ground state space is the spin $L/2$ representation of SU(2). For all $\Delta > 1$ and $A = \frac{1}{2}\sqrt{1 - 1/\Delta^2}$, the non-uniform ground states can be thought of as kink states, which are roughly described as the Ising kinks plus quantum fluctuations. In this picture the degeneracy corresponds to the possible locations of the kink. For $A = -\frac{1}{2}\sqrt{1 - 1/\Delta^2}$ the kinks have to be replaced by antikinks, i.e., the roles of $\uparrow$ and $\downarrow$ spins have to be interchanged (or, equivalently, one can interchange left and right). We refer to [17] and [16] for more details and explicit expressions.

In the thermodynamic limit the boundary terms disappear to infinity and the left-right symmetry of the model, broken by the particular boundary terms we have introduced, must be restored. It is therefore obvious that both the kink and antikink states appear as infinite volume ground states of the model.

For our purposes the most convenient way to describe the space of ground states of a chain of length L is to introduce deformed raising and lowering operators which, together with the third component of the spin, generate the algebra (quantum group) of $\mathrm{SU}_\mu(2)$. The concrete representation of $\mathrm{SU}_\mu(2)$ is *not* left-right symmetric, and is different for the boundary terms that produce kink and antikink ground states. In fact the two mutually non-commuting representations of $\mathrm{SU}_\mu(2)$ together generate the infinite-dimensional quantum affine symmetry algebra $\widehat{sl}(2)$ that lies at the basis of the integrability of the model (see e.g. [18]). We should stress, however, that a rigorous formulation of this infinite dimensional symmetry of the XXZ chain, has not yet been obtained. We will not use it here.

In our computations we will not need anything beyond some basic facts of the representation theory of $\mathrm{SU}_\mu(2)$. We therefore restrict the discussion of the quantum group symmetry of the XXZ model to the bare minimum. One can think of the quantum group symmetry as a systematic way to construct operators that commute with the Hamiltonians H^{μ}_L. The parallellism with the usual arguments in the "theory of angular momentum" in quantum mechanics (representations of SU(2)) is so perfect that the reader will hardly notice the difference.

For $0 < \mu < 1$ define the 2×2 matrix t by

$$t = \mu^{-2S^3} \tag{2.4}$$

and define as usual $S^{\pm} = S^1 \pm iS^2$. It is trivial to check that $S^{\pm}$ and t satisfy the following commutation relations

$$tS^{\pm} = \mu^{\mp 2} S^{\pm} t \tag{2.5-a}$$

$$[S^+, S^-] = \frac{t - t^{-1}}{\mu^{-1} - \mu} = 2S^3 \tag{2.5-b}$$

They are just the SU(2) commutation relations in a disguised form. The remarkable fact is that there is a simple definition of the tensor product (coproduct of the quantum group or pseudogroup [19, 20]) of any two representations of the commutation relations (2.5-a)–(2.5-b), yielding a new representation. Here we only need the total-spin operators for a chain of L spins, which are given by

$$S^3_{[1,L]} = \sum_{x=1}^{L} \mathbb{1}_1 \otimes \cdots \otimes S^3_x \otimes \mathbb{1}_{x+1} \otimes \cdots \mathbb{1}_L \tag{2.6-a}$$

$$S^+_{[1,L]} = \sum_{x=1}^{L} t_1 \otimes \cdots \otimes t_{x-1} \otimes S^+_x \otimes \mathbb{1}_{x+1} \otimes \cdots \mathbb{1}_L \tag{2.6-b}$$

$$S^-_{[1,L]} = \sum_{x=1}^{L} \mathbb{1}_1 \otimes \cdots \otimes S^-_x \otimes t^{-1}_{x+1} \otimes \cdots t^{-1}_L \tag{2.6-c}$$

where we used an index to identify the sites on which the tensor factors act. Note that, for $L \geq 2$, the operators $S^{\pm}_{[1,L]}$ depend on μ through t. One can easily check that the total "spin" operators as defined in (2.6-a)–(2.6-c) commute with the interaction terms $h^{\mu}_{x,x+1}$ and hence with the Hamiltonian H^{μ}_L itself.

3 Estimate of the mass gap for finite chains

We begin this section with the derivation of a simple lower bound for the spectral gap of finite chains for Hamiltonians that share some of the basic properties observed in the XXZ chain (Theorem 3.2). A more general version of this estimate was given in [1] where it was used to prove the existence of a spectral gap in arbitrary Generalized Valence Bond Solid chains with a finite number of ground states. As a strategy for obtaining lower bounds for the spectral gap of the generator of a spin dynamics, the method of proof is inspired by the work of Lu and Yau [21] on the gap in the spectrum of the Glauber and Kawasaki dynamics of the Ising model. The ingredients that go into the estimate are not very different from the ones in [22] and in fact similar elements underly the arguments in [23, 24, 25].

Here, we restrict ourselves to the simplest form of this estimate, which is sufficient for the application to the XXZ chain.

Consider an arbitrary spin chain of L sites and with Hilbert space $\mathcal{H}_L = \bigotimes_{x=1}^{L}(\mathbb{C}^d)_x$, where again we use the index x to associate the tensor factors with the sites in the chain. We assume that the Hamiltonian is of the following form:

$$H_L = \sum_{x=1}^{L-1} h_{x,x+1} \tag{3.1}$$

where $h_{x,x+1}$ is a translation of $h_{1,2}$, acting non-trivially only at the nearest neighbour pair $\{x, x+1\}$. Assume furthermore that $h_{1,2} \geq 0$ and that $\ker H_L \neq \{0\}$. We will denote by γ_2 the smallest nonzero eigenvalue of $h_{1,2}$, i.e., the gap of H_2. It is obvious that

$$\ker H_L = \bigcap_{x=1}^{L-1} \ker h_{x,x+1} \tag{3.2}$$

For an arbitrary subset Λ let G_Λ be the orthogonal projection onto

$$\ker \sum_{x,\{x,x+1\}\subset\Lambda} h_{x,x+1} \tag{3.3}$$

For intervals $[a,b]$, $1 \leq a < b \leq L$, $G_{[a,b]}$ is the orthogonal projection onto the zero eigenvectors of $\sum_{x=a}^{b-1} h_{x,x+1}$, and $G_{\{x\}} = \mathbb{1}$ for all x. From these definitions it immediately follows that the orthogonal projections G_Λ satisfy the following properties:

$$G_{\Lambda_2}G_{\Lambda_1} = G_{\Lambda_1}G_{\Lambda_2} = G_{\Lambda_2} \text{ if } \Lambda_1 \subset \Lambda_2 \tag{3.4-a}$$

$$G_{\Lambda_1}G_{\Lambda_2} = G_{\Lambda_2}G_{\Lambda_1} \text{ if } \Lambda_1 \cap \Lambda_2 = \emptyset \tag{3.4-b}$$

$$h_{x,x+1} \geq \gamma_2(\mathbb{1} - G_{[x,x+1]}) \tag{3.4-c}$$

Define operators E_n, $1 \leq n \leq L$, on $\mathcal{H}_L$ by

$$E_n = \begin{cases} \mathbb{1} - G_{[1,2]} & \text{if } n = 1 \\ G_{[1,n]} - G_{[1,n+1]} & \text{if } 2 \leq n \leq L-1 \\ G_{[1,L]} & \text{if } n = L \end{cases} \tag{3.5}$$

One can then easily verify, using the properties (3.4-a)-(3.4-c), that $\{E_n \mid 1 \leq n \leq L\}$ is a family of mutually orthogonal projections summing up to $\mathbb{1}$, i.e.:

$$E_n^* = E_n, \qquad E_n E_m = \delta_{m,n} E_n, \qquad \sum_{n=1}^{L} E_n = \mathbb{1} \tag{3.6}$$

The preceding paragraph applies directly to the XXZ chain. Next, we make a non-trivial assumption which we will verify for the XXZ chain later.

Assumption 3.1 *There exists a constant* ε, $0 \leq \varepsilon < 1/\sqrt{2}$, *such that for all* $1 \leq n \leq L-1$

$$E_n G_{[n,n+1]} E_n \leq \varepsilon^2 E_n \tag{3.7}$$

or, equivalently,

$$\|G_{[n,n+1]} E_n\| \leq \varepsilon \tag{3.8}$$

Note that, due to (3.4-a), $G_{[n,n+1]}E_n = G_{[n,n+1]}G_{[1,n]} - G_{[1,n+1]}$. This relates Assumption 3.1 with Lemma 6.2 in [22], where an estimate for $\|G_{[n,n+1]}G_{[1,n]} - G_{[1,n+1]}\|$ is given for general Valence Bond Solid chains with a unique infinite volume ground state. The same observation also implies that $[G_{[n,n+1]}, G_{[1,n]}] = [G_{[n,n+1]}, E_n]$, which, if (3.8) holds, is bounded above in norm by 2ε.

The next theorem is a special case of Theorem 2.1 in [1]. Just like Theorem 6.4 in [22] it provides a lower bound on the gap of the finite volume Hamiltonians, but it achieves this in a slightly more efficient way. We will repeat the proof for the particular case stated here, because it is simple, short, and instructive.

Theorem 3.2 *With the definitions of above and under Assumption 3.1 the following estimate holds for all* ψ *satisfying* $G_{[1,L]}\psi = 0$, *i.e.,* ψ *that are orthogonal to the space of ground states of* H_L :

$$\langle \psi \mid H_L \psi \rangle \geq \gamma_2 (1 - \sqrt{2}\varepsilon)^2 \|\psi\|^2 \tag{3.9}$$

i.e., the spectrum of H_L *has a gap of at least* $\gamma_2(1-\sqrt{2}\varepsilon)^2$ *above the lowest eigenvalue, which is* 0.

proof:

¿From the properties (3.6) of the E_n and the assumption that $G_{[1,L]}\psi = 0$, it immediately follows that

$$\|\psi\|^2 = \sum_{n=1}^{L-1} \|E_n \psi\|^2 \tag{3.10}$$

One can estimate $\|E_n\psi\|^2$ in terms of $\langle \psi \mid h_{n,n+1}\psi\rangle$ as follows. First insert $G_{[n,n+1]}$ and the resolution $\{E_m\}$:

$$\|E_n\psi\|^2 = \langle \psi \mid (\mathbb{1} - G_{[n,n+1]})E_n\psi\rangle + \langle \psi \mid \sum_{m=1}^{L-1} E_m G_{[n,n+1]} E_n \psi \rangle \tag{3.11}$$

Using (3.4-a) and (3.4-b) one easily veryfies that E_m commutes with $G_{n,n+1}$ if either $m \leq n-2$ or $m \geq n+1$. In these cases $E_m G_{[n,n+1]} E_n = G_{[n,n+1]} E_m E_n = 0$, because the E_n form an orthogonal family. By this observation we obtain the following estimate. For any choice of constants $c_1, c_2 > 0$:

$$\|E_n\psi\|^2 \;=\; \langle \psi \mid (\mathbb{1} - G_{[n,n+1]})E_n\psi\rangle + \langle (E_{n-1} + E_n)\psi \mid G_{[n,n+1]} E_n \psi \rangle$$

$$\leq \frac{1}{2c_1}\langle\psi \mid (\mathbb{1} - G_{[n,n+1]})\psi\rangle + \frac{c_1}{2}\langle\psi \mid E_n\psi\rangle \qquad (3.12)$$

$$+\frac{1}{2c_2}\langle\psi \mid E_n G_{[n,n+1]} E_n\psi\rangle + \frac{c_2}{2}\langle\psi \mid (E_{n-1}+E_n)^2\psi\rangle$$

where we have applied the inequality

$$|\langle\varphi_1 \mid \varphi_2\rangle| \leq \frac{1}{2c}\|\varphi_1\|^2 + \frac{c}{2}\|\varphi_2\|^2 \quad ,$$

for any $c > 0$, to both terms of (3.11). The first term in the right side of inequality (3.12) can be estimated with the interaction using (3.4-c). The third term can be estimated with (3.7). It then follows that

$$(2 - c_1 - \frac{\varepsilon^2}{c_2})\|E_n\psi\|^2 - c_2\|(E_{n-1}+E_n)\psi\|^2 \leq \frac{1}{c_1\gamma_2}\langle\psi \mid h_{n,n+1}\psi\rangle$$

The term containing E_{n-1} is absent for $n = 1$, and $E_n\psi = 0$ if $n = L$. We now sum over n and use (3.10) to obtain

$$(2 - c_1 - \frac{\varepsilon^2}{c_2} - 2c_2)\|\psi\|^2 \leq \frac{1}{c_1\gamma_2}\langle\psi \mid H_L\psi\rangle$$

Finally put $c_1 = 1 - \varepsilon\sqrt{2}$ and $c_2 = \varepsilon/\sqrt{2}$ and one obtains the estimate (3.9) stated in the theorem.

■

We now return to the XXZ chain and prove a lower bound on the gap of the finite volume Hamiltonians (1.1) as an application of Theorem 3.2.

Proposition 3.3 *For a spin 1/2 chain of length L, the Hamiltonians*

$$H_L^{XXZ} = A(S_L^3 - S_1^3) - \sum_{x=1}^{L-1}\{\frac{1}{\Delta}(S_x^1S_{x+1}^1 + S_x^2S_{x+1}^2) + S_x^3S_{x+1}^3 - \frac{1}{4}\} \qquad (3.13)$$

with $A = \pm\frac{1}{2}\sqrt{1-1/\Delta^2}$, $\Delta \geq 1$, *have an* $L+1$*-fold degenerate ground state with eigenvalue* 0 *and their next largest eigenvalue,* γ_L*, satisfies*

$$\gamma_L \geq (1 - \sqrt{\frac{2\mu}{\mu+\mu^{-1}}})^2 = (1-\sqrt{1-\tanh\lambda})^2 = \Gamma_{\text{low}}(\Delta) \qquad (3.14)$$

where $\Gamma_{\text{low}}(\Delta)$ *is defined in (1.7). If* $\Delta = 1$ *one has the lower bound*

$$\gamma_L \geq (1 - \sqrt{1 - \frac{1}{L-1}})^2 \geq \frac{1}{4L^2} \qquad (3.15)$$

proof:
Due to the reflection (left-right) symmetry of the interaction it is clearly sufficient to consider one sign of A in (3.13), say $A \geq 0$. The proof consists in giving a constant ε for which Assumption 3.1 holds. In fact, for the model under consideration one can simply compute the quantity $\|G_{[n,n+1]}E_n\|$. The spaces on which the $G_{[a,b]}$ project are described explicitly in Section 2. Here, we consider all operators as acting on $\mathcal{H}_L$. The quantity $\|G_{[n,n+1]}E_n\|$ is equal to C_n defined for $n \geq 1$ by

$$C_n = \sup_{\substack{0\neq\psi\in\mathcal{H}_{n+1} \\ E_n\psi=\psi}} \frac{\|G_{[n,n+1]}\psi\|}{\|\psi\|} \tag{3.16}$$

It follows from (3.4-a) that $\|E_n\psi\|^2 = \|G_{[1,n]}\psi\|^2 - \|G_{[1,n+1]}\psi\|^2$. Therefore, the requirement $E_n\psi = \psi$ can be expressed as

$$G_{[1,n]}\psi = \psi, \qquad G_{[1,n+1]}\psi = 0 \tag{3.17}$$

As long as $1 \leq n \leq L-1$, C_n does not depend on L.

The representation theory of $\mathrm{SU}_\mu(2)$ is isomorphic to the one of $\mathrm{SU}(2)$ [26]. The irreducible representations can be labeled by the half-integers $s = 0, 1/2, 1, 3/2, \ldots$, and we denote them by $D^{(s)}$. We will use a subscript to indicate the set of sites on which a particular representation acts. $D^{(s)}$ is realized on $\mathcal{H}_{2s} \cong (\bigotimes \mathbb{C}^2)^{2s}$ as the space of ground states of H^μ_{2s}. These elementary facts are sufficient to determine the vectors ψ satisfying (3.17). Indeed, $G_{[1,n]}\psi = \psi$ implies that $\psi \in D^{(n/2)}_{[1,n]} \otimes D^{(1/2)}_{\{n+1\}} \subset \mathcal{H}_{n+1}$. As $D^{(n/2)} \otimes D^{(1/2)} \cong D^{((n-1)/2)} \oplus D^{((n+1)/2)}$, $G_{[1,n+1]}\psi = 0$ implies that $\psi \in D^{((n-1)/2)}$.

The ratio of norms in (3.16) is invariant under the action of $\mathrm{SU}_\mu(2)$, and is therefore the same for all $\psi \in D^{((n-1)/2)}$. One shows this just as one would for a group representation. For completeness we include a detailed argument here. $G_{[n,n+1]} = \mathbb{1} - |\xi_\mu\rangle\langle\xi_\mu|$ commutes with $S^3_{[1,n+1]}$. Therefore, it is sufficient to consider ψ that are eigenvectors of $S^3_{[1,n+1]}$. Starting form one such eigenvector we can generate all others by application of $S^\pm_{[1,n+1]}$. Assuming that $S^\pm_{[1,n+1]}\psi \neq 0$ and using the fact that $G_{[n,n+1]}$ also commutes with $S^\pm_{[1,n+1]}$ we find

$$\frac{\|G_{[n,n+1]}S^\pm_{[1,n+1]}\psi\|^2}{\|S^\pm_{[1,n+1]}\psi\|^2} = \frac{\langle\psi \mid G_{[n,n+1]}(S^\pm_{[1,n+1]})^* S^\pm_{[1,n+1]}\psi\rangle}{\langle\psi \mid (S^\pm_{[1,n+1]})^* S^\pm_{[1,n+1]}\psi\rangle} \tag{3.18}$$

As $(S^+_{[1,n+1]})^* = \mu^{-1}t_{[1,n+1]}S^-_{[1,n+1]}$ (see (2.6-a) and (2.6-b)), $(S^\pm_{[1,n+1]})^* S^\pm_{[1,n+1]}\psi$ is an eigenvector of $S^3_{[1,n+1]}$ with the same eigenvalue as ψ. This vector also belongs to the same irreducible representation and hence it must be proportional to ψ. It follows that the ratio in (3.18) is the same as for ψ.

The computation of C_n is now straightforward, for we have to consider just one vector $0 \neq \psi \in D^{((n-1)/2)}$, e.g., one with $S^3_{[1,n+1]} = (n-1)/2$. Any such vector is of the form

$$\psi = \sum_{x=1}^{n+1} a_x D_x \tag{3.19}$$

where D_x denotes the usual basis vector with all spins up except at the site x where the spin is down. Up to normalization the coefficients a_x are uniquely determined by the conditions (3.17). A possible choice is

$$\psi = \frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}} D_{n+1} - \sum_{x=1}^{n} \mu^{-x} D_x \tag{3.20}$$

A short computation shows that

$$\|\psi\|^2 = \frac{(\mu^n - \mu^{-n})(\mu^n - \mu^{-n-2})}{(\mu - \mu^{-1})^2} \tag{3.21}$$

$$\|G_{[n,n+1]}\psi\|^2 = \frac{(\mu^{n-1} - \mu^{-n+1})(\mu^n - \mu^{-n-2})}{(\mu - \mu^{-1})^2(\mu + \mu^{-1})} \tag{3.22}$$

and therefore

$$C_n = \frac{\sqrt{\mu(1 - \mu^{2n-2})}}{\sqrt{(\mu + \mu^{-1})(1 - \mu^{2n})}} \tag{3.23}$$

which is increasing in n and $C_n < 1/\sqrt{2}$ for all $n \geq 1$. We conclude that the assumptions of Theorem 3.2 are satisfied with

$$\varepsilon = C_{L-1} < \sqrt{\frac{\mu}{\mu + \mu^{-1}}} \tag{3.24}$$

This proves the theorem for $0 < \mu < 1$. For $\mu = 1$ one has to consider the limit

$$\varepsilon = \lim_{\mu \uparrow 1} C_{L-1} = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{1}{L-1}} \tag{3.25}$$

which is straightforward to compute.

■

4 The infinite chain

In order to be able to prove rigorous statements about the spectrum of the infinite chain we need to introduce the mathematical objects that define the infinite system. Although all interesting properties of the infinite chain can be expressed as results for limits of quantities defined for finite chains, the reverse is not true. Not all limits of finite chain quantities give interesting or even sensible statements about the infinite chain. By using a clean definition of the infinite system we will have no difficulty in sorting out the relevant statements about the thermodynamic limit of the XXZ chain.

Let the symbols $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$ denote the four superselection sectors of the infinite XXZ chain with $\Delta > 1$, corresponding to *up, kinks, antikinks*, and *down* respectively. We can describe the GNS Hilbert spaces [27] of these four superselection sectors as the so-called incomplete tensor products [28] $\mathcal{H}_{\alpha\beta}$, for α and $\beta =\uparrow$ or $\downarrow$, defined by

$$\mathcal{H}_{\alpha\beta} = \overline{\bigcup_{\Lambda} \left(\bigotimes_{x\in\Lambda} \mathbb{C}^2 \otimes \bigotimes_{y\in\Lambda^c} \Omega_{\alpha\beta}(y) \right)} \tag{4.1}$$

where

$$\Omega_{\alpha\beta}(y) = \begin{cases} |\alpha\rangle & \text{if } y \leq 0 \\ |\beta\rangle & \text{if } y > 0 \end{cases} \tag{4.2}$$

We also define the vectors $\Omega_{\alpha\beta}$ as the infinite product vectors

$$\Omega_{\alpha\beta} = \bigotimes_{y\in\mathbf{Z}} \Omega_{\alpha\beta}(y) \in \mathcal{H}_{\alpha\beta} \tag{4.3}$$

Let $\mathcal{A}_\Lambda$ denote the local observables acting nontrivially only on the sites in the finite set Λ. Local observables $X \in \mathcal{A}_\Lambda$ act on $\mathcal{H}_{\alpha\beta}$ in the obvious way, e.g., the spin matrices at the site x act on the x^{th} factor of the tensor product (4.1). From the definitions above it is clear that vectors ψ of the form

$$\psi = X\Omega_{\alpha\beta}, \qquad X \in \bigcup_{\Lambda} \mathcal{A}_\Lambda \tag{4.4}$$

form a dense subspace of $\mathcal{H}_{\alpha\beta}$. Note that if $\alpha \neq \beta$, $\Omega_{\alpha\beta}$ is *not* the GNS vector representing one of the kink (or antikink) ground states.

The mass gap of the infinite chain is a property which is defined with respect to a particular ground state of the infinite system or, more precisely, with respect to a superselection sector. The Hamiltonian is represented on $\mathcal{H}_{\alpha\beta}$ as the generator $H_{\alpha\beta}$ of the Heisenberg dynamics of observables acting on $\mathcal{H}_{\alpha\beta}$. The dense subspace of the vectors ψ defined in (4.4) is in the domain of $H_{\alpha\beta}$, and the selfadjoint operator $H_{\alpha\beta}$ is uniquely determined by the requirement

$$H_{\alpha\beta} X\Omega = \lim_{\Lambda\to\mathbf{Z}} [H_\Lambda^{XXZ}, X]\Omega_{\alpha\beta} \tag{4.5}$$

We remark that $H_{\alpha\beta}$ does not depend on boundary terms such as $A(S_a^3 - S_b^3)$ added to the XXZ Hamiltonian for finite chains. It is well-known [27] that $H_{\alpha\beta}$ is a positive operator in general and in the present case this could not be more clear. An explicit formula for $H_{\alpha\beta}$ is

$$H_{\alpha\beta} X\Omega_{\alpha\beta} = \sum_{\{x,x+1\}\cap\Lambda\neq 0} h_{x,x+1}^{\alpha\beta} X\Omega_{\alpha\beta} \tag{4.6}$$

where $h_{x,x+1}^{\alpha\beta}$ can be taken to be $h_{x,x+1}^{\mu}$ if $\alpha\beta =\uparrow\uparrow, \downarrow\downarrow$, or $\uparrow\downarrow$. If $\alpha\beta =\downarrow\uparrow$ the sign of the boundary term has to be reversed. This is equivalent to replacing μ by μ^{-1}.

The mass gap $\gamma_{\alpha\beta}$ is then just the gap above 0 in the spectrum of $\mathcal{H}_{\alpha\beta}$. A formula for $\gamma_{\alpha\beta}$ is

$$\gamma_{\alpha\beta} = \inf_{\substack{0\neq\psi\perp\ker H_{\alpha\beta}\\ \psi\in\mathrm{dom}\, H_{\alpha\beta}}} \frac{\langle\psi \mid H_{\alpha\beta}\psi\rangle}{\langle\psi \mid \psi\rangle} \tag{4.7}$$

As $(\ker H_{\alpha\beta})^{\perp} = \mathrm{ran}\, H_{\alpha\beta}$ the infimum in (4.7) can be taken over vectors of the form $H_{\alpha\beta}\psi$, and it suffices to take ψ of the form (4.4) because they are a core for $H_{\alpha\beta}$.

There is no a priori reason why the spectrum of $H_{\alpha\beta}$ should be independent of the superselection sector, i.e. independent of $\alpha\beta$. We already know that the multiplicity of the lowest eigenvalue is different: it is 1 for $H_{\uparrow\uparrow}$ and $H_{\downarrow\downarrow}$ and infinite for $H_{\uparrow\downarrow}$ and $H_{\downarrow\uparrow}$. Therefore, a priori, we should not expect $\gamma_{\alpha\beta}$ to be independent of $\alpha\beta$. One can easily convince oneself, however, that $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}$ and that $\gamma_{\uparrow\uparrow} = \gamma_{\downarrow\downarrow}$. From a simple argument given in Section 4.2 it follows that $\gamma_{\alpha\beta} \leq \gamma_{\uparrow\uparrow}$. The upper and lower bounds that we will derive here are independent of $\alpha\beta$.

4.1 Proof of the lower bound of Theorem 1.1

In order to prove the lower bound of (1.5) we simply have to show that the lower bound on the finite volume gap obtained in Section 3 remains valid in the thermodynamic limit, irrespective of the particular zero energy ground state that we are considering. It is important that the finite volume gap estimates were obtained for the "correct" boundary conditions of (1.1). More explicitly we show that for any choice of $\alpha\beta$ and all local observabels X the following inequality holds:

$$\langle\Omega_{\alpha\beta} \mid X^* H_{\alpha\beta}^3 X\Omega_{\alpha\beta}\rangle \geq \Gamma_{\mathrm{low}}(\Delta)\langle\Omega_{\alpha\beta} \mid X^* H_{\alpha\beta}^2 X\Omega_{\alpha\beta}\rangle \tag{4.8}$$

If $\alpha = \beta$, $\Omega_{\alpha\beta}$ is the vector representing the unique ground state of $H_{\alpha\beta}$. If $\alpha \neq \beta$, $\Omega_{\alpha\beta}$ is not a ground state itself (except in the Ising limit $\Delta = \infty$). But all the kink (if $\alpha\beta = \uparrow\downarrow$) or antikink states (if $\alpha\beta = \downarrow\uparrow$) are represented as vectors in the Hilbertspace $\mathcal{H}_{\alpha\beta}$ defined by (4.1), and together these vectors span $\ker H_{\alpha\beta}$. A proof of these statements can be found in [16].

The inequality (4.8) follows from Proposition 3.3 when one observes that for $X \in \mathcal{A}_\Lambda$

$$\langle\Omega_{\alpha\beta} \mid X^* H_{\alpha\beta}^3 X\Omega_{\alpha\beta}\rangle = \langle\Omega_{\alpha\beta} \mid X^* (H_{\Lambda\pm 3}^{\mu})^3 X\Omega_{\alpha\beta}\rangle \tag{4.9}$$

Obviously, $X^*(H_{\Lambda\pm3}^{\mu})^3 X \in \mathcal{A}_{\Lambda\pm3}$. Therefore the expectation value in the right side of (4.9) can be computed in the density matrix $\rho_{\Lambda\pm3}$ which describes the state $\Omega_{\alpha\beta}$ in the finite volume $\Lambda \pm 3$. The same is true for the right side of (4.8). We conclude that it is sufficient to ascertain that

$$\mathrm{Tr}\, \rho_{\Lambda\pm3} X^* (H_{\Lambda\pm3}^{\mu})^3 X \geq \Gamma_{\mathrm{low}}(\Delta)\, \mathrm{Tr}\, \rho_{\Lambda\pm3} X^* (H_{\Lambda\pm3}^{\mu})^2 X \tag{4.10}$$

which immediately follows from the finite volume gap estimate of Proposition 3.3.

■

4.2 Proof of the upper bound of Theorem 1.1

First we argue that it suffices to prove the upper bound for $H_{\uparrow\uparrow}$. It is obvious that the gap of $H_{\downarrow\downarrow}$ will satisfy the same bound. For the gap of the model in the kink and antikink sectors we have an inequality which can be derived as follows. The translation invariant ground states can be obtained as weak limits of the kink or antikink states by letting the position of the kink (or antikink) tend to $\pm\infty$. We then have

$$\inf_{\Lambda, X\in\mathcal{A}_\Lambda} \frac{\langle \Omega_{\alpha\beta} \mid X^*(H^\mu_{\Lambda\pm 3})^3 X\Omega_{\alpha\beta}\rangle}{\langle \Omega_{\alpha\beta} \mid X^*(H^\mu_{\Lambda\pm 3})^2 X\Omega_{\alpha\beta}\rangle} \tag{4.11}$$

$$\leq \inf_{\Lambda, X\in\mathcal{A}_\Lambda} \lim_{n\to\pm\infty} \frac{\langle \Omega_{\alpha\beta} \mid \tau_n(X^*(H^\mu_{\Lambda\pm 3})^3 X)\Omega_{\alpha\beta}\rangle}{\langle \Omega_{\alpha\beta} \mid \tau_n(X^*(H^\mu_{\Lambda\pm 3})^2 X)\Omega_{\alpha\beta}\rangle} \tag{4.12}$$

$$= \inf_{\Lambda, X\in\mathcal{A}_\Lambda} \frac{\langle \Omega_{\uparrow\uparrow} \mid X^*(H^\mu_{\Lambda\pm 3})^3 X\Omega_{\uparrow\uparrow}\rangle}{\langle \Omega_{\uparrow\uparrow} \mid X^*(H^\mu_{\Lambda\pm 3})^2 X\Omega_{\uparrow\uparrow}\rangle} \tag{4.13}$$

where τ_n denotes the translation over n lattice units in the chain. It follows that $\gamma_{\alpha\beta} \leq \gamma_{\uparrow\uparrow}$.

For the proof of the upper bound it is convenient to present the dense subspace of $\operatorname{ran} H_{\alpha\beta}$ formed by the vectors of the form (4.4) in a slightly different way. Observe that the spaces $\ker H^\mu_\Lambda \subset \mathcal{H}_{\alpha\beta}$ are decreasing in Λ. Therefore, in order to assure that a certain ψ belongs to $\operatorname{ran} H_{\alpha\beta}$, it suffices to check that $\psi \perp \ker H^\mu_\Lambda$ for some suitable Λ.

We fix an interval $[1, n]$ and introduce the usual spin wave operators X_k, $k = 2\pi m/n, m = 0, \ldots, n-1$, given by

$$X_k = \frac{1}{\sqrt{n}} \sum_{x=1}^{n} e^{ikx} S^-_x \tag{4.14}$$

The normalization and the allowed values for k are chosen such that

$$\langle \Omega_{\uparrow\uparrow} \mid X^*_l X_k \Omega_{\uparrow\uparrow}\rangle = \delta_{k,l} \tag{4.15}$$

The vectors ψ we need for the upper bound are linear combinations of two spin waves, i.e. $\psi = (c_1 X_{k_1} + c_2 X_{k_2})\Omega_{\uparrow\uparrow}$. Due to (4.15) we have $\|\psi\|^2 = |c_1|^2 + |c_2|^2$. For any pair of distinct k_1, k_2, the coefficients c_1, c_2 can be chosen such that $G_{[1,n]}\psi = 0$, i.e. $\psi \perp \ker H^\mu_{[1,n]}$. This follows from the fact that $\ker H^\mu_{[1,n]}$ contains exactly one vector for each eigenvalue of $S^3_{[1,n]}$. All vectors $X_k\Omega_{\uparrow\uparrow}$ have $S^3_{[1,n]} = (n-2)/2$. It follows that any two-dimensional space of vectors ψ with fixed, distinct k_1, k_2 and arbitrary c_1, c_2 must contain a ray $\perp \ker H^\mu_{[1,n]}$. Hence the upper bound (1.6) of Theorem 1.1 can be proved by showing that

$$\inf_{n,k_1,k_2} \sup_{c_1,c_2} \frac{\langle \psi \mid H_{\uparrow\uparrow}\psi\rangle}{\langle \psi \mid \psi\rangle} = \Gamma_{\text{upp}}(\Delta) \equiv 1 - \frac{1}{\Delta} \tag{4.16}$$

which we do next.

¿From the definition (4.14) of the X_k it is clear that the only matrix elements of $H_{\uparrow\uparrow}$ we need are the $T_{x,y}$, $1 \le x, y \le n$, defined by

$$T_{x,y} = \langle \Omega_{\uparrow\uparrow} \mid S_x^+ H^\mu_{[0,n+1]} S_y^- \Omega_{\uparrow\uparrow} \rangle = \frac{1}{2\Delta} \left\{ 2\Delta \delta_{x,y} - \delta_{x,y-1} - \delta_{x,y+1} \right\} \tag{4.17}$$

It is then easily seen that the $\sup_{c_1,c_2}$ in the left side of (4.16) yields the norm of the 2×2 matrix $M(n, k_1, k_2)$ with matrix elements

$$M(n, k_1, k_2)_{i,j} = M_n(k_i, k_j) \tag{4.18}$$

where $M_n(k, l)$, for k, l of the form $2\pi m/n$, is the function

$$M_n(k,l) = \frac{1}{n} \sum_{x,y=1}^{n} e^{-ikx} T_{x,y} e^{iky} \tag{4.19}$$

$$= \delta_{k,l}(1 - \Delta^{-1} \cos k) + (e^{il} + e^{-ik})/(2\Delta n) \tag{4.20}$$

It is then obvious that $\inf_{n,k_1,k_2} \|M(n, k_1, k_2)\| = 1 - \Delta^{-1}$.

A similar calculation yields the L-dependent upper bound (1.8) in the case $\Delta = 1$. ■

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References

1. B. Nachtergaele, preprint archived as `cond-mat/9410110`
2. F.H.L. Essler, V.E. Korepin, and K. Schoutens, *J. Phys. A: Math. Gen.* **25**, 4115–4126 (1992)
3. A.N. Kirilov, *J. Sov. Math.* **30**, 2298–2310 (1985)
4. C.N. Yang and C.P. Yang, *Phys. Rev.* **150**, 321–327 (1966)
5. D. Babbitt and L. Thomas, *Commun. Math. Phys.* **54**, 255–278 (1977)

6. H. Araki, *Commun. Math. Phys.* **14**, 120–157 (1969)
7. J.D. Johnson, *J. Appl. Phys.* **52**, 1991–1992 (1981)
8. D.C. Mattis: *The Many-Body Problem*, World Scientific, Singapore, 1993
9. V. Pasquier and H. Saleur, *Nucl. Phys.* **B330**, 523-556 (1990)
10. J.D. Johnson, S. Krinsky, and B.M. McCoy, *Phys. Rev.* **A8**, 2516–2547 (1973)
11. J.D. Johnson and B.M. McCoy, *Phys. Rev.* **A6**, 1613–1626 (1972)
12. A. Klümper, *Z. Phys.* **B91**, 507–519 (1993)
13. G. Albertini, V.E. Korepin, and A. Schadschneider, preprint archived as `cond-mat/9411051`, revised version of 11 Jan. 1995.
14. F.C. Alcaraz, M.N. Barber, and M.T. Batchelor, *Ann. Phys. (NY)* **182**, 280–343 (1988)
15. T.Kennedy, and H. Tasaki, *Commun. Math. Phys.* **147**, 431–484 (1992)
16. C.-T.Gottstein and R.F. Werner, in preparation
17. F.C. Alcaraz, S.R. Salinas, and W.F. Wreszinski, preprint
18. B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki, *Commun. Math. Phys.* **151**, 89–153 (1993)
19. V. S. Drinfel'd, in *Proceedings of the ICM Berkeley*, Academic Press, 1987, volume 1, pp 798–820
20. S. L. Woronowicz, *Commun. Math. Phys.* **111**, 613–665 (1987)
21. S.-L. Lu and H.-T. Yau, *Commun. Math. Phys.* **156**, 399–433 (1993)
22. M. Fannes, B. Nachtergaele, and R.F. Werner *J. Phys. A: Math. Gen.* **24**, L185-L190 (1991), and *Commun. Math. Phys.* **144**, 443–490 (1992)
23. I. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki, *Phys. Rev. Lett.* **59**, 799–802 (1987), and *Commun. Math. Phys.* **115**, 477–528 (1988)
24. B. Zegarlinski, in *Proceedings of the Third Internatinal Conference on Stochastic Processes, Physics, and Geometry*, S. Albeverio et al. (Eds), World Scientific, Singapore, 1993
25. F. Martinelli and E. Olivieri, Parts I and II, *Commun. Math. Phys.* **161**, 447–486, 487–514 (1994)

26. S. L. Woronowicz, *Pub. R.I.M.S. Kyoto* **23**, 117–181 (1987)

27. O. Bratteli and D.W. Robinson: *Operator algebras and quantum statistical mechanics*, 2 volumes, Springer Verlag-Berlin-Heidelberg-New York, 1979 and 1981

28. A. Guichardet, Lecture Notes Series Matematisk Institut Aarhus Universitet, No 13, Aarhus, 1969