Connection between Different Function Theories in Clifford Analysis*

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Abstract

We describe an explicit connection between solutions to equations $Df = 0$ (the Generalized Cauchy-Riemann equation) and $(D+M)f = 0$, where operators D and M commute. The described connection allows to construct a "function theory" (the Cauchy theorem, the Cauchy integral, the Taylor and Laurent series etc.) for solutions of the second equation from the known function theory for solution of the first (generalized Cauchy-Riemann) equation.

As well known, many physical equations related to the orthogonal group of rotations or the Lorentz group (the Dirac equation, the Maxwell equation etc.) can be naturally formulated in terms of the Clifford algebra. For them our approach gives an explicit connection between solutions with zero and non-zero mass (or external fields) and provides with a family of formulas for calculations.

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1 Introduction

Clifford analysis [4, 6] (the function theory of nullsolutions of the generalized Cauchy-Riemann operator D) has a structure closer that of the complex analysis of one variable than to standard complex analysis of several variables. The success of Clifford analysis is mainly explained because the generalized Cauchy-Riemann operator D (2.2) factorizes the Laplace operator $\Delta = \sum_0^n \frac{\partial^2}{\partial x_j^2}$. The development of Clifford analysis is motivated not only by mathematical reasons but also by its high applicability to the various physic problem (the Maxwell equations, the Dirac equation and other, which is connected with the orthogonal group of rotations or the Lorentz group). Applications of Clifford analysis to the operator theory may be found at the [16, 9]. After the development of the function theory for nullsolutions of the generalized Cauchy-Riemann operator it is natural to look [5, 19] for an analogous function theory for λ-solutions of the generalized Cauchy-Riemann operator (or nullsolutions of the operator $D - \lambda I$), namely, such a function f that $Df = \lambda f$ (or $Df = f\lambda$ due to non-commutativity of the Clifford multiplication).

In the paper we describe (Section 2) an explicit connection between solutions to equations $Df = 0$ and $(D + M)f = 0$, where operators D and M commute. This particularly includes the mentioned above case of equation $Df = f\lambda$. The described connection allows (Section 3) to construct a "function theory" (the Cauchy theorem, the Cauchy integral, the Taylor and Laurent series etc.) for solutions of the second equation from the known function theory for solution of the first (generalized Cauchy-Riemann) equation. For the mentioned above physical applications our approach gives an explicit connection between solutions with zero and non-zero mass (or external fields) and provides with a family of formulas for calculations.

We would like to present here basic ideas rather than to achieve (even if it is possible) the final level of generality. Thus many presented results may be considered in another (sometimes wider) context. For example, this gives an alternative approach to the theory of operator D_α in quaternionic analysis [11].

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2 A Connection between M-Solutions to the Generalized Cauchy-Riemann Operator for Different M

Let now e_j be generators of the Clifford algebra $\mathbf{C}l(0,n)$ (we use books [4, 6] as a standard reference). This means that the following *anti-commutation* relations hold:

$$\{e_i, e_j\} := e_i e_j + e_j e_i = -2\delta_{ij} e_0, \tag{2.1}$$

where $e_0 = I$. We will referee to e_j as to operators (of orthogonal transformations) acting in a (finite-dimensional) Hilbert space [6, Chap. 0, § A.1].

Function $f : \mathbb{R}^n \to \mathbf{Cl}(0,n)$ is called *monogenic* if it satisfies the *generalized Cauchy-Riemann equation*

$$Df := \frac{\partial f(y)}{\partial y_0} - \sum_{j=1}^{n} e_j \frac{\partial f(y)}{\partial y_j} = 0 \quad \text{or} \quad \frac{\partial f(y)}{\partial y_0} = \sum_{j=1}^{n} e_j \frac{\partial f(y)}{\partial y_j}. \tag{2.2}$$

The success of Clifford analysis is mainly explained because the generalized Cauchy-Riemann operator (2.2) factorizes the Laplace operator $\Delta = \sum_0^n \frac{\partial^2}{\partial x_j^2}$.

Hörmander's remark from paper [1] gives us by the fundamental solution to the generalized Cauchy-Riemann equation in the form

$$\begin{aligned} K(y) &= \mathcal{F}^{\eta\to y} e^{-iy_0 \sum_{j=1}^n \eta_j e_j} \\ &= \int_{\mathbb{R}^n} e^{i\sum_{j=1}^n y_j \eta_j}\, e^{-iy_0 \sum_{j=1}^n \eta_j e_j}\, d\eta. \end{aligned}$$

("Simply take the Fourier transform with respect to the spatial variables, and solve the equation in y_0" [1]). Otherwise, any solution $f(y)$ to (2.2) is given by a convolution of some function $\widetilde{f}(y)$ on $\mathbb{R}^{n-1}$ and the fundamental solution $K(y)$. On the contrary, a convolution $K(y)$ with any function is a solution to (2.2). We have:

$$\begin{aligned} [K * f](y) &= \int_{\mathbb{R}^n} K(y-t)\, f(t)\, dt \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n (y_j - t_j)\eta_j}\, e^{-iy_0 \sum_{j=1}^n \eta_j e_j}\, d\eta f(t)\, dt \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n t_j \eta_j}\, e^{-i\sum_{j=1}^n \eta_j (y_0 e_j - y_j e_0)}\, d\eta f(t)\, dt \end{aligned}$$

$$= \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n \eta_j (y_0 e_j - y_j e_0)} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^n t_j \eta_j} f(t)\, dt\, d\eta$$
$$= \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n \eta_j (y_0 e_j - y_j e_0)} \widehat{f}(-\eta)\, d\eta. \qquad (2.3)$$

Equation (2.3) defines the Weyl functional calculus [1] for the function $f(-y)$ and the n-tuple of operators

$$\{\vec{y}_j = y_0 e_j - y_j e_0\},\ 1 \leq j \leq n. \qquad (2.4)$$

Thus any solution to the generalized Cauchy-Riemann equation (2.2) can be written as a function of n monomials (2.4):

$$\breve{f}(y_0, y_1, \ldots, y_n) = f(y_0 e_1 - y_1 e_0, y_0 e_2 - y_2 e_0, \ldots, y_0 e_n - y_n e_0). \qquad (2.5)$$

Another significant remark: if we fix the value $y_0 = 0$ in (2.5) we easily obtain:

$$\breve{f}(0, y_1, \ldots, y_n) = f(-y_1 e_0, -y_2 e_0, \ldots, -y_n e_0) = f(-y_1, -y_2, \ldots, -y_n) e_0.$$

Thus we may consider the function $\breve{f}(y_0, y_1, \ldots, y_n)$ in $n+1$ variables as *analytic (or Cauchy-Kovalevska) expansion* for the function $f(y_1, \ldots, y_n)$ in n variables (compare with [14]).

Using the power series decomposition for the exponent one can see that formula (2.3) defines the permutational (symmetric) product of monomials (2.4). The significant role of such monomials and functions generated by them were described for quaternionic analysis in [18], for Clifford analysis in [13, 15], for Fueter-Hurwitz analysis in [12]. But during our consideration we used only the commutation relation $[e_0, e_j] = 0$ and never used the anti-commutation relations (2.1). Thus formula (2.3) *is true and may be useful without Clifford analysis* as well.

Proposition 2.1 *Any solution to equation* (2.2)*, where e_j are arbitrary self-adjoint operators, is given as arbitrary function of n monomials* (2.4) *by the formula* (2.3).

Due to physical application we will consider equation

$$\frac{\partial f}{\partial y_0} = (\sum_{j=1}^n e_j \frac{\partial}{\partial y_j} + M) f, \qquad (2.6)$$

where e_j are arbitrary self-adjoint operators and M is a bounded operator commuting with all $e_j \frac{\partial}{\partial y_j}$.

Example 2.2 If e_j are generators (2.1) of the Clifford algebra and $M = M_\lambda$ is an operator of multiplication from the *right*-hand side by the Clifford number λ, differential operator

$$(\sum_{j=1}^{n} e_j \frac{\partial}{\partial y_j} + M) f, \tag{2.7}$$

factorizes the Helmholtz operator $\Delta + M_{\lambda^2}$. Equation (2.6) is known in quantum mechanics as the *Dirac equation for a particle with a non-zero rest mass* [2, §20], [3, §6.3] and [10]. We will specialize our results for the case $M = M_\lambda$, especially for the simplest (but still important!) case $\lambda \in \mathbb{R}$.

Example 2.3 It is well known [8, Chap. VI, § 1.2], that an operator commutes with differential operators (particularly with the generalized Cauchy-Riemann operator) if and only if it is an operator of convolution with a generalized function. The operator M can be used for an introducing into equation (2.6) a non-zero mass of the particle or external fields.

Notably, the operator M_λ from the previous Example is the convolution from the right-hand side (non-commutativity of Clifford multiplication!) with the Dirac function $\lambda\delta(x)$.

Thus the family of possible operators M is rather width and includes examples with interesting applications. Simple modification of the previous calculations (2.3) gives us the following result

Proposition 2.4 *Any solution to equation* (2.6)*, where e_j are arbitrary self-adjoint operators and M commutes with all $e_j \frac{\partial}{\partial y_j}$, is given by the formula*

$$e^{y_0 M} \int_{\mathbb{R}^n} e^{-i \sum_{j=1}^{n} \eta_j (y_0 e_j - y_j e_0)} \widehat{f}(-\eta)\, d\eta, \tag{2.8}$$

where f is an arbitrary function on $\mathbb{R}^{n-1}$.

Here

$$e^{y_0 M} = \sum_{j=0}^{\infty} \frac{(y_0 M)^j}{j!} \tag{2.9}$$

is well defined for all bounded M. Comparing (2.3) and (2.8) we obtain

Theorem 2.5 *The function $f(y)$ is a solution to the equation*

$$\frac{\partial f}{\partial y_0} = (\sum_{j=1}^{n} e_j \frac{\partial}{\partial y_j} + M_1) f$$

if and only if the function

$$g(y) = e^{y_0 M_2} e^{-y_0 M_1} f(y)$$

is a solution to the equation

$$\frac{\partial g}{\partial y_0} = (\sum_{j=1}^{n} e_j \frac{\partial}{\partial y_j} + M_2) g,$$

where M_1 and M_2 are bounded operators commuting with e_j.

Corollary 2.6 *The function $f(y)$ is a solution to the equation* (2.6) *if and only if the function $e^{y_0 M} f(y)$ is a solution to the generalized Cauchy-Riemann equation* (2.2).

In the case $M = M_\lambda$ we have $e^{y_0 M_\lambda} f(y) = f(y) e^{y_0 \lambda}$ and if $\lambda \in \mathbb{R}$ then $e^{y_0 M_\lambda} f(y) = f(y) e^{y_0 \lambda} = e^{y_0 \lambda} f(y)$.

3 Function Theory for M-Solutions of the Generalized Cauchy-Riemann Operator

In this section we construct a function theory (in the classic sense) for M-solutions of the generalized Cauchy-Riemann operator basing on Clifford analysis and Corollary 2.6. Proofs are very short and almost evident, but we present them for the sake of completeness.

The set of solutions to (2.2) and (2.6) in a nice domain Ω will be denoted by $\mathfrak{M}(\Omega) = \mathfrak{M}_0(\Omega)$ and $\mathfrak{M}_M(\Omega)$ correspondingly. In the case $M = M_\lambda$ we use the notation $\mathfrak{M}_\lambda(\Omega) = \mathfrak{M}_{M_\lambda}(\Omega)$ also. We suppose that all functions from $\mathfrak{M}_\lambda(\Omega)$ are continuous in the closure of Ω. Let

$$E(y-x) = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(m+1)/2}} \frac{\overline{y-x}}{|\, y-x \,|^{n+1}} \tag{3.1}$$

be the Cauchy kernel [6, p. 146] and

$$d\sigma = \sum_{j=0}^{n} (-1)^j e_j dx_0 \wedge \ldots \wedge [dx_j] \wedge \ldots \wedge dx_m. \tag{3.2}$$

be the differential form of the "oriented surface element" [6, p. 144]. Then for any $f(x) \in \mathfrak{M}(\Omega)$ we have the Cauchy integral formula [6, p. 147]

$$\int_{\partial\Omega} E(y-x)\, d\sigma_y\, f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}. \tag{3.3}$$

Theorem 3.1 (Cauchy's Theorem) *Let $f(y) \in \mathfrak{M}_M(\Omega)$. Then*

$$\int_{\partial\Omega} d\sigma_y\, e^{-y_0 M} f(y) = 0.$$

Particularly, for $f(y) \in \mathfrak{M}_\lambda(\Omega)$ we have

$$\int_{\partial\Omega} d\sigma_y\, f(y) e^{-y_0\lambda} = 0,$$

and

$$\int_{\partial\Omega} d\sigma_y e^{-y_0\lambda}\, f(y) = 0,$$

if $\lambda \in \mathbb{R}$.

Proof. It easily follows because $e^{-y_0 M} f(y) \in \mathfrak{M}(\Omega)$ and the corresponding result for the generalized Cauchy-Riemann equation [6, Chap. II, § 0.2.1]. □

Theorem 3.2 (Cauchy's Integral Formula) *Let $f(y) \in \mathfrak{M}_M(\Omega)$. Then*

$$e^{x_0 M} \int_{\partial\Omega} E(y-x)\, d\sigma_y\, e^{-y_0 M} f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}. \tag{3.4}$$

Particularly, for $f(y) \in \mathfrak{M}_\lambda(\Omega)$ we have

$$\int_{\partial\Omega} E(y-x)\, d\sigma_y\, f(y) e^{(x_0-y_0)\lambda} = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}.$$

and

$$\int_{\partial\Omega} E(y-x) e^{(x_0-y_0)\lambda}\, d\sigma_y\, f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}.$$

if $\lambda \in \mathbb{R}$.

PROOF. It easily follows from the fact that $e^{-y_0 M} f(y) \in \mathfrak{M}(\Omega)$ and the corresponding result for the generalized Cauchy-Riemann equation [6, Chap. II, § 0.2.2]. □

It is hard to expect that formula (3.4) may be rewritten as

$$\int_{\partial\Omega} E'(y-x)\, d\sigma_y\, f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}$$

with a simple function $E'(y-x)$.

Because an application of the bounded operator $e^{y_0 M}$ does not destroy uniformed convergency of functions we obtain (cf. [6, Chap. II, § 0.2.2, Theorem 2])

Theorem 3.3 (Weierstrass' Theorem) *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{M}_M(\Omega)$, which converges uniformly to f on each compact subset $K \Subset \Omega$. Then*

1. *$f \in \mathfrak{M}_M(\Omega)$.*
2. *For each multi-index $\beta = (\beta_0, \ldots, \beta_m) \in \mathbb{N}^{n+1}$, the sequence $\{\partial^\beta f_k\}_{k\in\mathbb{N}}$ converges uniformly on each compact subset $K \Subset \Omega$ to $\partial^\beta f$.*

Theorem 3.4 (Mean Value Theorem) *Let $f \in \mathfrak{M}_M(\Omega)$. Then for all $x \in \Omega$ and $R > 0$ such that the ball $\mathbb{B}(x,R) \Subset \Omega$,*

$$f(x) = e^{x_0 M} \frac{(n+1)\Gamma(\frac{n+1}{2})}{2R^{n+1}\pi^{(m+1)/2}} \int_{\mathbb{B}(x,R)} e^{-y_0 M} f(y)\, dy.$$

PROOF. We again referee to corresponding theorem [6, Chap. II, § 0.2.2, Theorem 3] for $\mathfrak{M}(\Omega)$ and Corollary 2.6. □

We skip Morera's, Painlevé's theorems [6, Chap. II, § 0.2.3] and Plemelj-Sokhotskij formulas formulated for $\mathfrak{M}_M(\Omega)$.

It is possible also to introduce the notion of the *differentiability* [15] for solutions to (2.2), namely, an increment of any solution to (2.2) may be approximated up to infinitesimals of the second order by a linear function of monomials $\vec{y}_j$ from (2.4). We now give λ-version[1] of this notion.

[1] At the rest of the paper we will consider only the case $M = M_\lambda$. The general case needs only obvious modifications.

Definition 3.5 The function $f(y)$ is called λ-*differentiable* from the left at y if there exist a λ-linear form

$$l(\vec{y}) = \sum_{j=1}^{n} \vec{y}_j A_j e^{y_0 \lambda}, \;\; A_j \in \mathbf{Cl}(n,0) \tag{3.5}$$

such that

$$\lim_{\Delta\vec{y}\to 0} \frac{|\, f(\vec{y}-\Delta\vec{y}) - f(\vec{y}) - l(\vec{y}) \,|}{\|\, \Delta\vec{y} \,\|} = 0. \tag{3.6}$$

The function is λ-differentiable in Ω if the function is λ-differentiable at all points of Ω. The λ-linear mapping $l(\vec{y})$ is called the λ-*derivative* of the function f at the point y.

We skip proofs of the following main results connected with the notion of λ-differentiability (compare with [15]).

Theorem 3.6 *If $f(\vec{y})$ is λ-differentiable then the corresponding λ-derivative is determined in a unique way.*

Theorem 3.7 *Let $f(\vec{y})$ be continuously real differentiable at a point $\vec{y}$. The function f is λ-differentiable at $\vec{y}$ if and only if $f \in \mathfrak{M}_\lambda$.*

Let us remind that symmetric products [15] is defined by the formula

$$a_1 \times a_2 \times \cdots \times a_k = \frac{1}{k!} \sum a_{j_1} a_{j_2} \cdots a_{j_k}, \tag{3.7}$$

where the sum is taken over all of permutations of $(j_1, j_2, \ldots, j_k)$. For a multi-index β the notation $\vec{y}^\beta$ denotes the corresponding symmetric product. If we consider the linear combination of such products then Clifford valued coefficients are written on the right-hand side.

Theorem 3.8 *Every $f(\vec{y}) \in \mathfrak{M}_\lambda(\Omega)$ is infinitely λ-differentiable and for any point $a \in \Omega$ can be presented by the Taylor series*

$$f(\vec{y}) = \sum_{\beta=0}^{\infty} (\vec{y} - \vec{a})^\beta c_\beta e^{-y_0 \lambda}$$

in some neighborhood of a.

An important property of Clifford analysis is the existence of the reproducing Bergman kernel [4, § 24]. We will give the λ-version of this result (compare with [5, 17]).

Theorem 3.9 *$\mathfrak{M}_\lambda(\Omega)$ has the reproducing formula*

$$\int_\Omega B(x,y)\, f(y) e^{(x_0-y_0)\lambda}\, dy = f(x), \tag{3.8}$$

where $B(x,y)$ is the Bergman kernel [4, § 24.1] from Clifford analysis.

Moreover, if $\lambda \in \mathbb{R}$ then formula (3.8) *takes the form of usual reproducing formula*

$$\int_\Omega B'_\lambda(x,y)\, f(y)\, dy = f(x), \tag{3.9}$$

where $B'_\lambda(x,y) = B(x,y)e^{(x_0-y_0)\lambda}$. For the unit ball $\mathbb{B}(0,1)$ centered at the origin the explicit formula for $B'_\lambda(x,y)$ is (compare with [17]):

$$\begin{aligned} B'_\lambda(x,y) &= \frac{\Gamma(\frac{n+1}{2})(n+1)e^{(x_0-y_0)\lambda}}{2\pi^{(n+1)/2}} \left(\frac{n+1}{(1-2\langle y,x\rangle + \mid y\mid^2 \cdot \mid x \mid^2)^{(n+1)/2}} \right. \\ &\quad - \frac{2\bar{x}y}{(1-2\langle y,x\rangle + \mid y\mid^2 \cdot \mid x \mid^2)^{(n+1)/2}} \\ &\quad \left. + \frac{(n+1)\overline{(y-x\mid y\mid^2)}(x-y\mid x\mid^2)}{(1-2\langle y,x\rangle + \mid y\mid^2 \cdot \mid x \mid^2)^{(n+3)/2}} \right). \end{aligned}$$

REMARK 3.10 Quaternionic analysis is not (formally speaking) a corollary of Clifford analysis. Thus quaternionic analysis needs an application the ideas of this paper rather than concrete results.

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