Singular continuous spectrum in ergodic theory

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Abstract

We prove that in the weak topology of measure preserving transformations, a dense G_δ has purely singular continuous spectrum in the orthocomplement of the constant functions. In the uniform topology, a dense G_δ of aperiodic transformations has singular continuous spectrum. We show that a dense G_δ of shift-invariant measures has purely singular continuous spectrum. These results stay true for $\mathbf{Z}^d$ actions of measure preserving transformations. There exist smooth unitary cocycles over an irrational rotation which have purely singular continuous spectrum.

1 Introduction

The spectral theory for the unitary operators belonging to measure-preserving transformations started with the works of Koopman and von Neumann and turned out to be very fruitful in ergodic theory (see for example [7, 35, 6]). A general problem in spectral theory for Schrödinger operators is to distinguish pure point (=discrete) spectrum (σ_{pp}), singular continuous spectrum (σ_{sc}) and absolutely continuous spectrum (σ_{ac}). As in the theory of Schrödinger operators, the spectral analysis of a transformation is in general not easy and evenso we will see that genericity results are easy to find, it is harder to find *concrete* examples of a measure preserving transformation with specific spectral properties. Such situations occur at other places in ergodic theory and mathematical physics: in the spectral theory for Schrödinger operators, singular continuous spectrum is often generic [44, 43, 37, 36] for concrete examples of operators L with $\sigma(L) = \sigma_{sc}(L)$ see [2, 20, 15]. Polygonal billiards are examples of difficult spectral problem there is no ergodic polygonal billiard known, evenso ergodicity is generic for such systems (see [13]).

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Discrete spectrum:
Transformations with discrete spectrum are known to be isomorphic to group translations on the dual group of the spectrum and the spectrum is always simple. On the other hand, the set of dynamical systems which have no eigenvalues (except the trivial eigenvalue 1 belonging to the constant eigenfunctions) coincides with the class of weakly mixing transformations. Therefore, pure point or absence of pure point spectrum is dynamically characterized.

Absolutely continuous spectrum:
Also absolutely continuous spectrum has relations with dynamical properties evenso no characterization is known. K-systems and ergodic automorphisms of commutative compact groups have absolutely continuous spectrum of countable multiplicity. Sufficient conditions for absence of σ_{ac} can be given in terms of periodic approximations [6]. If a transformation has positive metric entropy, a theorem of Sinai [6] assures that it contains a Bernoulli shift as a factor. This implies a result of Rohlin saying that such a transformation has some absolutely continuous spectrum of infinite multiplicity. Much research has been devoted to the multiplicity of the spectrum (see for example [21, 22, 7, 16]). Rohlin mentions ([39], p. 218) that the set of transformations T with $\sigma(T) = \sigma_{ac}(T)$ are meager both in the uniform and the weak topology of measure preserving transformations, a result which can also be proven by periodic approximation. There seems not to exist a necessary and sufficient condition for purely absolutely continuous spectrum. This can be illustrated by the still open question of finding substitution systems with purely absolutely continuous spectrum (see [4]).

Singular continuous spectrum.
While automorphisms with pure point spectrum or purely absolutely continuous spectrum are easy to give, examples of dynamical systems with purely singular continuous spectrum are harder to find. Robinson ([21] Theorem 7.1 and the remark in the intruduction) constructed interval exchange transformations with purely singular continuous spectrum. Katok-Stepin have shown ([26] p. 201) that a generic volume preserving homeomorphism of a manifold has purely singular continuous spectrum, strengthening so Oxtoby-Ulams's theorem about generic ergodicity. With their methods [25], it can be deduced (see [24] remark 4.1.1), that a generic measure preserving transformation has purely singular continuous spectrum. We will prove this here also with other methods.

Mixed spectrum:
Many systems have mixed spectrum. For example, for certain skew products, there is both countable absolutely continuous and discrete spectrum [18].
For substitution systems, there are conditions for distinguishing discrete and mixed discrete-continuous spectrum. For example, for the Thue-Morse system $0 \mapsto 01, 1 \mapsto 10$, there is mixed spectrum [35]. Substitutions systems are not strongly mixing and have therefore not purely absolutely continuous spectrum ([32] p. 50). Continuous spectrum and so some singular continuous spectrum are known for the substitution systems $0 \mapsto 01010, 1 \mapsto 011$ (see [35] p. 131) and for $0 \mapsto 001$, $1 \mapsto 11100$ [8]. The Rudin-Shapiro system is an example of a substitution containing some absolutely continuous spectrum of multiplicity 2 [35]. Toeplitz flows can have discrete spectrum [19], there are examples with fast periodic approximation and so no absolutely continuous spectrum [17] and examples with some absolutely continuous spectrum of finite multiplicity [28].
Given a transformation with mixed spectrum, the conditional expectation projection map leads to factors, which have pure spectrum of each occurring spectral component. Concrete examples with purely singular continuous spectrum can therefore be obtained by taking suitable factors of some substitution systems. There are many other candidates (see also the discussion).

For other generic spectral properties of measure preserving transformations, see [9]. In this note, spectral questions are discussed in the context of recent developments of operator theory [44, 43].

2 Measure preserving transformations

A measure preserving automorphism T of a probability space $(X, \mathcal{A}, m)$ defines the unitary Koopman operator $U_T : f \mapsto f(T)$ on $L^2(X)$. Since 1 is always an eigenvalue, it is convenient to restrict U_T to the orthogonal complement of the constant functions and to speak of the spectrum of this operator. Let $\mathcal{G}$ be the complete topological group of automorphisms of $(X, \mathcal{A}, m)$ with the weak topology: T_j converges to T weakly, if $m(T_j(A) \Delta T(A)) \to 0$ for all $A \in \mathcal{A}$ (see [14]).

Theorem 2.1 *In the weak topology, a dense G_δ of transformations T in $\mathcal{X}$ satisfies $\sigma(U_T) = \sigma_{sc}(U_T)$.*

Proof. By Halmos mixing theorem [14], a transformation $T \in \mathcal{G}$ is weekly mixing ($\Leftrightarrow T \times T$ is ergodic) if and only if it has continuous spectrum.
By Halmos second category theorem [14], the set of weakly mixing transformations (and so with continuous spectrum) is a dense G_δ in $\mathcal{G}$ in the weak topology.
Rohlin's lemma (see [14]) shows that the set of periodic transformations (and so with pure point spectrum) is dense in the uniform topology and so also dense in the weak topology.
If $T_n \to T$ weakly in $\mathcal{G}$, then the corresponding Koopman operators U_n converge to U in the strong operator topology and so in the strong resolvent topology. (We know that $U_n 1_A \to U 1_A$ for every $A \in \mathcal{A}$. This stays true for step functions $\sum_n a_n 1_{A_n}$ which are dense in $L^2(X,m)$. Therefore $U_n \to U$ in the strong operator topology and by general principles in the strong resolvent sense.)
We can now apply Simon's wonderland theorem [44, 43]) in the following form: let $\mathcal{X}$ be a complete metric space of unitary operators, where the metric is stronger than the strong resolvent topology. Assume there exists a dense set with continuous spectrum and a dense set with pure point spectrum, then there exists a dense G_δ of purely singular continuous spectrum. (The wonderland theorem is formulated for selfadjoint operators. However, the proofs that the two sets $\{A \in \mathcal{X} \mid , A$ has no eigenvalues $\}$, $\{A \in \mathcal{X} \mid , A$ has no absolutely continuous spectrum $\}$ are both G_δ', hold also of unitary operators. Alternatively, the result for unitary operators can be obtained from the result for selfadjoint operators by Cayley transforms.) □

Remarks.
1) As mentioned in the introduction, Theorem 2.1 is also reachable with Katok-Stepin's approximation methods and is deducable (while not explicitely stated) from the results in [25].
2) This theorem or the theorem of Katok-Stepin [26] saying that a generic volume preserving homeomorphism has purely singular continuous spectrum can be used to get generic purely singular continuous spectrum for other topological spaces of transformations: take two topological spaces $X \subset Y$ of transformations which are both strong enough to use the wonderland theorem. Assume X is dense in Y in the topology of Y and a generic set in X has purely singular continuous spectrum, then a generic set in Y has purely singular continuous spectrum.
2) Theorem 2.1 illustrates the proof of Halmos genericity result: it is enough

to show that weakly mixing is dense in order to conclude that it is a dense G_δ. The density of weakly mixing follows directly from the conjugacy lemma of Halmos (see [1] for more details about conjugates of antiperiodic transformations). In order to get generically weakly mixing, one needs a dense set with no σ_{ac} and a dense set with no σ_{pp}.

Theorem 2.1 can not true in the uniform topology $d(T,S) = m(S(x) \neq T(x))$ because ergodic transformations are then nowhere dense [14]. We can however restrict ourself to $\mathcal{K}$, the closure of the aperiodic transformations in the uniform topology.

Theorem 2.2 *In the uniform topology, $\mathcal{K}$ contains a dense G_δ of transformations T with $\sigma(U_T) = \sigma_{sc}(U_T)$.*

Proof. $\mathcal{G}_a$ is a complete metric space since it is a closed set of a complete metric space.
By Rohlin's lemma, it contains a dense set of periodic transformations.
The conjugacy class $[T]$ of one aperiodic transformation T is dense in $\mathcal{G}_a$ in the uniform topology (see [14, 11]). The existence of weakly mixing transformations, gives therefore a dense set of weakly mixing transformations.
The wonderland theorem implies again the claim. $\square$

Remarks.
1) It follows that $\mathcal{K}$ is nowhere dense in $\mathcal{G}$ in the uniform topology.
2) It follows that all pure spectral types σ_{sc}, σ_{ac}, σ_{pp} are dense in the weak topology in $\mathcal{X}$ and dense in the uniform topology in $\mathcal{K}$.
3) Singular continuous spectrum can say something about the dynamics. Theorem (2.1) implies a result of Rohlin [38] saying that generically, a transformation has zero metric entropy. To obtain this corollary, one uses Rohlin's theorem [38] stating that a transformation with positive metric entropy has some countable Lebesgue spectrum.
4) Let U be the unitary Koopman operator of a transformation T. Consider the free Schrödinger operator $L = U + U^*$ on $L^2(X)$. Since U can be written as a function of L as $L/2 + i\sqrt{1 - L^2/4}$, the operator L has the same spectral type as U (every eigenvalue of U corresponds to an eigenvalue of L and converse) and L has some absolutely continuous component if and only if U has. Theorem 2.1 shows that we can find such free Schrödinger operators on $L^2(X)$ with purely singular continuous spectrum.

5) With the induced product topology, the set $\mathcal{G}^d$ of $\mathbf{Z}^d$- actions is a complete metric space. We say, $T \in \mathcal{G}^d$ has purely singular continuous spectrum, if all the transformations $T^n, n \neq 0$ have purely singular continuous spectrum in the orthocomplement of the eigenspace of the eigenvalue 1. Since the projection from $\mathcal{G}^d$ to one coordinate is continuous and maps open sets into open sets, the inverse image of a generic set is generic. It follows that a dense G_δ of $\mathbf{Z}^d$-actions has purely singular continuous spectrum. Also in the multidimensional case, a K-system has countable Lebesgue spectrum and systems with positive metric entropy have some Lebesgue spectrum (see [23]). It follows that a generic $\mathbf{Z}^d$-action has zero entropy.

3 Shift invariant measures

Given a topological shift $(X = A^{\mathbf{Z}}, T)$ over a finite alphabet A. The space of shift-invariant measures is a complete metric space with respect to the weak-* topology.

Theorem 3.1 *There exists a generic set of T-invariant measures m for which the abstract dynamical system (X, T, m) has purely singular continuous spectrum.*

Proof. The claim follows again with wonderland:
according to Parthasarathy [33], there exists a dense set of periodic invariant measures.
Sigmund [42] showed that there exists a generic set of invariant measures for which the system is weakly mixing.
If $\mu_n \to \mu$, then $U_n f \to U f$ for every continuous function f and so $U_n \to U$ strongly and so in the strong resolvent sense. $\square$

Remarks. 1) As a corollary, we obtain an other result of Sigmund [41] saying that the set of shift-invariant measures with zero metric entropy is a dense G_δ.
2) Summarizing the other known results in literature (see [10]): A generic shift-invariant measure on X is ergodic, nonatomic, with support X, weakly mixing and not strongly mixing, has zero metric entropy and singular continuous spectrum.
3) An other corollary is that if f is an invertible Axiom-A diffeomorphism restricted to a basic set Ω_i, there is a generic set of f invariant measures on

Ω_i which have purely singular continuous spectrum (see [40, 42]).
4) Theorem (3.1) stays true if the finite alphabet A is replaced by a compact metric space Y. The reason is that there is a dense set of measures in $Y^{\mathbf{Z}}$ which have support on some $A^{\mathbf{Z}}$, where A is a finite subset of Y. We can especially take $Y = Z^{(\mathbf{Z}^{d-1})}$, where Z is a compact space, to obtain the corollary that a generic $\mathbf{Z}^d$-shift invariant measure in $Z^{(\mathbf{Z}^d)}$ has purely singular continuous spectrum and again with [23] zero metric entropy.

4 Circle-valued cocycles

Let T_α be the irrational rotation $x \mapsto x + \alpha$ mod 1 on the circle $X = \mathbf{T}^1$. Define the maps $a_{\beta,s} : X \to \{|z| = 1\}$ by $a_{\beta,s}(x) = e^{2\pi i s 1_{[0,\beta)}}$ and the unitary operators $U_{\beta,s,\alpha} f(x) = a_{\beta,s} f(x + \alpha)$.

Theorem 4.1 *For a generic set of pairs (α, β), the operator $U_{\beta,s,\alpha}$ has purely singular continuous spectrum.*

Proof. If α has bounded partial quotients and $s \neq 0$ and $\beta \notin \alpha\mathbf{R}/\mathbf{Z}$, then $U_{\beta,s,\alpha}$ has purely continuous spectrum ([31] Theorem 2.4), (use also [3] Proposition 2.1).
If α is rational, then $U_{\beta,s,\alpha}$ has pure point spectrum. $U_{\beta,s,\alpha}$ depends continuously on (α, β) in the strong operator topology. The claim follows again from the wonderland theorem. □

Let V be a measurable map from $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ to $\mathbf{R}$. Consider the evolution of a function $u \in L^2(\mathbf{T}^1)$ with time-dependent vector field

$$H_t = i(\frac{d}{d\theta} + V(\theta)\delta(t + 2\pi n\alpha)) .$$

Mapping over one period gives the Floquet map $Su(\theta) = u(\theta + 1)$ of a solution $\dot{u} = Hu$. This map is a circle-valued cocycle and we get

Corollary 4.2 *There are potentials $V = s \cdot 1_{[0,\beta)}$ for which the unitary Floquet operator of H_t has purely singular continuous spectrum.*

Given now $m \in \mathbf{Z} \setminus \{0\}$ and $a_{\phi,m}(x) = e^{2\pi i\phi(x)+mx}$, where ϕ is in the topological space $C(\mathbf{T})$ of continuous periodic functions. Denote with $U_{\phi,m,\alpha}$ the associated unitary weighted composition operator on $L^2(X)$: $(U_{\phi,m,\alpha} f)(x) = a_{\phi,m}(x) f(x + \alpha)$.

Theorem 4.3 *For $m \neq 0$, there exists a generic set of $(\phi, \alpha) \in C(\mathbf{T}) \times \mathbf{R}$ such that the operator $U_{\phi,m,\alpha}$ has purely singular continuous spectrum.*

Proof. The complete metric space $C(\mathbf{T})$ contains the dense set of functions ϕ which have a Fourier series $\hat{\phi}_n = o(1/n)$. For such ϕ and irrational α, the operator $U_{\phi,m\alpha}$ has no eigenvalues if $m \neq 0$ ([29] Corollary 2). If α is rational, then the operator U has pure point spectrum. Apply wonderland. $\square$

Remark. In Theorem 4.3, the space $C(\mathbf{T})$ can be replaced with any subset of $L^2(\mathbf{T})$ which is equiped with a topology in which functions with Fourier series of order $o(1/n)$ are dense. Especially, this gives smooth cocycles with purely singular continuous spectrum.

5 Discussion and questions

We have shown here how putting together known results in ergodic theory and operator theory leads to new results in ergodic theory. However, further investigations of the nature of transformations with purely singular continuous spectrum is necessary and many questions are open.

For every $f \in L^2(X)$ is defined the correlation sequence $c_n = (U^n f, f)$. The Fourier transform of this sequence is the spectral measure μ_f called the correlation measure. It has purely singular continuous spectrum if T has. How can one characterize sequences which are the Fourier transform of singular continuous probability measures on the circle? This question (see [30] and references therin for selfaffine measures in $\mathbf{R}^n$) would shed light onto the problem to characterize transformations T with $\sigma(T) = \sigma_{ac}(T)$ or $\sigma(T) = \sigma_{sc}(T)$.

If T is the irrational rotation on the circle, it is known that the set of $A \in \mathcal{A}$ for which the induced transformation T_A is weakly mixing is generic with respect to the metric $d(A, B) = m(A \Delta B)$ [11, 34]. Veech asked (see [34] p. 229), whether the set of A for which $\sigma_{pp}(T_A) = \sigma(T_A)$ is dense (see also [5]). If the answer to this question were yes, then the wonderland theorem would give a generic set of A's with $\sigma_{sc}(T_A) = \sigma(T_A)$.

It would be interesting to get smooth differential equations $\dot{u} = A(u,v), \dot{v} = B(u,v)$ on the torus with $A^2 + B^2 > 0$ and preserving a smooth measure μ with flows having purely singular continuous spectrum. If the rotation number $\lambda = \int A \, d\mu / \int B \, d\mu$ of the flow satisfies a Diophantine condition, Kolmogorov showed that the flow has purely discrete spectrum. Also Katok-Stepin (see [6]) show with the theory of periodic approximations that there exists a dense set of λ such that the flow has no absolutely continuous spectrum. According to [6], Kolmogorov (worked out by Sklover) constructed examples which are weakly mixing. Krygin [27] has given examples with mixed spectrum.

Consider a family of Sinai Billiards in a square table with a one convex scatterer (of fixed diameter and center) as a parameter. If we take for the scatterer the Hausdorff topology on the compact set of closed convex sets (compare [12]), we expect to have a dense set of mixing systems, when the scatterer is smooth and with positive curvature everywhere and a dense set with no absolutely continuous spectrum when the scatterer is a polygon. This would lead for a generic set of convex scattereres purely singular continuous spectrum and especially ergodicity.

The spectral question can also be asked in noncommutative ergodic theory: an automorphism S on the hyperfinite factor $(\mathcal{X}, \mathrm{tr})$ preserving the trace defines a unitary operator U_S on the completion of the pre-Hilbert space $(\mathcal{X}, \mathrm{tr}(\cdot,\cdot))$. An obvious problem is to ask for the existence of a noncommutative example S with $\sigma(U_S) = \sigma_{sc}(U_S)$.

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