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Integro-Difference Equation for a correlation function of the spin-$\frac{1}{2}$ Heisenberg XXZ chain

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ABSTRACT

We consider the *Ferromagnetic-String-Formation-Probability* correlation function (FSFP) for the spin-$\frac{1}{2}$ Heisenberg XXZ chain. We construct a completely integrable system of integro-difference equations (IDE), which has the FSFP as a τ-function. We derive the associated Riemann-Hilbert problem and obtain the large distance asymptotics of the FSFP correlator in some limiting cases.

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1 Introduction

In this paper we continue our investigation of a particular zero-temperature correlation function of the XXZ Heisenberg model in the critical regime $-1 < \Delta < 1$ in an external magnetic field. The XXZ hamiltonian is given by

$$\mathcal{H} = \sum_j \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta\ (\sigma_j^z \sigma_{j+1}^z - 1) - h\sigma_j^z\ , \tag{1.1}$$

where the sum is over all integers j, σ^α are Pauli matrices and h is an external magnetic field. For later convenience we define $\Delta = \cos(2\eta) = \frac{w+\frac{1}{w}}{2}$, where $\frac{\pi}{2} < \eta < \pi$. The FSFP correlation function is defined as follows

$$P(m) = \langle 0|\ \prod_{j=1}^{m} P_j\ |0\rangle, \tag{1.2}$$

where $|0\rangle$ is the antiferromagnetic ground state and $P_j = \frac{1}{2}(\sigma_j^z + 1)$ is the projection operator onto the state with spin up at site number j. The physical meaning of $P(m)$ is the probability of finding a ferromagnetic string (i.e. m (adjacent) parallel spins up) in the ground state $|0\rangle$ of the model (1.1) for a given value of the magnetic field h. From a mathematical point of view this correlator turns out the simplest one to be considered (see [1]).

In a recent paper [2] we have derived a representation of the correlation function (1.2) as a determinant of a Fredholm integral operator

$$P(m) = \frac{\left(0\left|\det\left(1 + \widehat{V}^{(m)}\right)\right|0\right)}{\det\left(1 - \frac{1}{2\pi}\widehat{K}\right)}\ . \tag{1.3}$$

Here $\widehat{K}$ and $\widehat{V}^{(m)}$ are integral operators acting on a function $f(\lambda)$ as

$$\left(\widehat{V}^{(m)} f\right)(\lambda) = \int_{-\Lambda}^{\Lambda} d\mu\ V^{(m)}(\lambda|\mu) f(\mu)\ , \quad \left(\widehat{K} f\right)(\lambda) = \int_{-\Lambda}^{\Lambda} d\mu\ K(\lambda|\mu) f(\mu)\ , \tag{1.4}$$

where Λ is a function of the anisotropy Δ and the magnetic field h. The kernels are given by (as compared to [2] we have performed a similarity transformation, which leaves $\det(1 + \widehat{V}^{(m)})$ invariant but changes the kernel $V^{(m)}(\lambda|\mu)$)

$$V^{(m)}(\lambda|\mu) = -\frac{1}{2\pi}\frac{\sin(2\eta)}{\sinh(\lambda-\mu)}\left(\frac{e_+^{(m)}(\lambda)e_-^{(m)}(\mu)}{(\sinh(\lambda-\mu+2i\eta)} - \frac{e_-^{(m)}(\lambda)e_+^{(m)}(\mu)}{\sinh(\mu-\lambda+2i\eta)}\right)\ , \tag{1.5}$$

$$K(\lambda|\mu) = \frac{\sin(4\eta)}{\sinh(\lambda-\mu+2i\eta)\sinh(\lambda-\mu-2i\eta)} \tag{1.6}$$

with functions $e_{\pm}^{(m)}(\lambda) = \left(\left(\frac{\sinh(\lambda+i\eta)}{\sinh(\lambda-i\eta)}\right)^m e^{\phi(\lambda)}\right)^{\pm\frac{1}{2}}$. Here $\phi(\lambda)$ is a *dual quantum field* acting in a bosonic Fock space with vacuum $|0)$. The fields ϕ are related to canonical Bose fields a and $a^\dagger$ (annihilating the vacuum and its dual respectively, *i.e.* $a(\lambda)|0) = 0 = (0|a^\dagger(\lambda))$ via the Bogoliubov transformation

$$\phi(\lambda) = a(\lambda) - \int_{-\infty}^{\infty} d\nu \ln\Big[h(\lambda,\nu)h(\nu,\lambda)\Big] a^\dagger(\nu)\ , \tag{1.7}$$

where $h(\lambda,\nu) = \frac{\sinh(\lambda-\nu+2i\eta)}{i\sin(2\eta)}$. The occurrence of the dual fields is a consequence of the interacting nature of the Heisenberg model (see Section XI.1 of [1]). By construction the fields $\phi(\lambda)$ commute for all values of spectral parameter $[\phi(\lambda), \phi(\mu)] = 0$. The contribution of the dual fields (to the expectation value $(0|\det(1+\hat{V}^{(m)})|0)$) can be visualized by decomposing all dual fields ϕ according to (1.7) and then moving all exponentials of annihilation operators $a(l)$ to the right, picking up contributions whenever passing by an $a^\dagger(\nu)$.

The purpose of the current paper is to derive a system of integro-difference equations that drive the Fredholm determinant of the operator $\widehat{V}^{(m)}$ in the correlation function $P(m)$ (1.3). This is done in Section 2 and can be considered as an extension of the idea of describing quantum correlation function by means of differential equations [Painlevé] due to E. Barouch, B.M. McCoy, T.T. Wu and C. Tracy and B.M. McCoy [3, 4]. The Fredholm determinant $\det\left(1 - \frac{1}{2\pi}\widehat{K}\right)$ in the denominator of (1.3) amounts merely to an overall normalization independent of the distance m and will not be considered here. In Section 3 the Riemann-Hilbert problem associated with the IDE is formulated. Finally, in Section 4, we determine the long-distance asymptotics of the FSFP in some limiting cases.

2 Integro-Difference Equations

We start by bringing the kernel of $\widehat{V}$ (1.5) to "standard" form [1]. We first perform a change of variables $z = \frac{e^{2\lambda}w-1}{e^{2\lambda}-w}$ (recall that $w = e^{2i\eta}$), which maps the real axis on the contour $C : \alpha \to z = \exp i\alpha$ (see Fig. 1) where $-\psi < \alpha < 2\pi + \psi$ ($\psi < 0$ by definition). The endpoints $\xi = e^{i\psi}$ and $\bar{\xi} = e^{-i\psi}$ (we integrate from $\bar{\xi}$ to ξ) of the contour are related to the magnetic field h *and* the anisotropy Δ. Using the identity (valid for $\frac{\pi}{2} < \eta < \pi$

and $z_1, z_2 \in C$)

$$\int_0^\infty ds\ e^{-i(w\frac{wz_1-1}{z_1-w}-\frac{1}{w}\frac{wz_2-1}{z_2-w})s} = \frac{-i}{w\frac{wz_1-1}{z_1-w}-\frac{1}{w}\frac{wz_2-1}{z_2-w}}\ , \tag{2.1}$$

the kernel of $\widehat{V}^{(m)}$ is found to be (up to a similarity transform which leaves the determinant unchanged)

$$V^{(m)}(z_1|z_2) = -\frac{i}{2\pi}\int_0^\infty ds\ \frac{e_+^{(m)}(z_1|s)e_-^{(m)}(z_2|s) - e_-^{(m)}(z_1|s)e_+^{(m)}(z_2|s)}{z_1 - z_2}\ . \tag{2.2}$$

where the functions $e_\pm^{(m)}$ are given by

$$\begin{aligned} e_+^{(m)}(z|s) &= \left(i(w-1/w)\ \frac{wz-1}{z-w}\ z^{-m}e^{-\phi(z)}\right)^{\frac{1}{2}} \exp\left(\frac{i}{w}\ \frac{wz-1}{z-w}\ s\right) \\ e_-^{(m)}(z|s) &= \left(i(w-1/w)\ \frac{wz-1}{z-w}\ z^{m}e^{\phi(z)}\right)^{\frac{1}{2}} \exp\left(-iw\ \frac{wz-1}{z-w}\ s\right) \end{aligned} \tag{2.3}$$

The integral operator $\widehat{V}^{(m)}$ now acts on a function $f(z)$ as

$$\left(\widehat{V}^{(m)}f\right)(z_1) = \int_C dz_2 V^{(m)}(z_1|z_2)f(z_2) \tag{2.4}$$

where the integration is to be performed along the contour C. We note that V is symmetric and nonsingular at $z_1 = z_2$. The resolvent $\widehat{R}^{(m)}$ of $\widehat{V}^{(m)}$ is defined by $(1+\widehat{V}^{(m)})(1-\widehat{R}^{(m)}) = 1$ and its kernel $R^{(m)}(z_1|z_2)$ can be written in a form similar to Eq. (2.2), namely [1]

$$R^{(m)}(z_1|z_2) = -\frac{i}{2\pi}\int_0^\infty ds\ \frac{f_+^{(m)}(z_1|s)f_-^{(m)}(z_2|s) - f_-^{(m)}(z_1|s)f_+^{(m)}(z_2|s)}{z_1 - z_2}\ . \tag{2.5}$$

Here $f_\pm^{(m)}$ are solutions of the linear integral equations $\left((1+\widehat{V}^{(m)})f_\pm^{(m)}\right)(z|s) = e_\pm^{(m)}(z|s)$. In terms of these functions we introduce the integral operators $B_{ab}^{(m)}$, $a, b = \pm$ acting as $(B_{ab}^{(m)}f^{(m)})(s) = \int_0^\infty dt\ B_{ab}^{(m)}(s,t)f(t)$ with the kernel

$$B_{ab}^{(m)}(s,t) = \frac{i}{2\pi}\int_C \frac{dz}{z} f_a^{(m)}(z|s)e_b^{(m)}(z|t)\ , \qquad a, b = \pm\ . \tag{2.6}$$

The *transpose* B^T acts like $\left(B^T f\right)(s) = \int_0^\infty dt\ B(t,s)f(t)$. We are now in the position to formulate the main

Theorem :

(i) The lattice logarithmic derivative of $\det\left(1+\widehat{V}^{(m)}\right)$ *is given in terms of the integral operator* $B_{ab}^{(m)}$ *as*

$$\frac{\det\left(1+\widehat{V}^{(m+1)}\right)}{\det\left(1+\widehat{V}^{(m)}\right)} = \det\left(1+B_{-+}^{(m)}\right) \tag{2.7}$$

(ii) The logarithmic derivative of $\det\left(1+\widehat{V}^{(m)}\right)$ *with respect to the boundaries of the contour* C *is expressed in terms of the functions* $F_\pm^{(m)}(s) = f_\pm^{(m)}(\xi|s)$ *and* $G_\pm^{(m)}(s) = f_\pm^{(m)}(\bar{\xi}|s)$ *as follows*

$$\begin{aligned} -i\partial_\psi \ln\det\left(1+\widehat{V}\right) &= \frac{1}{2\pi}\int_0^\infty ds\, \Big\{F_+(s)\partial_\psi F_-(s) - F_-(s)\partial_\psi F_+(s) \\ &\qquad + G_-(s)\partial_\psi G_+(s) - G_+(s)\partial_\psi G_-(s)\Big\} \\ &\quad + \frac{1}{4\pi^2}\,\frac{\xi+\bar{\xi}}{\xi-\bar{\xi}}\,\left(\int_0^\infty ds\;\left[F_+(s)G_-(s) - F_-(s)G_+(s)\right]\right)^2 \end{aligned} \tag{2.8}$$

(iii) The following set of completely integrable integro-difference equations for the unknowns F, G, B *in (i) and (ii) holds*

$$\begin{aligned} \frac{1}{\sqrt{\xi}} F_+^{(m+1)} &= \frac{1}{\xi}\left\{F_+^{(m)} - \left(1 - B_{+-}^{(m+1)}\right)\left(B_{++}^{(m)}\right)^T F_-^{(m)}\right\} \\ \frac{1}{\sqrt{\xi}} F_-^{(m+1)} &= \frac{1}{\xi}\left\{\left(\xi + B_{--}^{(m+1)}\left(B_{++}^{(m)}\right)^T\right) F_-^{(m)} - B_{--}^{(m+1)}\left(1 + \left(B_{-+}^{(m)}\right)^T\right) F_+^{(m)}\right\} \\ \frac{1}{\sqrt{\bar{\xi}}} G_+^{(m+1)} &= \frac{1}{\bar{\xi}}\left\{G_+^{(m)} - \left(1 - B_{+-}^{(m+1)}\right)\left(B_{++}^{(m)}\right)^T G_-^{(m)}\right\} \\ \frac{1}{\sqrt{\bar{\xi}}} G_-^{(m+1)} &= \frac{1}{\bar{\xi}}\left\{\left(\bar{\xi} + B_{--}^{(m+1)}\left(B_{++}^{(m)}\right)^T\right) G_-^{(m)} - B_{--}^{(m+1)}\left(1 + \left(B_{-+}^{(m)}\right)^T\right) G_+^{(m)}\right\} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} -i\frac{\partial}{\partial\psi} B_{ab}(s,t) &= \frac{i}{2\pi}\Big\{F_a(s)\left[F_b(t) + \left(F_+B_{-b} - F_-B_{+b}\right)(t)\right] \\ &\qquad + G_a(s)\left[G_b(t) + \left(G_+B_{-b} - G_-B_{+b}\right)(t)\right]\Big\} \end{aligned} \tag{2.10}$$

where $a, b = \pm$.

The additional restrictions necessary to solve the equations uniquely are provided by the requirements on analyticity and the asymptotic behaviour of the solutions of the corresponding linear system (2.12), (2.13), i.e. *by specification of the data in the corresponding Riemann-Hilbert problem (see Section 3).*

Proof: The proof is analogous to the one for the XXX-case [5] so that we only sketch the main steps. (i) is a direct consequence of the *shift-equation*

$$z_1^{-\frac{1}{2}}\, V^{(m+1)}(z_1|z_2)\, z_2^{\frac{1}{2}} = V^{(m)}(z_1|z_2) + \frac{i}{2\pi}\int_0^\infty ds\, \frac{e_+^{(m)}(z_1|s)e_-^{(m)}(z_2|s)}{z_1}\,. \tag{2.11}$$

which follows directly from (2.2) and (2.3). (ii) and (iii) follow from the Lax-representation

$$\begin{aligned}\frac{1}{\sqrt{z}} f_-^{(m+1)}(z|s) &= f_-^{(m)}(z|s) - \frac{1}{z} B_{--}^{(m+1)} \left(\left[1 + \left(B_{-+}^{(m)} \right)^T \right] f_+^{(m)} - \left(B_{++}^{(m)} \right)^T f_-^{(m)} \right)(z|s) \ , \\ \frac{1}{\sqrt{z}} f_+^{(m+1)}(z|s) &= \frac{1}{z} \left(f_+^{(m)} - \left[1 - B_{+-}^{(m+1)} \right] \left(B_{++}^{(m)} \right)^T f_-^{(m)} \right)(z|s) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\frac{\partial}{\partial \psi} f_\pm(z|s) &= \frac{1}{2\pi} \left\{ \frac{\xi}{z-\xi} f_\pm(\xi|s) \int_0^\infty dt \ \left(f_+(\xi|t) f_-(z|t) - f_-(\xi|t) f_+(z|t) \right) \right. \\ &\left. + \frac{\bar{\xi}}{z-\bar{\xi}} f_\pm(\bar{\xi}|s) \int_0^\infty dt \ \left(f_+(\bar{\xi}|t) f_-(z|t) - f_-(\bar{\xi}|t) f_+(z|t) \right) \right\} \ . \end{aligned} \tag{2.13}$$

q.e.d

It is a remarkable fact that (2.7)-(2.10) are formally identical to the corresponding expressions for the XXX case [5].

3 The operator-valued Riemann-Hilbert problem

In this section we show that all results of Section 2 can be reformulated in terms of an infinite-dimensional Riemann-Hilbert problem (RHP) for the integral operator valued function $\chi(z)$. From the solution $\chi(z)$ of this RHP one can then determine an asymptotic expansion of the correlator $P(m)$.

Let us now consider the following infinite dimensional Riemann-Hilbert problem for the integral operator-valued function $\chi(z)$:

1. $\chi(z)$ is analytic outside the contour C (Fig. 1).

2. $\chi^-(z) = \chi^+(z) \cdot L^{(m)}(z)$ for $z \in C$, and $\chi^\pm$ are the boundary values of the function $\chi(z)$ as indicated in Fig. 1, and where the (integral-operator valued) "conjugation matrix" $L^{(m)}(z)$ is given by

$$L^{(m)}(z) = I + \ell^{(m)}(z) \tag{3.1}$$

where $I(s,t) = \delta(s-t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\ell^{(m)}(z|s,t) = \begin{pmatrix} -e_+^{(m)}(z|s) e_-^{(m)}(z|t) & e_+^{(m)}(z|s) e_+^{(m)}(z|t) \\ -e_-^{(m)}(z|s) e_-^{(m)}(z|t) & e_-^{(m)}(z|s) e_+^{(m)}(z|t) \end{pmatrix} \tag{3.2}$$

3. $\chi(z) \to I$ as $z \to \infty$.

In order to simplify our notations we will from now on suppress the m-dependence in equations where all quantities are to be taken with the same m and write *e.g.* $\ell(z|s,t)$ instead of $\ell^{(m)}(z|s,t)$. In terms of the corresponding kernels the properties 1–3 can be rewritten in the following way:

P1. $\chi(z|s,t)$ is an analytic function of $z \notin C$ for all s,t.

P2. $\chi^-(z|s,t) = \chi^+(z|s,t) + \int_0^\infty ds' \chi^+(z|s,s')\ \ell(z|s',t)$ for $z \in C$

P3. $\chi(z|s,t) = I(s,t) + \frac{1}{z}\Psi_1(s,t) + \ldots$ as $z \to \infty$

The connection of the RHP 1–3 to the IDE of Section 2 is summarized in the following lemma:

Lemma 1: *Suppose now that the solution of the Riemann-Hilbert problem 1–3 exists and is unique. Then the function* $\Psi(z) = \chi(z) \cdot \begin{pmatrix} z^{-m} & 0 \\ 0 & 1 \end{pmatrix}$ *satisfies the integral operator-valued linear system (2.12), (2.13).*

Proof : Applying the standard arguments based on Liouville's theorem and on the m independence of the conjugation integral operator $L_0(z)$

$$L_0(z) = \begin{pmatrix} z^m & 0 \\ 0 & 1 \end{pmatrix} L(z) \begin{pmatrix} z^{-m} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

we obtain

$$\Psi^{(m+1)} \left[\Psi^{(m)}\right]^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot I \ + \ \frac{1}{z}\ U_0 \tag{3.4}$$

where $U_0 = \chi^{(m+1)}(0) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \left[\chi^{(m)}(0)\right]^{-1}$. As we will now show by determining U_0, (3.4) is the integral-operator valued analog of (2.12).

By construction the operator $\ell(z)$ (3.2) is nilpotent $\ell^2(z) = 0$, hence we have $L^{-1}(z) = I - \ell(z)$, which in turn implies the equation

$$\chi^{-1}(z|s,t) = \begin{pmatrix} \chi_{22}(z|t,s) & -\chi_{12}(z|t,s) \\ -\chi_{21}(z|t,s) & \chi_{11}(z|t,s) \end{pmatrix} \tag{3.5}$$

for the solution of the RH-Problem 1–3. Introducing notations

$$\chi(0) = \begin{pmatrix} 1 - B_{+-} & B_{++} \\ -B_{--} & 1 + B_{-+} \end{pmatrix} \tag{3.6}$$

we conclude from (3.5) that

$$\chi^{-1}(0) = \begin{pmatrix} 1 + B_{-+}^T & -B_{++}^T \\ B_{--}^T & 1 - B_{+-}^T \end{pmatrix} . \tag{3.7}$$

Substituting the expansion P3 for $\chi(z)$ as $z \to \infty$ into (3.4), we see that $(U_0)_{11} = 1$. This together with formulae (3.6) and (3.7) allow us to rewrite U_0 as

$$U_0 = \begin{pmatrix} 1 & -\left(1 - B_{+-}^{(m+1)}\right)\left(B_{++}^{(m)}\right)^T \\ -B_{--}^{(m+1)}\left(1 + \left(B_{-+}^{(m)}\right)^T\right) & B_{--}^{(m+1)}\left(B_{++}^{(m)}\right)^T \end{pmatrix} . \tag{3.8}$$

We now define functions $f_\pm(z|s)$ *via*

$$\begin{pmatrix} f_+(z|s) \\ f_-(z|s) \end{pmatrix} = \int_0^\infty dt\ \chi^+(z|s,t) \begin{pmatrix} e_+(z|t) \\ e_-(z|t) \end{pmatrix} . \tag{3.9}$$

It can be shown analogously to Section XV.6 of [1] and [6] that the functions $f_\pm$ defined this way are identical to the ones defined in Section 2 for Eq. (2.5). This then implies that the kernels $B_{ab}(s,t)$ from (3.6) are just the kernels introduced in (2.6). To prove this one has to consider the canonical integral representation for the solution of the problem 1–3

$$\chi(z_1) = I - \frac{1}{2\pi i}\int_C \frac{dz_2}{z_2 - z_1}\chi^+(z_2)\ \ell(z_2) , \tag{3.10}$$

which implies

$$\chi(0) = I + \frac{i}{2\pi}\int_C \frac{dz}{z}\chi^+(z)\ \ell(z) . \tag{3.11}$$

Taking into account the explicit formula for $\ell(z)$ and Eq. (3.9) we obtain the representations (2.6) for $B_{ab}(s,t)$ defined in (3.6). This shows that $\Psi(z)$ fulfills the integral-operator valued version of the system (2.12), which can be reobtained from (3.4) in the following way: rewriting (3.4) as

$$z^{-\frac{1}{2}}\chi^{(m+1)}(z) \cdot \begin{pmatrix} z^{-\frac{1}{2}} & 0 \\ 0 & z^{\frac{1}{2}} \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot I \ + \ \frac{1}{z}\, U_0\right] \chi^{(m)}(z) , \tag{3.12}$$

and then applying both sides of (3.12) to the vector $\begin{pmatrix} e_+^{(m)}(z|s) \\ e_-^{(m)}(z|s) \end{pmatrix}$ and taking into account (3.9) and that $e_\pm^{(m)} z^{\mp\frac{1}{2}} = e_\pm^{(m+1)}$ we arrive back at the equations (2.12).

Let us now study the ξ-derivative of the function $\chi(z)$ in order to connect (2.13) to our RHP. The corresponding analysis is very similar to the ones for the Bose gas [6] and the XXX chain [5]. In the neighbourhood of C the function $\chi(z)$ can be represented as

$$\chi(z) = \widehat{\chi}(z) \cdot \chi_0(z) \tag{3.13}$$

where $\widehat{\chi}(z)$ is single-valued, invertible and analytic in that neighbourhood, and

$$\chi_0(z) = I - \frac{i}{2\pi} \ln \frac{z-\bar{\xi}}{z-\xi}\, \ell(z)\ . \tag{3.14}$$

From (3.13) we conclude at once that

$$\frac{\partial \chi(z)}{\partial \psi} \cdot \chi^{-1}(z) = \frac{A_+}{z-\bar{\xi}} + \frac{A_-}{z-\xi} \tag{3.15}$$

where

$$A_\pm = \lim_{z\to z_\pm} (z-z_\pm)\, \widehat{\chi}(z) \cdot \frac{\partial \chi_0(z)}{\partial \psi} \cdot \chi_0^{-1}(z) \cdot \widehat{\chi}^{-1}(z)\ , \qquad z_+ = \bar{\xi}\ , \quad z_- = \xi\ . \tag{3.16}$$

Differentiation of (3.14) gives

$$\left(\frac{\partial \chi_0(z)}{\partial \psi} \chi_0^{-1}\right)(z) = \frac{1}{2\pi}\left[\frac{\bar{\xi}}{z-\bar{\xi}} + \frac{\xi}{z-\xi}\right]\, \ell(z)\ . \tag{3.17}$$

Using this together with $\ell^2 = 0$ and (3.2) and (3.9) in (3.16) we obtain

$$\begin{aligned} A_\pm(s,t) &= \frac{z_\pm}{2\pi}\left(\widehat{\chi} \cdot \ell \cdot \widehat{\chi}^{-1}\right)(z_\pm|s,t) = \frac{z_\pm}{2\pi}\left(\chi \cdot \ell \cdot \chi^{-1}\right)(z_\pm|s,t) \\ &= \frac{z_\pm}{2\pi}\begin{pmatrix} -f_+(z_\pm|s) f_-(z_\pm|t) & f_+(z_\pm|s) f_+(z_\pm|t) \\ -f_-(z_\pm|s) f_-(z_\pm|t) & f_-(z_\pm|s) f_+(z_\pm|t) \end{pmatrix}\ . \end{aligned} \tag{3.18}$$

Recalling the definition of the potentials $F_\pm(s)$, $G_\pm(s)$, we can rewrite the final formulae for $A_\pm$ as

$$\begin{aligned} A_+(s,t) &= \frac{\bar{\xi}}{2\pi}\begin{pmatrix} -G_+(s)G_-(t) & G_+(s)G_+(t) \\ -G_-(s)G_-(t) & G_-(s)G_+(t) \end{pmatrix}\ , \\ A_-(s,t) &= \frac{\xi}{2\pi}\begin{pmatrix} -F_+(s)F_-(t) & F_+(s)F_+(t) \\ -F_-(s)F_-(t) & F_-(s)F_+(t) \end{pmatrix}\ . \end{aligned} \tag{3.19}$$

From (3.15) it follows that $\Psi(z)$ fulfills the integral-operator valued version of (2.13)

$$\frac{\partial \Psi(z|s,t)}{\partial \psi} = \int_0^\infty dt' \left(\frac{A_+(s,t')}{z-\bar{\xi}} + \frac{A_-(s,t')}{z-\xi} \right) \Psi(z|t',t) \ . \tag{3.20}$$

Acting with (3.20) on $\begin{pmatrix} f_+(z|t) \\ f_-(z|t) \end{pmatrix}$ and taking (3.19) into account we arrive at the basic equation (2.13) for the ψ-derivative of $f_\pm(z|s)$. This completes the proof of the Lemma. **q.e.d.**

4 Asymptotics in some limiting cases

In this section we obtain the large distance asymptotics of the FSFP correlation functions in two limiting cases. The first case corresponds to the limit of very strong magnetic fild. In the second case, the magnetic field is arbitrary but the value of parameter η is chosen to be $3\pi/4$ corresponding to the free fermionic point.

For very large h close to the critical field $h_c = 4\cos^2\eta$ (for which the ground state turns into the saturated ferromagnetic state) the integration boundary Λ in (1.4) tends to zero acoording to

$$\Lambda = \frac{1}{2}|\tan\eta|\sqrt{h_c - h} + \mathcal{O}(h_c - h) \ . \tag{4.1}$$

In the limit $h \to h_c$, the kernel (1.5) can be expanded to order $\mathcal{O}(m(h_c-h)^{\frac{3}{2}})$ as

$$V^{(m)}(\lambda|\mu) \sim V_0(\lambda|\mu) = -\frac{1}{\pi}\frac{\sin(m|\cot\eta|(\lambda-\mu))}{\lambda-\mu} \ . \tag{4.2}$$

We note that the dual fields do not contribute at all because

$$\phi(l) = \phi(0) + \mathcal{O}((h_c-h)^{\frac{1}{2}}), \tag{4.3}$$

and the dual expectation value can simply be dropped. Thus we find that in the limit $h \to h_c$, $m \to \infty$ with $m(h_c - h) \ll 1$ [1]

$$P(m) \sim \det\left(1 + V_0\right) \ . \tag{4.4}$$

The long-distance asymptotics of this Fredholm determinant is known [7]

$$\ln\det\left[1 - \frac{1}{\pi}\frac{\sin(a(x-y))}{x-y}\right]\Big|_{-s}^{s} \sim -\frac{1}{2}(as)^2 \quad \text{for } as \to \infty. \tag{4.5}$$

[1]This inequality rather than $m(h_c-h)^{\frac{3}{2}} \ll 1$ is a consequence of a more accurate analysis of the contribution of the terms dropped in (4.2) to the determinant.

In this way we obtain a Gaussian decay of the FSFP for large magnetic fields $h \approx h_c$ at large distances

$$P(m) \sim e^{-\frac{1}{8}(h_c-h)m^2} \quad \text{for} \;\; h \to h_c,\; (h_c-h)^{-\frac{1}{2}} \ll m \ll (h_c-h)^{-1} \,. \tag{4.6}$$

This result complements the expression for *near* asymptotics obtained in [2]

$$P(m) \sim e^{-\frac{1}{\pi}\sqrt{h_c-h}\; m} \quad \text{for} \;\; h \to h_c,\; m \ll (h_c-h)^{-\frac{1}{2}} \,. \tag{4.7}$$

For the case of the XXX chain an analogous analysis can be performed (based on the determinant representation [8]) giving the same result (4.6) (note that $h_c = 4$ for the XXX case).

The above results (and more) can also be obtained from the Riemann-Hilbert problem 1–3 presented at the beginning of Section 3. As shown above, in the limit $h \to h_c$ the dual fields can be dropped and the integral kernel (1.5) can be approximated as

$$V^{(m)}(\lambda|\mu) = \frac{i}{2\pi}\,\frac{\tilde{e}_+(\lambda)\tilde{e}_-(\mu) - \tilde{e}_-(\lambda)\tilde{e}_+(\mu)}{\sinh(\lambda-\mu)}\,, \qquad \tilde{e}_\pm(\lambda) = \left(\frac{\sinh(\lambda+i\eta)}{\sinh(\lambda-i\eta)}\right)^{\pm\frac{m}{2}} \,. \tag{4.8}$$

It is worth emphasizing that this is the only approximation that we will use in this approach. Performing the same change of variables as in Sect. 2 this becomes an integral operator of the form (2.4) with kernel

$$V^{(m)}(z_1|z_2) = -\frac{i}{2\pi}\,\frac{e_+(z_1)e_-(z_2) - e_-(z_1)e_+(z_2)}{z_1-z_2}, \qquad e_\pm(z) = z^{\mp\frac{m}{2}} \tag{4.9}$$

and the RHP 1–3 above reduces to the ordinary 2×2 matrix RHP *without s-integration*:

$\widetilde{\mathrm{P}1}$. $\chi(z)$ is an analytic function of $z \notin C$

$\widetilde{\mathrm{P}2}$. $\chi^-(z) = \chi^+(z)\begin{pmatrix} 0 & z^{-m} \\ -z^m & 2 \end{pmatrix}$

$\widetilde{\mathrm{P}3}$. $\chi(\infty) = I$.

The functional determinant of the operator (4.9) can be rewritten as the determinant of an $m\times m$ matrix from which the behaviour for small m is found. The large-m asymptotics can be obtained in terms of the RHP $\widetilde{\mathrm{P}1}$–$\widetilde{\mathrm{P}3}$ and yields the following asymptotic formula for $\det(I+\hat{V}^{(m)})$ [9]

$$\begin{aligned} &\ln\det\left(I+\hat{V}^{(m)}\right) \sim m^2 \ln\delta \quad \text{for} \;\; m\to\infty \\ &\delta = -\sin\frac{\psi}{2}\,, \quad e^{i\psi} = \frac{e^{2\Lambda}w-1}{e^{2\Lambda}-w}\,, \end{aligned} \tag{4.10}$$

where $\Lambda \sim m^{-\frac{1}{2}-\epsilon}$. In terms of the magnetic field this condition translates into $m(h_c - h) \sim m^{-2\epsilon}$. The asymptotic equation (4.10) is actually equivalent to (4.6) as

$$\begin{aligned} \ln\delta &\sim -\frac{1}{2}\,\Lambda^2\cot^2\eta + O(\Lambda^4)\ , \quad \text{for}\ \ \Lambda\to 0 \\ &\sim -\frac{1}{8}\,(h_c - h)\ , \quad \text{for}\ \ h\to h_c \end{aligned} \tag{4.11}$$

Equations (4.6) and (4.10) describe the long-distance ($m \to \infty$) asymptotics of $P(m)$ in the XXZ model in the limit of small Λ.

Let us now consider the free fermionic point $\eta = 3\pi/4$ (XX0 model) in more detail. For this case the RHP $\widetilde{\text{P1}}$–$\widetilde{\text{P3}}$ is the *exact* RHP and (4.8), (4.9) are the *exact* integral operators for the FSFP for *any* values of Λ. Thus (4.10) with $\eta = 3\pi/4$ is the result for the long-distance asymptotics of the functional determinant of $I + \widehat{V}^{(m)}$ in the XX0 model for arbitrary Λ and thus arbitrary magnetic field h

$$\ln\det\left(I + \widehat{V}^{(m)}\right)_{XX0} \sim \frac{m^2}{2}\ \ln\left(\frac{1}{2} + \frac{1}{2\cosh 2\Lambda}\right) \quad \text{for}\ \ m\to\infty\ . \tag{4.12}$$

For the XX0 model we have $\cosh 2\Lambda = 2/h$. Hence we finally obtain for the asymptotic behaviour of the FSFP at the free fermionic point as a function of the magnetic field

$$P_{XX0}(m) \sim \left(\frac{2+h}{4}\right)^{\frac{m^2}{2}} \tag{4.13}$$

which coincides with (4.6) for $h \to h_c = 2$.

On the basis of the above results we conjecture that the FSFP exhibits a Gaussian decay for any value of η and any value of the magnetic field $h < h_c$.

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Figure 1:

Contour C on the unit circle for the integration with respect to z in the integral operator $\widehat{V}$. '+' and '−' indicate direction in which the limit $z \to C$ has to be taken to obtain the boundary values $\chi^{\pm}(z)$ for the Riemann-Hilbert problem in Section 3.