Exact Results on Stationary Turbulence in 2D: Consequences of Vorticity Conservation

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Abstract

We establish a series of exact results for a model of stationary turbulence in two-dimensions: the incompressible Navier-Stokes equation with stochastic force white-noise in time. Essentially all of our conclusions follow from the simple consideration of the simultaneous conservation of energy and enstrophy by the inertial dynamics. Our main results are as follows: (1) we show the blow-up of mean energy as $\sim \ell_0^2 \frac{\varepsilon}{\nu}$ for $\nu \to 0$ when there is no IR-dissipation at the large length-scale ℓ_0; (2) with an additional IR-dissipation, we establish the validity of the traditional cascade directions and magnitudes of flux of energy and enstrophy for $\nu \to 0$, assuming finite mean energy in the limit; (3) we rigorously establish the balance equations for the energy and vorticity invariants in the 2D steady-state and the forward cascade of the higher-order vorticity invariants assuming finite mean values; (4) we derive exact inequalities for scaling exponents in the 2D enstrophy range, as follows: if $\langle|\Delta_{\mathbf{l}}\omega|^p\rangle \sim \ell^{\zeta_p}$, then $\zeta_2 \leq \frac{2}{3}$ and $\zeta_p \leq 0$ for $p \geq 3$. If the minimum Hölder exponent of the vorticity $h_{\min} < 0$, then we establish a 2D analogue of the refined similarity hypothesis which improves these bounds. The most novel and interesting conclusion of this work is the connection established between "intermittency" in 2D and "negative Hölder singularities" of the vorticity: we show that the latter are necessary for deviations from the 1967 Kraichnan scaling to occur.

1 Introduction

An essential difference between the incompressible Euler equations in 2D and 3D is that the vorticity in each fluid element is conserved in the former but not the latter. This conservation is expressed by the fact that the vorticity field in 2D satisfies the equation

$$D_t\omega(\mathbf{r},t) = 0, \tag{1.1}$$

where $D_t = \partial_t + \mathbf{v}(\mathbf{r},t)\boldsymbol{\cdot}\boldsymbol{\nabla}$ is the convective or material derivative. It follows that each of the integrals

$$\Omega^{(n)}(t) = \frac{1}{n!}\int_\Lambda d^2\mathbf{r}\ \omega^n(\mathbf{r},t), \tag{1.2}$$

for integers $n > 0$ is an invariant of the dynamics, in addition to the energy integral

$$E(t) = \frac{1}{2}\int_\Lambda d^2\mathbf{r}\ v^2(\mathbf{r},t). \tag{1.3}$$

These statements are true in 2D not only for smooth, classical solutions, but also for a wide class of *weak solutions*, relevant to the description of turbulence. For example, the weak solutions constructed by Yudovich [1] for the initial data $\omega_0 \in L^\infty(\Lambda)$ conserve each of the above integrals.

However, it has long been recognized that the presence of an additional quadratic invariant, the *enstrophy* $\Omega(t) = \Omega^{(2)}(t)$, changes drastically the nature of 2D turbulence in comparison to the 3D case: see the works of Lee [2] and Fjortoft [3]. In the late 1960's it was suggested by Kraichnan [4], Leith [5], and Batchelor [6] that this extra constraint in 2D results in two distinct inertial ranges which can exist simultaneously: an energy- transferring range in which energy is passed down to smaller wavenumbers and an enstrophy-transferring range in which enstrophy is passed up to larger wavenumbers. To maintain such a steady state, energy and enstrophy need to be fed into the fluid at an intermediate range. More recently, such steady-states driven by random external forces have been rigorously constructed by Vishik and Fursikov: see Theorem XI.2.1 in [7]. Nevertheless, much of the conventional picture of "dual cascades" of energy and enstrophy remains conjectural. Furthermore, a new attempt by Polyakov [8, 9] to construct stationary statistical solutions of the Euler equations from the hierarchy of correlation functions in suitable 2D conformal field theories has prompted some questions [10] concerning the role of the higher invariants in Eq.(1.2) for $n \geq 3$.

Our goal in this work is to derive—either *a priori* or under suitable hypotheses —exact results in the theory of 2D turbulence. We have in mind two audiences. On the one hand, the paper should be helpful to fluid mechanicians, statistical physicists and engineers working in this area in order to provide rigorous results as constraints and checks on physical theory-building. On the other hand, the paper should be useful for mathematicians interested in 2D turbulence in order to provide a mathematical statement of known results and open problems in the area, which are generally formulated less precisely. In view of these objectives, many of the results of this paper are "old," as the conclusions have long been believed but without careful proofs. In such cases, we have found that the traditional arguments can often be converted to proofs without much difficulty. However, the paper also contains several new results, including, in particular, exact bounds on scaling exponents of vorticity. To make the paper readable to both audiences we have adopted a compromise between the styles of the different communities. We do not state results as theorems in the text and many of technical details of the proofs are relegated to three long Appendices, for readers interested in those details. Hopefully mathematicians reading the paper will be able to formulate precise theorem statements for themselves from the textual discussion and the technical appendices.

The precise contents of this work are as follows: in Section 2 we introduce the balance equations for energy and the vorticity invariants in 2D, which form the basis of most of our analysis. We discuss the relation between regularity of the 2D Euler solutions and the validity of the naive conservation laws. We also introduce here the models with random stirring forces, whose stationary measures are our main focus in this work. In Section 3 we prove results on the cascade directions of energy and enstrophy in these steady states: that is, we establish the existence of scale ranges of constant mean flux, as well as the signs and magnitudes of the fluxes. These results in general require hypotheses on bounded means of energy and enstrophy as viscosity tends to zero, which are interesting open problems. Finally, we discuss here the fluxes of the higher-order vorticity invariants, and establish their cascade direction as toward the ultraviolet, appealing to stronger (but still reasonable) assumptions of finite means. In Section 4 we use the results of the previous sections to derive rigorous bounds on the scaling exponents of the vorticity difference in the 2D enstrophy cascade range. These bounds all appear to be new, except for the energy spectrum bound which was previously observed by Sulem and Frisch (who, however, never published a complete argument.) Our bounds establish a precise connection between "anomalous scaling" in 2D, or corrections to the 1967 Kraichnan scaling theory, and the existence of "negative Hölder singularities" of the vorticity. Although it follows easily from a heuristic "multifractal model" of 2D turbulence, the connection does not seem to have been pointed out before.

Even more than the 3D energy inertial range, the 2D cascades are a "theoreticians' turbulence." From a practical point of view the subject has interest mostly from the possibility that certain geophysical flows, such as atmospheres on scales of 100-10,000 km, are predominantly two-dimensional in nature, from the similiarity to certain problems in plasma physics, and from the relation to the rapid rotation limit of 3D turbulence, which, by consideration of the Taylor-Proudman theorem, ought to behave as two-dimensional for small Rossby number. From the purely theoretical point of view the 2D steady-state is a useful model on which to test ideas for the 3D case, since it is generally more amenable to exact analysis but is still a strongly nonlinear, statistically nonequilibrium system with highly nontrivial behavior. It is mostly this latter point of view which motivates our work here.

2 Vorticity Balance and Transfer Equations

(i) Balance Equations for Free Evolution

We here consider the balance equations governing the local conservation of the vorticity invariants in space and in scale. To introduce the concepts in the simplest context, we discuss first free evolution, i.e. equations without any external forcing. Thus, our starting point is the 2D Euler equations in the "vorticity formulation"

$$\partial_t \omega(\mathbf{r},t) + \boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}(\mathbf{r},t)\omega(\mathbf{r},t)) = 0, \tag{2.1}$$

in which the velocity field is recovered instantaneously from the vorticity field via the integral equation

$$\mathbf{v}(\mathbf{r},t) = \int_\Lambda d^2\mathbf{r}'\ \mathbf{B}_\Lambda(\mathbf{r}-\mathbf{r}')\omega(\mathbf{r}',t). \tag{2.2}$$

Here $\mathbf{B}_\Lambda(\mathbf{r}) \equiv \boldsymbol{\nabla}\times(\hat{\mathbf{e}}_3 D_\Lambda(\mathbf{r}))$ and $D_\Lambda(\mathbf{r})$ is the Green function of the laplacian in the 2D domain Λ with Dirichlet b.c. To avoid technically-involved discussions of the boundary conditions, we shall actually consider here always $\Lambda = \mathbf{T}^2$, the 2D torus of sidewidth ℓ_0, and adopt periodic b.c. There are, in our opinion, more substantial differences between free and forced turbulent flow in 2D than there are in 3D. This is mostly connected with the fact that the 2D enstrophy cascade is "non-accelerated," so that no finite-time singularities occur, and "non-local," so that the statistics of the largest eddies play a more substantial role. We hope to treat elsewhere in more detail the subject of enstrophy cascade in freely-decaying 2D turbulence. Here we shall just discuss those aspects that will be relevant to our main topic, the forced steady-states.

To define a precise notion of "scale of motion" we use the *filtering approach* which is common in the turbulence modelling method of large-eddy simulation and already exploited by us for an exact analysis of energy transfer in the incompressible Euler equations for arbitrary dimension [11, 12]. The basic idea of the filtering technique is to define a "large-scale" vorticity field $\overline{\omega}_\ell(\mathbf{r},t)$ and a "small-scale" field $\omega'_\ell(\mathbf{r},t)$ instantaneously by the equations

$$\overline{\omega}_\ell(\mathbf{r},t) = \int_\Lambda d^2\mathbf{r}'\ G_\ell(\mathbf{r}-\mathbf{r}')\omega(\mathbf{r}',t) \tag{2.3}$$

and

$$\omega'_\ell(\mathbf{r},t) = \omega(\mathbf{r},t) - \overline{\omega}_\ell(\mathbf{r},t). \tag{2.4}$$

The "filter function" $G_\ell(\mathbf{r}) = \ell^{-d}G(\ell^{-1}\mathbf{r})$ is assumed to be smooth with rapid decay, both in physical and in Fourier space. The parameter ℓ fixes the arbitrary length in the division of the field into "large-scale" and "small-scale" components. If the filter is convoluted with the equation of motion, Eq.(2.1), an equation for the large-scale vorticity field is obtained of the form

$$\partial_t \overline{\omega}_\ell(\mathbf{r},t) + \boldsymbol{\nabla}\cdot(\overline{\mathbf{v}}_\ell(\mathbf{r},t)\overline{\omega}_\ell(\mathbf{r},t) + \boldsymbol{\sigma}_\ell(\mathbf{r},t)) = 0. \tag{2.5}$$

The large-scale velocity field is defined here in the same manner by $\overline{\mathbf{v}}_\ell = G_\ell * \mathbf{v}$. Furthermore

$$\boldsymbol{\sigma}_\ell(\mathbf{r},t) \equiv \overline{(\mathbf{v}\omega)}_\ell(\mathbf{r},t) - \overline{\mathbf{v}}_\ell(\mathbf{r},t)\overline{\omega}_\ell(\mathbf{r},t) \tag{2.6}$$

plays the same role as the usual small-scale stress tensor $\boldsymbol{\tau}(\mathbf{r},t)$ in the analogous equation for the large-scale velocity. Thus, we refer to it as the *vortical stress.* Observe that $\sigma_i = \epsilon_{jk}\partial_j\tau_{ik}$ in terms of the usual stress, with ϵ_{ij} the antisymmetric Levi-Civita tensor in 2D.

The *enstrophy* integral

$$\Omega(t) = \frac{1}{2}\int_\Lambda d^2\mathbf{r}\ \omega^2(\mathbf{r},t) \tag{2.7}$$

is formally conserved by the dynamics Eq.(2.1). From the Eq.(2.5) for the large-scale vorticity it is straightforward to derive by the standard methods of nonequilibrium thermodynamics a *local balance equation* for its large-scale density

$$h_\ell(\mathbf{r},t) \equiv \frac{1}{2}\overline{\omega}_\ell^2(\mathbf{r},t). \tag{2.8}$$

It has the form

$$\partial_t h_\ell(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{K}_\ell(\mathbf{r},t) = -Z_\ell(\mathbf{r},t) \tag{2.9}$$

in which the current

$$\mathbf{K}_\ell(\mathbf{r},t) \equiv h_\ell(\mathbf{r},t)\overline{\mathbf{v}}_\ell(\mathbf{r},t) + \overline{\omega}_\ell(\mathbf{r},t)\boldsymbol{\sigma}_\ell(\mathbf{r},t) \tag{2.10}$$

represents space-transport of the large-scale enstrophy, and the *enstrophy flux*

$$Z_\ell(\mathbf{r},t) \equiv -\boldsymbol{\nabla}\overline{\omega}_\ell(\mathbf{r},t)\cdot\boldsymbol{\sigma}_\ell(\mathbf{r},t) \tag{2.11}$$

represents the enstrophy transfer to the small-scale modes. Note that the large-scale *vorticity-gradient* $\overline{\boldsymbol{\xi}}_\ell \equiv \boldsymbol{\nabla}\overline{\omega}_\ell$ enters into the enstrophy transfer process in two different ways: the term

$-\mathbf{v}'\cdot\nabla\overline{\omega}$ in the equation for ω' produces a growth in that quantity at the expense of the large-scale vorticity $\overline{\omega}$, and the term $-\overline{\xi}_j \cdot \partial_i \overline{v}_j$ in the equation for $\overline{\xi}_i$ itself produces a steepening of the large-scale vortex-gradient (hence a redistribution of the vorticity to higher wavenumbers) by nonlinear stretching due to the strain at that same scale. Both of these processes contribute to the dynamical development of the enstrophy flux Z_ℓ.

An identical analysis can be made of the balance for the local densities

$$h_\ell^{(n)}(\mathbf{r},t) = \frac{1}{n!}\overline{\omega}_\ell^n(\mathbf{r},t) \tag{2.12}$$

of the contribution to the invariants $\Omega^{(n)}$ in the large-scale modes. (Observe that $h_\ell^{(n)}$ has wavenumber support inside a spectral radius $\sim n/\ell$.) By a similar calculation as before it follows that

$$\partial_t h_\ell^{(n)}(\mathbf{r},t) + \nabla\cdot\mathbf{K}_\ell^{(n)}(\mathbf{r},t) = -Z_\ell^{(n)}(\mathbf{r},t) \tag{2.13}$$

with

$$\mathbf{K}_\ell^{(n)}(\mathbf{r},t) \equiv h_\ell^{(n)}(\mathbf{r},t)\overline{\mathbf{v}}_\ell(\mathbf{r},t) + h_\ell^{(n-1)}(\mathbf{r},t)\boldsymbol{\sigma}_\ell(\mathbf{r},t) \tag{2.14}$$

and

$$Z_\ell^{(n)}(\mathbf{r},t) \equiv h_\ell^{(n-2)}(\mathbf{r},t)Z_\ell(\mathbf{r},t). \tag{2.15}$$

The latter represents the flux of the invariant quantity $\Omega^{(n)}$ from the large-scale to the small-scale modes. It is of some interest that for $n > 2$ it is simply proportional to the enstrophy flux itself.

Vorticity conservation and regularity of initial data

Although each of the invariants $\Omega^{(n)}$ is *formally* conserved by the 2D Euler equations, they need not be conserved for general weak solutions. In the terminology of Polyakov [8, 9] these naive conservation laws may be vitiated by "anomalies." This corresponds exactly to the traditional picture of the ultraviolet enstrophy cascade in [4, 5, 6], which posits some nonvanishing flux of enstrophy escaping to infinite wavenumber. The analogous possibility for failure of naive energy conservation was already noted by Onsager [14], who pointed out that a minimal degree of "irregularity" is required of the velocity field for this to occur: in fact, the velocity field must have Hölder index $h \leq \frac{1}{3}$, for otherwise the flux will vanish asymptotically and energy will be conserved. A corresponding statement holds in the case of 2D enstrophy conservation [13]: the Hölder index of the *vorticity field* must take a value $h \leq 0$, or otherwise enstrophy is conserved.

The proof of this conservation statement is very similar to that for energy conservation [15, 11]. Indeed, an expression exists for the "vortical stress vector" $\boldsymbol{\sigma}$ quite similar to one discovered by Constantin et al. [15] for the ordinary stress:

$$\boldsymbol{\sigma}(\mathbf{r},t) = [\Delta\omega(\mathbf{r},t)\Delta\mathbf{v}(\mathbf{r},t)]_\ell - [\Delta\omega(\mathbf{r},t)]_\ell[\Delta\mathbf{v}(\mathbf{r},t)]_\ell. \tag{2.16}$$

Here $\Delta_{\mathbf{s}}f(\mathbf{r}) \equiv f(\mathbf{r}) - f(\mathbf{r}-\mathbf{s})$ denotes backward-difference and $[f]_\ell = \int d^2\mathbf{s}\; G_\ell(\mathbf{s})f(\mathbf{s})$ is the average over the separation-vector $\mathbf{s}$ with respect to the filter function. Since

$$\Delta_{\mathbf{l}}\omega(\mathbf{r},t) = O(\ell^h) \tag{2.17}$$

implies

$$\Delta_{\mathbf{l}}\mathbf{v}(\mathbf{r},t) = O(\ell) \tag{2.18}$$

and

$$\boldsymbol{\nabla}\overline{\omega}_\ell(\mathbf{r},t) = O(\ell^{h-1}), \tag{2.19}$$

it follows directly from the expression Eq.(2.11) for the enstrophy flux that

$$Z_\ell(\mathbf{r},t) = O(\ell^{2h}). \tag{2.20}$$

Hence, $Z_\ell \to 0$ for $\ell \to 0$ when $h > 0$, and enstrophy is conserved. For that matter, it follows from the like expressions Eq.(2.15) that $Z_\ell^{(n)} = O(\ell^{2h})$ and the higher-order vorticity invariants are conserved as well when $h > 0$. This Hölder criterion for enstrophy conservation is actually contained in Yudovich' more general result [1], since for $h > 0$ it obviously holds that $C^h(\Lambda) \subset L^\infty(\Lambda)$.

These results allow some classification of decaying turbulence in 2D according to the regularity of the initial data. It is known from the work of Wolibner [16], Yudovich [1] and Kato [17], that a unique solution of the Euler equations exists globally in time for initial data $\omega_0 \in C^{k+s},\ k \in \mathbf{Z}_0^+, 0 < s < 1$. Furthermore, the solution remains Hölder continuous at any finite time t with positive exponent $h(t) = k + \exp[-C\|\omega_0\|_\infty t]s$. Therefore, it follows that, regardless of the size of the Reynolds number initially, there will be no turbulence in 2D— in the sense of a long enstrophy cascade range— if the initial data $\omega_0 \in C^{k+s}$ with $k \geq 1$. This might be called initial data of Type O. On the other hand, if $\omega_0 \in C^s,\ 0 < s < 1$—which we may refer to as Type A—then turbulent cascades may develop over time. Although fluxes of enstrophy must vanish asymptotically for $\ell \to 0$ at any finite time $t > 0$ starting from such data, nevertheless mean constant fluxes of these invariants may occur over growing ranges of ℓ as $t \to \infty$. This accords very well with the traditional picture of the 2D enstrophy cascade, which postulates a "non-accelerated cascade" in which constant fluxes develop at time t down to an exponentially small, but non-zero, length-scale $\sim \ell_0 e^{-C\overline{\sigma}t}$ [18, 19]. The situation for $\omega \in L^\infty$ is the borderline case, but Yudovich' results show that it is essentially similar and can still be considered Type A.

However, it seems possible that the enstrophy conservation could be violated in finite time (or immediately) for less regular initial data, which we refer to as Type B. This phenomenon would be the 2D analogue of the Onsager conjecture on existence of energy-dissipative solutions of Euler in 3D. Weak solutions of 2D Euler have been constructed under less restrictive assumptions on the initial data than those assumed by Yudovich. In fact, existence has now been proved within successively larger classes: for $\omega \in L^2$ by Vishik and Komech [20], for $\omega \in L^p$ with any $1 < p < \infty$ by DiPerna and Majda [21], for $\omega \in L\log L$ by Chae [22], and, most generally, for $\omega \in L^1$ by Delort [23]. Nothing is known about uniqueness of solutions in these classes, nor about conservation properties. We conjecture that enstrophy conservation can be violated at finite time for suitable initial data $\omega_0 \in L^p$ with any $p < \infty$. In the case $\omega \in L^2$ it was argued by Sulem and Frisch [24] that enstrophy will be conserved if the energy spectrum obeys a suitable decay estimate $E(k) = O\left(k^{-(\frac{11}{3}+\epsilon)}\right)$ for any $\epsilon > 0$. We shall derive this result below in the context of forced, steady-states. However, it remains possible that enstrophy conservation may fail in the class $\omega \in L^2$ with spectrum decaying less steeply.

The previous results on weak solutions do not cover all the cases of physical interest. For example, a common situation considered for numerical simulation is decaying turbulence with initial data chosen from a Gaussian random ensemble of initial vorticity fields ω_0 with the Kraichnan k^{-3} energy spectrum: see Farge et al. [25, 26], Benzi and Vergassola [27]. In that case, the initial vorticity fields have *infinite enstrophy* with finite probability, and $\omega_0 \in L^2$ fails. In this case, as well as others, it is more natural to consider $\omega \in B_p^{s,\infty}$, the *Besov spaces*, with suitable s and p. These spaces will be very important for our later analyses, but here we just remark that they correspond to classes of functions with Hölder index s in the space L^p-mean sense. It is a consequence of Theorem 4 in [28] for the case $p = 2$ that the individual realizations of a homogeneous ensemble of random vorticity fields with enstrophy spectrum $\Omega(k) = O\left(k^{-(1+2s)}\right)$ will belong to $B_2^{s-,\infty}$ with probability one. Therefore, with the Kraichnan spectrum, for which $s = 0$, it will be true that $\omega_0 \in B_2^{0-,\infty}$. This space is "marginally worse" than L^2 and existence within this class is not given by any of the previous theorems.

Recently, Shnirelman [29] has established solvability of the Euler equations for $\omega \in B_2^{s,\infty}$ with $s > 2$. Actually, solutions with such large values of s are classical and Shnirelman's results are only a slight improvement of the well-known results of Ebin and Marsden [31] and others on solvability with $\omega \in H^s,\ \ s > 2$. The result on enstrophy conservation of Sulem and Frisch [24] is best stated in terms of the Besov spaces, and implies that $\omega \in B_2^{s,\infty},\ \ s > \frac{1}{3}$ is a sufficient condition for enstrophy conservation to hold. Clearly, the Besov index in Shnirelman's construction is too large to be of interest for 2D enstrophy cascades, but we conjecture that Euler solutions in fact exist globally in $B_2^{0,\infty}$. This conjecture has a very natural physical interpretation that a bound of the enstrophy spectrum by Kraichnan's spectrum, $\Omega(k) = O\left(k^{-1}\right)$, will be dynamically preserved in time. It would be very natural to look for enstrophy dissipation at finite time within this class.

There is an issue raised in the works [26, 27] which shall be important in our discussion later of the steady-states. Both of these works reported that "negative exponents" occur due to "cusps" in vortex cores, with $h \approx -1/2$ the most typical value and $h_{\min} \approx -1$. Since there seemed to be no internal mechanism in 2D for generation of such singularities from regular initial data, it was conjectured in [26, 27] that the "cusps" observed in those works were present initially. Actually, a homogeneous *Gaussian* random field with enstrophy spectrum $\sim k^{-(1+2s)}$, from which initial data were selected in [26, 27] for $s = 0$, have for *all* $s \in \mathbf{R}$ realizations $\omega \in B_{\infty}^{s-,\infty}$ a.s. and not merely $\omega \in B_2^{s-,\infty}$ as given by Theorem 4 of [28]. This is a simple generalization to nonpositive $s \in \mathbf{R}$ of the well-known Wiener-Lévy theorem, which states that the minimum Hölder singularity for realizations of such a Gaussian ensemble is $h_{\min} = s-$ a.s. (G. Eyink, unpublished). [1] Therefore, the initial data of [26, 27] are distributional, but with no singularities as severe as $h = -1/2$ and the "cusps" observed, if real and not numerical artefacts, must have been produced in the course of the dynamical evolution. Such a phenomenon would not be dissimilar to what happens for $h > 0$. As discussed above, the exponent in that case can decay, but at most to $[\![h]\!]$, the greatest integer less than h [16, 1, 17]. Similarly, one might suppose that if the initial data $\omega \in B_{\infty}^{0,\infty}(\mathbf{T}^2)$, then a solution exists globally in $B_{\infty}^{-1,\infty}(\mathbf{T}^2)$. Again, the exponent may deteriorate in magnitude by 1. This behavior would be consistent with our conjecture that a solution exists globally in $B_2^{0,\infty}(\mathbf{T}^2)$, because $B_2^{0,\infty}(\mathbf{T}^2) \subset B_{\infty}^{-1,\infty}(\mathbf{T}^2)$ in 2D as a consequence of Besov space embedding theorems (see Appendix III).

Remark on nonlocality

A few remarks are in order regarding the issue of "locality" of the enstrophy cascade. It is possible to make an analysis of the contributions to the enstrophy flux from wavevector triads in distinct octave bands, analogous to that made for energy transfer in [32, 12]. Except for a single class of contributions, all triads yield an enstrophy flux Z_{ℓ_K} for $\ell_K = 2^{-K}$ which is actually

[1] Because of this fact, our classification of initial data into "Type A" and "Type B" for vorticity fields chosen at random from Gaussian ensembles coincides with that used by She et al. (for velocity fields) in the study of decaying Burgers turbulence [30].

$O(\ell_K^{3h})$. This is the magnitude of the flux contributed by the "local triads" with all wavevectors from octave bands near the Kth. However, the nonlocal class of terms like $\omega_N(\mathbf{v}_M{\boldsymbol\cdot}\boldsymbol{\nabla})\omega_L$ with $N, L \approx K$ and $M \ll K$ contributes at the order

$$Z_{\ell_K}^{\text{nloc.}} \sim 2^{K(1-2h)}2^{-M(1+h)}\frac{2^M}{2^K} = 2^{-hM}2^{-2hK}. \tag{2.21}$$

Note that $\mathbf{v}_M \sim 2^{-(1+h)M}$ since the velocity field is in C^{1+h} when $\omega \in C^h$ and that the factor $2^M/2^K$ comes from cancellations due to detailed conservation, as described in [32, 12]. This class of contributions is dominated by the triads with $M \approx 0$, i.e. by the largest-scale modes, and gives the leading term in the flux $\sim \ell_K^{2h}$. Hence, the enstrophy cascade is *infrared-dominated* and nonlocal when $0 < h < 1$. [2]

The preceding analysis was made in Fourier language, but the same result may be obtained in physical space. In fact, the estimate $\sim \ell^{2h}$ obtained in Eq.(2.20) is of the same magnitude, and its origin is similar. Locality is violated because $\Delta_{\mathbf{l}}\mathbf{v}$ is not a small-scale quantity when $\mathbf{v} \in C^{1+h}$ and $h > 0$! The proper small-scale quantity in this circumstance is instead

$$\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r}) - (\mathbf{l}{\boldsymbol\cdot}\boldsymbol{\nabla})\mathbf{v}(\mathbf{r}) = O(\ell^{1+h}), \tag{2.22}$$

requiring an additional subtraction $\sim O(\ell)$. The subtracted term still appears in $\Delta_{\mathbf{l}}\mathbf{v}(\mathbf{r})$, which is dominated by the large-scales. For this reason the "vortical stress" $\boldsymbol{\sigma}$ in 2D is not a truly small-scale quantity when $\omega \in C^h,\ 0 < h < 1$, but instead gets its dominant contribution from the infrared. The infrared domination of enstrophy transfer was already remarked by Kraichnan in [4] from his "K41-style" theory. Here it is demonstrated from the fluid equations themselves.

(ii) Statistical Steady States

Up until this point we have considered only the formally-conservative inertial dynamics. However, to produce stationary cascade states there must be added to the equations both external forces, to provide an input of energy and enstrophy, and viscous damping, to provide a

[2] Notice that our rigorous analysis is entirely in terms of *upper bounds*. Therefore, we have not really established nonlocal domination of the cascade unless we can show that these upper bounds are achieved. We expect this to be true but have no proof.

source of dissipation. It is particularly convenient to produce the turbulence with a zero-mean, Gaussian random force, white-noise in time:

$$\langle f_i(\mathbf{r},t) f_j(\mathbf{r}',t')\rangle = 2F_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{2.23}$$

Here $F_{ij}(\mathbf{r}-\mathbf{r}')$ is the integral kernel of the covariance operator, a positive, trace-class operator on $L^2(\mathbf{T}^d)$. In this case it is possible to construct in finite domain Λ statistical solutions of the Navier-Stokes dynamics: see the monograph of Vishik and Fursikov [7], Theorems X.1.1 and X.1.2. Those authors demonstrated also for this case the existence of stationary statistical solutions with finite mean energy and enstrophy, by explicit construction through a rigorous application of the Bogolyubov-Krylov averaging method (Theorem XI.2.1 of [7].) Aside from the possibility of rigorous proofs of existence, the most convenient aspect of the white-noise force case is the fact that energy and enstrophy inputs are entirely external control parameters without any dependence upon the detailed statistics of the fluid. This was first demonstrated formally by Novikov [36]. The precise statement of his result is that

$$\varepsilon \equiv \nu\langle|\boldsymbol{\nabla}\mathbf{v}(\mathbf{r},t)|^2\rangle = F_{ii}(\mathbf{0}). \tag{2.24}$$

This is essentially an energy balance relation, expressing equality of mean dissipation on the left with mean energy input by the force on the right. Novikov's result has been rigorously verified by Vishik and Fursikov in 2D, but in 3D only an inequality was established:

$$\varepsilon \equiv \nu\langle|\boldsymbol{\nabla}\mathbf{v}(\mathbf{r},t)|^2\rangle \leq F_{ii}(\mathbf{0}). \tag{2.25}$$

(see Theorem XI.2.2 of [7]). There is a real physical issue involved here, since singularities might provide an extra sink of energy in addition to regular viscous dissipation if they can occur for small but positive viscosity. This is ruled out in 2D but not so far in 3D. Unless this can be done, we do not expect that the balance relation with equality, Eq.(2.24), can be proved in 3D. At least it would need to be shown that such "dissipative singularities" occurred with zero probability with respect to the steady-state measure (see Appendix I).

In this section we discuss the steady-state balance equations for the vorticity-invariants $\Omega^{(n)}$, $n \geq 1$ in 2D, analogous to Novikov's energy balance relation Eq.(2.24). These can be demonstrated to hold in 2D as exact equalities by the methods of Vishik and Fursikov, but we postpone the rigorous proofs to Appendix I of this paper. Here we shall simply give the formal arguments in the manner of Novikov's work. It may be useful to recall here the basic elements of his derivation of the energy balance, before extending it to the vorticity invariants. The starting point is the randomly-forced Navier-Stokes system

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p + \nu\nabla^2\mathbf{v} + \mathbf{f}, \tag{2.26}$$

where $\mathbf{f}$ is the random force defined above. Actually, let us for the moment consider a slight generalization to a force with the more general covariance

$$\langle f_i(\mathbf{r},t) f_j(\mathbf{r}',t')\rangle = 2F_{ij}(\mathbf{r}-\mathbf{r}',t-t'), \tag{2.27}$$

no longer delta-correlated in time but still space-time homogeneous. Furthermore, we still assume its statistics to be *Gaussian* with zero mean. If one assumes the existence of a stationary measure for this system with regular realizations, then there is a balance of viscous dissipation with energy input by the force, as:

$$\nu\langle|\boldsymbol{\nabla}\mathbf{v}(\mathbf{r},t)|^2\rangle = \langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{f}(\mathbf{r},t)\rangle. \tag{2.28}$$

However, the Gaussian integration-by-parts identity may be used to rewrite the input term as

$$\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{f}(\mathbf{r},t)\rangle = 2\int_{-\infty}^{t} dt' \int_{\Lambda} d^2\mathbf{r}'\ F_{ij}(\mathbf{r}-\mathbf{r}',t-t')G_{ij}(\mathbf{r},t;\mathbf{r}',t'), \tag{2.29}$$

where

$$G_{ij}(\mathbf{r},t;\mathbf{r}',t') = \langle\frac{\delta v_i(\mathbf{r},t)}{\delta f_j(\mathbf{r}',t')}\rangle \tag{2.30}$$

is the *mean response function* to the stochastic force. [3] What is special to the case of delta-correlation in time is that only the time-diagonal part of the individual response function

[3]It is interesting to observe that the previous relation, which is exact for a Gaussian force, holds for a general stochastic driving force in the DIA approximation: see [37].

appears, which satisfies

$$\begin{aligned}\widehat{G}_{ij}(\mathbf{r},t;\mathbf{r}',t) &\equiv \frac{\delta v_i(\mathbf{r},t)}{\delta f_j(\mathbf{r}',t)} \\ &= \frac{1}{2}\delta_{ij}\delta^2(\mathbf{r}-\mathbf{r}'),\end{aligned} \tag{2.31}$$

in every realization. Notice that this is the response to the *random* force and the result differs by a factor of $\frac{1}{2}$ from the familiar result for infinitesimal response to a deterministic force. A full justification of this prescription involves the use of the Ito stochastic calculus (see [7]). Inserting Eq.(2.31) into Eq.(2.29) with the delta-correlation Eq.(2.23), the balance equation in its original form, Eq.(2.24), results.

We now extend these considerations to vorticity conservation. By taking the curl of the Eq.(2.26), we derive the equation for the vorticity

$$\partial_t\omega + (\mathbf{v}\cdot\boldsymbol{\nabla})\omega = \nu\nabla^2\omega + q, \tag{2.32}$$

with

$$q = \hat{\mathbf{e}}_3\cdot(\boldsymbol{\nabla}\times\mathbf{f}). \tag{2.33}$$

If we assume for simplicity isotropic forcing, i.e. $F_{ij} = \delta_{ij}F$ in Eq.(2.27), then it follows easily that

$$\langle q(\mathbf{r},t)q(\mathbf{r}',t')\rangle = 2Q(\mathbf{r}-\mathbf{r}',t-t'), \tag{2.34}$$

with $Q = -\nabla^2 F$ and the random force q is still Gaussian. ¿From the Eq.(2.32) we derive for $n \geq 2$ the local balance relation for individual solutions:

$$\partial_t h^{(n)} + \boldsymbol{\nabla}\cdot(h^{(n)}\mathbf{v} - \nu h^{(n-1)}\boldsymbol{\nabla}\omega) = -\nu h^{(n-2)}|\boldsymbol{\nabla}\omega|^2 + h^{(n-1)}q, \tag{2.35}$$

with $h^{(n)} \equiv \frac{1}{n!}\omega^n$. In the steady state we therefore find that

$$\eta^{(n)} \equiv \nu\langle h^{(n-2)}|\boldsymbol{\nabla}\omega|^2\rangle = \langle h^{(n-1)}q\rangle. \tag{2.36}$$

This is the average balance equation for dissipation of the nth vorticity invariant $\Omega^{(n)}$, and its input by the force q for the general case.

However, with the Gaussian integration-by-parts identity we can derive in the same manner as before

$$\langle h^{(n-1)}(\mathbf{r},t)q(\mathbf{r},t)\rangle = 2\int_{-\infty}^{t} dt' \int_{\Lambda} d^2\mathbf{r}'\ Q(\mathbf{r}-\mathbf{r}',t-t')\langle h^{(n-2)}(\mathbf{r},t)\widehat{H}(\mathbf{r},t;\mathbf{r}',t')\rangle, \tag{2.37}$$

in which

$$\widehat{H}(\mathbf{r},t;\mathbf{r}',t') = \frac{\delta\omega(\mathbf{r},t)}{\delta q(\mathbf{r}',t')} \tag{2.38}$$

is the individual response operator of the vorticity to the stochastic force q. For the particular case $n=2$, with $\eta=\eta^{(2)}$, we derive the result

$$\eta = 2\int_{-\infty}^{t} dt' \int_{\Lambda} d^2\mathbf{r}'\ Q(\mathbf{r}-\mathbf{r}',t-t')H(\mathbf{r},t;\mathbf{r}',t'), \tag{2.39}$$

analogous to Eqs.(2.28),(2.29), with $H(\mathbf{r},t;\mathbf{r}',t')\equiv\langle\widehat{H}(\mathbf{r},t;\mathbf{r}',t')\rangle$. In particular, for *white-noise* force in time we see that

$$\eta = Q(\mathbf{0}) \tag{2.40}$$

and is also independent of the fluid statistics. As we shall see, this fact has profound consequences. Another remarkable feature of this case is that

$$\eta^{(n)} = \langle h^{(n-2)}\rangle\eta \tag{2.41}$$

for $n\geq 2$, so that all of the higher-order inputs are proportional to the input of enstrophy.

For our discussion in the next section we shall need a slight generalization of the previous results which we mention without a detailed derivation. It can be given as above. We consider the local-balance of vorticity invariants in the large-scales, Eq.(2.9), with the addition of viscous dissipation and random stirring. The balance equation now becomes

$$\partial_t h_\ell^{(n)}(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{K}_\ell^{(n)}(\mathbf{r},t) = -Z_\ell^{(n)}(\mathbf{r},t) - D_\ell^{(n)}(\mathbf{r},t) + h_\ell^{(n)}(\mathbf{r},t)q_\ell(\mathbf{r},t) \tag{2.42}$$

in which the current

$$\mathbf{K}_\ell^{(n)}(\mathbf{r},t) \equiv h_\ell^{(n)}(\mathbf{r},t)\overline{\mathbf{v}}_\ell(\mathbf{r},t) + h_\ell^{(n-1)}(\mathbf{r},t)\boldsymbol{\sigma}_\ell(\mathbf{r},t) - \nu h_\ell^{(n-1)}(\mathbf{r},t)\boldsymbol{\nabla}\omega_\ell(\mathbf{r},t), \tag{2.43}$$

and

$$D_\ell^{(n)}(\mathbf{r},t) = \nu h_\ell^{(n-2)}(\mathbf{r},t)|\boldsymbol{\nabla}\omega_\ell(\mathbf{r},t)|^2, \tag{2.44}$$

is the dissipation of the nth invariant in the large-scales only. Note that $Z_\ell^{(n)}$ is the flux given in Eq.(2.15). If we now define

$$\eta_\ell^{(n)} \equiv \langle h_\ell^{(n-1)} q_\ell \rangle \tag{2.45}$$

to be the *input* of the *nth* invariant into the length-scales $> \ell$, then we obtain the balance equation

$$\eta_\ell^{(n)} = \langle Z_\ell^{(n)} \rangle + \langle D_\ell^{(n)} \rangle. \tag{2.46}$$

Remark on a curious independence property

We just wish to point out here a property of the force white-noise in time, which follows from Eq.(2.41) and the general result Eq.(2.36). In combination they yield, for each $n \geq 0$,

$$\langle \omega^n(\mathbf{r}) \cdot |\boldsymbol{\nabla}\omega(\mathbf{r})|^2 \rangle = \langle \omega^n(\mathbf{r}) \rangle \langle |\boldsymbol{\nabla}\omega(\mathbf{r})|^2 \rangle. \tag{2.47}$$

Observe that this implies, under some technical assumptions, that the vorticity at a point, $\omega(\mathbf{r})$, and the magnitude of its gradient at that same point, $\xi(\mathbf{r}) = |\boldsymbol{\nabla}\omega(\mathbf{r})|$, are statistically independent! Some implications this curious fact will be discussed later.

3 Cascade Directions of Energy and Enstrophy

Preliminaries

We consider here a somewhat more general situation than before, in which we have an infrared dissipation in some range $[k_0, k_e]$ above the lowest wavenumber k_0, and also ultraviolet dissipation in the range $[k_v, \infty]$. More precisely, the dynamical equation in Fourier space is taken to be

$$\partial_t a_i(\mathbf{k}) + \sum_{\mathbf{p}j,\mathbf{q}k} B_{\mathbf{k}i,\mathbf{p}j,\mathbf{q}k} a_j(\mathbf{p}) a_k(\mathbf{q}) = f_i(\mathbf{k}) - \alpha_{(r)} k^{-r} \theta_{[k_0,k_e]}(k) a_i(\mathbf{k}) - \nu_{(s)} k^s \theta_{[k_v,\infty]}(k) a_i(\mathbf{k}). \tag{3.1}$$

Note that $B_{\mathbf{k}i,\mathbf{p}j,\mathbf{q}k} = \frac{i}{2}(k_j P_{ik}(\mathbf{k}) + k_i P_{jk}(\mathbf{k}))\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}}$ is the usual coefficient of the Euler nonlinearity. The force $f_i(\mathbf{k})$ is assumed to have support in a range $[\frac{1}{2}k_f, 2k_f]$. We consider an arbitrary sth-order hyperviscosity with coefficient $\nu_{(s)}$, acting in the ultraviolet above some wavenumber k_v. Of course, even without the cutoff k_v this dissipation would be substantial only for large k when s was large enough. However, it will be useful in our discussion below to put in the cutoff wavenumber. Likewise, we consider an arbitrary "hypoviscosity" with coefficient $\alpha_{(r)}$ acting in the infrared below some wavenumber k_e. Similar dissipation mechanisms are often argued to occur at the largest length-scales in atmospheric flows. Special cases of this model have been the subject of a number of recent high Reynolds number simulations [38, 39, 40].

The primary issue we address in this section is the support of the energy spectrum and the direction of cascades of energy and enstrophy in the steady-states of these models. There is a simple heuristic argument for the cascade directions which is part of the "folklore" of the subject: in essence it goes back to Kraichnan [4]. If the energy input ε and enstrophy input η are divided into infrared and ultraviolet fluxes, as

$$\varepsilon = \varepsilon_{ir} + \varepsilon_{uv}, \tag{3.2}$$

and

$$\eta = \eta_{ir} + \eta_{uv}, \tag{3.3}$$

then it is possible to *define* two wavenumbers in some sense characteristic of the infrared and and ultraviolet ranges, as

$$k_{ir}^2 \equiv \eta_{ir}/\varepsilon_{ir}, \tag{3.4}$$

and

$$k_{uv}^2 \equiv \eta_{uv}/\varepsilon_{uv}. \tag{3.5}$$

Of course, it is also true that

$$\eta = k_f^2 \varepsilon \tag{3.6}$$

up to an immaterial constant of order one. Solving these relations yields

$$\frac{\varepsilon_{uv}}{\varepsilon_{ir}} = \frac{k_f^2 - k_{ir}^2}{k_{uv}^2 - k_f^2}, \tag{3.7}$$

and

$$\frac{\eta_{uv}}{\eta_{ir}} = \frac{k_{uv}^2(k_f^2 - k_{ir}^2)}{k_{ir}^2(k_{uv}^2 - k_f^2)}. \tag{3.8}$$

A particular limit of interest is $k_{uv} \to \infty$, in which case $\varepsilon_{uv}/\varepsilon_{ir} \to 0$ and $\eta_{uv}/\eta_{ir} \to (k_f/k_{ir})^2 - 1$. Hence, in that limit all of the energy flows to the low wavenumbers, i.e. there is an *inverse cascade* of energy. In that same limit, a fraction $(k_{ir}/k_f)^2\eta$ of the enstrophy input goes to the low wavenumbers and a fraction $\left(1 - \left(\frac{k_{ir}}{k_f}\right)^2\right)\eta$ goes to high wavenumbers. If the limit $k_{ir} \ll k_f$ is considered subsequently, then there is predominantly a *direct cascade* of enstrophy. As we shall show in detail below, this argument is basically correct and leads to the proper directions and magnitudes of transfer. Its weakness is that it does not tell how the characteristic wavenumbers k_{ir}, k_{uv} depend upon physical parameters of the model (such as viscosity ν, etc.) Furthermore, it makes implicit assumptions on the existence of intervals of constant flux, finiteness of energy, etc. It is one of our purposes below to give, as much as is possible, an *a priori* treatment of the problem and, if additional assumptions are required, to make them explicit.

Notation: moments of the energy spectrum

Although somewhat clumsy because of the generality of our model, we use the following notations:

$$E_{(p)} \equiv \int_{k_0}^{\infty} k^p E(k)dk, \tag{3.9}$$

$$E_{(p)}^{<\ell} \equiv \int_{2\pi/\ell}^{\infty} k^p E(k)dk, \tag{3.10}$$

$$E_{(p)}^{>\ell} \equiv \int_{k_0}^{2\pi/\ell} k^p E(k)dk, \tag{3.11}$$

$$E_{(p)}^{(\ell_1,\ell_2)} \equiv \int_{2\pi/\ell_1}^{2\pi/\ell_2} k^p E(k)dk, \tag{3.12}$$

if $\ell_1 > \ell_2$, and otherwise $\equiv 0$.

Similar abbreviations will be used later for moments of the enstrophy spectrum.

Energy and enstrophy balance equations

Equations for energy balance in the scales $> \ell$ are derived, formally by the methods used above, or, rigorously as in Appendix I:

For $\ell_e > \ell > \ell_f$:

$$\alpha_{(r)} E_{(-r)}^{>\ell_e} + \nu_{(s)} E_{(s)}^{(\ell_v,\ell)} + \langle \Pi_\ell \rangle = 0. \tag{3.13}$$

For $2^{-1}\ell_f > \ell$:

$$\alpha_{(r)} E_{(-r)}^{>\ell_e} + \nu_{(s)} E_{(s)}^{(\ell_v,\ell)} + \langle \Pi_\ell \rangle = \varepsilon. \tag{3.14}$$

In the same way, equations are derived for enstrophy balance in the scales $> \ell$:

For $\ell_e > \ell > \ell_f$:

$$\alpha_{(r)} E_{(-r+2)}^{>\ell_e} + \nu_{(s)} E_{(s+2)}^{(\ell_v,\ell)} + \langle Z_\ell \rangle = 0. \tag{3.15}$$

For $2^{-1}\ell_f > \ell$:

$$\alpha_{(r)} E_{(-r+2)}^{>\ell_e} + \nu_{(s)} E_{(s+2)}^{(\ell_v,\ell)} + \langle Z_\ell \rangle = \eta. \tag{3.16}$$

I. Case Without Infrared Dissipation

Divergence of energy as viscosity tends to zero

Here we take also $k_v = k_0$. This was the model constructed by Vishik and Fursikov for $s = 2$ [7]. They noted that $\nu\langle|\mathbf{\nabla v}|^2\rangle = F(\mathbf{0})$ is independent of viscosity $\nu = \nu_{(2)}$, and interpreted that result as verifying Kolmogorov's hypothesis of dissipation non-vanishing with viscosity going to zero. However, this is a misinterpretation. Kolmogorov's hypothesis was for the 3D case where there is an ultraviolet energy cascade. On the contrary, in the 2D case there will be an inverse cascade. Hence, energy will remain constant as $\nu \to 0$ in the unforced case (decaying turbulence) and the mean energy will *diverge* in the randomly forced case as $\nu \to 0$. We present a proof of this result (which was found by the author and Z.-S. She):

In this case we have

$$\varepsilon = \nu E_{(2)}, \tag{3.17}$$

and

$$\eta = \nu E_{(4)}. \tag{3.18}$$

By the Cauchy-Schwartz inequality,

$$E_{(2)} \leq \sqrt{E_{(4)} E_{(0)}}. \tag{3.19}$$

Thus,

$$\varepsilon \leq \sqrt{\nu \eta E_{(0)}}, \tag{3.20}$$

or,

$$E_{(0)} \geq \frac{\varepsilon^2}{\nu \eta}. \tag{3.21}$$

Hence, *energy must diverge like* ν^{-1} *as* $\nu \to 0$. Observe from $\eta = \varepsilon / \ell_f^2$ that

$$E_{(0)} \geq \ell_f^2 \frac{\varepsilon}{\nu}. \tag{3.22}$$

A similar *upper bound* is obtained from

$$\varepsilon \geq \nu k_0^2 E_{(0)}, \tag{3.23}$$

so that

$$E_{(0)} \leq \ell_0^2 \frac{\varepsilon}{\nu}. \tag{3.24}$$

This shows, by the way, that the energy is finite for finite viscosity. These bounds are consistent with the picture that most of the dissipation of energy, and also of enstrophy, occurs in the large-scales, at wavenumbers $\leq k_f$, so that

$$\varepsilon \sim \nu k_f^2 E_{(0)}, \quad \eta \sim \nu k_f^4 E_{(0)}. \tag{3.25}$$

The amplitude of the velocity fluctuations is forced to rise to a high enough level in the low wavenumbers to achieve energy balance with the input from the force.

Localization of energy

As a simple application of the Chebychev inequality, note that

$$\begin{aligned}\int_K^\infty E(k)dk &\leq \int_{k_0}^\infty \left(\frac{k}{K}\right)^4 E(k)dk \\ &= \frac{\eta/\nu}{K^4} \\ &= (\text{const.})\frac{E_{(0)}}{(K\ell_f)^4}.\end{aligned} \tag{3.26}$$

Flux directions

Define

$$\varepsilon_{uv}(\ell) = \langle \Pi_\ell \rangle, \tag{3.27}$$

for $\ell < 2^{-1}\ell_f$. By energy balance,

$$\begin{aligned}\varepsilon_{uv}(\ell) &= \varepsilon - \nu E_{(2)}^{>\ell} \\ &= \nu E_{(2)}^{<\ell}.\end{aligned} \tag{3.28}$$

Since $E_{(2)}^{<\ell} \leq \ell^2 \cdot E_{(4)}$, by Chebychev again,

$$\begin{aligned}\varepsilon_{uv}(\ell) &\leq \ell^2 \cdot \eta \\ &= \left(\frac{\ell}{\ell_f}\right)^2 \varepsilon.\end{aligned} \tag{3.29}$$

Note the similarity to Fjortoft's old result [3] on the amount of energy able to reach small length-scales in freely decaying turbulence.

We may also define the UV enstrophy flux as

$$\eta_{uv}(\ell) = \langle Z_\ell \rangle \tag{3.30}$$

for $\ell < 2^{-1}\ell_f$. For this enstrophy flux the same argument yields

$$\begin{aligned}\eta_{uv}(\ell) &= \nu E_{(4)}^{(<\ell)} \\ &= \eta \cdot \left(\frac{E_{(4)}^{<\ell}}{E_{(4)}}\right).\end{aligned} \tag{3.31}$$

It seems possible that a finite fraction of the enstrophy input η goes to the high wavenumbers.

II. Case With Infrared Dissipation

Finiteness of Energy

Since we have seen that energy *must* diverge for vanishing viscosity without some low-wavenumber dissipation, it seems impossible that any $\nu \to 0$ limit of that state can exist. Therefore, we consider here the general model of Eq.(3.1) (with $s = 2$.) Although we believe it to be true, so far we have not been able to prove that energy remains finite even if the IR dissipation is added! For some of our arguments below we will use that fact. Therefore, we state it as a

Hypothesis 1 $E_{(0)} = O(1)$ *as* $\nu \to 0$.

Some notation we will also use below is

$$\varepsilon_e \equiv \alpha_{(r)} E_{(-r)}^{>\ell_e} \tag{3.32}$$

for the IR energy dissipation, and

$$\eta_e \equiv \alpha_{(r)} E_{(-r+2)}^{>\ell_e} \tag{3.33}$$

for the IR enstrophy dissipation, and likewise

$$\varepsilon_v \equiv \nu E_{(2)}^{<\ell_v} \tag{3.34}$$

for the UV energy dissipation, and

$$\eta_v \equiv \alpha_{(r)} E_{(-r+2)}^{<\ell_v} \tag{3.35}$$

for the UV enstrophy dissipation. Now we have the balance relations

$$\varepsilon = \varepsilon_e + \varepsilon_v, \tag{3.36}$$

and

$$\eta = \eta_e + \eta_v. \tag{3.37}$$

For the most part we shall consider the case $\ell_v = \ell_0$, but consider the more general situation at one point later on.

Asymptotic locations of energy and enstrophy dissipation

It is possible here to use the same method of argument as before. Thus

$$\begin{aligned} 0 \le \varepsilon - \varepsilon_e = \varepsilon_v &= \nu E_{(2)} \\ &\le \sqrt{\nu^2 E_{(4)} E_{(0)}} \\ &= \sqrt{\nu \eta E_{(0)}}. \end{aligned} \tag{3.38}$$

¿From this we can draw several conclusions. First,

$$\varepsilon_v = O\left(\sqrt{\nu E_{(0)}}\right). \tag{3.39}$$

and

$$\varepsilon_e = \varepsilon + O\left(\sqrt{\nu E_{(0)}}\right), \tag{3.40}$$

so that asymptotically all of the energy dissipation is infrared for $\nu \to 0$, under our hypothesis. Note actually that we need only the weaker result $E_{(0)} = o(1/\nu)$ to draw that conclusion. Secondly,

$$\Omega = E_{(2)} = O\left(\sqrt{\frac{E_{(0)}}{\nu}}\right). \tag{3.41}$$

Thus, enstrophy may diverge as $\nu \to 0$, but at most as $\nu^{-1/2}$ if the hypothesis is true. Finally, since $\eta_e = k_e^2 \varepsilon_e$ (up to an unimportant constant factor), it follows from Eq.(3.40) also that

$$\eta_e = \left(\frac{k_e}{k_f}\right)^2 \cdot \eta + O\left(\sqrt{\nu E_{(0)}}\right). \tag{3.42}$$

Consequently, it follows also that

$$\eta_v = \left[1 - \left(\frac{k_e}{k_f}\right)^2\right] \eta + O\left(\sqrt{\nu E_{(0)}}\right). \tag{3.43}$$

Thus $\eta_v \approx \eta$ when $k_f \gg k_e$ and $\eta_v \approx 0$ when $k_f \approx k_e$.

Fluxes of energy and enstrophy

Now set

$$\varepsilon_{ir}(\ell) \equiv -\langle \Pi_\ell \rangle \tag{3.44}$$

for $\ell > 2\ell_f$, and

$$\varepsilon_{uv}(\ell) \equiv +\langle \Pi_\ell \rangle \tag{3.45}$$

for $\ell < 2^{-1}\ell_f$. Thus,

$$\varepsilon_{ir}(\ell) = \varepsilon_e + \nu E^{>\ell}_{(2)}. \tag{3.46}$$

and

$$\varepsilon_{uv}(\ell) = \varepsilon - \varepsilon_e - \nu E^{>\ell}_{(2)}. \tag{3.47}$$

Note that

$$E^{(>\ell)}_{(2)} \le \frac{\text{(const.)}}{\ell^2} E_{(0)}. \tag{3.48}$$

Along with our previous result for ε_e in Eq.(3.40) ,this gives

$$\varepsilon_{ir}(\ell) = \varepsilon + O\left(\sqrt{\nu E_{(0)}}, \frac{\nu E_{(0)}}{\ell^2}\right), \tag{3.49}$$

and

$$\varepsilon_{uv}(\ell) = O\left(\sqrt{\nu E_{(0)}}, \frac{\nu E_{(0)}}{\ell^2}\right). \tag{3.50}$$

Under our hypothesis, or assuming even $E_{(0)} = o(1/\nu)$, we see that in the limit $\nu \to 0$ that $\varepsilon_{ir}(\ell) = \varepsilon,\ \ \ell_e > \ell > 2\ell_f$, and $\varepsilon_{uv}(\ell) = 0,\ \ \ell < 2^{-1}\ell_f$. Notice that this argument *proves* the constancy of flux over the relevant ranges.

For enstrophy we can likewise set

$$\eta_{ir}(\ell) \equiv -\langle Z_\ell \rangle \tag{3.51}$$

for $\ell > 2\ell_f$, and

$$\eta_{uv}(\ell) \equiv +\langle Z_\ell \rangle \tag{3.52}$$

for $\ell < 2^{-1}\ell_f$. Thus,

$$\eta_{ir}(\ell) = \eta_e + \nu E^{>\ell}_{(4)}. \tag{3.53}$$

and

$$\eta_{uv}(\ell) = \eta - \eta_e - \nu E^{>\ell}_{(4)}. \tag{3.54}$$

Also,

$$E_{(4)}^{(>\ell)} \leq \frac{(\text{const.})}{\ell^4} E_{(0)}. \tag{3.55}$$

We conclude finally that

$$\eta_{ir}(\ell) = \left(\frac{k_e}{k_f}\right)^2 \cdot \eta + O\left(\sqrt{\nu E_{(0)}}, \frac{\nu E_{(0)}}{\ell^4}\right), \tag{3.56}$$

and

$$\eta_{uv}(\ell) = \left[1 - \left(\frac{k_e}{k_f}\right)^2\right] \eta + O\left(\sqrt{\nu E_{(0)}}, \frac{\nu E_{(0)}}{\ell^4}\right). \tag{3.57}$$

In the limit $\nu \to 0$ we recover the results of the traditional argument, with $k_{ir} \equiv k_e$. Note that we can get $\eta_{uv} \to \eta$ by taking $k_f \gg k_e$ and $\eta_{uv} \to 0$ by taking $k_f \approx k_e$.

Extent of the enstrophy inertial range

As before, we have proved that the enstrophy flux is really finite over the relevant UV range:

$$\eta_{uv}(\ell) \approx \eta_v, \tag{3.58}$$

for

$$2^{-1}\ell_f \geq \ell \gg \ell_d \equiv \frac{\nu^{1/4} E_{(0)}^{1/4}}{\eta^{1/4}}. \tag{3.59}$$

In contrast, the classical Kraichnan dissipation length is

$$\ell_d^{\text{Kr}} = \frac{\nu^{1/2}}{\eta^{1/6}}. \tag{3.60}$$

Since $E_{(0)} \gg \nu \cdot \eta^{1/3}$ in the limit $\nu \to 0$, it follows that $\ell_d \gg \ell_d^{\text{Kr}}$. Thus, this argument does not establish constant flux down to the classical dissipation scale.

In this context, it may be noted that it is possible to define a dissipation wavenumber intrinsically for the steady-state, by

$$\kappa_d^2 \equiv \frac{E_{(4)}}{E_{(2)}}. \tag{3.61}$$

Many of our previous conclusions would follow if it were possible to prove

Hypothesis 2 $\lim_{\nu \to 0} \kappa_d = \infty$.

Indeed, by its definition,

$$\begin{aligned} \varepsilon_v &= \frac{\eta_v}{\kappa_d^2} \\ &\leq \frac{\eta}{\kappa_d^2}, \end{aligned} \tag{3.62}$$

so that $\varepsilon_v \to 0$ would follow under this hypothesis as $\nu \to 0$. Of course, in this case $\varepsilon_e \to \varepsilon$. Furthermore, since $\eta_e = k_e^2 \varepsilon_e$, it would then follow that $\eta_e \to (k_e/k_f)^2 \eta$ for $\nu \to 0$. As a consequence, $\eta_v \to [1-(k_e/k_f)^2]\eta$ in that same limit. These conclusions all involve dissipation. Note that the corresponding statements about fluxes would still require the finite-energy assumption (Hypothesis 1). Unfortunately, Hypothesis 2 does not seem at the moment to be susceptible to proof. In [41] some similar wavenumbers are studied. However, for those wavenumbers it is proved that k_d^{Kr} is an *upper bound* in 2D, which does not suffice for our purpose. One situation where the results can be obtained cheaply is by keeping the cutoff k_v in the model, and taking it by hand a function of ν which goes to infinity as $\nu \to 0$. Since

$$\varepsilon_v \leq \frac{\eta}{k_v^2} \tag{3.63}$$

in this circumstance, we get the desired conclusions.

Both of the previous two hypotheses follow from the even stronger one:

Hypothesis 3 $\Omega_{(0)} = O(1)$ *as* $\nu \to 0$.

Here we introduce the notation $\Omega_{(p)} = E_{(p+2)}$ for the pth moments of the enstrophy spectrum. Obviously, it implies Hypothesis 1 on finiteness of energy. Note that the Hypothesis 3 is true when the energy spectrum is steeper than the classical Kraichnan spectrum $E(k) \sim k^{-3}$ but invalid at that spectrum. However, it implies an even stronger result than Hypothesis 2, namely, that the wavenumber κ_d introduced there equals the classical Kraichnan wavenumber:

$$\kappa_d = \kappa_d^{\mathrm{Kr}}. \tag{3.64}$$

In fact, an alternative way of writing κ_d is as

$$\kappa_d^2 = \frac{\Omega_{(2)}}{\Omega_{(0)}} \sim \frac{\eta/\nu}{\Omega_{(0)}}, \tag{3.65}$$

with $\eta_{uv} \sim \eta$. However, if Hypothesis 3 holds then

$$\Omega_{(0)} \propto \eta^{2/3}, \tag{3.66}$$

as $\nu \to 0$. This is a consequence of dimensional analysis, since, if the zero-viscosity limit exists, then all statistical quantities must be functions only of η, with units of (time)^3, and the model wavenumbers k_0, k_f. Because Ω has units of (time)^2, it must then be $\eta^{2/3}$ times some function of the fixed wavenumber ratio k_0/k_f. Hence, $\kappa_d^2 \sim \eta^{1/3}/\nu$, or

$$\kappa_d \sim \frac{\eta^{1/6}}{\nu^{1/2}} = \kappa_d^{\mathrm{Kr}}. \tag{3.67}$$

In particular, the dissipation wavenumber diverges to infinity as $\nu \to 0$.

With the stronger Hypothesis 3 it is not hard, in fact, to show that the enstrophy flux will be constant up to the Kraichnan wavenumber κ_d^{Kr}. This validates the significance of the latter as the dissipation wavenumber. In fact, with the Hypothesis 3, our earlier estimates on the energy and enstrophy dissipation, Eqs. (3.39),(3.40),(3.42), (3.43), may be improved to

$$\varepsilon_v = \nu\Omega_{(0)}, \tag{3.68}$$

$$\varepsilon_e = \varepsilon - \nu\Omega_{(0)}, \tag{3.69}$$

$$\eta_e = \left(\frac{k_e}{k_f}\right)^2 \cdot \eta + O\left(\nu\Omega_{(0)}\right), \tag{3.70}$$

and

$$\eta_v = \left[1 - \left(\frac{k_e}{k_f}\right)^2\right] \eta + O\left(\nu\Omega_{(0)}\right). \tag{3.71}$$

Likewise, the estimates of the fluxes in Eqs.(3.49), (3.50),(3.56), and (3.57), may be improved under the stronger hypothesis to give

$$\begin{aligned} \varepsilon_{uv}(\ell) &= \nu\Omega_{(0)}^{<\ell} \\ &= O\left(\nu\Omega_{(0)}\right), \end{aligned} \tag{3.72}$$

$$\varepsilon_{ir}(\ell) = \varepsilon - \nu\Omega_{(0)}^{<\ell}$$

$$= \quad \varepsilon + O\left(\nu\Omega_{(0)}\right), \tag{3.73}$$

$$\begin{aligned}\eta_{ir}(\ell) &= \eta_e + \nu\Omega_{(2)}^{>\ell} \\ &= \left(\frac{k_e}{k_f}\right)^2 \cdot \eta + O\left(\nu\Omega_{(0)}, \frac{\nu\Omega_{(0)}}{\ell^2}\right),\end{aligned} \tag{3.74}$$

and

$$\begin{aligned}\eta_{uv}(\ell) &= \eta - \eta_e - \nu\Omega_{(2)}^{>\ell} \\ &= \left[1 - \left(\frac{k_e}{k_f}\right)^2\right]\eta + O\left(\nu\Omega_{(0)}, \frac{\nu\Omega_{(0)}}{\ell^2}\right).\end{aligned} \tag{3.75}$$

In particular, it follows from the last estimate that $\eta_{uv}(\ell) \approx \eta$ down to a lengthscale ℓ_d which is determined by

$$\eta \sim \frac{\nu\Omega_{(0)}}{\ell_d^2} \sim \frac{\nu\eta^{2/3}}{\ell_d^2}, \tag{3.76}$$

whose solution yields

$$\ell_d \sim \frac{\nu^{1/2}}{\eta^{1/6}} \equiv \ell_d^{\mathrm{Kr}}. \tag{3.77}$$

Therefore, the ultraviolet end of the inertial range is demarcated by the wavenumber κ_d^{Kr} when the Hypothesis 3 is valid.

Fluxes of the higher-order vorticity invariants

Notice that our arguments for constancy of fluxes do not *necessarily* work for the higher-order vorticity invariants. In fact, as already noted by Falkovich and Hanany in [10], the inputs for those quantities need not be supported just in the spectral range $[2^{-1}k_f, 2k_f]$. In fact, this can be easily seen from our expression Eq.(2.45), since $q_\ell = q$ when $\ell < 2^{-1}\ell_f$ but the same is not true for $h_\ell^{(n-1)}$. Thus, the input of these invariants can be spread over the entire inertial range. In the white-noise force case we can derive an exact expression for the input into the length-scales $> \ell$, generalizing our earlier expression for total input. It is

$$\eta_\ell^{(n)} = \langle h_\ell^{(n-2)}\rangle \cdot \eta, \tag{3.78}$$

when $\ell \leq \beta \cdot \ell_f$, where β is some fraction < 1 that depends upon the choice of filter function G_ℓ. In fact, by applying the Gaussian integration-by-parts identity to Eq.(2.45), we obtain the expression for the input

$$\eta_\ell^{(n)} = 2\int_\Lambda d^2\mathbf{r}'\ Q(\mathbf{r}-\mathbf{r}')\langle h_\ell^{(n-2)}(\mathbf{r},t)\frac{\delta\overline{\omega}_\ell(\mathbf{r},t)}{\delta q(\mathbf{r}',t)}\rangle, \tag{3.79}$$

where from the definitions and the Ito rule

$$\frac{\delta\overline{\omega}_\ell(\mathbf{r},t)}{\delta q(\mathbf{r}',t)} = \frac{1}{2}G_\ell(\mathbf{r}-\mathbf{r}'). \tag{3.80}$$

Hence,

$$\eta_\ell^{(n)} = \langle Q, G_\ell\rangle_{L^2}\cdot\langle h_\ell^{(n-2)}\rangle. \tag{3.81}$$

However, if the filter has a Fourier transform $\widehat{G}(\mathbf{k})\equiv 1$ in some neighborhood of $\mathbf{k}=\mathbf{0}$, then for a small enough $\ell<\beta\cdot\ell_f$ it follows that $\langle Q, G_\ell\rangle_{L^2} = Q(\mathbf{0})$, and thus Eq.(3.78) follows.

This formula shows explicitly how the input is distributed over length-scales in the white-noise case. As one application, let us observe that the inputs of all of the *even-order* invariants must diverge if there is an ultraviolet divergence of the enstrophy, or if

$$\begin{aligned}\frac{1}{2}\langle\overline{\omega}_\ell^2\rangle &= \int_{k_0}^{\infty}|\widehat{G}_\ell(k)|^2\Omega(k)dk\\ &\approx \int_{k_0}^{2\pi/\ell}\Omega(k)dk.\end{aligned} \tag{3.82}$$

diverges for $\ell\to 0$ (after the limit $\nu\to 0$.) In fact, by the previous formula and the Hölder inequality

$$\begin{aligned}\eta_\ell^{(2m)} &= \frac{1}{(2m-2)!}\langle\overline{\omega}_\ell^{2(m-1)}\rangle\cdot\eta\\ &\geq \frac{1}{(2m-2)!}\langle\overline{\omega}_\ell^2\rangle^{m-1}\cdot\eta.\end{aligned} \tag{3.83}$$

If the effects of viscosity can be ignored, then the flux of these invariants must get increasingly large at small-scales (or else the flux in the infrared direction must be infinite) when the enstrophy itself diverges.

This is the scenario proposed by Falkovich and Hanany [10], who considered energy spectra of the k^{-3}-form with log-corrections

$$E(k)\sim\eta^{2/3}k^{-3}\ln^{-s}(k\ell_f) \tag{3.84}$$

(as proposed much earlier by Kraichnan [42] with the precise value $s = \frac{1}{3}$.) It is easy to calculate for these spectra that

$$\frac{1}{2}\langle\overline{\omega}_\ell^2\rangle \sim \ln^{1-s}\left(\frac{\ell_f}{\ell}\right). \tag{3.85}$$

Therefore, the proposal of [10] that

$$\langle Z_\ell^{(2m)}\rangle \sim \ln^{(m-1)(1-s)}\left(\frac{\ell_f}{\ell}\right) \tag{3.86}$$

appears consistent for the white-noise force case. Our rigorous argument with the Hölder inequality shows that the "reducible contribution" —claimed in [10] to be dominant—is at least a lower-bound.

However, if $\langle\omega^2\rangle < \infty$, then it is possible to have constant-flux inertial ranges of the higher-order invariants. In fact, let us consider for $n > 2$ a more general

Hypothesis 4 $\Omega^{(n-2)} = O(1)$ *as* $\nu \to 0$.

Under this assumption the inputs of the higher invariants all have limits

$$\lim_{\ell\to 0}\eta_\ell^{(n)} = \Omega^{(n-2)}\cdot\eta < \infty, \tag{3.87}$$

and thus are restricted to essentially a finite wavenumber range around k_f. It seems most likely that these invariants will behave as the enstrophy and develop *ultraviolet cascades.* This, in fact, follows if we make an additional

Hypothesis 5 $\Omega^{(2n-2)} = O(1)$ *as first* $\nu \to 0$ *and then* $\frac{k_e}{k_f} \to 0$.

Indeed, it is easy to derive a bound for the infrared dissipation $\eta_e^{(n)}$ of the nth invariant:

$$\eta_e^{(n)} = O\left(\sqrt{\alpha_{(r)}k_e^{-r}\Omega^{(2n-2)}\eta_e}\right), \tag{3.88}$$

as a direct consequence of its definition

$$\begin{aligned}\eta_e^{(n)} &= \alpha_{(r)}\langle h^{(n-1)}(-\triangle)^{-r/2}P^{<k_e}\omega\rangle \\ &= \alpha_{(r)}\langle(-\triangle)^{-r/4}P^{<k_e}h^{(n-1)}\cdot(-\triangle)^{-r/4}P^{<k_e}\omega\rangle,\end{aligned} \tag{3.89}$$

and the Cauchy-Schwartz inequality. In conjunction with Eq.(3.70) and the Hypothesis 5, this yields

$$\begin{aligned} \eta_e^{(n)} &= O\left(\frac{k_e}{k_f}\sqrt{\alpha_{(r)} k_e^{-r} \Omega^{(2n-2)} \eta}\right) \\ &\to 0, \end{aligned} \tag{3.90}$$

as $\nu \to 0, (k_e/k_f) \to 0$ in that order. Therefore, all of the higher-order vorticity-invariants must cascade to high wavenumbers. This conclusion supports the argument of Falkovich and Hanany [10] against the conformal "minimal models" of Polyakov [8, 9]. As pointed out in [10], the constraints of constant fluxes of all the higher-order invariants would be satisfied in Polyakov's scheme by fixing—for each one—the dimension of a primary field. This could not be done in the minimal models with a finite, closed algebra of primaries. Polyakov has suggested [9] that the difficulty may be avoided if the higher-order invariants all cascade to low-wavenumbers. However, it is easy to check that his conformal solutions have all $\Omega^{(n)}_{\rm conf} < \infty$ for $n \geq 2$, and thus the previous argument implies non-vanishing ultraviolet fluxes of the higher-order invariants. [4] The only way that we can see for the minimal models to survive as viable descriptions of 2D turbulence is to appeal to a non-conformal effect: namely, the contribution $\Omega^{(n)}_{\rm cond}$ of the "infrared condensates" must diverge for all $n > 2$ as $\nu \to 0, (k_e/k_f) \to 0$. However, we see no likely reason for this to occur and, in the absence of concrete matching conditions between the conformal and nonconformal ranges, it is a difficult question to investigate in Polyakov's approach.

Our Hypothesis 5 implies not only that there will be ultraviolet cascades of the higher-order invariants but also it implies that the corresponding dissipation wavenumbers $\kappa_d^{(2m)}$, defined as

$$[\kappa_d^{(2m)}]^2 \equiv \frac{\langle h^{(2m-2)} |\boldsymbol{\nabla}\omega|^2\rangle}{\langle h^{(2m)}\rangle}, \tag{3.91}$$

all coincide with the Kraichnan wavenumber $\kappa_d^{\rm Kr}$. (Note that the definition only makes sense

[4] However, it is not clear to us that Polyakov takes into account correctly the contribution of "negative dimension" operators to the nth order correlator of the vorticity. This might lead to spectra $\Omega^{(n)}(k)$ of the higher invariants with slower decay at high-wavenumber than the enstrophy spectrum $\Omega(k) = \Omega^{(2)}(k)$.

a priori for even $n = 2m$ when both sides are guaranteed to be positive.) This follows as a consequence of the curious "independence" property of the vorticity and its gradient magnitude, noted at the end of Section 2*(ii)*. In fact, appealing to Eq.(2.47), it follows that

$$\begin{aligned} [\kappa_d^{(2m)}]^2 &= \frac{\langle h^{(2m-2)}\rangle\langle|\nabla\omega|^2\rangle}{\langle h^{(2m)}\rangle}, \\ &= \frac{\Omega^{(2m-2)}}{\Omega^{(2m)}}\cdot\frac{\eta}{\nu}. \end{aligned} \tag{3.92}$$

By the same reasoning which led to Eq.(3.66), it follows under Hypothesis 5 that

$$\frac{\Omega^{(2m)}}{\Omega^{(2m-2)}} \sim \eta^{2/3}, \tag{3.93}$$

and thus

$$\kappa_d^{(2m)} \sim C^{(2m)}\frac{\eta^{1/6}}{\nu^{1/2}} = C^{(2m)}\kappa_d^{\mathrm{Kr}}. \tag{3.94}$$

The "independence" property is special to the case of Gaussian force, white-noise in time and it is not clear that the result Eq.(3.94) is more general than that.

4 Scaling Indices in the Ultraviolet Enstrophy Range

We present here exact bounds on scaling exponents of the vorticity field in forced, stationary turbulence in 2D. All of the estimates are based upon the condition of constant mean enstrophy flux η at small scales, which was established in the last section under various reasonable hypotheses. As observed by Paladin and Vulpiani [43], the velocity field is a poor indicator of small-scale structure in 2D and numerical studies have instead properly focused on the vorticity field [25, 26, 27]. Therefore, the scaling laws we consider are those for the *vorticity structure functions*, presumed in the form:

$$\langle|\Delta_1\omega|^p\rangle \sim \ell^{\zeta_p}. \tag{4.1}$$

Here "$\sim$" is interpreted to mean that the logarithm of the LHS divided by $\log\ell$ goes to the limit ζ_p as $\ell \to 0$. This type of "multiscaling" behavior is indicative of intermittency in the vorticity distribution. We are not aware of any numerical work which directly verifies such scaling laws in 2D steady-states. (The papers [25, 26, 27] make some related analyses of vorticity in freely-decaying 2D turbulence via wavelets, which is discussed further below.)

We shall actually assume these scaling laws in just the weak sense of big-O bounds:

$$\langle|\Delta_{\mathbf{l}}\omega|^p\rangle = O(\ell^{\zeta_p}). \tag{4.2}$$

More properly, we take ζ_p to be the supremum of the ζ's for which Eq.(4.2) holds for each fixed p value, $p \geq 1$. The average $\langle\cdot\rangle$ could be interpreted either as a space-average over the torus $\mathbf{T}^2$, or as an ensemble average. The two averages ought to be distinguished. To make clear the difference, we shall use z_p for the (maximal) exponent in the estimate Eq.(4.2) for space-averages. If this is in fact considered to be a space average, then it just a so-called "Besov condition," the defining criterion that $\omega \in B_p^{s,\infty}$, $s = \frac{z_p}{p}$, one of the scale of so-called *Besov spaces* (a standard reference is [47].) Note that Eq.(4.2) for space-averages characterizes $\omega \in B_p^{s,\infty}$ only if $s > 0$, whereas for $s \leq 0$ it is just a sufficient condition. For further discussion, see Appendix III. We shall make use of some simple parts of the theory of Besov spaces in our proofs below. Of course, to appeal to the constant flux condition, it is the ensemble-average that ought to be considered. However, it is a consequence of Theorem 4 in [28] that Eq.(4.2) for ensemble-averages implies that the same estimate holds almost surely for space-averages with ζ_p replaced by $z = \zeta_p - \epsilon$, for any $\epsilon > 0$. In other words, the realizations $\omega \in B_p^{\sigma_p-\epsilon,\infty}$ with probability one, for $\sigma_p = \zeta_p/p$. [5] This will suffice for our purposes.

We shall generally consider below the idealized inertial-range of infinite extent, obtained as the $\nu \to 0$ limit of the steady-states discussed in Section 2 (ii). A cautionary remark ought to be made that this limit has never been established to exist, and it could be false. The only cases that we know for which the existence of the zero-viscosity limit has been established are that of freely-decaying, homogeneous turbulence in infinite Euclidean space $\mathbf{R}^d$ for $d = 3$ or $d = 2$ ([7],Theorem VIII.3.1) and that of evolving turbulence on the $2D$ periodic domain $\mathbf{T}^2$ with

[5] Under the additional condition that $\langle\|\omega\|_{B_p^{s_p(1-\epsilon),\infty}}\rangle < \infty$ for all $\epsilon > 0$ it can even be seen that $\sigma_p = \text{ess.inf}_\omega s_p(\omega)$, in which $s_p(\omega)$ is the maximal index s in the realization ω of the ensemble.

suitable deterministic forces and with random initial data of finite mean enstrophy ([7],Theorem VIII.4.2). Therefore, we shall give below some consideration of another formulation of the results, in which the big-O bounds Eq.(4.2) are assumed valid only for $\nu > 0$ but with constants that are independent of ν. This allows us to make statements about scaling exponents that are valid for the mathematically constructed (and more physically realistic) ensembles with $\nu > 0$.

Kolmogorov Relation and Exact Bounds

One important exact result is a scaling law analogous to the von Karman-Howarth-Kolmogorov relation in 3D. The result in 2D, derived in detail in Appendix II, is:

$$\eta = -\frac{1}{4}\boldsymbol{\nabla}_{\mathbf{l}}\boldsymbol{\cdot}\langle[\Delta_{\mathbf{l}}\mathbf{v}][\Delta_{\mathbf{l}}\omega]^2\rangle. \tag{4.3}$$

The RHS is a "physical-space enstrophy flux," $\eta(\mathbf{l})$, defined by the time-derivative of the vorticity 2-point correlation under the inertial dynamics (cf. [9], Eq.(20)) and the relation is simply an expression of the constant ultraviolet enstrophy flux η for the $\nu \to 0$ limit. The physical-space flux is related to the usual Fourier-space flux $Z(k)$ via

$$Z(k) = \frac{k}{2\pi}\int_{\mathbf{R}^2} d^2\mathbf{l}\ \frac{J_1(k\ell)}{\ell}\eta(\mathbf{l}). \tag{4.4}$$

To make our arguments simpler, we will assume also homogeneity and isotropy, which follow from the analogous properties for the force. In that case, Eq.(4.3) simplifies to

$$\langle[\Delta_{\mathbf{l}}v_{\|}(\mathbf{r})][\Delta_{\mathbf{l}}\omega(\mathbf{r})]^2\rangle = -2\eta\ell, \tag{4.5}$$

for $\ell < \ell_f$, with $\Delta_{\mathbf{l}}v_{\|}(\mathbf{r}) = \hat{\mathbf{l}}\boldsymbol{\cdot}\Delta_{\mathbf{l}}\mathbf{v}(\mathbf{r})$. This is analogous to the Kolmogorov "4/5-law" in 3D for the 3rd-order velocity structure function.

We first use this relation to derive a bound on the exponent x in the scaling law for the energy spectrum $E(k) \sim k^{-x}$. Of course, this is related to the enstrophy spectral exponent y in the law $\Omega(k) \sim k^{-y}$ and to the vorticity structure-function exponent ζ_2, as

$$x = 2 + y = 3 + \zeta_2. \tag{4.6}$$

We note that there are a number of predictions for the exponent x. The original 1967 theory of Kraichnan gave $x = 3$ [4], modified later with a logarithmic correction $E(k) \sim k^{-3}\log^{-1/3}(k\ell_f)$ [42] (see also [44]). Saffman proposed instead $x = 4$ [45], Moffatt $x = 11/3$ [46], and, most recently, Polyakov has considered an infinite number of possible spectra with $x > 3$ corresponding to solutions of various 2D conformal field theories [8, 9]. An exact bound was proposed in 1975 by Sulem and Frisch that $x \leq 11/3$ [24]. [6] Interestingly, this yields Moffatt's spectrum as the upper limit. In particular, the Saffman prediction $x = 4$ is ruled out as an ultraviolet spectrum in the enstrophy range.

Here we shall give a simple derivation of the Sulem-Frisch bound,

$$\zeta_2 \leq \frac{2}{3}, \tag{4.7}$$

based upon Eq.(4.5) and the fundamental Besov space embedding theorem [47]. We first give the proofs taking the average as over space, and afterward we shall formulate the argument for the (more proper) ensemble-average. Thus, we assume that the exponent $z_2 > 2/3$, or $s_2 > 1/3$. This implies that $\mathbf{v} \in B_2^{1+s,\infty}$ for each $s < s_2$ (see Appendix III). In particular, it holds for an $s > 1/3$. However, the fundamental embedding theorem (see Section 2.7.1, [47] and Appendix III) states that

$$B_p^{s,\infty}(\mathbf{T}^d) \subset B_{p'}^{s',\infty}(\mathbf{T}^d), \tag{4.8}$$

(continuous embedding) with

$$s - \frac{d}{p} = s' - \frac{d}{p'}. \tag{4.9}$$

Applying this for $d = 2$ with $p = 2$ and $p' = \infty$, and noting that $B_\infty^{s,\infty} = C^s$ (the space of Hölder functions of index s) it follows that $\mathbf{v} \in C^s(\mathbf{T}^2)$, or $\|\Delta_{\mathbf{l}}\mathbf{v}\|_\infty = O(\ell^s)$. Then, applying the Hölder inequality

$$|\langle[\Delta_{\mathbf{l}} v_\parallel(\mathbf{r})][\Delta_{\mathbf{l}}\omega(\mathbf{r})]^2\rangle| \quad \leq \quad \|\Delta_{\mathbf{l}}\mathbf{v}\|_\infty \cdot \|\Delta_{\mathbf{l}}\omega\|_2^2$$

[6] It should be noted that the original Sulem-Frisch paper gave the estimate $x \leq 4$ as a rigorous bound and only a heuristic argument for $x \leq 11/3$. Later, the authors suggested that the sharper bound should hold rigorously as well (U. Frisch, private communication, 1992).

$$= \quad O\left(\ell^{3s}\right). \tag{4.10}$$

However, Eq.(4.10) leads to a contradiction when $s > 1/3$ since its LHS is $\sim \ell$ by Eq.(4.5) but, by the above argument, is $O\left(\ell^{3s}\right)$. Thus, we conclude that $z_2 \leq 2/3$.

To transpose the argument to ensemble averages—for which the constant flux condition was proved under reasonable conditions in Section 3.II—we appeal to Theorem 4 of [28]. This states that Eq.(4.2) for ensemble averages implies that for any $s < \sigma_p$,

$$\|\Delta_{\mathbf{l}}\omega\|_{L^p} = O(\ell^s) \qquad \text{a.s.} \tag{4.11}$$

In that case, the previous argument may be applied to infer that if $\zeta_2 > 2/3$, then

$$\lim_{\ell\to 0}\frac{1}{\ell}\int_{\mathbf{T}^2} d^2\mathbf{r}\ [\Delta_{\mathbf{l}}v_{\|}(\mathbf{r})][\Delta_{\mathbf{l}}\omega(\mathbf{r})]^2 = 0 \qquad \text{a.s.} \tag{4.12}$$

This is not consistent with Eq.(4.5) as long as the quantity

$$\mathcal{Z}_\ell \equiv \frac{1}{\ell}\int_{\mathbf{T}^2} d^2\mathbf{r}\ [\Delta_{\mathbf{l}}v_{\|}(\mathbf{r})][\Delta_{\mathbf{l}}\omega(\mathbf{r})]^2 \tag{4.13}$$

is "uniformly integrable," or when

$$\lim_{A\to\infty}\sup_{\ell>0}\langle|\mathcal{Z}_\ell| 1_{\{|\mathcal{Z}_\ell|>A\}}\rangle = 0. \tag{4.14}$$

In other words, if the contributions of the tails of the distribution to the average of $|\mathcal{Z}_\ell|$ go to zero uniformly in ℓ, then

$$\lim_{\ell\to 0}\frac{1}{\ell}\langle[\Delta_{\mathbf{l}}v_{\|}][\Delta_{\mathbf{l}}\omega]^2\rangle = \lim_{\ell\to 0}\langle\mathcal{Z}_\ell\rangle = 0, \tag{4.15}$$

contradicting Eq.(4.5).

One can expect the assumption Eq.(4.14) to be true, since the quantity $\mathcal{Z}_\ell$ has ℓ-independent expectation -2η, providing uniformity in ℓ, and, since it is a space-average, making unlikely any "intermittency" as $\ell \to 0$. In other words, it is possible in principle that $\mathcal{Z}_\ell$ has a small probability $\sim \ell$ to take on a very large magnitude $\sim \eta\cdot\ell^{-1}$ for each $\ell > 0$ and a large probability $\sim 1-\ell$ to be zero. That would be consistent with both $\langle\mathcal{Z}_\ell\rangle = -2\eta$ and $\lim_{\ell\to 0}\mathcal{Z}_\ell = 0$ a.s. (the last as a consequence of the Borel-Cantelli lemma) but it would violate the uniform integrability assumption. No contradiction could then be derived. However, we do not expect a space-average quantity such as $\mathcal{Z}_\ell$ to show such strong intermittency as $\ell \to 0$, in which the constant mean would be achieved by large, rare fluctuations. Therefore, the uniform integrability assumption Eq.(4.14) seems to us reasonable.

Another rather different condition would also suffice for the proof, namely

$$\langle \|\omega\|^3_{B^{s,\infty}_2} \rangle < \infty. \tag{4.16}$$

This means that the constants in the big-O estimate Eq.(4.2), as a space-average, can be chosen for each realization ω so that the ensemble-average of their cube is finite. This also seems reasonable. A careful examination of the argument in the previous paragraph shows that

$$|\mathcal{Z}_\ell| \leq (\text{const.})\|\omega\|^3_{B^{s,\infty}_2}. \tag{4.17}$$

See Appendix III. Since the latter is assumed integrable, the dominated convergence theorem would yield Eq.(4.15) and lead to a contradiction with constant flux. Thus, our bound $\zeta_2 \leq 2/3$, Eq.(4.7), holds for the ensemble-average exponent under quite reasonable assumptions.

Bounds can obtained in the same way for ζ_p with $p = 3$ and higher. The simplest such estimate is

$$\zeta_p \leq 0, \tag{4.18}$$

for every $p \geq 3$. We shall give proofs of this assuming averages as over space. The extension to ensemble-averages can be made in the same way as before, assuming uniform integrability of $\mathcal{Z}_\ell$, Eq.(4.14), or finite means of constants, such as

$$\langle \|\omega\|^3_{B^{s,\infty}_p} \rangle < \infty. \tag{4.19}$$

Therefore, we do not explicitly consider the extension. For transparency of notation we shall use the more conventional ensemble-average notation, with the understanding that the argument is really made first for space-averages and then extended as above.

To obtain the estimate first on ζ_3, we use a condition on the velocity,

$$\|\Delta^2_{\mathbf{l}}\mathbf{v}\|_3 = O\left(\ell^{\sigma_3+1}\right), \tag{4.20}$$

equivalent to Eq.(4.1) for $p = 3$ with $\zeta_3 = 3\sigma_3 > 0$. The proof of this equivalence requires some potential theory estimates which are discussed in Appendix III. Note that the second-difference is necessary when $\sigma_3 > 0$, since then $\|\Delta_{\mathbf{l}}\mathbf{v}\|_3 = O(\ell)$. However, we show that, in fact, this assumption is not consistent with constant flux and that

$$\zeta_3 \leq 0. \tag{4.21}$$

For the proof we just observe that by a different application of the Hölder inequality

$$\begin{aligned} |\langle[\Delta_{\mathbf{l}}v_{\|}(\mathbf{r})][\Delta_{\mathbf{l}}\omega(\mathbf{r})]^2\rangle| &\leq \|\Delta_{\mathbf{l}}\mathbf{v}\|_3 \cdot \|\Delta_{\mathbf{l}}\omega_\ell\|_3^2 \\ &= O\left(\ell^{2\sigma_3+1}\right). \end{aligned} \tag{4.22}$$

Again, this contradicts Eq.(4.5) if $\sigma_3 > 0$ and yields the bound Eq.(4.21). Since the Hölder inequality also implies that the combination $\sigma_p = \frac{\zeta_p}{p}$ is non-increasing in p, it follows at once that $\zeta_p \leq 0$ for every $p \geq 3$. [7]

Refined Similarity Hypothesis and Improved Bounds

The same results as above can be obtained using the condition

$$\langle Z_\ell(\mathbf{r})\rangle = \eta, \tag{4.23}$$

in terms of the local flux variable introduced in Section 2 (i). For example, assuming $\zeta_3 = 3\sigma_3 > 0$, we can estimate in the same way with the Hölder inequality that

$$\begin{aligned} \|Z_\ell\|_1 &\leq (\text{const.})\|\boldsymbol{\nabla}\overline{\omega}_\ell\|_3 \cdot \|[\Delta\mathbf{v}]_\ell\|_3 \cdot \|[\Delta\omega]_\ell\|_3 \\ &= O\left(\ell^{\sigma_3-1}\cdot\ell\cdot\ell^{\sigma_3}\right) = O\left(\ell^{2\sigma_3}\right), \end{aligned} \tag{4.24}$$

which gives Eq.(4.21). Note that the estimate $\|\boldsymbol{\nabla}\overline{\omega}_\ell\|_3 = O(\ell^{\sigma_3-1})$ follows from the representation

$$\boldsymbol{\nabla}\overline{\omega}_\ell(\mathbf{r}) = -\int d^2\mathbf{l}\ \boldsymbol{\nabla}G_\ell(\mathbf{l})\Delta_{\mathbf{l}}\omega(\mathbf{r}) \tag{4.25}$$

[7] Incidentally, this gives another proof of the bound $\zeta_2 \leq 2/3$, in conjunction with the Besov space embedding theorem. Indeed, $B_2^{s,\infty} \subset B_3^{s',\infty}$ with $s' = s - \frac{1}{3}$, so that $s_3 > 0$ if $s_2 > \frac{1}{3}$.

and the assumed bound on $\|\Delta_1\omega\|_3$.

However, these estimates can now be further improved. The basic idea is that a scaling relation should exist between the *local enstrophy flux* variable $Z_\ell(\mathbf{r})$ and the vorticity difference at the same point $\mathbf{r}$:

$$Z_\ell(\mathbf{r}) \sim \frac{[\Delta_\ell \mathbf{v}(\mathbf{r})][\Delta_\ell \omega(\mathbf{r})]^2}{\ell}. \tag{4.26}$$

This is an analogue of the *refined similarity hypothesis* (RSH) in 3D which—in the version of Kraichnan [48]—states that local energy flux scales as $\Pi_\ell(\mathbf{r}) \sim [\Delta_\ell v(\mathbf{r})]^3/\ell$ in terms of the velocity difference at the same point. The Eq.(4.26) will imply relations between corresponding scaling exponents of the vorticity differences and the enstrophy flux, in the same way as the ordinary RSH in 3D. More precisely, if we assume

$$\langle |Z_\ell|^p \rangle = O(\ell^{\tau_p}), \tag{4.27}$$

then there follow heuristically from Eq.(4.26) relations between the exponents ζ_p and τ_p :

$$\zeta_p = \tau_{p/3}. \tag{4.28}$$

These relations are proved here as inequalities and yield our main estimates on exponents. The proofs given below follow closely methods used in our discussion of the 3D RSH in [10]. However, unlike there, they are formulated directly in terms of ensemble-averages. Thus a statistical hypothesis is required, either a moment condition of the form

$$\langle \|\omega\|^p_{B^{s,\infty}_p} \rangle < \infty, \tag{4.29}$$

for all $s < \sigma_p$, or else uniform integrability of flux, when $\zeta_p < 0$. [8]

One useful observation for the proof is that the assumption Eq.(4.2) can be reformulated using L^p-spaces based upon joint space & ensemble-averaging. In other words, we define an L^p-norm on random functions of the torus, as

$$|\!|\!| f |\!|\!|_p \equiv \left[\frac{1}{|\Lambda|}\int_\Lambda d^2\mathbf{r}\langle |f(\mathbf{r})|^p\rangle\right]^{1/p}. \tag{4.30}$$

[8] We have already seen that $\zeta_p \leq 0$ is required for $p \geq 3$. If the minimum Hölder singularity of the vorticity is negative, $-1 < h_{\min} < 0$, then $\zeta_p < 0$ for large enough p: see [13].

Appealing to statistical homogeneity of the ensemble, we can write the condition Eq.(4.2) as

$$\|\Delta_{\mathbf{l}}\omega\|_p = O\left(\ell^{\sigma_p}\right). \tag{4.31}$$

The next key point is that for $1 < p < \infty$, with the usual space-L^p norms,

$$\|\Delta_{\mathbf{l}}\mathbf{v}\|_p \leq C \cdot \ell^{1+s} \cdot \|\omega\|_{B_p^{s,\infty}} \tag{4.32}$$

with a constant C which is *independent of* ω whenever

$$\|\Delta_{\mathbf{l}}\omega\|_p = O\left(\ell^{s}\right) \tag{4.33}$$

with $-1 < s < 0$. This result is shown in Appendix III to be true as a consequence of the Calderón-Zygmund inequality and a theorem on characterization of Besov spaces by differences. Note that it is *not* true when $s \geq 0$.[9] Furthermore, Eq.(4.32) will hold a.s. for the realizations of the ensemble by a generalization of Theorem 4 of [28] (Appendix III). Note that Theorem 4 does not apply when $\sigma_p < 0$. Taking the pth power of Eq.(4.32), ensemble-averaging, and taking finally the pth root, we obtain the inequality

$$\|\Delta_{\mathbf{l}}\mathbf{v}\|_p \leq C \cdot \ell^{1+s} \cdot \langle \|\omega\|^p_{B_p^{s,\infty}}\rangle^{1/p} \tag{4.34}$$

for the joint space& ensemble norm. In conjunction with Eq.(4.31) we then derive that

$$\|\Delta_{\mathbf{l}}\mathbf{v}\|_p = O\left(\ell^{s+1}\right). \tag{4.35}$$

In that case, we can use again the Hölder inequality to derive for every $s < \sigma_{3p}$,

$$\begin{aligned} \|Z_\ell\|_p &\leq \|\nabla\overline{\omega}_\ell\|_{3p} \cdot \|[\Delta\mathbf{v}]_\ell\|_{3p} \cdot \|[\Delta\omega]_\ell\|_{3p} \\ &= O\left(\ell^{s-1}\cdot\ell^{s+1}\cdot\ell^{s}\right) = O\left(\ell^{3s}\right). \end{aligned} \tag{4.36}$$

However, using Eq.(4.27) for $\ell \to 0$ with the *maximal* choice of τ_p, we obtain

$$\zeta_p \leq \tau_{p/3}. \tag{4.37}$$

[9] The corresponding statement then is that $\|\Delta_{\mathbf{l}}\mathbf{v}\|_p \leq C\cdot\ell\cdot\|\omega\|_{B_p^{s,\infty}}$, which was used to establish our previous results.

This is our main estimate. Note from $\|Z_\ell\|_1 \leq \|Z_\ell\|_p$ for $p \geq 1$ that

$$\tau_p \leq 0. \tag{4.38}$$

Hence, Eq.(4.37) can improve upon Eq.(4.18). The exponent $\tau_p < 0$ if the *local* enstrophy flux is increasingly intermittent as $\ell \to 0$. We emphasize that the local flux is distinct from the global quantity obtained by space-averaging, which is presumably *not* intermittent (see above).

A Formulation Valid For Nonzero Viscosity

All of the previous results on exponent bounds have assumed the existence of a zero-viscosity limit for the stationary ensemble. However, it is worthwhile to formulate versions of these results which are valid at finite $\nu > 0$. The way to reformulate the premise Eq.(4.2) of the argument in that context is that

$$\langle |\Delta_1 \omega|^p \rangle \leq C_{\sigma,p} \ell^\zeta, \;\; \sigma = \zeta/p, \tag{4.39}$$

with a *viscosity-independent* constant $C_{\sigma,p}$. Of course, at positive viscosity, the vorticity is presumably analytic in 2D. Therefore, Eq.(4.39) will hold with an arbitrarily large σ, but with a constant which in general increases as $\nu \to 0$. Therefore, our assumption will be that for each fixed $p \geq 1$ Eq.(4.39) holds for some real σ with a viscosity-independent constant $C_{\sigma,p}$. We then take σ_p to be the supremum of the σ's for which this is true. In that case, we can still show that the previous derived bounds on the exponents σ_p are valid. To see this, recall that the enstrophy flux was shown in Section 3.II to be constant down to a length-scale ℓ_d which *vanishes* as $\nu \to 0$, if either of two reasonable hypotheses is satisfied. Under the assumption of uniformly bounded mean energy (Hypothesis 1) it was shown that $\ell_d \sim \nu^{1/4}$ and under the stronger assumption of uniformly bounded mean enstrophy (Hypothesis 3) it was shown that $\ell_d \sim \nu^{1/2}$. In either case, we may consider a length-scale λ intermediate between ℓ_0 and ℓ_d, for example

$$\lambda \equiv \sqrt{\ell_0 \ell_d}. \tag{4.40}$$

It then follows from the results of Section 3.II that

$$\lim_{\nu\to 0}\langle Z_\lambda\rangle = \eta. \tag{4.41}$$

However, if we assume that Eq.(4.2) holds for $p = 3$ with $\sigma_3 > 0$—just to name one case—then we obtain the bound

$$|\langle Z_\lambda\rangle| = O(\lambda^{\zeta_3}), \tag{4.42}$$

and that is inconsistent with Eq.(4.41). Hence, we still conclude under our modified assumptions that $\zeta_p \leq 0$ for $p \geq 3$.

It is not completely clear, however, that this reformulation avoids the assumption that the $\nu \to 0$ limit exists. Notice that the best constants in the big-O bounds of Eq.(4.2) are one choice of a seminorm for the Besov spaces,

$$||\omega||^*_{B^{s,\infty}_p} \equiv \sup_{\ell>0}\frac{\|\Delta_{\mathbf{l}}\omega\|_{L^p}}{\ell^s}, \tag{4.43}$$

which provides along with the L^p-norms a Banach space norm for $B^{s,\infty}_p,\ \ s > 0, p \geq 1$:

$$\|\omega\|_{B^{s,\infty}_p} \equiv \|\omega\|_{L^p} + ||\omega||^*_{B^{s,\infty}_p}. \tag{4.44}$$

Consider the case $p = 2$ (since we showed above that it is not possible that $s > 0$ for $p \geq 3$). In that case, $C_s = ||\omega||^*_{B^{s,\infty}_2}$ is the best constant that can be chosen in every realization for the enstrophy-spectrum decay estimate

$$\Omega(k) \leq C_s \cdot k^{-(1+2s)}. \tag{4.45}$$

If this constant C_s has finite ensemble-average $\langle C_s\rangle_\nu < \infty$ and also $\langle\omega^2\rangle_\nu < \infty$, with as well *viscosity-independent* upper bounds on the averages, then

$$\sup_{\nu>0}\langle\|\omega\|_{B^{s,\infty}_p}\rangle_\nu < \infty. \tag{4.46}$$

That is enough to infer from the Prokhorov theorem the existence (at least along a subsequence) of a $\nu \to 0$ limit supported on $B_2^{s-\epsilon,\infty}$. [10] Therefore, a strong enough form of the assumption on viscosity-independent bounds implies the existence of a zero-viscosity limiting ensemble.

Multifractal Model and Negative Exponents of Vorticity

An interpretation of "multiscaling" laws of the form of Eq.(4.1), with ζ_p a nonlinear function of p, was proposed by Parisi and Frisch in [49]. Their hypothesis was that the scaling could be attributed to the function having local Hölder exponents h on sets $S(h)$ of "fractal dimension" $D(h)$. A simple heuristic argument led to

$$\zeta_p = \inf_{h \geq h_{\min}} [ph + (d - D(h))]. \tag{4.47}$$

It was shown by Jaffard [50] that if the scaling holds in just big-O sense with a space-average (for even a *single* p), then at least a part of the "multifractal hypothesis" is correct, namely, the function is locally Hölder continuous and the bound holds

$$D_H(h) \leq ph + (d - \zeta_p) \tag{4.48}$$

for the Hausdorff dimension of the set $S(h)$. In [28] an extension is made of these results to ensemble-averages and to "negative Hölder exponents," which may occur if $\zeta_p < d/p$. It was also shown in [28] that Eq.(4.47) holds when $D(h)$ is taken to be a certain "box-counting dimension spectrum" (which is *not* the dimension of a physical-space set such as $S(h)$ in general.) These results are unrestricted in dimension and thus hold for the vorticity in $d = 2$. It was already proposed by Benzi and Vergassola in [27] that the "multifractal model" should hold in 2D for the vorticity field. They also made a very preliminary attempt to determine $D(h)$ from numerical data on freely decaying turbulence in 2D with initial data chosen from a Gaussian random field of energy spectrum k^{-3}. An interesting conclusion of their analysis—discussed

[10] Since $B_2^{s,\infty}$ is not separable, this is not immediate. However, $B_2^{s,\infty}$ is compactly imbedded in the separable Banach space $H_2^{s-\epsilon}$ as a consequence of the continuous imbedding $B_2^{s,\infty} \subset H^{s-(\epsilon/2)}$ and the Rellich compactness lemma. This is enough to infer that the measures $\overline{\mu}_\nu,\ \nu > 0$ are tight as distributions on $H_2^{s-\epsilon}$; see Theorem II.3.1 of [7]. The continuous imbedding $H^s \subset B_2^{s,\infty}$ for all real s concludes the argument.

already above—was that "negative exponents" $h \approx -1/2$ occur due to "cusps" in vortex cores, confirming earlier work of Farge and Holschneider [26].

The conditions considered by those authors are different than ours, since we discuss forced steady-states rather than free decay. However, from our previous results, we can establish a condition for steady-states requiring the "negative exponents" to occur there. Namely, we can show that such singularities are necessary for "intermittency" in the 2D enstrophy cascade. It is easy to see that $h_{\min} > 0$ can never be consistent with the enstrophy flux condition. However, it seems possible that $h_{\min} = 0$. Supposing that is true, the formula Eq.(4.47) shows that $\zeta_p \geq 0$ for $p \geq 0$, and, in conjunction with Eq.(4.18),

$$\zeta_p = 0 \tag{4.49}$$

for $p \geq 3$. Hence, the 1967 theory of Kraichnan [1] must be exact for $p \geq 3$—up to possible logarithmic corrections—if $h_{\min} = 0$! This result can be extended to the case $p = 2$ if a somewhat stronger assumption is made, that is, we can show that as well

$$\zeta_2 = 0. \tag{4.50}$$

It was another theorem of Yudovitch [1] that, if the initial vorticity is bounded, then at any later time under Euler evolution, $\|\Delta_{\mathbf{l}}\mathbf{v}\|_\infty \leq c(\Lambda) \cdot \|\omega_0\|_\infty \ell(1 + \log \ell)$. Consider the following (strong) form of this as an assumption for the steady-state:

$$\begin{aligned} \|\Delta_{\mathbf{l}}\mathbf{v}\|_\infty &\equiv \text{ess.}\sup_{\{\mathbf{r}\in\Lambda,\mathbf{v}\}}|\Delta_{\mathbf{l}}\mathbf{v}(\mathbf{r})| \\ &= O(\ell(1+\log \ell)), \end{aligned} \tag{4.51}$$

where the supremum is over all realizations of the statistical ensemble and points of the space domain Λ. Notice that the assumption just means that the individual velocity fields are *quasi-Lipschitz* uniformly over the ensemble and space. In fact, we obtain then

$$\begin{aligned} \|Z_\ell\|_1 &\leq (\text{const.})\|\boldsymbol{\nabla}\overline{\omega}_\ell\|_2 \cdot \|[\Delta\mathbf{v}]_\ell\|_\infty \cdot \|[\Delta\omega]_\ell\|_2 \\ &= O\left(\ell^{s_2-1} \cdot \ell \log \ell \cdot \ell^{s_2}\right) = O\left(\ell^{2s_2}\log \ell\right), \end{aligned} \tag{4.52}$$

which yields Eq.(4.50). In this case, we obtain also the k^{-3} energy spectrum as exact, up to possible logarithmic terms. It may seem surprising that "monoscaling" is exact in 2D for $h_{\min} = 0$, since dynamically one should expect that Hölder exponents of the vorticity $h > 0$ will occur at least at some points in space. However, it is important to appreciate that the 1967 scaling is consistent with a *nontrivial* spectrum of singularities $D(h)$ over $h \geq 0$. It follows from Eq.(4.47) that the condition $\zeta_p = 0$ for $p \geq 0$ is equivalent to the condition that $D(0) = 2$. This still allows $D(h) \neq 0$ for $h > 0$.

For the zero-viscosity limit of the forced steady-state it seems even more probable that such negative Hölder singularities may form, due to the effects of the random forcing building up over time. If, on the other hand, the hypothesis $h_{\min} = 0$ is correct, it requires a universality in the ultraviolet range of 2D turbulence, which is not strongly confirmed by simulations. Although one simulation by V. Borue [14] has confirmed the Kraichnan theory, a plethora of energy spectra, generally steeper than k^{-3}, have been observed (see references in [14].) Without any definitive explanation of this fact, we will just suggest that the results may be due to a limited range of Reynolds numbers and a consequent contamination of scaling from the infrared range. In fact, large-scale, coherent vortex structures are commonly observed in such simulations which have very steeply-decaying spectra in Fourier space. This would also explain why spectra much steeper than allowed by the bound $x \leq 11/3$ are commonly observed, and the lack of universality in such scaling, since the vortex structures depend upon the details of forcing in the large-scales. It should be clear that the energy spectrum itself is not the quantity most sensitive to the possible presence of "negative Hölder singularities." To clear up the issue whether $h_{\min} = 0$ or not it would be far preferable to measure $\langle (\Delta_{\mathbf{l}}\omega)^p \rangle$ for $p \geq 3$, the higher order vorticity structure functions.

5 Appendices

Appendix I: Rigorous Proof of the Steady-State Balance Equations

We give here the proof of the balance equations in the steady-state with white-noise force in time. The proof of the energy-balance equation, Eq.(2.24), was already given as Theorem XI.2.2 of Vishik and Fursikov [7]. We shall discuss therefore the corresponding equations for the vorticity invariants, Eq.(2.41) in the text. In the case $n = 2$, corresponding to the enstrophy balance, we will outline a complete proof. However, for $n > 2$ we cannot give a complete proof. Instead we shall show that the balance equations for (even) $n > 2$ follow from expected regularity of the steady-state measure. As we discuss below, the regularity properties assumed should follow by a generalization of the standard energy-estimates to the higher-order vorticity invariants. The difficulty in the argument arises from the fact that simple Fourier-Galerkin truncations of the 2D Euler dynamics preserve only the quadratic invariants, energy and enstrophy, and not the higher invariants (which was already observed some time ago by Kraichnan [51].) We emphasize again that the balance equations do require some proof, since they might be false if singularities were to occur at a viscosity which is positive but small enough. Of course, that is not expected to occur in 2D.

As in the proof of Theorem XI.2.2 in [7], our main tool is the *stationary Kolmogorov equation*

$$\int \overline{\mu}(d\omega)\ \exp(i\langle\omega,\varphi\rangle)K(\omega,\varphi) = 0, \tag{5.1}$$

in which $\overline{\mu}$ is the stationary measure, $\varphi \in H^s,\ s > 2$ and

$$K(\omega,\varphi) = i\langle\nu \bigtriangleup \omega - B(\mathbf{v},\omega),\varphi\rangle - \langle\varphi, H\varphi\rangle. \tag{5.2}$$

Note in the last equation that $B(\mathbf{v},\omega) = (\mathbf{v}\cdot\boldsymbol{\nabla})\omega$ and $\mathbf{v} = (\mathrm{rot})^{-1}\omega$. This may be proved in the same manner as Eq.(XI.2.7) of [7], along with the additional *a priori* estimate for $\alpha \geq 0$,

$$\overline{\mu}\left(\|\omega\|_{L^2}^{2\alpha}\|\boldsymbol{\nabla}\omega\|_{L^2}^2\right) \leq C_\alpha < \infty. \tag{5.3}$$

Although this result is not stated in [7], it follows very directly from the methods developed there. Hence, we shall simply sketch here the proof. Time-dependent statistical solutions are obtained for 2D Navier-Stokes with force white-noise in time in Theorem XI.3.1 of [7]. However, the *a priori* bounds (XI.3.9) derived there are not sufficient to discuss the steady-state, because they are exponential in time. Instead, the bounds

$$E\left(\|\omega(t)\|_{L^2}^{2+2\alpha}+\nu\int_0^t ds\ \|\omega(s)\|_{L^2}^{2\alpha}\|\boldsymbol{\nabla}\omega(s)\|_{L^2}^2\right)\leq A_\alpha+B_\alpha\cdot t \tag{5.4}$$

are required. These can be proved by exactly the same argument used in [7] to prove Theorem X.4.1, but with vorticity substituting for velocity. It then follows by the Bogolyubov-Krylov averaging method used to prove Theorem XI.2.2 that a stationary statistical solution exists in 2D for each $\nu>0$, which obeys the estimate Eq.(5.3). Note that the uniqueness of the solutions $\mu(t,\cdot)$ to the time-dependent Kolmogorov equation is proved in 2D under certain conditions (Theorem XI.4.2 of [7]), but the uniqueness of the steady-state $\overline{\mu}$ seems to be open. We expect it to be true. The Kolmogorov equation Eq.(5.1) can also be directly obtained from Eq.(XI.2.7) of [7] by setting

$$\nu_i=-\epsilon_{ij}\partial_j\varphi=-\partial_i^{\top}\varphi. \tag{5.5}$$

It is valid then for all $\varphi\in H^s,\ s>3$, which suffices just as well for our purposes.

We give next the proof of the enstrophy balance equation,

$$\overline{\mu}(\nu\|\boldsymbol{\nabla}\omega\|_{L^2}^2)=H(\mathbf{0}), \tag{5.6}$$

following closely the proof of the energy balance in [7]. As there, we take

$$\varphi(\mathbf{r})=\varphi_{\mathbf{k}}\cdot e^{i\mathbf{k}\cdot\mathbf{r}}+\varphi_{\mathbf{k}}^*\cdot e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{5.7}$$

in the Kolmogorov equation Eq.(5.2). Differentiating twice with respect to $\varphi_{\mathbf{k}},\varphi_{\mathbf{k}}^*$, we obtain

$$\int\overline{\mu}(d\omega)\ \exp(i\langle\omega,\varphi\rangle)\left[|\widehat{\omega}_{\mathbf{k}}|^2K+2i\mathrm{Re}\left(\widehat{\omega}_{\mathbf{k}}^*\frac{\partial K}{\partial\varphi_{\mathbf{k}}}\right)+\frac{\partial^2K}{\partial\varphi_{\mathbf{k}}^*\partial\varphi_{\mathbf{k}}}\right]=0. \tag{5.8}$$

Summing over all wavenumbers $\mathbf{k}$ in the sphere of radius 2^N, we obtain

$$\int\overline{\mu}(d\omega)\ [-2\langle\nu\triangle\omega-B(\mathbf{v},\omega),P_N\omega\rangle-2H_N]=0, \tag{5.9}$$

where P_N is the projection onto the space spanned by the eigenmodes $\exp(i\mathbf{k}\cdot\mathbf{r})$ with $|\mathbf{k}|\leq 2^N$ and H_N is the partial trace $\mathrm{Tr}(P_NH)=H_N(\mathbf{0})$. Recall that for any $\omega\in H^1$ the following statements are true: $\omega_N \equiv P_N\omega \to \omega$ strongly in H^1;

$$|\langle \nu \triangle \omega - B(\mathbf{v},\omega), \omega_N\rangle| \leq C \cdot (1 + \|\omega\|_{L^2})\|\omega\|_{H^1}\|\omega_N\|_{H^1}; \tag{5.10}$$

and, $\langle B(\mathbf{v},\omega),\omega\rangle = 0$. Since $\omega \in H^1$ a.s. with respect to $\overline{\mu}$, it therefore follows that

$$\langle \nu \triangle \omega - B(\mathbf{v},\omega), P_N\omega\rangle \to -\nu\|\boldsymbol{\nabla}\omega\|_{L^2} \qquad \overline{\mu} - \text{a.s.}. \tag{5.11}$$

On the other hand, it follows as well from Eq.(5.10) that the function in the above limit for each N is bounded by $C \cdot (1 + \|\omega\|_{L^2})\|\omega\|_{H^1}^2$ and the latter is integrable by the *a priori* estimate Eq.(5.3). It therefore follows by the Lebesgue dominated convergence theorem upon taking the limit $N \to +\infty$ in Eq.(5.9) that

$$\overline{\mu}(2\nu\|\boldsymbol{\nabla}\omega\|^2 - 2H(\mathbf{0})) = 0,. \tag{5.12}$$

This is equivalent to Eq.(5.6), as claimed.

We would like to follow the same pattern of argument to prove the higher-order balance equations

$$\overline{\mu}(\nu\omega^{p-2}\|\boldsymbol{\nabla}\omega\|^2) = \overline{\mu}(\omega^{p-2})H(\mathbf{0}) \tag{5.13}$$

for even integers $p > 2$ (or Eq.(2.41), $n = p$.) For this we shall employ the following *assumed estimates*:

$$\overline{\mu}\left(\|\omega\|_{L^p}^{(1+\alpha)p}\right) \leq C_p < \infty, \tag{5.14}$$

and

$$\overline{\mu}\left(\|\omega\|_{L^p}^{\alpha p} \cdot \int \omega^{p-2}|\boldsymbol{\nabla}\omega|^2\right) \leq C_p < \infty \tag{5.15}$$

for $\alpha \geq 0$ and for even $p > 2$, as well as

$$\overline{\mu}\left(\|\omega\|_{L^p}^{\alpha p}\|\boldsymbol{\nabla}\omega\|_{L^2}^2\right) \leq C_p < \infty. \tag{5.16}$$

We shall afterward discuss the motivation for adopting these particular hypotheses by making generalized energy-estimates. (Note the third has a somewhat different status, but is still quadratic in $\boldsymbol{\nabla}\omega$.) Here we shall use these conditions to rigorously demonstrate the balance equations Eq.(5.13). [11] The first step is to set $\varphi(\mathbf{r}) = \sum_{i=1}^{p} \varphi_{\mathbf{k}_i} e^{i\mathbf{k}_i\cdot\mathbf{r}}$ in the stationary Kolmogorov equation Eq.(5.1), apply the differential operator $\frac{1}{p!}(\partial^p/\partial\varphi^*_{\mathbf{k}_1}\cdots\partial\varphi^*_{\mathbf{k}_p})$, and then take all $\varphi_{\mathbf{k}} = 0$. The result is

$$\int \overline{\mu}(d\omega) \left[\frac{i^p}{p!}\sum_{i=1}^{p} \widehat{\omega}_{\mathbf{k}_1}\cdots\underline{\widehat{\omega}_{\mathbf{k}_i}}\cdots\widehat{\omega}_{\mathbf{k}_p} \times \left(-\nu k_i^2 \widehat{\omega}_{\mathbf{k}_i} - \langle e^{i\mathbf{k}_i\cdot\mathbf{r}}, B(\mathbf{v},\omega)\rangle\right)\right.$$
$$\left. + \frac{i^p}{p!}\sum_{i\neq j=1}^{p} \widehat{\omega}_{\mathbf{k}_1}\cdots\underline{\widehat{\omega}_{\mathbf{k}_i}}\cdots\underline{\widehat{\omega}_{\mathbf{k}_j}}\cdots\widehat{\omega}_{\mathbf{k}_p} \times \widehat{H}(\mathbf{k}_i)\delta_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{0}}\right] = 0, \tag{5.17}$$

where underlining indicates omission of those terms.

The next step is to multiply through the preceding equation by the factor $\prod_{i=1}^{p} \widehat{\phi}(\mathbf{k}_i/2^N)$, where ϕ is a smooth function on $\mathbf{R}^2$ to be chosen, and then to sum over all $\mathbf{k}_i$, $i = 1, ..., p$ with $|\mathbf{k}_i| \leq 2^N$ for each i and $\mathbf{k}_1 + \cdots + \mathbf{k}_p = \mathbf{0}$. The reason for introducing the smoothing factor is that we wish to appeal to L^p-convergence and, as is well-known, Fourier series on $\mathbf{T}^d$ do not converge in L^p-norm for $d > 1$ (e.g. [52], Section VII.4). On the other hand, if the function ϕ considered above satisfies the modest properties that

$$|\phi(\mathbf{r})| \leq A\cdot(1+r)^{-(d+\epsilon)}, \tag{5.18}$$

and

$$|\widehat{\phi}(\mathbf{k})| \leq A\cdot(1+k)^{-(d+\epsilon)}, \tag{5.19}$$

for some constant $A < \infty$ and $\epsilon > 0$, and also

$$\widehat{\phi}(\mathbf{0}) = \int d^d\mathbf{r}\ \phi(\mathbf{r}) = 1, \tag{5.20}$$

then the *generalized means*

$$(S_\epsilon f)(\mathbf{r}) \equiv \sum_{\mathbf{k}} \widehat{\phi}(\epsilon\mathbf{k}) \widehat{f}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{5.21}$$

[11] It is worth observing that a similar discussion can be made for the 3D energy balance. For example, the energy-balance Eq.(2.24) would follow in 3D by the argument of [7] if the *a priori* estimate could be established that $\int \overline{\mu}(d\mathbf{v})\, \|\boldsymbol{\nabla}\mathbf{v}\|^{5/2}\|\mathbf{v}\|^{1/2} < \infty$, improving Eq.(XI.2.8). This would be consistent with violation of the energy equality (Ito formula) for a set of individual realizations with zero probability.

have the properties that S_ϵ is bounded on $L^p(\mathbf{T}^d)$ and that $S_\epsilon f \to f$ in strong L^p-sense as $\epsilon \to 0$. See Section VII.2 of [52]. Let us denote $S_N \equiv S_{2^{-N}}$ and $\omega_N = S_N\omega$, etc. Then the result of the aforementioned operations is the equation

$$\int \overline{\mu}(d\omega)\,\left[\frac{1}{(p-1)!}\int_{\mathbf{T}^2} d^2\mathbf{r}\;\omega_N^{p-1}(\mathbf{r})\,(-\nu\,\triangle\,\omega_N(\mathbf{r}) + S_N B(\mathbf{v},\omega))\right]$$
$$= \int \overline{\mu}(d\omega)\,\left[\frac{1}{(p-2)!}\int_{\mathbf{T}^2} d^2\mathbf{r}\;\omega_N^{p-2}(\mathbf{r})\right]\cdot H_N. \tag{5.22}$$

We study now the convergence as $N \to \infty$ of each term.

It follows from the first estimate Eq.(5.14) that $\omega \in L^p$ for each even $p \geq 2$ a.s. with respect to $\overline{\mu}$. By the L^p-boundedness of S_N,

$$\int_{\mathbf{T}^2} d^2\mathbf{r}\;\omega_N^{p-2}(\mathbf{r}) \leq C\cdot\int_{\mathbf{T}^2} d^2\mathbf{r}\;\omega^{p-2}(\mathbf{r}) \tag{5.23}$$

for even $p > 2$ and the righthand side is $\overline{\mu}$-integrable by Eq.(5.14). Thus we may apply the Lebesgue theorem to infer for the last term that

$$\lim_{N\to\infty}\overline{\mu}\left[\int_{\mathbf{T}^2} d^2\mathbf{r}\;\omega_N^{p-2}(\mathbf{r})\right] = \overline{\mu}\left[\int_{\mathbf{T}^2} d^2\mathbf{r}\;\omega^{p-2}(\mathbf{r})\right]. \tag{5.24}$$

Similar considerations can be made for the middle term. In fact, for all integers $p \geq 1$

$$\begin{aligned} |\langle\omega_N^p, S_N B(\mathbf{v},\omega)\rangle| &\leq \|\omega_N^p\|_{L^2}\|S_N B(\mathbf{v},\omega)\|_{L^2} \\ &\leq \|S_N\omega\|_{L^{2p}}^p\|B(\mathbf{v},\omega)\|_{L^2} \\ &\leq \|\omega\|_{L^{2p}}^p\left[1 + \int v^2|\boldsymbol{\nabla}\omega|^2\right], \end{aligned} \tag{5.25}$$

The last function is $\overline{\mu}$-integrable by the rigorous estimate Eq.(5.3). To see this, we observe that the velocity may be represented by the line-integral

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}(\mathbf{r}') + \int_0^{|\mathbf{r}-\mathbf{r}'|} ds\,(\hat{\mathbf{n}}\boldsymbol{\cdot}\boldsymbol{\nabla})\mathbf{v}, \tag{5.26}$$

where $\hat{\mathbf{n}} = (\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$, valid when $|\boldsymbol{\nabla}\mathbf{v}| \in L^1$. We can average this equation over the torus $\mathbf{T}^2$ with respect to $\mathbf{r}'$ and use the fact that

$$\int \mathbf{v} = \mathbf{0} \qquad \overline{\mu}-\text{a.s.}, \tag{5.27}$$

when the force spectrum has support bounded away from the origin. In that case,

$$\mathbf{v}(\mathbf{r}) = \frac{1}{\ell_0^2}\int_{\mathbf{T}^2} d^2\mathbf{r}' \int_0^{|\mathbf{r}-\mathbf{r}'|} ds\, (\hat{\mathbf{n}}\cdot\boldsymbol{\nabla})\mathbf{v}, \tag{5.28}$$

and

$$\begin{aligned} |\mathbf{v}(\mathbf{r})| &\leq C(\ell_0)\cdot \|\boldsymbol{\nabla}\mathbf{v}\|_{L^1} \\ &\leq C'(\ell_0)\cdot \|\boldsymbol{\nabla}\mathbf{v}\|_{L^2} \\ &\leq C''(\ell_0)\cdot \|\omega\|_{L^2}, \end{aligned} \tag{5.29}$$

in which the dependence of the constants on the finite box-size ℓ_0 is made explicit. Note the last line is a consequence of the well-known Calderón-Zygmund inequality [53]. Thus,

$$\int v^2|\boldsymbol{\nabla}\omega|^2 \leq C'''(\ell_0)\cdot \|\omega\|_{L^2}^2\|\boldsymbol{\nabla}\omega\|_{L^2}^2 \tag{5.30}$$

and the righthand side is $\overline{\mu}$-integrable by Eq.(5.3), as claimed. Furthermore, since $\omega_N \to \omega$ in L^{2p}, also $\omega_N^p \to \omega^p$ in L^2. Because $S_N B(\mathbf{v},\omega) \to B(\mathbf{v},\omega)$ in L^2 as well, we conclude that

$$\begin{aligned} \lim_{N\to 0}\langle \omega_N^p, S_N B(\mathbf{v},\omega)\rangle &= \langle \omega^p, B(\mathbf{v},\omega)\rangle \\ &= 0 \qquad \overline{\mu}-\text{a.s.} \end{aligned} \tag{5.31}$$

for all integers $p \geq 1$. Therefore, the Lebesgue theorem allows us to conclude that

$$\lim_{N\to 0}\overline{\mu}\left[\langle \omega_N^{p-1}, S_N B(\mathbf{v},\omega)\rangle\right] = 0. \tag{5.32}$$

However, we must treat the first term in Eq.(5.22) differently, since $\omega_N \to \omega$ in H^1 and thus $|\boldsymbol{\nabla}\omega_N|^2 \to |\boldsymbol{\nabla}\omega|^2$ only in L^1. Let us set

$$I_p[\omega] = \int \omega^{p-2}|\boldsymbol{\nabla}\omega|^2, \tag{5.33}$$

and then, for some $R > 0$, define also

$$I_{p,R}[\omega] = \int_{\{|\omega|<R\}} \omega^{p-2}|\boldsymbol{\nabla}\omega|^2, \tag{5.34}$$

and

$$\overline{I}_{p,R}[\omega] = \int_{\{|\omega| \geq R\}} \omega^{p-2} |\boldsymbol{\nabla}\omega|^2 \tag{5.35}$$

so that $I_p[\omega] = I_{p,R}[\omega] + \overline{I}_{p,R}[\omega]$. We shall show that it is enough to establish that

$$\lim_{N\to\infty} \overline{\mu}\left(I_{p,R}[\omega_N]\right) = \overline{\mu}\left(I_{p,R}[\omega]\right) \tag{5.36}$$

in order to take the limit $N \to \infty$ in the first term.

In fact, we shall show that by choosing R sufficiently large, independent of N, we may guarantee that

$$|\overline{\mu}\left(I_p[\omega_N]\right) - \overline{\mu}\left(I_{p,R}[\omega_N]\right)| < \epsilon \tag{5.37}$$

and

$$|\overline{\mu}\left(I_p[\omega]\right) - \overline{\mu}\left(I_{p,R}[\omega]\right)| < \epsilon \tag{5.38}$$

for any preselected $\epsilon > 0$. To prove this, we observe first that for even $p \geq 2$,

$$\begin{aligned} \overline{\mu}\left(\overline{I}_{p,R}[\omega_N]\right) &\leq \frac{1}{R^2}\overline{\mu}\left(\int \omega_N^p |\boldsymbol{\nabla}\omega_N|^2\right) \\ &\leq \frac{1}{R^2}\overline{\mu}\left(\|\omega\|_{L^p}^p \|\boldsymbol{\nabla}\omega\|_{L^2}^2\right) \\ &\leq \frac{C_p}{R^2} < \infty. \end{aligned} \tag{5.39}$$

The last line follows from the validity of the third assumed estimate, Eq.(5.16), for arbitrary even p. The middle line is a consequence of the fact that, for a suitable choice of ϕ, and for $f, g \geq 0$,

$$\int f_N g_N \leq \|f\|_{L^1} \|g\|_{L^1} \tag{5.40}$$

for each $N \geq 1$. It is enough to take a real-valued $\widehat{\phi} \in C^\infty$ with compact support and with also ϕ real and positive. It is easy to construct examples with all these properties. In that case, the square $|\widehat{\phi}|^2$ has the same properties and defines another operator T_N (through multiplication of the Fourier coefficients by $|\widehat{\phi}_N(\mathbf{k})|^2$) with the same properties as S_N. Thus,

$$\int d^2\mathbf{r}\ f_N(\mathbf{r}) g_N(\mathbf{r}) = \sum_{\mathbf{k}} |\widehat{\phi}_N(\mathbf{k})|^2 \widehat{f}_{\mathbf{k}}^* \widehat{g}_{\mathbf{k}}$$

$$
\begin{aligned}
&= \|T_N(f * g)\|_{L^1} \\
&\leq \|f * g\|_{L^1} \\
&\leq \|f\|_{L^1}\|g\|_{L^1}. \qquad (5.41)
\end{aligned}
$$

Next we use Jensen's inequality to obtain $\omega_N^p \leq S_N(\omega^p)$ for $p \geq 1$ and $|\nabla\omega_N|^2 = |S_N(\nabla\omega)|^2 \leq S_N\left(|\nabla\omega|^2\right)$. In that case, taking $f = \omega^p$ with even $p \geq 2$ and $g = |\nabla\omega|^2$ in Eq.(5.41), we find that

$$
\int \omega_N^p |\nabla\omega_N|^2 \leq \|\omega^p\|_{L^1}\|(\nabla\omega)^2\|_{L^1} = \|\omega\|_{L^p}^p \|\nabla\omega\|_{L^2}^2 \qquad (5.42)
$$

just as required in Eq.(5.39). The consequence of this equation, is, in short, that

$$
|\overline{\mu}\left(I_p[\omega_N]\right) - \overline{\mu}\left(I_{p,R}[\omega_N]\right)| < \frac{C_p}{R^2}, \qquad (5.43)
$$

and R may be chosen so large that $C_p/R^2 < \epsilon$, giving Eq.(5.37). On the other hand,

$$
\lim_{R\to\infty} \overline{\mu}\left(I_{p,R}[\omega]\right) = \overline{\mu}\left(I_p[\omega]\right) \qquad (5.44)
$$

by the monotone convergence theorem. Since $\overline{\mu}(I_p[\omega]) < \infty$, we may pick R large enough that also Eq.(5.38) is valid.

Therefore, as claimed, we must only show that

$$
\lim_{N\to\infty} \overline{\mu}\left(I_{p,R}[\omega_N]\right) = \overline{\mu}\left(I_{p,R}[\omega]\right). \qquad (5.45)
$$

Since

$$
I_{p,R}[\omega_N] \leq R^{p-2}\|\nabla\omega_N\|_{L^2}^2 \leq R^{p-2}\|\nabla\omega\|_{L^2}^2 \qquad (5.46)
$$

and the righthand side is $\overline{\mu}$-integrable, it is enough to prove that

$$
\lim_{N\to\infty} I_{p,R}[\omega_N] = I_{p,R}[\omega] \qquad \overline{\mu} - \text{a.s.} \qquad (5.47)
$$

However, we see that the functions $f_N \equiv \omega_N^{p-2}\chi_{\{\omega_N<R\}}, f \equiv \omega^{p-2}\chi_{\{\omega<R\}}$ and $g_N \equiv |\nabla\omega_N|^2, g \equiv |\nabla\omega|^2$ satisfy the following properties $\overline{\mu}$-a.s.: (*i*) $|f_N| \leq R^{p-2} < \infty$, (*ii*) $f_N(\mathbf{r}) \to f(\mathbf{r})$ pointwise a.e. with respect to Lebesgue measure, and (iii) $g_N \to g$ in L^1. This is enough to conclude by a triangle argument that $I_{p,R}[\omega_N] = \int f_N g_N \to \int fg = I_{p,R}[\omega]$, as required. We conclude that

$$\lim_{N\to\infty} \overline{\mu}\left(\int \omega_N^{p-2}|\boldsymbol{\nabla}\omega_N|^2\right) = \overline{\mu}\left(\int \omega^{p-2}|\boldsymbol{\nabla}\omega|^2\right), \tag{5.48}$$

which allows the limit to be taken in the first term of the Eq.(5.22).

Therefore, from all of the preceding argument, we obtain from Eq.(5.22) in the limit $N \to 0$ that

$$\begin{aligned}\int \overline{\mu}(d\omega)&\left[\frac{1}{(p-2)!}\int_{\mathbf{T}^2} d^2\mathbf{r}\ \nu\omega^{p-2}(\mathbf{r})|\boldsymbol{\nabla}\omega|^2\right] \\ &= \int \overline{\mu}(d\omega)\left[\frac{1}{(p-2)!}\int_{\mathbf{T}^2} d^2\mathbf{r}\ \omega^{p-2}(\mathbf{r})\right]\cdot H,\end{aligned} \tag{5.49}$$

for even $p > 2$, which are exactly the higher-order balance equations Eq.(5.13).

The motivation for the estimates Eqs.(5.14),(5.15) we have assumed is the following: Let again $\omega_N = P_N\omega$ and consider the Fourier-Galerkin approximation to the dynamics

$$\partial_t\omega_N + P_N B(\mathbf{v}_N, \omega_N) = \nu \bigtriangleup \omega_N + q_N. \tag{5.50}$$

It is unfortunately the case that

$$\int \omega_N^{p-1} P_N B(\mathbf{v}_N, \omega_N) \neq 0 \tag{5.51}$$

for $p > 2$. If we ignore this fact, assuming for the moment that $\|\omega_N(t)\|_{L^p}^p$ is conserved by the approximate dynamics, then the Ito formula gives for each $\alpha \geq 0$,

$$\begin{aligned} d\|\omega_N(t)\|_{L^p}^{(\alpha+1)p} &= \frac{-4\nu(\alpha+1)}{q}\|\omega_N(t)\|_{L^p}^{\alpha p}\int \omega_N^{p-2}(t)|\boldsymbol{\nabla}\omega_N(t)|^2\cdot dt + dM_N(t) \\ &\quad +\frac{H_N}{2}\Big[(\alpha+1)\alpha p^2\|\omega_N(t)\|_{L^p}^{(\alpha-1)p}\|\omega_N(t)\|_{L^{2p-2}}^{2p-2}dt \\ &\quad +(\alpha+1)p(p-1)\|\omega_N(t)\|_{L^p}^{\alpha p}\|\omega_N(t)\|_{L^{p-2}}^{p-2}dt\Big], \end{aligned} \tag{5.52}$$

where

$$M_N(t) \equiv p\int_0^t \langle \omega_N^{p-1}(s), dq_N(s)\rangle \tag{5.53}$$

is a martingale part. It is not too difficult to make this equation the basis of an inductive proof of the bounds Eqs.(5.14),(5.15). We do not give the argument here because the above equation is not really correct. Notice that

$$\langle \omega_{N-1}^{p-1}, P_N B(\mathbf{v}_{N-1}, \omega_{N-1})\rangle = \langle \omega_{N-1}^{p-1}, B(\mathbf{v}_{N-1}, \omega_{N-1})\rangle = 0, \tag{5.54}$$

despite Eq.(5.51). Therefore, the violation of the higher-order conservation laws for the truncated dynamics is, in some sense, a "boundary effect" in Fourier space associated to the interactions near the wavenumber cutoff at 2^N. If some control of these boundary terms can be obtained, then the estimates Eqs.(5.14),(5.15) should follow. It seems to us an interesting open mathematical problem to establish these estimates as *a priori* bounds. More sophisticated approximations to 2D Euler may be useful here, such as the finite-mode dynamics based upon $SU(N)$ Lie algebras which have $O(N)$ integrals of motion for $O(N^2)$ modes [54]. However, we would be very surprised if the presumed bounds should not be valid and we regard the balance equations therefore as rather convincingly demonstrated.

Appendix II: Kolmogorov Relation for the 2D Enstrophy Cascade

We shall derive here the Kolmogorov-type relation for 2D, Eq.(4.5). The method we use is patterned closely after U. Frisch's derivation of the "4/5-law" in Section 6.2 of [55].

The first step is to define a *physical-space enstrophy flux* as

$$\eta(\mathbf{l}) \equiv -\frac{1}{2}\frac{d}{dt}\langle \omega(\mathbf{r},t)\omega(\mathbf{r}+\mathbf{l},t)\rangle\Big|_{\mathrm{Euler},t=0}, \tag{5.55}$$

where the time-derivatives are to be evaluated using the inviscid Euler equations. (Compare this definition with Polyakov's "point-splitting" method in [9].) A simple calculation, using statistical homogeneity and incompressibility, yields

$$\eta(\mathbf{l}) = -\frac{1}{4}\boldsymbol{\nabla}_{\mathbf{l}}\boldsymbol{\cdot}\langle \Delta_{\mathbf{l}}\mathbf{v}(\Delta_{\mathbf{l}}\omega)^2\rangle. \tag{5.56}$$

If we now consider the randomly-forced NS equation, Eq.(2.32),

$$\partial_t\omega + (\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\omega = \nu\nabla^2\omega + q, \tag{5.57}$$

then we obtain *formally*, for the steady-state, the balance-equation

$$\eta(\mathbf{l}) = \frac{1}{2}\langle\omega(\mathbf{r})[q(\mathbf{r}+\mathbf{l}) + q(\mathbf{r}-\mathbf{l})]\rangle + \nu\nabla_{\mathbf{l}}^2\langle\omega(\mathbf{r})\omega(\mathbf{r}+\mathbf{l})\rangle. \tag{5.58}$$

For a Gaussian force, white-noise in time, this becomes

$$\eta(\mathbf{l}) = \frac{1}{2}(Q(\mathbf{l}) + Q(-\mathbf{l})) + \nu\nabla_{\mathbf{l}}^2 R(\mathbf{l}), \tag{5.59}$$

where $Q(\mathbf{r}-\mathbf{r}')$ is the covariance function of q and $R(\mathbf{l}) = \langle\omega(\mathbf{r})\omega(\mathbf{r}+\mathbf{l})\rangle$ is the vorticity covariance function. We have sketched a formal derivation of the last balance equation, Eq.(5.59), but it should be possible to prove rigorously in 2D by the methods of Appendix I.

Note that the viscous term will be negligible when ν is small and $\ell = |\mathbf{l}| \gg \ell_d^{\mathrm{Kr}}$. This can be easily checked to be true using arguments like those in Section 3.II. In fact, for the case of isotropic forcing, this can be inferred from the well-known representation

$$R(\mathbf{l}) = \int_0^\infty J_0(k\ell)\Omega(k)dk. \tag{5.60}$$

(See Theorem 2.5.2 of [56].) However, if the force spectrum has support only in a finite wavenumber range around k_f, then

$$Q(\mathbf{l}) \approx Q(\mathbf{0}) \tag{5.61}$$

when $\ell \ll \ell_f$. We therefore obtain that

$$-\frac{1}{4}\boldsymbol{\nabla}_{\mathbf{l}}\boldsymbol{\cdot}\langle\Delta_{\mathbf{l}}\mathbf{v}(\Delta_{\mathbf{l}}\omega)^2\rangle \approx \eta \tag{5.62}$$

with $\eta \equiv Q(\mathbf{0})$, when $\ell_f \gg \ell \gg \ell_d^{\mathrm{Kr}}$. Alternatively, we may consider idealized limits, first $\nu \to 0$ and subsequently $\ell \to 0$, always assuming that such limits exist.

For completeness we note that this relation can be obtained in another way by relating the "physical-space flux" $\eta(\mathbf{l})$ to the usual spectral flux Z_K, as

$$Z_K = \frac{K}{2\pi}\int d^2\mathbf{l}\frac{J_1(K\ell)}{\ell}\eta(\mathbf{l}). \tag{5.63}$$

By its definition

$$Z_K \equiv -\frac{1}{2}\frac{d}{dt}\int_{|\mathbf{k}|\leq K}\frac{d^2\mathbf{k}}{(2\pi)^2}\langle|\widehat{\omega}(\mathbf{k})|^2\rangle\Big|_{Euler}$$

$$= \int_{|\mathbf{k}|\le K} \frac{d^2\mathbf{k}}{(2\pi)^2} \int d^2\mathbf{l}\; e^{i\mathbf{k}\cdot\mathbf{l}}\eta(\mathbf{l}). \tag{5.64}$$

However,

$$\begin{aligned}\int_{|\mathbf{k}|\le K} d^2\mathbf{k}\; e^{i\mathbf{k}\cdot\mathbf{l}} &= 2\int_0^K kdk \int_0^\pi d\theta\; e^{ik\ell\cos\theta} \\ &= 2\pi \int_0^K J_0(k\ell)\, kdk. \end{aligned}\tag{5.65}$$

We have used a common integral representation of the Bessel function J_0 (Eq.(9.1.18) of [57].) It is also a well-known relation for the Bessel functions that

$$\int_0^z t^\nu J_{\nu-1}(t)dt = z^\nu J_\nu(z) \tag{5.66}$$

when Re$\nu > 0$. (See Eq.(11.3.20) of [57].) Applying this for $\nu = 1, z = K\ell, t = k\ell$ we obtain Eq.(5.63) as claimed. It is then easy to see that in the range of scales where $\eta(\mathbf{l}) \approx \hat{\eta}$, a constant, that also $Z_K \approx \hat{\eta}$. We just use

$$\int_0^\infty J_\nu(t)dt = 1, \tag{5.67}$$

valid for Re$\nu > -1$ (Eq.(11.4.17) of [52]), in the case $\nu = 1$. Therefore, we see that we must have $\hat{\eta} = \eta$, and we re-derive Eq.(5.62).

We obtain the special form in Eq.(4.5),

$$\langle [\Delta_{\mathbf{l}} v_\parallel(\mathbf{r})][\Delta_{\mathbf{l}}\omega(\mathbf{r})]^2\rangle = -2\eta\ell, \tag{5.68}$$

under the condition of statistical isotropy. This can be easily guaranteed if the forcing is chosen to be isotropic.[12] In that case we may represent the vector function

$$\mathbf{A}(\mathbf{l}) \equiv \langle \Delta_{\mathbf{l}}\mathbf{v}(\Delta_{\mathbf{l}}\omega)^2\rangle \tag{5.69}$$

[12] More properly, this requires one to consider the infinite-volume setting. In that case, it is easy to show for isotropic forcing that at least one homogeneous and isotropic stationary statistical solution exists. Of course, it is not known whether the steady-state is unique. In particular, it may be that there is a "spontaneous-breaking" of $SO(2)$ symmetry, so that the space-ergodic measures, or "pure phases," are not rotation-invariant. Barring this unlikely possibility, the symmetries of the stationary measure follow from those of the force.

in terms of two functions $A_\parallel(\ell)$ and $A_\perp(\ell)$ depending only upon the magnitude $\ell = |\mathbf{l}|$, as

$$A_i(\mathbf{l}) = \hat{l}_i A_\parallel(\ell) + \epsilon_{ij}\hat{l}_j A_\perp(\ell). \tag{5.70}$$

It is then easy to calculate that

$$\boldsymbol{\nabla}_{\mathbf{l}}\boldsymbol{\cdot}\mathbf{A} = \frac{1}{\ell}A_\parallel(\ell) + \frac{dA_\parallel}{d\ell}(\ell). \tag{5.71}$$

Thus, we obtain from Eq.(5.62) that

$$\frac{dA_\parallel}{d\ell}(\ell) + \frac{1}{\ell}A_\parallel(\ell) = -4\eta. \tag{5.72}$$

The only solution of this equation regular at $\ell = 0$ is

$$A_\parallel(\ell) = -2\eta\ell. \tag{5.73}$$

This is exactly Eq.(5.68). Observe that $A_\perp \equiv 0$ if the steady-state is invariant as well under space-reflection (i.e. the full invariance under the improper rotation group $O(2)$.) In that case, we have even

$$\mathbf{A}(\mathbf{l}) = -2\eta\mathbf{l} \tag{5.74}$$

as a vector identity.

Appendix III: Potential Theory Estimates and Results for Besov Spaces in 2D

We prove here the estimates from potential theory and the embedding theorems used in Section 4 to derive bounds on 2D scaling exponents. We shall consider these in essentially the order of their appearance in that section.

First, we give for completeness a derivation of the Besov space embedding theorem Eq.(4.8), since the proof is very elementary. The defining criterion for $f \in B_p^{s,q}(\mathbf{T}^d)$ is that the sequence

$$(2^{sN}\|f * \varphi_N\|_p : N \geq 0) \in \ell^q, \tag{5.75}$$

where $(\widehat{\varphi}_N : N \geq 0)$ is a smooth partition of unity with $\mathrm{supp}\widehat{\varphi}_N \subset [2^{N-1}, 2^{N+1}]$. If a function g has its Fourier transform supported in the unit ball $\{\mathbf{k} : |\mathbf{k}| \leq 1\}$, then it can be written as $g = \phi * g$, where ϕ is a function with C^∞ Fourier transform $\widehat{\phi}$ supported in the larger ball $\{\mathbf{k} : |\mathbf{k}| \leq 2\}$ and $= 1$ in the unit ball. For each $p \geq 1$ it follows by the Hölder inequality that

$$\|g\|_\infty = \|\phi * g\|_\infty \leq C_p \|g\|_p, \tag{5.76}$$

where $C_p = \|\phi\|_q$, $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, for each $p' \geq p$,

$$\begin{aligned} \|g\|_{p'} &\leq \|g\|_\infty^{1-\frac{p}{p'}} \cdot \|g\|_p^{\frac{p}{p'}} \\ &\leq C\|g\|_p, \end{aligned} \tag{5.77}$$

with $C = (C_p)^{1-\frac{p}{p'}}$. A simple scaling argument shows that if g has instead its Fourier transform supported in the ball $\{\mathbf{k} : |\mathbf{k}| \leq 2^N\}$, then

$$\|g\|_{p'} \leq C \cdot 2^{N\left(\frac{d}{p} - \frac{d}{p'}\right)} \cdot \|g\|_p \tag{5.78}$$

for each $p' \geq p$. If we apply this observation to the function $g = f * \varphi_N$, then we see that for each $N \geq 0$,

$$2^{s'N} \|f * \varphi_N\|_{p'} \leq C \cdot 2^{sN} \|f * \varphi_N\|_p, \tag{5.79}$$

when

$$s' - \frac{d}{p'} = s - \frac{d}{p}. \tag{5.80}$$

This implies that $\|f\|_{B^{s',q}_{p'}} \leq C\|f\|_{B^{s,q}_p}$, concluding the proof.

We now discuss the relation between the two conditions $\|\Delta_{\mathbf{l}}\omega\|_p = O(\ell^s)$ and $\|\Delta_{\mathbf{l}}^2 \mathbf{v}\|_p = O(\ell^{1+s})$. Since $\omega = \boldsymbol{\nabla}\times\mathbf{v}$,

$$\mathbf{v} = \mathbf{B} * \omega, \tag{5.81}$$

as in Eq.(2.2), where $\mathbf{B}$ is the integral kernel of the inverse operator to rot $= \boldsymbol{\nabla}\times\cdot$. The relation therefore involves potential theory estimates, which are readily available in the literature. We first observe that for any $s \in \mathbf{R}$,

$$\|\Delta_{\mathbf{l}}\omega\|_p = O(\ell^s) \longrightarrow \|\Delta_{\mathbf{l}}^2 \mathbf{v}\|_p = O(\ell^{1+s}). \tag{5.82}$$

Here the derivative $\boldsymbol{\nabla}\mathbf{v}$ is taken in the distribution sense, and a straightforward calculation gives

$$\boldsymbol{\nabla}\mathbf{v} = P(\boldsymbol{\nabla}\mathbf{B}) * \omega + \frac{1}{2}\varepsilon \cdot \omega, \tag{5.83}$$

where the first term is a principal part convolution integral for the singular kernel $\boldsymbol{\nabla}\mathbf{B}$ and ε in the second term is the Levi-Civita matrix. Observe, furthermore, that the difference operator $\Delta_{\mathbf{l}}$ commutes with convolution operators, so that

$$\boldsymbol{\nabla}(\Delta_{\mathbf{l}}\mathbf{v}) = P(\boldsymbol{\nabla}\mathbf{B}) * (\Delta_{\mathbf{l}}\omega) + \frac{1}{2}\varepsilon \cdot (\Delta_{\mathbf{l}}\omega). \tag{5.84}$$

Applying the Calderón-Zygmund inequality [53] to the first term, it then follows that

$$\|\boldsymbol{\nabla}(\Delta_{\mathbf{l}}\mathbf{v})\|_p \leq C\|\Delta_{\mathbf{l}}\omega\|_p \tag{5.85}$$

for $1 < p < \infty$. However, if $\Delta_{\mathbf{l}} f$ is represented by the line-integral

$$\Delta_{\mathbf{l}} f(\mathbf{r}) = \int_0^{\ell} ds\ \hat{\mathbf{l}} \cdot \boldsymbol{\nabla} f(\mathbf{r} + s\hat{\mathbf{l}}), \tag{5.86}$$

valid for $|\boldsymbol{\nabla} f| \in L^1$, then a simple application of Jensen's inequality gives in general that

$$\|\Delta_{\mathbf{l}} f\|_p \leq \ell \cdot \|\boldsymbol{\nabla} f\|_p \tag{5.87}$$

for $p \geq 1$. Applying this to $f = \Delta_{\mathbf{l}}\mathbf{v}$, we obtain that

$$\|\Delta_{\mathbf{l}}^2\mathbf{v}\|_p \leq C \cdot \ell \cdot \|\Delta_{\mathbf{l}}\omega\|_p = O\left(\ell^{1+s}\right), \tag{5.88}$$

as claimed.

However, if also $0 < s < 1$, then we actually have equivalence

$$\|\Delta_{\mathbf{l}}\omega\|_p = O(\ell^s) \longleftrightarrow \|\Delta_{\mathbf{l}}^2\mathbf{v}\|_p = O(\ell^{1+s}), \tag{5.89}$$

for $p > 1$. In fact, if $\mathbf{v} \in L^p$, then

$$\|\Delta_{\mathbf{l}}^2\mathbf{v}\|_p = O(\ell^{1+s}) \longleftrightarrow \mathbf{v} \in B_p^{1+s,\infty} \longleftrightarrow \boldsymbol{\nabla}\mathbf{v} \in B_p^{s,\infty}. \tag{5.90}$$

The first equivalence follows from Theorem 2.5.12 in [47] on characterization of Besov spaces by differences. The second equivalence is a well-known result for all $s \in \mathbf{R}$. For $0 < s < 1$ it follows in the $\longleftarrow$ direction from the characterization by differences applied to $\boldsymbol{\nabla}\mathbf{v}$, $\|\Delta_{\mathbf{l}}\boldsymbol{\nabla}\mathbf{v}\|_p = O(\ell^s)$, and the inequality Eq.(5.87). It follows in the $\longrightarrow$ direction from the defining criterion Eq.(5.75) applied to $\mathbf{v}$, and the following consequence

$$\|\varphi_N * (\boldsymbol{\nabla}\mathbf{v})\|_p = \|\boldsymbol{\nabla}(\varphi_N * \mathbf{v})\|_p \leq C \cdot 2^N \cdot \|\varphi_N * \mathbf{v}\|_p \tag{5.91}$$

of the Nikol'skij inequality (see Remark 1, Section 1.3.2 of [47]).

On the other hand, if $-1 < s < 0$, then there is no longer equivalence but it is still true that

$$\begin{aligned} \|\Delta_{\mathbf{l}}\omega\|_p = O(\ell^s) \quad &\longrightarrow \quad \|\Delta_{\mathbf{l}}^2\mathbf{v}\|_p = O(\ell^{1+s}) \\ &\longleftrightarrow \quad \|\Delta_{\mathbf{l}}\mathbf{v}\|_p = O(\ell^{1+s}) \\ &\longrightarrow \quad \omega \in B_p^{s,\infty}. \end{aligned} \tag{5.92}$$

for $p > 1$. The middle equivalence is another consequence of Theorem 2.5.12 in [47] on characterization by differences, which states that both are equivalent to $\mathbf{v} \in B_p^{1+s,\infty}$ when $0 < 1+s < 1$. To obtain the last $\longrightarrow$ implication, we recall $\mathbf{v} \in B_p^{1+s,\infty} \longleftrightarrow \boldsymbol{\nabla}\mathbf{v} \in B_p^{s,\infty}$ and ω is, of course, just a component of $\boldsymbol{\nabla}\mathbf{v}$. We should emphasize that the Theorem 2.5.12 in [47] on characterization by differences does not apply when $s \leq d\left(\frac{1}{\min\{1,p\}} - 1\right)$ and, in particular, when $s \leq 0$ for $p \geq 1$. Therefore, $\|\Delta_{\mathbf{l}}\omega\|_p = O(\ell^s)$ is only a sufficient condition for $\omega \in B_p^{s,\infty}$ and not a necessary one when $s \leq 0$.

We next show that the condition

$$\langle|\Delta_{\mathbf{l}}\omega|^p\rangle = O(\ell^{\zeta_p}) \tag{5.93}$$

with $\zeta_p < 0$ implies that $\mathbf{v} \in B_p^{1+s,\infty}$ a.s. for each $s < \sigma_p$ and, therefore,

$$\|\Delta_{\mathbf{l}}\mathbf{v}\|_p \leq \ell^{1+s} \cdot \|\mathbf{v}\|_{B_p^{s,\infty}} \qquad \overline{\mu} - \text{a.s.} \tag{5.94}$$

Theorem 4 in [28] does not apply when $\sigma_p < 0$. Instead we shall show directly that $\mathbf{v} \in B_p^{1+s,\infty}$ by means of the following characterization. Define the *truncated-ball means*

$$(\mathcal{V}_\ell^M f)(\mathbf{r}) \equiv \overline{(\Delta_{\ell\cdot\mathbf{h}}^M f)(\mathbf{r})}, \tag{5.95}$$

where the line indicates a space-average over the "truncated-ball" $\{\mathbf{h} : 1 < |\mathbf{h}| < 2\}$. Then it is known that

$$\|f\|^{\#}_{B_p^{s,\infty}} = \sup_{N\geq 0}\left(2^{sN}\|\mathcal{V}^M_{2^{-N}}f\|_p\right) \tag{5.96}$$

defines a seminorm for the Besov space $B_p^{s,\infty}$ with any integer $M > s > 0$. See Sections 2.5.11-12 of [47]. To employ this result, we first average the condition Eq.(5.93) over the truncated-ball:

$$\overline{\langle|\Delta_{\ell\cdot\mathbf{h}}\omega|^p\rangle} = O(\ell^{\zeta_p}). \tag{5.97}$$

This implies for each fixed ℓ, or for any fixed countable set of ℓ's, that

$$\Delta_{\ell\cdot\mathbf{h}}\omega \in L_p \qquad \overline{\mu} - \text{a.e. } \omega, \lambda - \text{a.e. } \mathbf{h}, \tag{5.98}$$

where λ is Lebesgue measure on the truncated-ball. We can then invoke Eq.(5.88) to obtain

$$\|\Delta^2_{\ell\cdot\mathbf{h}}\mathbf{v}\|_p \leq C\cdot\ell\cdot\|\Delta_{\ell\cdot\mathbf{h}}\omega\|_p \qquad \overline{\mu} - \text{a.e. } \omega, \lambda - \text{a.e. } \mathbf{h}. \tag{5.99}$$

This may be averaged jointly over the truncated-ball and the ensemble to obtain, with Eq.(5.97),

$$\overline{\langle\|\Delta^2_{\ell\cdot\mathbf{h}}\mathbf{v}\|_p\rangle} = O(\ell^{1+\sigma_p}). \tag{5.100}$$

However, it follows from Jensen's inequality that

$$\|\mathcal{V}^2_\ell\mathbf{v}\|_p \leq \overline{\|\Delta^2_{\ell\cdot\mathbf{h}}\mathbf{v}\|_p} \tag{5.101}$$

and thus

$$\langle\|\mathcal{V}^2_\ell\mathbf{v}\|_p\rangle \leq \overline{\langle\|\Delta^2_{\ell\cdot\mathbf{h}}\mathbf{v}\|_p\rangle} = O(\ell^{1+\sigma_p}) \tag{5.102}$$

for each ℓ in a countable set $\mathcal{L}$. Let us choose the latter set to be $\mathcal{L} = \{\ell_N = 2^{-N} : N \geq 0\}$, or, in other words,

$$\langle\|\mathcal{V}^2_{2^{-N}}\mathbf{v}\|_p\rangle = O\left(2^{-N(1+\sigma_p)}\right) \tag{5.103}$$

for all $N \geq 0$. Then the same Borel-Cantelli argument used in the proof of Theorem 4 of [28] implies that

$$\|\mathcal{V}^2_{2^{-N}}\mathbf{v}\|_p = O\left(2^{-N(1+s)}\right) \qquad \overline{\mu} - \text{a.s.} \tag{5.104}$$

for each $s < \sigma_p$. We conclude, by the characterization theorem, that $\mathbf{v} \in B_p^{1+s,\infty}$ a.s. and that Eq.(5.94) holds.

The last result we needed in our arguments is the inequality

$$\|\mathbf{v}\|_{B_p^{1+s,\infty}} \leq C\|\omega\|_{B_p^{s,\infty}} \tag{5.105}$$

for some constant $C > 0$, which was used in establishing sufficiency of the moment conditions. For Eq.(5.105) we may refer to the literature. Since Eq.(5.83) relates $\boldsymbol{\nabla}\mathbf{v}$ to ω by a classical singular-integral operator T, we may appeal to Theorem A of Lemarie [58]:

$$\|T\omega\|_{B_p^{s,q}} \leq C'\|\omega\|_{B_p^{s,q}} \tag{5.106}$$

for $1 > s > 0$ and $p, q \in [1, +\infty]$. This is a result of Calderón-Zygmund type in the Besov spaces. See also [59]. In our case,

$$\|\boldsymbol{\nabla}\mathbf{v}\|_{B_p^{s,\infty}} \leq C''\|\omega\|_{B_p^{s,\infty}}. \tag{5.107}$$

¿From this we conclude at once Eq.(5.105).

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