Reversible Anosov diffeomorphisms and large deviations.

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Abstract: the volume contraction obeys a large deviation rule.[1]

§*1. – Anosov maps and thermodynamic formalism.*

This section reviews the basic results of Sinai's theory of transitive Anosov diffeomorphisms. It is meant for the reader with some familiarity with the subject and it should be used for notations only. Expert readers will probably find the rest of the paper self contained, without this section.

Let $\mathcal{C}$ be a d–dimensional, C^∞, compact manifold and let S be a C^∞, transitive Anosov diffeomorphism, [AA], [S]. If W^u_x, W^s_x denote the *unstable* or *stable* manifold at $x \in \mathcal{C}$, we call $W^{u,\delta}_x, W^{s,\delta}_x$ the connected parts of W^u_x, W^s_x containing x and contained in the sphere with center x and radius δ. Let d_u, d_s be the *dimensions* of W^u_x, W^s_x: $d = d_u + d_s$. It is convenient to take δ always smaller than the smallest curvature radius of W^u_x, W^s_x for $x \in \mathcal{C}$. Transitivity means that W^u_x, W^s_x are dense in $\mathcal{C}$ for all $x \in \mathcal{C}$.

The map S can be regarded, locally near x, either as a map of $\mathcal{C}$ to $\mathcal{C}$ or of W^u_x to W^u_{Sx}, or of W^s_x to W^s_{Sx}. The *jacobian matrices* of the "three" maps will be $d \times d$, $d_u \times d_u$ and $d_s \times d_s$ matrices denoted respectively $\partial S(x)$, $\partial S(x)_u$, $\partial S(x)_s$. The absolute values of the respective determinants will be denoted, respectively, $\Lambda(x)$, $\Lambda_u(x)$, $\Lambda_s(x)$ and are Hölder continuous functions, strictly positive (in fact $\Lambda(x)$ is C^∞), [S], [AA]. Likewise one can define the jacobians of the n–th iterate of S which are denoted by appending a label n to $\Lambda, \Lambda_u, \Lambda_s$ and are related to the latter by:

$$\begin{aligned}\Lambda_n(x) =& \prod_{j=0}^{n-1} \Lambda(S^j x), \quad \Lambda_{u,n}(x) = \prod_{j=0}^{n-1} \Lambda_u(S^j x), \quad \Lambda_{s,n}(S^j x) = \prod_{j=0}^{n-1} \Lambda_{s,n}(S^j x)\\ \Lambda_n(x) =&\Lambda_{u,n}(x)\,\Lambda_{s,n}(x)\chi_n(x)\end{aligned} \tag{1.1}$$

and $\chi_n(x) = \frac{\sin\alpha(S^n x)}{\sin\alpha(x)}$ is the ratio of the sines of the *angles* $\alpha(S^n x)$ and $\alpha(x)$ between W^u and W^s at the points $S^n x$ and x. Hence $\chi_n(x)$ is bounded above and below in terms of a constant $B > 0$: $B^{-1} \le \chi_n(x) \le B$, for all x (by the transversality of W^u and W^s).

A set E is a *rectangle* with *center* x and *axes* Δ^u, Δ^s if:

1) Δ^u, Δ^s are two connected surface elements of W^u_x, W^s_x containing x.

2) for any choice of $\xi \in \Delta^u$ and $\eta \in \Delta^s$ the local manfolds $W^{s,\delta}_\xi$ and $W^{u,\delta}_\eta$ intersect in one

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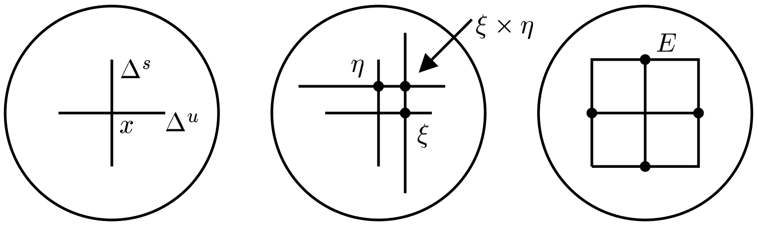
and only one point $x(\xi,\eta)=W^{s,\delta}_\xi\cap W^{u,\delta}_\eta$. The point $x(\xi,\eta)$ will also be denoted $\xi\times\eta$.
3) the boundaries $\partial\Delta^u$ and $\partial\Delta^s$ (regarding the latter sets as subsets of W^u_x and W^s_x) have zero surface area on W^u_x and W^s_x.
4) E is the set of points $\Delta^u\times\Delta^s$.

Note that *any* $x'\in E$ can be regarded as the center of E because there are Δ'^u,Δ'^s both containing x' and such that $\Delta^u\times\Delta^s\equiv\Delta'^u\times\Delta'^s$. Hence each E can be regarded as a rectangle centered at any $x'\in E$ (with suitable axes). See figure.



The circle is a small neighborhood of x; the first picture shows the axes; the intermediate picture shows the $\times$ operation and $W^{u,\delta}_\eta, W^{s,\delta}_\xi$; the third picture shows the rectangle E with the axes and the four marked points are the boundaries $\partial\Delta^u$ and $\partial\Delta^s$. The picture refers to a two dimensional case and the stable and unstable manifolds are drawn as flat, *i.e.* the Δ's are very small compared to the curvature of the manifolds. The center x is drawn in a central position, but it can be *any* other point of E provided Δ^u and Δ^s are correspondingly redefined. One should meditate on the symbolic nature of the drawing in the cases of higher dimension.

The *unstable boundary* of a rectangle E will be the set $\partial_u E=\Delta^u\times\partial\Delta^s$; the *stable boundary* will be $\partial_s E=\partial\Delta^u\times\Delta^s$. The boundary ∂E is therefore $\partial E=\partial_s E\cup\partial_u E$. The set of the *interior points* of E will be denoted E^0. A *pavement* of $\mathcal{C}$ will be a covering $\mathcal{E}=(E_1,\ldots,E_\mathcal{N})$ of $\mathcal{C}$ by $\mathcal{N}$ rectangles with pairwise disjoint interiors. The *stable (or unstable) boundary* $\partial_s\mathcal{E}$ (or $\partial_u\mathcal{E}$) of $\mathcal{E}$ is the union of the stable (or unstable) boundaries of the rectangles E_j: $\partial_u\mathcal{E}=\cup_j\partial_u E_j$ and $\partial_s\mathcal{E}=\cup_j\partial_s E_j$.

A pavement $\mathcal{E}$ is called *markovian* if its stable boundary $\partial_s\mathcal{E}$ retracts on itself under the action of S and its unstable boundary retracts on itself under the action of S^{-1}, [S], [Bo]; this means:

$$S\partial_s\mathcal{E}\subseteq\partial_s\mathcal{E},\qquad S^{-1}\partial_u\mathcal{E}\subseteq\partial_u\mathcal{E} \tag{1.2}$$

Setting $M_{j,j'}=0$, $j,j'\in\{1,\ldots,\mathcal{N}\}$, if $SE^0_j\cap E^0_{j'}=\emptyset$ and $M_{j,j'}=1$ otherwise we call C the set of sequences $\underline{j}=(j_k)^\infty_{k=-\infty}$, $j_k\in\{1,\ldots,\mathcal{N}\}$ such that $M_{j_k,j_{k+1}}\equiv1$. The transitivity of the system $(\mathcal{C},S)$ implies that M is *transitive*: *i.e.* there is a power of the matrix M with all entries positive. The space C will be called the space of the *compatible symbolic sequences*. If $\mathcal{E}$ is a markovian pavement and δ is small enough the map:

$$X:\underline{j}\in C\ \to\ x=\bigcap_{k=-\infty}^{\infty}S^{-k}E_{j_k}\in\mathcal{C} \tag{1.3}$$

is continuous and $1-1$ between the complement $\mathcal{C}_0\subset\mathcal{C}$ of the set $N=\cup^\infty_{k=-\infty}S^k\partial\mathcal{E}$ and the complement $C_0\subset C$ of $X^{-1}(N)$. This map is called the *symbolic code* of the points of

$\mathcal{C}$: it is a code that associates with each $x \notin N$ a sequence of symbols $\underline{j}$ which are the labels of the rectangles of the pavement that are successively visited by the motion $S^j x$.

The symbolic code X transforms the action of S into the *left shift* ϑ on C: $SX(\underline{j}) = X(\vartheta \underline{j})$. A key result, [S], is that it transforms the *volume measure* μ_0 on $\mathcal{C}$ into a *Gibbs distribution*, [LR], [R], $\overline{\mu}_0$ on C with formal hamiltonian:

$$H(\underline{j}) = \sum_{k=-\infty}^{-1} h_-(\vartheta^k \underline{j}) + h_0(\underline{j}) + \sum_{k=0}^{\infty} h_+(\vartheta^k \underline{j}) \tag{1.4}$$

where, see (1.1):

$$h_-(\underline{j}) = -\log \Lambda_s(X(\underline{j})), \quad h_+(\underline{j}) = \log \Lambda_u(X(\underline{j})), \quad h_0(\underline{j}) = \log \sin \alpha(X(\underline{j})) \tag{1.5}$$

If F is smooth on $\mathcal{C}$ the function $\overline{F}(\underline{j}) = F(X(\underline{j}))$ can be represented in terms of suitable functions $\Phi_k(j_{-k}, \ldots, j_k)$ as:

$$\overline{F}(\underline{j}) = \sum_{k=1}^{\infty} \Phi_k(j_{-k}, \ldots, j_k), \qquad |\Phi_k(j_{-k}, \ldots, j_k)| \le \varphi e^{-\lambda k} \tag{1.6}$$

where $\varphi > 0, \lambda > 0$. In particular $h_\pm$ (and h_0) enjoy the property (1.6) (*short range*).

If $\overline{\mu}_+, \overline{\mu}_-$ are the Gibbs states with formal hamiltonians:

$$\sum_{k=-\infty}^{\infty} h_+(\vartheta^k \underline{j}), \qquad \sum_{k=-\infty}^{\infty} h_-(\vartheta^k \underline{j}) \tag{1.7}$$

the distributions $\mu_\pm$ on $\mathcal{C}$, images of $\overline{\mu}_\pm$ via the code X in (1.3), will be the *forward* and *backward statistics* of the volume distribution μ_0 (corresponding to $\overline{\mu}_0$ via the code X), [S]. This means that:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} F(S^{\pm k} x) = \int_{\mathcal{C}} \mu_\pm(dy) F(y) \equiv \mu_\pm(F) \tag{1.8}$$

for all smooth F and for μ_0–almost all $x \in \mathcal{C}$. The distributions $\mu_\pm$ are often called the *SRB distributions*, [ER]; the above statements and (1.7),(1.8) constitute the content of a well known theorem by Sinai, [S].

An approximation theorem for μ_+ can be given in terms of the *coarse graining* of $\mathcal{C}$ generated by the markovian pavement $\mathcal{E}_T = \bigvee_{k=-T}^{T} S^{-k}\mathcal{E}$.[2] If $E_{j_{-T},\ldots,j_T} \equiv \cap_{k=-T}^{T} S^{-k} E_{j_k}$ and if $x_{j_{-T},\ldots,j_T}$ is a point, arbitrarily chosen, in the coarse grain set $E_{j_{-T},\ldots,j_T}$ we define the distribution $\mu_{T,\tau}$ by setting:

[2] Where $\vee$ denotes the operation which, given two pavements $\mathcal{E}, \mathcal{E}'$ generates a new pavement $\mathcal{E} \vee \mathcal{E}'$: the rectangles of $\mathcal{E} \vee \mathcal{E}'$ simply consist of all the intersections $E \cap E'$ of pairs of rectangles $E \in \mathcal{E}$ and $E' \in \mathcal{E}'$.

$$\mu_{T,\tau}(F) \equiv \int_{\mathcal{C}} \mu_{T,\tau}(dx)F(x) = \frac{\sum_{j_{-T},\ldots,j_T} \overline{\Lambda}_{u,\tau}^{-1}(x_{j_{-T},\ldots,j_T})F(x_{j_{-T},\ldots,j_T})}{\sum_{j_{-T},\ldots,j_T} \overline{\Lambda}_{u,\tau}^{-1}(x_{j_{-T},\ldots,j_T})}$$
$$\overline{\Lambda}_u(x) \stackrel{def}{=} \prod_{k=-\tau/2}^{\tau/2-1} \Lambda_u(S^k x) \tag{1.9}$$

Then for all smooth F it is: $\lim_{T\ge\tau/2,\,\tau\to\infty} \mu_{T,\tau}(F) = \mu_+(F)$. Note that equation (1.9) can also be written:

$$\mu_{T,\tau}(F) = \frac{\sum_{j_{-T},\ldots,j_T} e^{-\sum_{k=-\tau/2}^{\tau/2-1} h_+(\vartheta^k \underline{j}^0)} F(X(\underline{j}^0))}{\sum_{j_{-T},\ldots,j_T} e^{-\sum_{k=-\tau/2}^{\tau/2-1} h_+(\vartheta^k \underline{j}^0)}} \tag{1.10}$$

where $\underline{j}^0 \in C$ is a compatible sequence, *arbitrarily chosen*, which agrees with $j_{-T},\ldots,j_T$ between $-T$ and T. (*i.e.* $X(\underline{j}^0) = x_{j_{-T},\ldots,j_T} \in E_{j_{-T},\ldots,j_T}$).

Notation: to simplify the notations we shall write $\underline{q}$ for the elements $\underline{q} = (j_{-T},\ldots,j_T)$ of $\{1,\ldots,N\}^{2T+1}$; and $E_{\underline{q}}$ will denote $E_{j_{-T},\ldots,j_T}$ and $x_{\underline{q}}$ a point of $E_{\underline{q}}$.

Remark: Note that the weights in (1.10) depend on the special choices of the centers $x_{\underline{q}}$ (*i.e.* of $\underline{j}^0$); but if $x_{\underline{q}}$ varies in $E_{\underline{q}}$ the weight of $x_{\underline{q}}$ changes by at most a factor, bounded above by some $B < \infty$ and below by B^{-1}, for all $T \ge 0$: this is a consequence of the short range properties of h_+ expressed by (1.6)).

The last formula shows that the forward statistics of μ_0 can be regarded as a Gibbs state for a *short range one dimensional spin chain with a hard core interaction.* The spin at x is the value of $j_x \in \{1,\ldots,\mathcal{N}\}$; the short range refers to the fact that, $\Lambda_u(x)$ being smooth, the function $h_+(\underline{j}) \equiv \log \Lambda_u(X(\underline{j}))$ can be represented as in (1.6) where the Φ_k play he role of "many spins" interaction potentials and the hard core refers to the fact that the only spin configurations $\underline{j}$ allowed are those with $M_{j_k,j_{k+1}} \equiv 1$ for all integers k.

§*2. – Reversible dissipative systems and results.*

Let $(\mathcal{C},S)$ be a transitive, smooth Anosov system, see §1, and let $\Lambda(x) = |\det \partial S(x)|$; let $\mu_\pm$ be the forward and backward statistics of the volume measure μ_0 (*i.e.* the SRB distributions for S and S^{-1}).

Definition (A): The system $(\mathcal{C},S)$ *is dissipative if:*

$$-\int_{\mathcal{C}} \mu_\pm(dx) \log \Lambda(x)^{\pm 1} = \sigma_\pm > 0 \tag{2.1}$$

Definition (B): The system $(\mathcal{C},S)$ *is reversible if there is an isometric involution* $i : \mathcal{C} \longleftrightarrow \mathcal{C}$,

($i^2 = 1$), such that: $iS = S^{-1}i$.

We consider from now on only dynamical systems $(\mathcal{C}, S)$ verifying (A),(B).

Remarks:
1) (A),(B) imply $\sigma_+ = \sigma_-$ and $\Lambda(x) = \Lambda(ix)^{-1}$; furthermore $iW^u_x = W^s_{ix}$ and the dimensions of the stable and unstable manifolds d_s, d_u are equal: $d_u = d_s$ and $d = d_u + d_s$ is even.
2) if $\Lambda_u(x), \Lambda_s(x)$ denote the absolute values of the jacobian determinants of S as a map of W^u_x to W^u_{Sx} and of W^s_x to W^s_{Sx}, then $\Lambda_u(x) = \Lambda_s(ix)^{-1}$.

Definition: The dimensionless entropy production rate or the phase space contraction rate at $x \in \mathcal{C}$ and over a time τ is the function $\varepsilon_\tau(x)$:

$$x \to \varepsilon_\tau(x) = \frac{1}{\sigma_+ \tau} \sum_{j=-\tau/2}^{\tau/2-1} \log \Lambda^{-1}(S^j x) = \frac{1}{\sigma_+ \tau} \log \overline{\Lambda}^{-1}(x) \tag{2.2}$$

with $\overline{\Lambda}(x) \stackrel{def}{=} \prod_{-\tau/2}^{\tau/2-1} \Lambda(S^j x)$ (see (1.9)).

Remarks:

i) by definition (see (2.1)):

$$\langle \varepsilon_\tau \rangle_+ = \lim_{T \to +\infty} \frac{1}{T} \sum_{j=0}^{T-1} \varepsilon_\tau(S^j x) \equiv \int \mu_+(dy) \varepsilon_\tau(y) = 1 \tag{2.3}$$

with μ_0–probability 1.

ii) Note that, on the other hand, $\lim_{\tau \to \infty} \varepsilon_\tau(x) = 0$ with μ_0–probability 1, by the reversibility (B).

Here we prove the following *fluctuation theorem*:

Fluctuation theorem: There exists $p^* > 0$ such that the SRB distribution μ_+ verifies:

$$p - \delta \le \lim_{\tau \to \infty} \frac{1}{\sigma_+ \tau} \log \frac{\mu_+(\{\varepsilon_\tau(x) \in [p-\delta, p+\delta]\})}{\mu_+(\{\varepsilon_\tau(x) \in -[p-\delta, p+\delta]\})} \le p + \delta \tag{2.4}$$

for all p, $|p| < p^*$.

The above theorem was first informally proved in [GC1],[GC2] where its interest for nonequilibrium statistical mechanics was pointed out. Although I think that the physical interest of the theorem far outweighs its mathematical aspects, see also [G1], it appears that it might be useful to write the explicit and formal proof (more detailed than [G2]) described below. The theorem can be regarded as a large deviation result for the probability distribution μ_+.

The strategy that I shall follow to take advantage of the existing literature is the following. First the function (2.2) is converted to a function on the spin configurations $\underline{j} \in C$ (see §1):

$$\tilde{\varepsilon}_\tau(\underline{j}) = \varepsilon_\tau(X(\underline{j})) = \frac{1}{\tau} \sum_{k=-\tau/2}^{\tau/2-1} L(\vartheta^k \underline{j}) \tag{2.5}$$

where $L(\underline{j}) \equiv \frac{1}{\sigma_+}\log\Lambda(X(\underline{j}))^{-1}$ has a *short range* representation of the type (1.6).

The SRB distribution μ_+ can be regarded as a Gibbs state $\overline{\mu}_+$ with short range potential on the space C of the compatible symbolic sequences, associated with a Markov partition $\mathcal{E}$. Therefore there is a function $\zeta(s)$ real analytic in s for $s\in(-p^*,p^*)$ for a suitable $p^*>0$, strictly convex and such that if $p<p^*$ and $[p-\delta,p+\delta]\subset(-p^*,p^*)$ it is:

$$\frac{1}{\tau}\log\overline{\mu}_+(\{\tilde{\varepsilon}_\tau(\underline{j})\in[p-\delta,p+\delta]\})\xrightarrow[\tau\to\infty]{}\max_{s\in[p-\delta,p+\delta]}-\zeta(s) \tag{2.6}$$

and the difference between the r.h.s. and the l.h.s. tends to 0 bounded by $D\tau^{-1}$ for a suitable constant D. The function $\zeta(s)$ is the Legendre transform of the function $\lambda(\beta)$ defined as:

$$\lambda(\beta)=\lim_{\tau\to\infty}\frac{1}{\tau}\log\int e^{\tau\beta\tilde{\varepsilon}_\tau(\underline{j})}\,\overline{\mu}_+(d\underline{j}) \tag{2.7}$$

i.e. $\lambda(\beta)=\max_{s\in(-p^*,p^*)}(\beta s-\zeta(s))$, where p^* can be taken $p^*=\lim_{\beta\to+\infty}\beta^{-1}\lambda(\beta)$ and the function $\lambda(\beta)$ is a real analytic, [CO], strictly convex function of $\beta\in(-\infty,\infty)$ and asymptotically linear: $\beta^{-1}\lambda(\beta)\xrightarrow[\beta\to\pm\infty]{}\pm p^*$.

The above (2.6) is a "large deviations theorem" for one dimensional spin chains with short range interactions, [L].

Hence it will be sufficient to prove the following; if $I_{p,\delta}=[p-\delta,p+\delta]$:

$$\frac{1}{\sigma_+\tau}\log\frac{\overline{\mu}_+(\{\tilde{\varepsilon}_\tau(x)\in I_{p,\delta\pm\eta(\tau)}\})}{\overline{\mu}_+(\{\tilde{\varepsilon}_\tau(x)\in I_{-p,\delta\mp\eta(\tau)})\}}\begin{cases}<p+\delta+\eta'(\tau)\\>p-\delta-\eta'(\tau)\end{cases} \tag{2.8}$$

with $\eta(\tau),\eta'(\tau)\xrightarrow[\tau\to\infty]{}0$.

§*3. – Thermodynamic formalism (proof the fluctuation theorem).*

Let, for n odd, $\underline{j}_X=(j_x,j_{x+1},\ldots,j_{x+n-1})$ if $X=(x,x+1,\ldots,x+n-1)$, and call $\overline{X}=x+(n-1)/2$ the *center* of X. If $\underline{j}\in C$ is an infinite spin configuration we also denote $\underline{j}_X$ the set of the spins with labels $x\in X$. The left shift of the interval X will be denoted by ϑ; *i.e.* by the same symbol of the left shift of a (infinite) spin configuration $\underline{j}$.

Let $l_X(\underline{j}_X)=l^{(n)}(j_x,j_{x+1},\ldots,j_{x+n-1})$, $h^+_X(\underline{j}_X)=h^{(n)}_+(j_x,j_{x+1},\ldots,j_{x+n-1})$ be translation invariant functions, *i.e.* functions such that $l_{\vartheta X}(\underline{j})\equiv l_X(\underline{j})$ and $h^+_{\vartheta X(\underline{j})}=h^+_X(\underline{j})$, and such that the functions $h_+(\underline{j})$, see (1.6), and $L(\underline{j})$, see (2.5), can be written:

$$\begin{aligned}L(\underline{j})=&\sum_{\overline{X}=0}l_X(\underline{j}_X), & h_+(\underline{j})=\sum_{\overline{X}=0}h^+_X(\underline{j}_X)\\ |l_X(\underline{j}_X)|\le& b_1e^{-b_2n}, & |h^+_X(\underline{j}_X)|\le be^{-b'n}\end{aligned} \tag{3.1}$$

for suitable constants b_1,b_2,b,b'. Then $\tau\tilde{\varepsilon}_\tau(\underline{j})$ can be written as $\sum_{\overline{X}\in[-\tau/2,\tau/2-1]}l_X(\underline{j}_X)$.

Hence $\tau\tilde{\varepsilon}_\tau(\underline{j})$ can be approximated by $\tau\tilde{\varepsilon}^M_\tau(\underline{j})=\sum^{(M)}l_X(\underline{j}_X)$ where $\sum^{(M)}$ means summation over the sets $X\subseteq[-\frac12\tau-M,\frac12\tau+M]$, while $\overline{X}$ is in $[-\frac12\tau,\frac12\tau-1]$. The approximation is described by:

$$|\tau\tilde{\varepsilon}^M_\tau(\underline{j})-\tau\tilde{\varepsilon}_\tau(\underline{j})|\le b_3 e^{-b_4 M}\equiv\eta_0(M) \tag{3.2}$$

for suitable[3] b_3, b_4 and for all $M\ge 0$. Therefore if $I_{p,\delta}=[p-\delta,p+\delta]$ and $M=0$ we have:

$$\mu_+(\{\varepsilon_\tau(x)\in I_{p,\delta}\})\begin{cases}\le\overline{\mu}_+(\{\tilde{\varepsilon}^0_\tau\in I_{p,\delta+b_3/\tau}\})\\ \ge\overline{\mu}_+(\{\tilde{\varepsilon}^0_\tau\in I_{p,\delta-b_3/\tau}\})\end{cases} \tag{3.3}$$

It follows from the general theory of 1–dimensional Gibbs distributions, [R2], that the $\overline{\mu}_+$–probability of a spin configuration coinciding with $\underline{j}_{[-\tau/2,\tau/2]}$ in the interval $[-\frac12\tau,\frac12\tau]$,[4] is:

$$\frac{\left[e^{-\sum^* h^+_X(\underline{j}_X)}\right]}{\sum_{\underline{j}'_{[-\tau/2,\tau/2]}}\left[\,\cdot\,\right]}\,P(\underline{j}_{[-\tau/2,\tau/2]}) \tag{3.4}$$

where $\sum^*$ denotes summation over all the $X\subset[-\tau/2,\tau/2-1]$; the denominator is just the sum of terms like the numerator, evaluated at a generic (compatible) spin configuration $\underline{j}'_{[-\tau/2,\tau/2]}$; finally P verifies the bound, [R2]:

$$B_1^{-1}<P(\underline{j}_{[-\tau/2,\tau/2]})<B_1 \tag{3.5}$$

with B_1 a suitable constant independent of $\underline{j}_{[-\tau/2,\tau/2]}$ and of τ (B_1 can be explicitly estimated in terms of b,b'). Therefore from (3.3) and (3.4) we deduce for any $T\ge\tau/2$:

$$\begin{aligned}&\mu_+(\varepsilon_\tau(x)\in I_{p,\delta})\le\overline{\mu}_+(\tilde{\varepsilon}^0_\tau\in I_{p,\delta+b_3/\tau})\le\\ &\le B_2\,\mu_{T,\tau}(\tilde{\varepsilon}^0_\tau\in I_{p,\delta+b_3/\tau})\le B_2\,\mu_{T,\tau}(\tilde{\varepsilon}_\tau\in I_{p,\delta+2b_3/\tau})\end{aligned} \tag{3.6}$$

for some constant $B_2>0$; and likewise a lower bound is obtained by replacing B_2 by B_2^{-1} and b_3 by $-b_3$.

Then if $p<p^*$ and $I_{p,\delta}\subset(-p^*,p^*)$ the set of the rectangles $E\in\bigvee_{-T}^T S^{-k}\mathcal{E}$ with center x such that $\varepsilon_\tau(x)\in I_{p,\delta}$ is *not empty.*

We immediately deduce the lemma:

Lemma 1: the distributions μ_+ and $\mu_{T,\tau}$, $T\ge\frac12\tau$, verify:

$$\frac{1}{\tau\sigma_+}\log\frac{\mu_+(\varepsilon_\tau(x)\in I_{p,\delta\mp 2b_3/\tau})}{\mu_+(\varepsilon_\tau(x)\in -I_{p,\delta\pm 2b_3/\tau})}\;\begin{cases}<\frac{\log B_2^2}{\tau\sigma_+}+\frac{1}{\tau\sigma_+}\log\frac{\mu_{T,\tau}(\tilde{\varepsilon}_\tau\in I_{p,\delta})}{\mu_{T,\tau}(\tilde{\varepsilon}_\tau\in -I_{p,\delta})}\\ >-\frac{\log B_2^2}{\tau\sigma_+}+\frac{1}{\tau\sigma_+}\log\frac{\mu_{T,\tau}(\tilde{\varepsilon}_\tau\in I_{p,\delta})}{\mu_{T,\tau}(\tilde{\varepsilon}_\tau\in -I_{p,\delta})}\end{cases} \tag{3.7}$$

for $I_{p,\delta}\subset[-p^*,p^*]$ and for τ so large that $p+\delta+2b_3/\tau<p^*$.

Hence (2.8) will follow if we can prove:

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[3] One can check from (3.1), that the constants b_3, b_4 can be expressed as simple functions of b_1, b_2.

[4] *i.e.* the spin configurations $\underline{j}'$ such that $j'_x=j_x$, $x\in[-\frac12\tau,\frac12\tau]$.

Lemma 2: there is a constant $\overline{b}$ such that the approximate SRB distribution $\mu_{T,\tau}$ verifies:

$$\frac{1}{\sigma_+\tau}\log\frac{\mu_{T,\tau}(\tilde\varepsilon\in I_{p,\delta})}{\mu_{T,\tau}(\tilde\varepsilon\in -I_{p,\delta})}\begin{cases}\le p+\delta+\overline{b}/\tau\\ \ge p-\delta-\overline{b}/\tau\end{cases}\tag{3.8}$$

for τ large enough (so that $\delta+\overline{b}/\tau<p^*-p$) and for all $T\ge\tau/2$.

The latter lemma will be proved in §4 and it is the only statement that does not follow from the already existing literature.

§4. – *Time reversal symmetry and large deviations.*

The relation (3.7) holds for any choice of the Markov partition $\mathcal{E}$. Note that if $\mathcal{E}$ is a Markov pavement also $i\mathcal{E}$ is such (because $iS=S^{-1}i$ and $iW^u_x=W^s_{ix}$); furthermore if $\mathcal{E}_1$ and $\mathcal{E}_2$ are Markov pavements also $\mathcal{E}=\mathcal{E}_1\vee\mathcal{E}_2$ are markovian. Therefore:

Lemma 3: there exists a time reversal Markov pavement $\mathcal{E}$, i.e. a Markov pavement such that $\mathcal{E}=i\mathcal{E}$.

This can be seen by taking any Markov pavement $\mathcal{E}_0$ and setting $\mathcal{E}=\mathcal{E}_0\vee i\mathcal{E}_0$. Alternatively one could construct the Markov pavement in such a way that it verifies automatically the symmetry [G2]. Since the center of a rectangle $E_{\underline{q}}\in\mathcal{E}_T$ can be taken to be any point $x_{\underline{q}}$ in the rectangle $E_{\underline{q}}$ we can and shall suppose that the centers of the rectangles in $\mathcal{E}_T$ have been so chosen that the center of $iE_{\underline{q}}$ is $ix_{\underline{q}}$, i.e. the time reversal of the center $x_{\underline{q}}$ of $E_{\underline{q}}$.

For τ large enough the set of configurations $\underline{q}=\underline{j}_{[-T,T]}$ such that $\varepsilon_\tau(x)\in I_{p,\delta}$ for all $x\in E_{\underline{q}}$ is not empty[5] and the ratio in (3.8) can be written, if $x_{\underline{q}}$ is the center of $E_{\underline{q}}\in\bigvee_{-T}^{T}\mathcal{E}$, as:

$$\frac{\sum_{\varepsilon_\tau(x_{\underline{q}})\in I_{p,\delta}}\overline{\Lambda}^{-1}_{u,\tau}(x_{\underline{q}})}{\sum_{\varepsilon_\tau(x_{\underline{q}})\in -I_{p,\delta}}\overline{\Lambda}^{-1}_{u,\tau}(x_{\underline{q}})}=\frac{\sum_{\varepsilon_\tau(x_{\underline{q}})\in I_{p,\delta}}\overline{\Lambda}^{-1}_{u,\tau}(x_{\underline{q}})}{\sum_{\varepsilon_\tau(x_{\underline{q}})\in I_{p,\delta}}\overline{\Lambda}^{-1}_{u,\tau}(i\,x_{\underline{q}})}\tag{4.1}$$

But the time reversal symmetry implies that $\overline{\Lambda}_{u,\tau}(x)=\overline{\Lambda}^{-1}_{s,\tau}(ix)$, see remark 2) following definition (B), §2.[6] This permits us to change (4.1) into:

$$\frac{\sum_{\varepsilon_\tau(x_{\underline{q}})\in I_{p,\delta}}\overline{\Lambda}^{-1}_{u,\tau}(x_{\underline{q}})}{\sum_{\varepsilon_\tau(x_{\underline{q}})\in I_{p,\delta}}\overline{\Lambda}_{s,\tau}(i\,x_{\underline{q}})}\begin{cases}<\max_{\underline{q}}\overline{\Lambda}^{-1}_{u,\tau}(x_{\underline{q}})\overline{\Lambda}^{-1}_{s,\tau}(x_{\underline{q}})\\ >\min_{\underline{q}}\overline{\Lambda}^{-1}_{u,\tau}(x_{\underline{q}})\overline{\Lambda}^{-1}_{s,\tau}(x_{\underline{q}})\end{cases}\tag{4.2}$$

where the maxima are evaluated as $\underline{q}$ varies with $\varepsilon_\tau(x_{\underline{q}})\in I_{p,\delta}$.

By (1.1) we can replace $\overline{\Lambda}^{-1}_{u,\tau}(x)\overline{\Lambda}^{-1}_{s,\tau}(x)$ with $\overline{\Lambda}^{-1}_{\tau}(x)B^{\pm1}$, see (1.9), (2.2); thus noting that

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[5] Note that $p^*=\sup_x\limsup_{\tau\to+\infty}\varepsilon_\tau(S^{\tau/2}x)$ and let $p\in(-p^*+\delta,p^*-\delta)$; furthermore $\zeta(s)$ is smooth, hence $>-\infty$, for all $|s|<p^*$.

[6] here it is essential that $\overline{\Lambda}_{u,\tau}(x)$ is the expansion of the unstable manifold between the initial point $S^{-\tau/2}x$ and the final point $S^{\tau/2}x$ which is a trajectory of time length τ, which at its central time is in x.

by definition of the set of $\underline{q}$'s in the maximum operation in (4.2) it is $\frac{1}{\sigma_+\tau}\log\overline{\Lambda}_\tau^{-1}(x_{\underline{q}})\in I_{p,\delta}$, we see that (3.8) follows with $\overline{b}=\frac{1}{\sigma_+\tau}\log B$.

Corollary: the above analysis gives us a concrete bound on the speed at which the limits in (2.4) are approached. Namely the error has order $O(\tau^{-1})$.

This is so because the limit (2.6) is reached at speed $O(\tau^{-1})$; furthermore the regularity of $\lambda(s)$ in (2.6) and the size of $\eta(\tau),\eta'(\tau)$ and the error term in (3.8) have all order $O(\tau^{-1})$.

The above analysis proves a large deviation result for the probability distribution μ_+: since m_+ is a Gibbs distribution, see (1.7), various other large deviations theorems hold fo them, [DV], [E], [O], but unlike the above they are not related to the time reversal symmetry.

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