Multiplicative cascade models and multifractality

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Abstract. We construct a (chaotic) deterministic variant of random multiplicative cascade models of turbulence. It preserves the hierarchical tree structure, thanks to the addition of infinitesimal noise or finite-state Markov approximations of chaotic maps. The zero-noise limit can be handled by Perron-Frobenius theory, just as the zero-diffusivity limit for the fast dynamo problem. We prove also the absence of phase transitions in conservative random multiplicative cascade models, corresponding to the non divergence of statistical moments.

Keywords. Multiplicative cascade, chaotic map, phase transition

1 Introduction

One popular way to describe the small scale activity of fully developed turbulence is to suppose that energy is transferred from the injection scale to the viscous scales, throughout a multi-steps process along the inertial range. In spite of the fact that this idea has been often used to predict many important features of turbulent flows, its relations with the structure of Navier-Stokes equations are still poorly understood. We construct a theory describing *deterministic* multiplicative cascade models and prove the absence of phase transitions in conservative *random* multiplicative cascade models.

In order to construct the deterministic model for the multiplicative cascade of energy we use shell models for fully developed turbulence and we connect the well known case of random independent multiplicative cascade model (the simplest model of the multifractality) and our case (deterministic) in two steps through introducing at first Markov random model and then by means of a special limit construction going to the deterministic model. There are two ways of this construction. The first one is to add a small amount of noise to the

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deterministic model and to use it as a usual Markov chain in the Markov random multiplicative cascade model. Then the deterministic model is the zero noise limit of this construction. The second way is to use an approximation of a chaotic dynamics by a finite-state random Markov chain, which states corresponds to elements of a partition of the phase space. The deterministic model in that case is obtained as a limit of the finer and finer partitions.

Random multiplicative models were introduced by Novikov and Stewart and by Yaglom as a simple way to describe stochastic transfer of energy along the inertial range. To define a random multiplicative model, a binary tree structure, obtained by hierarchically partitioning the original volume of size l_0 in subvolumes of size $l_n = 2^{-n} l_0$, they used to describe fluctuations at different scales. The energy dissipation, ε_n, associated to a cube at scale l_n, is multiplicatively linked to the energy dissipation, ε_{n-1}, at the larger scale, l_{n-1}, through a random variable ξ_n:

$$\varepsilon_n = \xi_n \varepsilon_{n-1} = \xi_n \xi_{n-1} \xi_{n-2} .... \xi_1 \bar{\varepsilon},$$

where $\{\xi_n\}$'s are identically and independently distributed positive random variables. The structure functions are now defined as

$$S_n(q) := E\{\varepsilon_n^{q/3}\} l_n^{q/3}, \tag{1}$$

where $E\{.\}$ denotes the mathematical expectation.

2 Construction of the deterministic multiplicative cascade

Our aim now is to construct a deterministic variant of the previous model. We shall do it in two steps. First, we consider, instead of independent random variables $\{\xi_n\}$, successive points on the orbit of a Markov process. This means that we consider a Markov process on a phase space X with a transition probability operator P and an observable $h : X \to R^1_+$. The (generalized) structure function at the scale $l_n = 2^{-n} l_0$ is then defined as:

$$S_n(q) = S_n(q, h, P) := E\left\{\prod_{k=1}^{n} h^q(x_k)\right\}, \tag{2}$$

where $x_k \in X$ are points of an orbit of the Markov process. Introducing now:

$$\zeta_n(q) = \prod_{k=1}^{n} h^q(x_k), \tag{3}$$

and the conditional expectation of $\zeta_n(q)$ with respect to the initial distribution density, $\rho(x)$, is given by:

$$E\{\zeta_n(q)|\rho\} = \int_X h^q(x)\rho(x) P^n_{h,q}\, 1(x)\, dx.$$

Here, the operator $P_{h,q}$ is defined by the relation $P_{h,q}\phi(x) = h^q(x)P\phi(x)$. The second step is to change from a Markov *random* dependence to a *deterministic* dependence by use of a chaotic map $f : X \to X$. It seems that this can be done in an obvious way, replacing the transition probability operator by the transfer-matrix (Perron-Frobenius operator) P_f of the chaotic map. However, with a deterministic map, we should give the same value to the two off-springs of the next generation, thereby trivializing the whole tree-structure.

To solve this problem, we build up this branching-deterministic process by inserting a small amount of noise at each node and by considering the total process as the superposition of the deterministic transfer along consecutive levels plus the noise. The final map will be obtained by taking the zero-noise limit. For a fixed value of $\varepsilon > 0$ it follows that the large-n limit of the structure functions is governed by the spectrum of the operator:

$$P_{f,h,q,\varepsilon} = h^q Q_\varepsilon P_f.$$

Here, Q_ε is the transition operator for the random perturbation.

There is also another possibility to define a deterministic (chaotic) process on a tree structure. Using a method of finite-state Markov approximations of chaotic maps, proposed by Ulam, we obtain a Markov chain with transition probabilities given by

$$p_{ij} = \frac{|X_i \cap f^{-1}X_j|}{|X_i|},$$

where the set of X_i's defines a suitable finite partition of the original phase space X. With this Markov process, we construct the analog of the operator $P_{h,q} = h^q P$. (Here P is the transition operator with matrix elements p_{ij}.)

3 Stability of statistics

We now observe that the construction based on taking the zero-noise limit appearing above is similar to that of the mathematical theory of *fast dynamos.* Fast dynamo theory describes the phenomenon by which rapid magnetic field growth can be sustained in the presence of a prescribed velocity field, when taking the zero-diffusivity limit. From a formal point of view, fast dynamo theory involves a combination of two operators: a transfer-matrix for some deterministic map, associated to a deterministic velocity field, and a diffusion-like operator, or equivalently small-amplitude noise.

In our paper [1], assuming that the Perron-Frobenius operator, corresponding to the map f is stochastically stable, and the map is chaotic and ergodic, we computed the structure functions as averages along-the-orbit for the deterministic map, i.e.

$$S_n(q) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} h^q(x_{i+k})$$

with $x_{i+1} = f(x_i)$. However, in the general case this is not true and the aim of this paper is to give assumptions under which the structure function could be calculated in the way above. Let f be a piecewise expanding (PE) one-dimensional map, i.e. the function f is piecewise monotonic and the derivative of some its power is strictly larger than 1 (see for details [2]). By singular points of the map f we shall mean all the points $x \in X$, where the derivative $f'(x)$ is not well defined (for instance, all boundary points of its monotonicity intervals). Denote by $q_\varepsilon(x,y)$ the transition probability density of random perturbations. Here ε is the perturbation "amplitude". We shall suppose that $\mathrm{var}(q_\varepsilon(x,.)) < C/\varepsilon$. This assumption is quite general and clearly is valid for independent uniformly distributed perturbations.

Theorem 1 *Let a map f be a PE map without periodic singular points. Then the map f is stochastically stable.*

Theorem 2 *Let a map f be an arbitrary PE map. If random perturbations satisfy the condition*

$$\int_{-\infty}^{x-C\varepsilon} q_\varepsilon(x,y)\,dy > \delta \quad \text{and} \quad \int_{x+C\varepsilon}^{\infty} q_\varepsilon(x,y)\,dy > \delta$$

for some positive C, δ and any $\varepsilon > 0$, then the map f is stochastically stable. If this condition is not satisfied, then there exists a family of small random perturbations, stabilizing unstable periodic orbits of the map.

Theorem 3 *Finite-state Markov approximations of PE maps are stochastically stable (even for the case of periodic singular points).*

4 Absence of phase transitions in conservative multiplicative cascade models

The shortcoming of our construction is that we, following the tradition in the physical literature, consider the mathematical expectation of the structure function (1,2), rather than the random structure function itself (3). We discuss now this question for random measures constructed by means of multiplicative cascade models. Consider a sequence of nonnegative random values $\{\xi_{ij}\}$ with $E\{\xi^q_{n,2k+1}\} = x_q$, $E\{\xi^q_{n,2k}\} = y_q$, $E\{\xi^q_{n,2k+1}\xi^q_{n,2k+2}\} = z_{2q}$, $X_q = x_q + y_q$, distributed on nodes of the binary tree:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \xi_{11} & & \xi_{12} & & \\ & \xi_{21} & \xi_{22} & & \xi_{23} & \xi_{24} & \\ \xi_{31} & \xi_{32} & & \dots & & \xi_{37} & \xi_{38} \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \end{array} \qquad (4)$$

Our assumptions correspond to the nature of the cascade process, because subtrees of the binary tree (4) of the same volume should have the same statistical properties. However, in the general case there is no need to suppose that these random values are uniquely identically distributed, or independent.

On the n-th level of the tree (4) there are 2^n nodes. We correspond these nodes to elements of the hierarchical partition of the initial volume Δ into 2^n disjoint subvolumes Δ_i of equal size. Consider a random measure μ_n, such that for any i the value $\mu_n(\Delta_i)$ is equal to the multiplication of all random values on the nodes along the branch from the root of the tree to the considered node. Under some weak assumptions (see, for example, [3]) these random measures μ_n converge (in the weak topology) to a deterministic limit measure μ_∞. This construction is known as a *multiplicative cascade model.*

Consider now the following functionals depending on a variable q:

$$\zeta_n(q) := \zeta_n(\mu_n, q) = \sum_i \mu_n^q(\Delta_i), \quad \zeta_\infty(q) := \zeta_n(\mu_\infty, q) \tag{5}$$

$$\Phi(q) := \lim_{n\to\infty} \frac{\log(\zeta_n(q))}{\log(2^n)}, \quad \Phi_\infty(q) := \lim_{n\to\infty} \frac{\log(\zeta_n(\mu_\infty, q))}{\log(2^n)}. \tag{6}$$

Our aim is to investigate the existence of the limit above and the properties of these functionals as functions of the variable q. We shall consider phase transitions in the weak sense as the divergence of the series (6), while in the strict sense the latter means that the function $\Phi(q)$ differs from $\Phi_\infty(q)$ (see [3]) and thus the statistics of the random measures μ_n do not converge to the corresponding statistics of the limit measure μ_∞. In [3] it was shown that the problem under consideration may be considered as a problem of statistical mechanics, where the measure μ_∞ plays the role of the *free energy* and the parameter q corresponds to the *inverse temperature.* The absence of phase transitions corresponds to the self-averaging property of the free energy, which characterizes the unique random state.

The multiplicative model is well known for the case of independent uniquely distributed random values ξ_{ij} with $E\{\xi_{ij}\} = 1/2$ and $x_q = y_q$, $z_{2q} = x_q^2$, and it is known [3] that there exists a critical value q_c (may be infinite), such that only while $q < q_c$ the convergence takes place. It is not very hard to construct a binomial distribution for ξ_{ij}, such that the critical value q_c will be really small (of order 1). So in multiplicative cascades with independent random values, the phase transitions are unavoidable.

Two first statistical moments of the random variable $\zeta_n(q)$ are equal to

$$E\{\zeta_n(q)\} = X_q^n \tag{7}$$

$$E\{\zeta_n^2(q)\} = X_{2q}^n + 2z_{2q}\Big(X_{2q}^{n-1} + X_{2q}^{n-2}X_q^2 + \ldots + X_{2q}^{n-k}X_q^{2(k-1)} + \ldots$$
$$+X_{2q}X_q^{2(n-2)} + X_q^{2(n-1)}\Big) = X_{2q}^n + 2z_{2q}\frac{X_{2q}^n - X_q^{2n}}{X_{2q} - X_q^2}. \tag{8}$$

Thus

$$D\{\zeta_n(q)\} := E\{(\zeta_n(q))^2\} - (E\{\zeta_n(q)\})^2 = 2z_{2q}\frac{X_{2q}^n - X_q^{2n}}{X_{2q} - X_q^2}, \qquad (9)$$

where $D\{.\}$ is the second central moment.

If $\zeta_n(q)$ converges to its mathematical expectation and one can exchange the latter with the limit operation, we have:

$$\Phi(q) := \lim_{n\to\infty}\frac{\log(\zeta_n(q))}{\log(2^n)} = \lim_{n\to\infty}\frac{\log((x_q + y_q)^n)}{\log(2^n)} = \log_2(x_q + y_q). \qquad (10)$$

In the case of independent uniquely distributed random variables one can suppose only that $X_1 = 2x_1 = 1$, which shows that the statistical moments can diverge for large q. Indeed, $X_q = 2x_q$ may be larger than 1, and thus the second central moments will diverge. Actually this does not prove the absence of the limit in (10), which was proved using quite different martingale technics in [3]. However, due to Chebishev's inequality, (10) shows that while $X_q^n \to 0$ as $n \to \infty$ we have the convergence the second central moments to zero, and thus $\zeta_n(q)$ converges to its mathematical expectation.

Consider now a *conservative* cascade. This means that for any n the measure μ_n is probabilistic, i.e. $\mu_n(\Delta) = 1$, which is not the case with independent random values (one only can assume there that $E\{\mu_n(\Delta)\} = 1$). To construct such a conservative cascade suppose that $\xi_{i,2k+2} = 1 - \xi_{i,2k+1}$ for any k, and random values $\xi_{i,2k+1} \in (0,1)$ are independent uniquely distributed for all i and k. Actually, every time we put a random value ξ to the left node, and $1 - \xi$ to the right node. In this case for any $q > 1$ we have

$$X_q = x_q + y_q = E\{\xi^q\} + E\{(1-\xi)^q\} < 1, \qquad (11)$$

which gives the convergence of the second central moments to zero, and thus, by Chebishev's inequality, the convergence of $\zeta_n(q)$ to its mathematical expectation, and thus leads to the absence of phase transitions.

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