THE $1/N$-EXPANSION AS A PERTURBATION ABOUT THE MEAN FIELD THEORY: A One-Dimensional Fermi Model

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Abstract

We examine a family of probability measure on $\mathbf{R}^L$ with real parameter $\zeta > 0$ and integer parameters $N, L > 0$. Each such measure is equivalent to the lattice version of a one-dimensional chiral-invariant Fermion quantum field theory with quartic interaction, with N the number of flavours. After applying the Matthews-Salam formula, the model becomes a statistical mechanical model of a chain of continuous Gaussian spins, coupled with a certain non-standard weight function. Finally, the model can also be considered as a probability measure

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on the set of tridiagonal matrices with fixed off-diagonal and random diagonal entries.

Our analysis shows how to develop an asymptotic expansion in $1/N$, valid for all L and ζ, for the fundamental expectation values. In particular the analysis proves that the model behaves like a one-dimensional Ising model as $L \to \infty$ for N large, and thus remains in a pure phase for all values of ζ. It is also shown that the Fermion model has a mass gap which agrees to the leading order in $1/N$ with the mean field value calculated by the argument of Gross-Neveu. The analytical technique we develop in essence combines transfer matrix method with Laplace method (steepest descent) for asymptotics of integrals.

1 Introduction

It was argued some years ago [NJ, GN] that a chiral invariant fermion theory with quartic interaction will acquire a mass dynamically by spontaneous symmetry breaking. The effective potential shows degenerate minima in the one-loop approximation, leading to a ground state with non-vanishing expectation of $\overline{\psi}\psi$.

Rigorous control of multi-phase theories was developed by Glimm-Jaffe-Spencer [GJS] who devised an expansion about the mean field theory by combining a cluster expansion with an expansion in phase boundaries.

This paper is part of a program which aims at understanding the mechanism by which mass is generated dynamically in models with four-fermion interactions. We here consider a one-dimensional chiral lattice model and show that a fermion mass is generated without symmetry breaking. Our procedure combines transfer matrices with a steepest descent method to produce $1/N$-corrections to the mean field theory and can be viewed as an alternative to the expansion proposed in reference [GJS].

1.1 The model

For fixed $\zeta > 0$ and integers $N > 0$, L even, we consider the joint probability measure $\mu_L(\phi)$ on $\mathbf{R}^L$, whose density $g_L(\phi)$ is given by

$$g_L(\phi) = \frac{1}{Z_L} \frac{e^{-\frac{1}{2}(\phi,\phi)}}{(2\pi)^{L/2}} \det\left(\Gamma_L + (\zeta/\sqrt{N})\ \Phi_L\right)^{2N} \tag{1.1}$$

where $(\phi, \phi) = \sum_{i=1}^{L} |\phi_i|^2$ and

$$\Gamma_L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \Phi_\mathrm{L} = \begin{pmatrix} \phi_1 & 0 & 0 & \cdots & 0 \\ 0 & \phi_2 & 0 & \cdots & 0 \\ 0 & 0 & \phi_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_L \end{pmatrix} \tag{1.2}$$

are $L \times L$ matrices. The partition function

$$Z_L = \int \prod_{i=1}^{L} \left(\frac{d\phi_i}{\sqrt{2\pi}}\right)\ e^{-\frac{1}{2}(\phi,\phi)} \det\left(\Gamma_L + (\zeta/\sqrt{N})\ \Phi_L\right)^{2N} \tag{1.3}$$

normalizes the measure.

We will be interested in the large N behavior of the limiting measure μ obtained by μ_L as $L \to \infty$. The existence of this limit will be a consequence of our analysis.

The measure μ_L arises from the application of the Matthews-Salam formula to the one dimensional fermionic system on a lattice $\Lambda \subset a\mathbf{Z}$ with $|\Lambda| = L$, described by the action

$$\mathcal{A}(\overline{\psi}, \psi) = a\left\{\sum_{x\in\Lambda} \overline{\psi}_x \cdot (i\mathbf{D}\psi)_x + \frac{\lambda^2}{2N} \sum_{x\in\Lambda} \left(\overline{\psi}_x \cdot \psi_x\right)^2\right\}. \tag{1.4}$$

Here $\mathcal{A}(\overline{\psi}, \psi)$ is a functional on the Grassman algebra generated by the fermionic field $(\overline{\psi}, \psi)$ with scalar product, $\overline{\psi} \cdot \chi = \sum_{j,\alpha} \overline{\psi}^j_\alpha \chi^j_\alpha$, running over "flavors" $j = 1, \ldots, N$, and spins, $\alpha =\uparrow, \downarrow$. $i\mathbf{D}$ is the symmetric difference operator whose kernel is given by

$$(i\mathbf{D})_{xy} = \frac{1}{a}\,(\Gamma_L)_{\frac{x}{a}\frac{y}{a}}\ \ \sigma_3$$

where a is the lattice spacing and σ_3 the diagonal Pauli matrix. $a\lambda^2 = \zeta^2$ is a dimensionless coupling constant.

To see this, we set $\mathbf{\Phi} = \Phi_L 1_2$ where 1_n is the $n \times n$ identity and notice that, due to the fact $\Gamma_L^T = -\Gamma_L$,

$$\det\left(ia\mathbf{D} + (\zeta/\sqrt{N})\ \mathbf{\Phi}\right)^N = \det\left(\Gamma_L + (\zeta/\sqrt{N})\ \Phi_L\right)^{2N} \tag{1.5}$$

can be written as the Grassman integral

$$\int \prod_{x\in\Lambda} d\overline{\psi}_x d\psi_x\ \exp\left\{\sum_{x\in\Lambda} \overline{\psi}_x \cdot \left[\left(ia\mathbf{D} + (\zeta/\sqrt{N})\ \mathbf{\Phi}\right)\psi\right]_x\right\}. \tag{1.6}$$

The Matthews-Salam formula is based on the identity

$$e^{(\zeta^2/2N)\ (\overline{\psi}\cdot\psi)^2} = \frac{1}{\sqrt{2\pi}}\int d\phi\ e^{-\frac{1}{2}\phi^2}\ e^{(\zeta/\sqrt{N})\ (\overline{\psi}\cdot\psi)\phi}. \tag{1.7}$$

Using (1.5)-(1.7), the partition function (1.3) can thus be written as a Grassman integration over the weight $e^{\mathcal{A}(\overline{\psi},\psi)}$, i.e.,

$$Z_L = \int \prod_{x\in\Lambda} d\overline{\psi}_x d\psi_x\ e^{\mathcal{A}(\overline{\psi},\psi)}. \tag{1.8}$$

Remarks: 1. Action (1.4) is the one-dimensional Euclidean lattice version (the only direction is time) of the Gross-Neveu model [GN]. The equivalence between (1.3) and (1.8) can be easily extended to d-dimensional regular lattices with $d > 1$, Cayley trees and other graphs [FHM]: If $\mathcal{G}$ is a graph with L sites labeled by $\{1, \ldots, L\}$, Φ_L is as in (1.2) and Γ_L is a conveniently defined skew-symmetric matrix with $(\Gamma_L)_{ij} = \pm 1$ if i, j are linked sites in $\mathcal{G}$ and 0 otherwise.

2. The model in $a\mathbf{Z}^2$ is of special interest since the massive version of (1.4) is perturbatively renormalizable and asymptotically free in the ultraviolet [GK, FMRS] (provided $N > 1$), which allows one to remove the lattice cutoff by a limiting procedure. Our results in $d = 1$ are uniform in the lattice spacing a provided $\lambda = \lambda(a) = \zeta/\sqrt{a}$ for any $0 < \zeta < \infty$.

3. Most of our results are valid if we consider a measure ν_L defined in $\mathbf{C}^L$ whose density is given by (1.1) with $\det(\Gamma + (\zeta/\sqrt{N})\Phi)$ replaced by

$\left|\det(\Gamma+(\zeta/\sqrt{N})\Phi)\right|$. By the Matthews-Salam formula, this measure represents a fermion system described by the action (1.4) with the local interaction $\left(\overline{\psi}_x \cdot \psi_x\right)^2$ replaced by

$$\left(\overline{\psi}_x \cdot \psi_x\right)^2 - \left(\overline{\psi}_x \cdot \sigma_2 \psi_x\right)^2 \tag{1.9}$$

where σ_2 is the imaginary Pauli matrix. Notice that (1.9) is invariant under the continuous chiral transformation

$$\psi \longrightarrow e^{i\frac{\theta}{2}\sigma_2}\,\psi \quad , \quad \overline{\psi} \longrightarrow \overline{\psi}\,e^{i\frac{\theta}{2}\sigma_2}.$$

ν_L is thus $U(1)$ invariant whereas (1.1) is invariant only under the discrete transformation $\phi \longrightarrow -\phi$ (corresponding to $\theta=\pi$).

1.2 The mean field approximation and Peierls estimate

As usual, the first step at understanding a probability measure in equilibrium statistical mechanics is to consider the mean field approximation. It consists here in conditioning the measure $\mu_L(\phi)$ on constant configurations $\{\phi_i=\varphi\}_{i=1}^L$. Denoting the mean field density by $h_L(\varphi)$, it follows that:

Proposition 1.1 *For any $\zeta>0$ and $L>2/\zeta^2-2$,*

(i) $h_L(\varphi)=h_L(-\varphi)$,

(ii) *$h_L(\varphi)$ achieves its maximum at $\varphi=\pm\varphi^*\equiv\pm m_L\sqrt{N}/\zeta$ where $m_L=m_L(\zeta)$ is the unique positive solution of the "mean-field" equation*

$$m = 2\zeta^2\frac{1}{L}\frac{f_L'(m)}{f_L(m)} \tag{1.10}$$

where $f_L(m)=\det(\Gamma_L+m1_L)$ (note (1.10) does not depend on N).

(iii) *For large L, m_L is given by*

$$m_L = m(\zeta)+\mathrm{O}(1/L)$$

where $m(\zeta)=\left(2(1+\zeta^4)^{1/2}-2\right)^{1/2}$. Furthermore, $\varphi=0$ is a local minimum of $h_L(\varphi)$ and

$$\frac{h_L(\varphi^*)}{h_L(0)} \le e^{cLN}$$

for some constant $c=c(\zeta)>0$.

¿From Proposition 1.1, the effective potential (see details in section 3)

$$V(\varphi) = \lim_{L\to\infty} \frac{-1}{L} \ln\ h_L(\varphi) \tag{1.11}$$

is a symmetric double-well-shaped potential with minima separated by a barrier of width $O(\sqrt{N})$ and height $O(N)$ at $\varphi = 0$. When $N \to \infty$, the density $h_L(\varphi)$ can be well approximated by a sum of two Gaussians centred at $\varphi = \pm\varphi^*$, corresponding to two pure phases. The expectation with respect to the mean field measure of any function $K(\varphi)$ is of the form of a Laplace integral. It is thus given, in the limit as $L \to \infty$, by

$$\mathbf{E}_h[K] \equiv \lim_{L\to\infty} \int_{-\infty}^{\infty} d\varphi\ h_L(\varphi)\ K(\varphi) = \frac{1}{2}\left(K(\varphi^*) + K(-\varphi^*)\right) \tag{1.12}$$

We shall consider the mean field approximation as the leading term of an asymptotic expansion in $1/N$. The next proposition shows that the Glimm-Jaffe-Spencer technique of controlling multi-phase problems is not suitable for the one-dimensional model.

Proposition 1.2 *(Peierls Estimate) Given an Ising-like configuration* $\{\sigma_i = \pm m\}$, $i = 1, \ldots, L$, *let us denote by* σ_L *the diagonal matrix whose entries are* $\sigma_i\delta_{ij}$ *and set* $f_L(\sigma) = \det(\Gamma_L + \sigma_L)$. *Then there exists* $\delta = \delta(m) < 1$, *uniformly in* L, *such that*

$$\left|\frac{f_L(\sigma)}{f_L(m)}\right| \leq \delta^p$$

where p *is the number of flips in the spin configuration* σ_L.

In a multi-phase system the stability of the Peierls mechanism [G] is determined by a competition between the symmetry breaking term and the entropy factor. The Peierls estimate is the basic ingredient in this mechanism. Proposition 1.2 thus shows that, although this model is non-local, it exhibits the same behavior as the short-range one-dimensional Ising model. Since the Peierls estimate is always overcomed by the entropy in this case, symmetry breaking does not occur.

Mass generation is expected to occur in the two or higher dimensional lattice Gross-Neveu model (1.4) as a consequence of symmetry breaking and the Glimm-Jaffe-Spencer technique may be useful to these cases. Bałaban and Gawedzki [BG], for example, have established spontaneous symmetry breaking in the two-dimensional pseudoscalar Yukawa model using these techniques. In their work, fluctuations around Ising-like configurations and the Peierls estimate were controlled by the factor $\exp\{-\frac{1}{2}\int(\nabla\phi)^2\}$. In our present problem, no gradient factor is present in the measure (1.1). As will be clear from our analysis, a direct proof of a mass gap turns out to be easier to obtain than to establish symmetry breaking (more details in [FHM]). We note that symmetry breaking has been proved recently by Kopper, Magnen and Rivasseau [KMR] in the two-dimensional Gross-Neveu model at large N and with a different choice of ultraviolet cutoff.

Before we turn to the original measure (1.1), we shall write the mass gap value in the mean field approximation. For this, we introduce the two-point function

$$\left\langle \overline{\psi}_x \cdot \psi_y \right\rangle_L = \frac{1}{Z_L} \int \prod_{x\in\Lambda} d\overline{\psi}_x d\psi_x \; e^{\mathcal{A}(\overline{\psi},\psi)} \overline{\psi}_x \cdot \psi_y \tag{1.13}$$

and denote by $\left\langle \overline{\psi}_x \cdot \psi_y \right\rangle$ its thermodynamic limit. The mass gap $\widehat{m}$ (inverse correlation length) is then defined by

$$\widehat{m} = \lim_{|x-y|\to\infty} \frac{-1}{|x-y|} \ln \left\langle \overline{\psi}_x \cdot \psi_y \right\rangle . \tag{1.14}$$

The mean field mass gap is obtained by setting $\phi_i = \varphi, \; i = 1, \ldots, L$, in the Matthews-Salam representation of $\langle \bar{\psi}_x \cdot \psi_x \rangle$.

Proposition 1.3 *The mean field mass gap $\widehat{m}_{MF}$, is given by*

$$\widehat{m}_{MF} = \frac{1}{2a} \ln \left[\zeta^2 + \left(1 + \zeta^4\right)^{1/2} \right] \tag{1.15}$$

1.3 The main results

Our first result controls the partition function of the model.

Theorem 1.4 *Given $\zeta \in (0, \infty)$, there exist $N_0 = N_0(\zeta)$ and $c_0 = c_0(\zeta) > 0$, such that, if $N > N_0$ the following holds*

$$Z_L = \left[c_0 \; e^{\nu N} \left(1 + \mathrm{O}(1/N)\right) \right]^L \left[1 + \mathrm{O}\left((\lambda_1^{(0)})^{-L} \right) \right] .$$

Here $\nu > 0$ is given by

$$\nu = \sup_{(x,y)\in I} f(x,y) - \ln \frac{e}{2\zeta^2} > 0 \tag{1.16}$$

where $\mathcal{I} = \{(z,z') \in [0,1]^2 : z + z' \leq 1\}$,

$$f(x,y) = g(x) + g(y) - (1-x-y)\ln(1-x-y) \tag{1.17}$$

with

$$g(z) = (1-z)\ln(1-z) - z\left(\ln z - \ln \frac{e}{2\zeta^2}\right)$$

and $\lambda_1^{(0)} = \lambda_1^{(0)}(\zeta)$ is an eigenvalue of an associated integral equation (4.16).

Remark: $f(x,y)$ is a concave symmetric function with maximum at $(\overline{x},\overline{x}) \in \mathcal{I}$ where

$$\overline{x} = \frac{1}{2}\left(1 + \frac{1}{\zeta^2}\right) - \frac{1}{2}\left(1 + \frac{1}{\zeta^4}\right)^{\frac{1}{2}} \tag{1.18}$$

varies from 0 to $\frac{1}{2}$ as ζ varies from ∞ to 0.

The positivity of ν leads to a free energy $F = \lim_{L\to\infty}(1/L)\ln Z_L = O(N)$. If symmetry breaking occurs, there is long range order (i.e. the correlation between ϕ_i and ϕ_j does not decay as $|i-j| \to \infty$) and the measure would attain to one of its maximums with $O(1)$ fluctuations around it. The next theorem shows exponential decay of the covariance, and implies the existence of a unique measure $\mu = \lim_{L\to\infty}\mu_L$ for the whole range of the coupling constant ζ.

Theorem 1.5 *For any $\zeta \in (0,\infty)$, there exists $N_0 = N_0(\zeta)$, such that, for $N > N_0$, the following bounds on the covariance (recall $\int d\mu_L(\phi)\phi_i \equiv 0$)*

$$C_1\ e^{-|i-j|/\ell} \leq \int d\mu(\phi)\phi_i\ \phi_j \leq C_2\ e^{-|i-j|/\ell}$$

holds with inverse correlation length $\ell^{-1} = c/N + O(1/N^2)$, $c = c(\zeta) > 0$ and $C_i = C_i(N,\zeta) < \infty$, $i = 1,2$.

We recall that the one-dimensional Ising model has exponential decay of the correlation at any temperature. In (1.1) ζ plays the role of the inverse temperature.

Theorem 1.6 *Under the conditions of Theorem 1.5, the following upper and lower bounds*

$$C_1 \ e^{-\widehat{m}|x-y|} \le \left\langle \overline{\psi}_x \cdot \psi_y \right\rangle \le C_2 \ e^{-\widehat{m}|x-y|}$$

holds with $\widehat{m} = \widehat{m}(\zeta) = \widehat{m}_{MF} + O(1/N)$.

The proof of Theorems 1.4, 1.5 and 1.6 is based on a formula which expresses $\det(\Gamma_L + (\zeta/\sqrt{N}) \ \Phi_L)$ as the generating function of a monomer-dimer problem (see e.g. [L]) with the monomer activity at site i proportional to the random field ϕ_i (Theorem 2.3). This provides us with a way to write the partition function Z_L as the trace of a $(N+1) \times (N+1)$ symmetric matrix T raised to the power L (for this simplification, we change to periodic boundary conditions). For small N, T can be explicitly diagonalized with its largest eigenvalue governing the behavior of expectation values in the measure μ_L. When N is large, T is approximated by an integral transfer matrix whose eigenvalues can be evaluated by the Laplace method.

The main results of this paper, Theorems 1.4, 1.5 and 1.6, give the sense in which the $1/N$-expansion is an expansion about the mean field theory.

This paper is organized as follows. In Section 2 we present our determinant expansion. We prove Propositions 1.1, 1.2 and 1.3 in Section 3. Theorem 1.4 will be proven in Section 4. The covariance of μ is studied in Section 5 where Theorem 1.5 is proven. The generation of a mass gap will be analyzed in Section 6.

2 The Monomer-Dimer Expansion

We start by writing the determinant at (1.1) in a power series of $\{\phi_j\}_{j=1}^{L}$. Let $\mathcal{L} = [1, L] \cap \mathbf{Z}$, L even, and set

$$F_L(\vartheta) = \det\left(\Gamma_L + \vartheta_L\right) \tag{2.1}$$

(later we shall set $\vartheta_i = (\zeta/\sqrt{N})\phi_i, \ i = 1, \ldots, L$). Since determinant is a multilinear function, the Taylor expansion of F_L is given by

$$F_L(\vartheta) = \sum_{Y \subseteq \mathcal{L}} \frac{\partial F_L}{\partial \vartheta_Y}(0) \ \vartheta_Y \tag{2.2}$$

where $\partial/\partial\vartheta_Y$ is shorthand for the multiple derivative $\prod_{i \in Y} \partial/\partial\vartheta_i$ and $\vartheta_Y = \prod_{i \in Y} \vartheta_i$.

To write F_L as a monomer-dimer problem, we introduce some notation. For a given $X \subseteq \mathcal{L}$, $\overline{X}$ denotes the subset of $[-1/2, L+1/2]$ obtained by replacing each $i \in X$ by the unit interval centred at site i. Let $(\overline{X}_1, \ldots, \overline{X}_n)$ be the decomposition of $\overline{X}$ into disjoint subsets and let $(X_1, \ldots, X_n)$ be the partition of X with $X_i = \overline{X}_i \cap \mathbf{Z}$. Calling this the regular partition of X, it is such that all sites in X_i are in consecutive order and $\mathrm{dist}(X_i, X_j) \geq 2$, $i \neq j$.

Lemma 2.1 *For $X \subseteq \mathcal{L}$, let $(X_1, \ldots, X_n)$ be the regular partition of X. If $Y = \mathcal{L} \backslash X \equiv X^c$, then*

$$\frac{\partial F_L}{\partial \vartheta_Y}(0) = \det \Gamma_{|X_1|} \ \det \Gamma_{|X_2|} \ \cdots \ \det \Gamma_{|X_n|} \tag{2.3}$$

is equal to 1 *provided $|X_i|$ is even for all $i = 1, \ldots, n$ and* 0 *otherwise.*

Proof. To prove (2.3) it suffices to show that

$$\frac{\partial F_L}{\partial \vartheta_i}(\vartheta) = F_{i-1}(\vartheta_{\{<i\}}) \ F_{L-i-1}(\vartheta_{\{>i\}}).$$

This follows by noting that $\partial F_L / \partial \vartheta_i = \det A_i$, where A_i is the cofactor matrix of the element ϑ_i. Lemma 2.1 is then proved since $\det \Gamma_k = 1$ if k is even and 0 otherwise.

□

We now consider objects in $\mathcal{L}$ called *dimers* which consist of a bond and its end points. To each non-overlapping collection of dimers in $\mathcal{L}$, the set X of sites covered by this collection is such that (2.3) is nonvanishing. Conversely, to each $X \in \mathcal{L}$ whose regular partition $(X_1, \ldots, X_n)$ is such that (2.3) is different from 0, there exists a unique dimer configuration with X being its covered set. This one-to-one correspondence leads us to the following definition:

Definition 2.2 *X is a compatible set of sites if it coincides with the covered set of a dimer configuration in $\mathcal{L}$.*

We have proven the following theorem:

Theorem 2.3

$$\det\left(\Gamma_L + (\zeta/\sqrt{N})\ \Phi_L\right) = \sum_{\text{compatible}X} \prod_{j\in X^c} \left(\zeta/\sqrt{N}\right)\ \phi_j \tag{2.4}$$

is the generating function of a stochastic monomer-dimer problem on $\mathcal{L}$ *whose monomer activity at site* i *is given by the Gaussian random variable* $(\zeta/\sqrt{N})\phi_i$.

The remainder of the paper is based on equation (2.4).

3 Mean Field and Peierls Estimate

Proof of Proposition 1.1: By setting $\phi_i = \varphi$, $i = 1, \ldots, L$ in equation (2.4), the effective potential (1.11) can be written as

$$V(\varphi) = N\left\{\frac{1}{2N}\varphi^2 - \lim_{L\to\infty}\frac{2}{L}\ln f_L\left((\zeta/\sqrt{N})\ \varphi\right)\right\} \tag{3.1}$$

where

$$f_L(z) = \sum_{\text{compatible}X} z^{|X^c|} \tag{3.2}$$

The critical points of (3.1) are of the form $\varphi^* = m\sqrt{N}/\zeta$ with m satisfying the self-consistency equation (1.10) which can be written as

$$\frac{m^2}{2\zeta^2} = \frac{1}{f_L(m)}\sum_X \frac{|X^c|}{L} m^{|X^c|}. \tag{3.3}$$

Since L is an even number, $f_L(z)$ is an even polynomial in z . This implies $V(-\varphi) = V(\varphi)$ and item (i) of Proposition 1.1.

In order to prove items (ii) and (iii), we shall derive the set of all solutions of (3.3) along with their respective stabilities.

Equation (3.3) can be written as

$$\frac{m^2}{2\zeta^2} = E_{L,m}(\mathrm{X}). \tag{3.4}$$

Here $E_{L,m}(\mathrm{X})$ is the mean of a discrete probability distribution on [0,1]:

$$\Pr_{L,m}(\mathrm{X} = l/L) = \frac{1}{f_L(m)}C_L(l)\ m^l \quad , \quad l = 0, 1, \ldots, L \tag{3.5}$$

and $C_L(l)$ is a combinatorial factor which accounts for the number of different ways l monomers can be located in a lattice of size L, occupied by $(L-l)/2$ dimers. Notice that $C_L(l) = 0$ if $L + l$ is odd.

$C_L(l)$ satisfies the recurrence relation

$$C_L(l) = C_{L-1}(l-1) + C_{L-2}(l)$$

since the left-most site is occupied either by a monomer or a dimer. Note also that

$$C_L(L) = 1.$$

It follows that

$$C_L(l) = \begin{cases} \binom{(L+l)/2}{l} & \text{if } L+l \text{ is even} \\ \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

¿From this one can see that

1. $E_{L,m}(\mathrm{X})$ increases monotonically from 0 to 1 as m increases from 0 to infinity;

2. for small m,
$$E_{L,m}(\mathrm{X}) = m^2 (L+2)/4 + O(m^4). \tag{3.7}$$

Therefore the self-consistency equation (1.10) has a unique solution if $\zeta^2 > 2/(L+2)$. Since the correction to the leading term in (3.7) is negative, one can show there is no solution if $\zeta^2 < 2/(L+2)$. This proves (ii).

To prove (iii) we use Stirling's formula for any $l \in \{1, \ldots, L\}$ to get:

$$\frac{1}{L} \ln C_L(l) = \frac{L+l}{2L} \ln \frac{L+l}{2L} - \frac{l}{L} \ln \frac{l}{L} - \frac{L-l}{2L} \ln \frac{L-l}{2L} + O\left(\frac{\ln L}{L}\right)$$

as $L \to \infty$.

Therefore

$$C_L(l)\; m^l = \exp\left\{L\, I_m(l/L) + O(\ln L)\right\}$$

where

$$I_m(x) = \frac{1+x}{2} \ln \frac{1+x}{2} - x \ln \frac{x}{m} - \frac{1-x}{2} \ln \frac{1-x}{2}.$$

is a concave function with maximum at $\widehat{x}$ determined by the equation

$$\frac{4\widehat{x}^2}{m^2\left(1-\widehat{x}^2\right)}=1. \tag{3.8}$$

The Laplace asymptotic method can now be used to approximate the probability distribution of X:

$$\begin{aligned} f_L(m)=\sum_{l=0}^{L} C_L(l)\ m^l \ &\sim\ \frac{L}{2}\int_{-\infty}^{\infty}\exp\left\{L\, I_m(x)\right\}dx \\ &\sim\ \left[\frac{\pi L}{2\left|I''(\widehat{x})\right|}\right]^{1/2}\exp\left\{L\, I_m(\widehat{x})\right\} \end{aligned} \tag{3.9}$$

and

$$E_{L,m}(\mathrm{X})=\frac{1}{f_L(m)}\sum_{l=0}^{L}\frac{l}{L}C_L(l)\ m^l\sim\widehat{x}.$$

In these formulas, the fractional error is $O(1/L)$.

The self-consistency equation (3.4) can thus be written as

$$\frac{m^2}{2\zeta^2}=\left(\frac{m^2}{4+m^2}\right)^{\frac{1}{2}} \tag{3.10}$$

which has one unstable solution $m_0=0$ and two stable solutions

$$m_{\pm}=\pm\left[2\left(1+\zeta^4\right)^{1/2}-2\right]^{1/2}. \tag{3.11}$$

The profile of $V(\varphi)$, in the limit as $L\to\infty$, follows from the expression (see eq. (3.1))

$$\lim_{L\to\infty}\frac{1}{L}\ln f_L\left(\left(\zeta/\sqrt{N}\right)\ \varphi\right)=\sup_{x\in[0,1]} I_{(\zeta/\sqrt{N})\varphi}(x)=\frac{1}{2}\ln\frac{1+\widehat{x}(\varphi)}{1-\widehat{x}(\varphi)} \tag{3.12}$$

where $\widehat{x}(\varphi)=\left(1+\frac{4N}{\zeta^2\varphi^2}\right)^{-1/2}$. This diverges logarithmically as $\varphi\to\pm\infty$ leading to a parabolic growth of $V(\varphi)$ for large φ. For small φ, the logarithm dominates the quadratic term and $V(\varphi)/N\simeq-\widehat{x}(\varphi)\simeq-(\zeta/2\sqrt{N})\,|\varphi|$

approaches 0 from negative values. $V(\varphi)$ attains to its minimum value at $\pm\varphi^* = m_\pm\sqrt{N}/\zeta$ which, by (3.11), behaves as

$$\varphi^* \simeq \begin{cases} \sqrt{N}\zeta & \text{for } \zeta \text{ small} \\ \sqrt{2N} & \text{for } \zeta \text{ large} \end{cases} .$$

This implies

$$V(\varphi^*) \simeq \begin{cases} -N\zeta^2/2 & \text{for } \zeta \text{ small} \\ -N(\ln 2\zeta^2 - 1) & \text{for } \zeta \text{ large} \end{cases} \tag{3.13}$$

which concludes the proof of Proposition 1.1.

Proof of Proposition 1.2: Each Ising-like configuration, $\{\sigma_i\}_{i=1}^L$, determines a sequence $\{\ell_i\}_{i=1}^{p+1}$, of integers $\ell_i > 0$ with $\sum_{i=1}^{p+1} \ell_i = L$, corresponding to the number of sites between two successive flips (ℓ_1 and ℓ_{p+1} being the number of sites from the boundary until closest flip).

¿From the equation (2.4)

$$f_L(\sigma) = \sum_{\text{compatible}X} \prod_{i\in X^c} \sigma_i \tag{3.14}$$

For each compatible set X we distinguish dimers, whose covering set is X, according to whether they occupy flip positions or not. If the latter dimers are summed first, (3.14) can be written as

$$f_L(\sigma) = \sum_{\mathbf{n}} f_{\ell_1-n_1}(m_+)\, f_{\ell_2-n_2-n_1}(m_-)\, \cdots\, f_{\ell_p-n_p-n_{p-1}}(m_-)\, f_{\ell_{p+1}-n_p}(m_+) \tag{3.15}$$

where the summation is over all p-tuples $\mathbf{n} = (n_1, \ldots, n_p) \in \{0,1\}^p$ with $n_j = 1$ if the position of the j-th flip is occupied by a dimer and 0 otherwise. Notice that

$$f_k(m_-) = (-1)^k\, f_k(m_+). \tag{3.16}$$

For simplicity we assume that $\ell_i \geq \ell_0,\ i = 1, \ldots, p$, for some ℓ_0 so large that

$$\frac{f_{\ell_0-n}(m)}{f_{\ell_0}(m)} \sim \left(\frac{2\widehat{x}}{m(1+\widehat{x})}\right)^n \tag{3.17}$$

with $\widehat{x}$ given by (3.8). Equation (3.17) can be obtained from (3.9) by iteratively solving the equation

$$f_\ell(m) = m\ f_{\ell-1}(m) + f_{\ell-2}(m)$$

for $\Delta_\ell = f_\ell(m)/f_{\ell-1}(m)$.

In view of (3.16), (3.17) and (3.8), (3.15) can be written as

$$\begin{aligned} f_L(\sigma) &\sim (-1)^{\ell_2+\ell_4+\cdots+\ell_p} \left\{ \sum_{n=0,1} \left(\frac{-4\widehat{x}^2}{m^2(1+\widehat{x})^2} \right)^n \right\}^p \\ &\sim (-1)^{\ell_2+\ell_4+\cdots+\ell_p} \left(1 - \frac{1-\widehat{x}}{1+\widehat{x}}\right)^p . \end{aligned} \tag{3.18}$$

Similarly,

$$f_L(m) \sim \left(1 + \frac{1-\widehat{x}}{1+\widehat{x}}\right)^p . \tag{3.19}$$

Equations (3.18) and (3.19) lead to

$$\left| \frac{f_L(\sigma)}{f_L(m)} \right| \sim \widehat{x}^p \tag{3.20}$$

with $\widehat{x}$, from (3.8) and (3.11), strictly less than 1 for any $0 < \zeta < \infty$. This proves Proposition 1.2 when the distances between flips are large enough. With a little more effort this condition can be removed and (3.20) obtained with $\widehat{x}$ replaced by $\delta = \delta(m) < 1$.

□

Proof of Proposition 1.3: We start by writing the two-point function (1.13) in the Matthews-Salam representation. Using (1.5)-(1.7) with (1.6) replaced by

$$\int \prod_{z\in\Lambda} d\overline{\psi}_z d\psi_z\ \overline{\psi}_x \cdot \psi_y\ \exp\left\{ \sum_{z\in\Lambda} \overline{\psi}_z \cdot \left[\left(ia\mathbf{D} + (\zeta/\sqrt{N})\ \mathbf{\Phi} \right) \psi \right]_z \right\}$$

$$= 2N \left(\Gamma_L + (\zeta/\sqrt{N})\ \Phi_L\right)^{-1}_{kl} \det\left(\Gamma_L + (\zeta/\sqrt{N})\ \Phi_L\right)^{2N} \quad ;\ k = x/a,\ l = y/a \tag{3.21}$$

and recalling (1.1) and (1.3), the two-point function can be written as

$$\left\langle \overline{\psi}_x \cdot \psi_y \right\rangle_L = 2N \int \prod_{i=1}^{L} d\phi_i \; g_L(\phi) \; \left(\Gamma_L + (\zeta/\sqrt{N})\Phi_L \right)_{kl}^{-1} \tag{3.22}$$

(We have used the evenness of L in (3.21) to write $(-\Gamma_L + \vartheta_L)_{kl}^{-1} = (\Gamma_L + \vartheta_L)_{kl}^{-1}$).

In the mean field approximation $\phi_i = \varphi, \; i = 1, \ldots, L$ and the density g_L is replaced by h_L. The mean field two-point function is then given by

$$\left\langle \overline{\psi}_x \cdot \psi_y \right\rangle_L = 2N \int d\varphi \; h_L(\varphi) \; \left(\Gamma_L + (\zeta/\sqrt{N}) \; \varphi 1_L \right)_{kl}^{-1}$$

which can be written, from (1.12), as

$$\left\langle \overline{\psi}_x \cdot \psi_y \right\rangle_L = 2N \; \left(\Gamma_L + m 1_L \right)_{kl}^{-1} + O\left(1/L\right) \tag{3.23}$$

where $m = m_+$ is given by (3.11) .

To avoid boundary influences, it is convenient to replace $\mathcal{L}$ by $\mathcal{L}' = [-L/2 + 1, L/2] \cap \mathbf{Z}$. The inverse matrix in (3.23) can be readily evaluated by the method of cofactors. It is given by

$$\left(\Gamma_L + m 1_L \right)_{kl}^{-1} = (-1)^{k+l} \; \frac{f_{L/2+k-1}(m) \; f_{L/2-l-1}(m)}{f_L(m)} \tag{3.24}$$

We now use Laplace asymptotic method to evaluate the right hand side of (3.24). In the limit as $L \to \infty$, expression (3.9) can be used to obtain the following behavior

$$\begin{aligned} \left(\Gamma_L + m 1_L \right)_{kl}^{-1} &= (-1)^{k+l} \; e^{-|l-k| I_m(\widehat{x})} + O\left(1/L\right) \\ &= (-1)^{k+l} \left(\frac{1-\widehat{x}}{1+\widehat{x}} \right)^{|l-k|/2} + O\left(1/L\right) \end{aligned}$$

which, by using (3.8) and (3.11), gives

$$\left(\Gamma_L + m 1_L \right)_{kl}^{-1} = (-1)^{k+l} \left[(1+\zeta^4)^{1/2} - \zeta^2 \right]^{|l-k|/2} + O\left(1/L\right) \tag{3.25}$$

The proof of Proposition 1.3 follows from (3.23) and (3.25).

$\square$

4 Proof of Theorem 1.4

Asymptotic analysis and a transfer matrix technique will now be combined to evaluate the partition function Z_L.

Let $\mu_{L,0}$ be the free measure given by (1.1) with $\zeta = 0$ and notice that $\mu_{L,0}(\phi) = \prod_j \mu_0(\phi_j)$ is a product measure with

$$d\mu_0(\phi) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\phi^2}\, d\phi$$

(recall $\det \Gamma_L = 1$).

Inserting the determinant expansion (2.4) into (1.3), we have

$$Z_L = \sum_{\mathbf{X}} \int d\mu_{L,0}(\phi) \prod_{i=1}^{2N} \left\{ \prod_{y \in X_i^c} \left(\zeta \big/ \sqrt{N}\right) \phi_y \right\} \tag{4.1}$$

where $\mathbf{X} = (X_1, \ldots, X_{2N})$ runs over all collections of $2N$ compatible sets.

Absence of long range order implies insensitivity to boundary conditions (this will be established in the next section). For simplicity, we change to periodic boundary conditions and wrap $\mathcal{L}$ into the torus $\mathbf{Z}/L\mathbf{Z}$.

Performing the Gaussian integrals in (4.1), this equation can be written as

$$\begin{aligned} Z_L = \sum_{\mathbf{n} \in \mathcal{N}_L} & \binom{2\,(N - n_L)}{2n_1} \binom{2(N - n_1)}{2n_2} \cdots \binom{2(N - n_{L-1})}{2n_L} \times \\ & \alpha(2(N - n_1 - n_L))\alpha(2(N - n_2 - n_1)) \, \ldots \, \alpha(2(N - n_L - n_{L-1})) \end{aligned} \tag{4.2}$$

where

$$\mathcal{N}_L = \left\{ \mathbf{n} \in \{0, 1, \ldots, N\}^L : 0 \le n_{k+1} - n_k \le N,\ k = 1, \ldots .L, n_{L+1} = n_1 \right\}$$

and

$$\alpha(k) = \int d\mu_0(\phi) \left(\frac{\zeta}{\sqrt{N}} \phi \right)^k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \left(\dfrac{\zeta^2}{2N} \right)^{k/2} \dfrac{k!}{(k/2)!} & \text{if } k \text{ is even} \end{cases} \tag{4.3}$$

In the expression (4.2), the occupation numbers $\mathbf{n} = (n_1, \ldots, n_L)$ count how many pairs of copies of each dimer there are in the collection $\{X_i\}$. To each configuration $\mathbf{n}$ a weight is associated due to the integration (4.3) at each site (the second line in (4.2)). The sum over $\mathbf{X}$ with $\mathbf{n}$ held fixed is counted by the binomial factors.

Expression (4.2) can be symmetrized and written in the following compact form

$$Z_L = \alpha^L \ \mathrm{Tr}\left(\mathrm{T^L}\right) \tag{4.4}$$

where $\alpha = \alpha(2N)$ and T is a $(N+1)\times(N+1)$ symmetric matrix whose elements are given by

$$T(n,n') = S(n)R(n,n')S(n') \tag{4.5}$$

with

$$S(n) = \left(\frac{2N^2}{e\zeta^2}\right)^n \left(\frac{(2(N-n))!}{(2N)!(2n)!}\right)^{1/2}$$

and

$$R(n,n') = \left(\frac{e}{N}\right)^{n+n'} \frac{N!}{(N-n-n')!} \ \chi(n+n'\le N)$$

where $\chi(P)=1$ if the relation P is satisfied and 0 otherwise.

Theorem 1.4 will follow from the usual transfer matrix method: let $\{\lambda_i\}_{j=0}^N$ be the eigenvalues of T and let us assume for a while $\lambda_0 > \overline{\lambda} \equiv \sup_{j\ge1}|\lambda_j|$. The partition function can thus be written as

$$Z_L = \alpha^L \sum_{i=0}^{N} \lambda_i^L = (\alpha\,\lambda_0)^L \left[1 + O\left(\left(\overline{\lambda}/\lambda_0\right)^L\right)\right] \tag{4.6}$$

The following result will be proven by using asymptotic analysis. For convenience in section 5, this will be stated in a form stronger than necessary for this section.

Lemma 4.1 *For any $0<\zeta<\infty$, there exist $N_0 = N_0(\zeta)$ and $c_0 = c_0(\zeta)$ such that, for $N > N_0$, the eigenvalues of T can be written in the following form*

$$\lambda_j = c_0\, e^{N\overline{f}} \ \left(\lambda_j^{(0)} + \frac{1}{N}\lambda_j^{(1)} + \mathrm{O}\left(\frac{1}{N^{3/2}}\right)\right) \ , \qquad j = 0,1,\ldots,n_0 \tag{4.7}$$

with $n_0 = n_0(\zeta,N) = \mathrm{O}(\ln N)$. Here $\overline{f} \equiv \max_{(x,y)\in I} f(x,y) = f(\overline{x},\overline{x})$ is defined in the Theorem 1.4. The quantities $\lambda_0^{(0)} = 1$ and $\lambda_j^{(0)}$, $j=1,\ldots,N$, are determined by an associated integral equation (4.16). Moreover $\lambda_0 > |\lambda_1| > \ldots > |\lambda_{n_0}| > \sup_{n>n_0}|\lambda_n|$.

In view of (4.3) and Lemma 4.1, (4.6) can be written as

$$Z_L = \left[c_0\, e^{\nu N}\left(1 + O(1/N)\right)\right]^L \left[1 + O\left(\left(\lambda_1^{(0)}\right)^{-L}\right)\right]$$

with

$$\nu = f(\overline{x}, \overline{x}) - \ln\frac{e}{2\zeta^2} = \ln\frac{(1-\overline{x})}{\overline{x}} > 0$$

for any $0 < \zeta < \infty$ (recall $0 < \overline{x} < 1/2$ by (1.18)). This concludes the proof of Theorem 1.4.

$\square$

Proof of Lemma 4.1: To apply Stirling's formula to expression (4.5) it is convenient to introduce a set $\mathcal{I}_{N,M} \in (1/N\ \mathbf{Z})^2$ given by

$$\mathcal{I}_{N,M} = \left\{(z, z') \in [1/M, 1]^2 \cap (1/N\ \mathbf{Z})^2 : z + z' \le 1 - 1/M\right\},$$

where M will be taken depending on N. We set $\mathcal{I}_N = \mathcal{I}_{N,\infty}$ and notice that $\mathcal{I}_{N,M}$ excludes points closer $1/M$ to the border of $\mathcal{I}_N$.

Writing $(n, n') = (Nx, Ny)$, M can be chosen such that if $(x, y) \in \mathcal{I}_{N,M}$,

$$K! = \sqrt{2\pi K}\ e^{-K}\, K^K \left(1 + \frac{1}{12K} + O\left(\frac{1}{N^2}\right)\right) \tag{4.8}$$

with K being any of the functions Nx, Ny or $N(1-x-y)$. The transfer matrix (4.5) can thus be written as

$$T(Nx, Ny) = \frac{1}{\sqrt{4\pi N}} L(x, y)\ \exp\{N f(x, y)\}\left(1 + \frac{l(x, y)}{N} + O\left(\frac{1}{N^2}\right)\right) \tag{4.9}$$

where f is given by (1.17),

$$L(x, y) = \left(\frac{1-x}{x}\right)^{\frac{1}{4}}\left(\frac{1}{1-x-y}\right)^{\frac{1}{2}}\left(\frac{1-y}{y}\right)^{\frac{1}{4}} \tag{4.10}$$

and

$$l(x, y) = \frac{1}{12}\left[\frac{1}{4x(1-x)} + \frac{1}{1-x-y} + \frac{1}{4y(1-y)}\right] \tag{4.11}$$

When $N \to \infty$, T can be approximated by an integral transfer matrix. The next proposition gives us the properties of $f(x, y)$ which will be needed.

Proposition 4.2 *Let f be given by (1.17). It follows that*

(i) *the critical points of f lie in the circle with centre $((1+\zeta^{-2})/2\,,(1+\zeta^{-2})/2)$ and radius $((1+\zeta^{-4})/2)^{1/2}$;*

(ii) *f is a symmetric function and concave on the domain $\mathcal{I} \equiv \mathcal{I}_\infty$. It attains to a maximum at $(\overline{x},\overline{x}) \in \mathcal{I}$, where $\overline{x}$ is the solution (1.18) of*

$$2\zeta^2\,\frac{\overline{x}(1-\overline{x})}{1-2\overline{x}} = 1 \tag{4.12}$$

which is given asymptotically by

$$\overline{x} \sim \left\{ \begin{array}{ll} \frac{1}{2\zeta^2}\left(1-\frac{1}{2\zeta^2}\right) & \zeta \gg 1 \\ \frac{1}{2}\left(1-\frac{\zeta^2}{2}\right) & \zeta \ll 1 \end{array} \right.$$

(Notice that $(\overline{x},\overline{x}) \in \mathcal{I}_{N,M}$ provided $2/M < \zeta^2 < M/2$);

(iii) *the Hessian of f at (x,y) is given by*

$$\mathbf{h}(x,y) = \frac{-1}{1-x-y}\begin{pmatrix} 1+\frac{(1-2x)(1-x-y)}{x(1-x)} & 1 \\ 1 & 1+\frac{(1-2y)(1-x-y)}{y(1-y)} \end{pmatrix} \tag{4.13}$$

Proof: Items (i) and (iii) follow by simple calculation. It follows from (4.13) that f is concave if and only if

$$(1-2x)y(1-y)+(1-2y)x(1-x)+(1-2x)(1-2y)(1-x-y) > 0 \tag{4.14}$$

holds. This condition is fulfilled for all $(x,y) \in \mathcal{I}$ since, by reorganizing (4.14), it can be written as

$$(1-x-y+2xy)(1-x-y).$$

This shows that $\mathbf{h}$ is a negative definite matrix, proving item (ii) and Proposition 4.2.

□

To facilitate approximating the discrete operator $T : \mathbf{R}^{N+1} \to \mathbf{R}^{N+1}$ by a continuum operator $\mathcal{T} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$, we introduce the following injection of the vector space $\mathbf{R}^{N+1}$ into piecewise constant functions in the vector space $L^2(\mathbf{R})$. Define intervals

$$J_n = \left(\frac{n-1/2}{\sqrt{N}} - \sqrt{N}\overline{x}, \frac{n+1/2}{\sqrt{N}} - \sqrt{N}\overline{x}\right] \tag{4.15}$$

for $n = 0, 1, \ldots, N$ and then let

$$(i(u))(z) = \begin{cases} N^{1/4}u(n) & z \in J_n \\ 0 & z \notin \bigcup_n J_n \end{cases}$$

The adjoint $i^\dagger$ is given by

$$\left(i^\dagger(v)\right)(n) = N^{1/4}\int_{J_n} v(x)\ dx$$

Note that $i^\dagger i$ is the identity matrix.

The operator kernel of $iTi^\dagger$ is a step function with step-width $1/\sqrt{N}$, which approximates

$$c_0 e^{N\bar{f}}\mathcal{T}(x,y)$$

where the integral operator $\mathcal{T}$ is defined by

$$\mathcal{T}v(z) = \sqrt{\frac{\rho}{2\pi}}\int_{-\infty}^{\infty} e^{\frac{1}{2}q(z,z')}\ v(z')\ dz'. \tag{4.16}$$

Here q is the (negative definite) quadratic form defined by the Hessian matrix (4.13) at the critical point, $\overline{\mathbf{h}} \equiv \mathbf{h}(\overline{x}, \overline{x})$,

$$c_0 = \frac{1}{\sqrt{2\rho}}L(\overline{x}, \overline{x}) = \frac{1}{2\rho^{1/2}\zeta\overline{x}}.$$

and

$$\begin{aligned}\rho &= -\overline{\mathbf{h}}_{11} + \sqrt{\det\overline{\mathbf{h}}} \\ &= \frac{\zeta^2}{(1+\zeta^4)^{1/2}-1}\left\{3 - 2(1+\zeta^4)^{1/2}\left[1 + (1-(1+\zeta^4)^{-1/2})^{1/2}\right]\right\}\end{aligned}$$

are convenient constants.

The following proposition is a well known result (see e.g. [CE]):

Proposition 4.3 *$\mathcal{T}$ is a self-adjoint compact operator in $L^2(\mathbf{R})$. The spectrum of $\mathcal{T}$ consists of the sequence $\{\lambda_j^{(0)} \equiv (\overline{\mathbf{h}}_{12}/\rho)^j\}_{j=0}^{\infty}$ and the corresponding eigenfunctions,*

$$\psi_j(z) = \exp\left\{-\frac{1}{2}\sqrt{\det \overline{\mathbf{h}}}\; z^2\right\} H_j(\gamma z) \quad , \quad j = 0, 1, \dots \tag{4.17}$$

where

$$\gamma^2 = \frac{\rho}{2}\left(1 - \left(\overline{\mathbf{h}}_{12}/\rho\right)^2\right) = \sqrt{\det \overline{\mathbf{h}}}.$$

and H_j is the jth normalized Hermite polynomial, are orthonormal with respect to the inner product $\langle \,\cdot\,,\,\cdot\,\rangle$ in $L^2(\mathbf{R}, dx)$ with weight 1.

To prove that the discrete eigenvalue problem is actually well approximated by the continuum problem, we shall apply the following version of Temple's inequality (for proof see Problem 15, p. 365 of ref. [RS]):

Proposition 4.4 *Let A be a symmetric $(N+1)\times(N+1)$ matrix. Let $I_j = (a_j, b_j) \subset \mathbf{R}$ and $\varphi_j \in \mathbf{R}^{N+1}$, $j = 0, 1, \dots, N$, be open intervals and normalized trial vectors such that*

$$\sigma(A) \cap I_j = \{\mu_j\}$$

and

$$\eta_j < (\gamma_j - a_j)\,(b_j - \gamma_j)$$

where $\gamma_j = (\varphi_j, A\varphi_j)$ and $\eta_j = (\varphi_j, (A-\gamma_j)^2\varphi_j)$. Then the following upper and lower bounds hold

$$\gamma_j - \frac{\eta_j}{(b_j - \gamma_j)} < \mu_j < \gamma_j + \frac{\eta_j}{(\gamma_j - a_j)}. \tag{4.18}$$

First, we shall apply this result to the matrix $A = c_0^{-1} e^{-N\overline{f}} T$ to show that the largest eigenvalues are given by

$$\mu_j = \lambda_j^{(0)} + O(1/\sqrt{N}) \quad , \quad j = 0, 1, \dots, n_0 \tag{4.19}$$

with $n_0 = n_0(N) = O(\ln N)$. We take trial vectors φ_j given in terms of the approximating eigenfunctions ψ_j (4.17):

$$\varphi_j = c_j i^{\dagger} \psi_j \tag{4.20}$$

with normalization $c_j^{-2} = \left(i^\dagger\psi_j, i^\dagger\psi_j\right) = \left\langle\psi_j, ii^\dagger\psi_j\right\rangle$.

The trial intervals are given by $a_j = \lambda_j^{(0)} - O(1/\sqrt{N})$, $b_j = \lambda_j^{(0)} + O(1/\sqrt{N})$. We must show each interval contains exactly one eigenvalue of A. The variances η_j must be shown to be $O(1/N)$. For this, we need the basic bounds:

Lemma 4.5 *1. The operator norm of the error $\Delta T = iAi^\dagger - \mathcal{T}$ is bounded by*

$$\|\Delta T\| = \mathrm{O}(1/\sqrt{N}).$$

2. For each eigenfunction ψ_j, $\quad j = 0, 1, \ldots, n_0$, we have

$$\|(1 - ii^\dagger)\psi_j\| \leq \mathrm{O}(1/\sqrt{N})$$

Proof: For 1., we need only estimate the L^2-norm of the kernel $\Delta T(x, y)$: this can be shown to be $O(1/\sqrt{N})$. The proof of 2. is similar, using properties of the Hermite polynomials.

□

Using these ingredients, we can verify the hypotheses of Proposition 4.4. For example,

$$\begin{aligned} (\varphi_j, A\varphi_j) &= c_j^2 \left\langle\psi_j, i\, A\, i^\dagger\psi_j\right\rangle \\ &= c_j^2 \left\langle\psi_j, \mathcal{T}\,\psi_j\right\rangle + O(1/\sqrt{N}) \\ &= \lambda_j^{(0)} + O(1/\sqrt{N}) \end{aligned} \tag{4.21}$$

since $c_j^2 = 1 + O(1/\sqrt{N})$ due to 2. in Lemma 4.5.

To show that the intervals I_j each contain exactly one eigenvalue, we need to verify that the remainder of the spectrum of A lays inside $(-|\lambda_{n_0}^{(0)}|, \lambda_{n_0}^{(0)}|)$. This is done by showing

$$|(\varphi, A\varphi)| \leq \left|\lambda_{n_0}^{(0)}\right| + O(1/\sqrt{N})$$

whenever φ is orthogonal to $\varphi_0, \ldots, \varphi_{n_0}$. Notice that n_0 is such that $\Delta\lambda_{n_0} = O(1/\sqrt{N})$ with

$$\Delta\lambda_j = \inf_{i\neq j} \mathrm{dist}(\lambda_{\mathrm{j}}^{(0)}, \lambda_{\mathrm{i}}^{(0)}) = \left(\frac{-\overline{\mathbf{h}}_{12}}{\rho}\right)^{\mathrm{j}} \left(1 - \left(\frac{\overline{\mathbf{h}}_{12}}{\rho}\right)^2\right).$$

Temple's inequality allows us to improve these eigenvalue estimates to any order in $1/N$. This is done by improving the approximate eigenfunctions, using Stirling's formula and the Laplace asymptotic method. One also needs the following result from numerical analysis [H] to estimate the error in replacing Riemann sums by Riemann integrals:

$$\left| \int_a^b f(y)\ dy - \frac{1}{N} \sum_{y \in [a,b] \cap \frac{1}{\sqrt{N}} \mathbf{Z}} f(y) \right| \leq \frac{(b-a)\ \|f\|_{\mathcal{C}^k}}{N^{k/2}}$$
$$+ \sup_{a - \frac{(b-a)k}{\sqrt{N}} < y < a} (b-a)\ |f(y)| + \sup_{b < y < b + \frac{(b-a)k}{\sqrt{N}}} (b-a)\ |f(y)| \quad (4.22)$$

with $||f||_{\mathcal{C}^k} = \sum_{n=0}^k \sup_{y \in [a,b]} |d^n f / dy^n(y)|$. This formula can be applied to the functions $\mathcal{T}(x,y)$ and $\psi_j(x)$ for k chosen arbitrarily large.

The $1/N$ corrections to the eigenvalues arise from two sources. Firstly, the Stirling's formula gives a factor

$$\bar{l} = l(\overline{x}, \overline{x}) = \frac{1+\zeta^2}{12(1-2\overline{x})}$$

due to (4.11). Secondly, following Laplace's method, the first and second order terms in the Taylor expansion of $L(x,y)$ about $(\overline{x}, \overline{x})$ also may produce $1/N$ contributions. Writing

$$L(\overline{x} + z/\sqrt{N}, \overline{x} + z'/\sqrt{N}) - L(\overline{x}, \overline{x}) = \frac{1}{\sqrt{N}} L'(\overline{x}, \overline{x})\ (z + z') + \frac{1}{2N} Q(z, z') + S(z, z') \quad (4.23)$$

where Q is the quadratic form obtained by the Hessian of L at $(\overline{x}, \overline{x})$ and S is the Taylor remainder, the next approximation $\mathcal{T}^{(1)}$ to the matrix A in the eigenvalue equation $A\varphi = \mu\,\varphi$, can be written as

$$\mathcal{T}^{(1)}\psi = (1 - \bar{l}/N)\mathcal{T}\psi + (1/\sqrt{N})\mathcal{V}_1\psi + (1/N)\mathcal{V}_2\psi + O(1/N^{3/2}) \quad (4.24)$$

where $\mathcal{V}_i$, $i = 1, 2$, are integral operators defined by

$$\mathcal{V}_i u(z) = \frac{1}{2\sqrt{\pi} c_0} \int_{-\infty}^{\infty} K_i(z, z')\ e^{\frac{1}{2} q(z,z')}\ u(z')\ dz' \quad (4.25)$$

with $K_1(z, z') = L'(\overline{x}, \overline{x})\ (z + z')$ and $K_2(z, z') = (1/2)Q(z, z')$.

This yields, by standard perturbation theory,

$$\lambda_j^{(1)} = \langle \psi_j, \mathcal{V}_2 \psi_j \rangle - \bar{l}\lambda_j^{(0)} + \sum_{i \neq j} \frac{\langle \psi_i, \mathcal{V}_1 \psi_j \rangle^2}{\lambda_j^{(0)} - \lambda_i^{(0)}} \tag{4.26}$$

with ψ_j given by (4.17).

Notice that there is no $O(1/N^{1/2})$ contribution to the eigenvalues since $< \psi_j, \mathcal{V}_1 \psi_j >$ vanishes for all j.

This concludes the proof of Lemma 4.1.

□

5 Proof of Theorem 1.5

We here extend the procedure used for treating the partition function Z_L to evaluate the covariance of μ_L.

Let us begin by defining

$$\int d\mu_L(\phi)\ \phi_i \phi_j \equiv \frac{N}{\zeta^2} \frac{Z_L^{ij}}{Z_L}. \tag{5.1}$$

where

$$Z_L^{ij} = \int d\mu_{L,0}(\phi)\ \det\left(\Gamma_L + \left(\zeta/\sqrt{N}\right)\phi\right)\ (\zeta/\sqrt{N})\phi_i\ (\zeta/\sqrt{N})\phi_j \quad ; \quad i,j \in \mathcal{L} \tag{5.2}$$

is a modified partition function.

Following the procedure in Section 4, (5.2) can be written as

$$Z_L^{ij} = \sum_{\mathbf{X}} \int d\mu_{L,0}(\phi) \prod_{i=1}^{2N} \left[\prod_{z \in X_i^c} \left(\zeta/\sqrt{N}\right) \phi_z \right] (\zeta/\sqrt{N})\phi_i\ (\zeta/\sqrt{N})\phi_j \tag{5.3}$$

with $\mathbf{X} = (X_1, \ldots, X_{2N})$ running, as before, over all collections of $2N$ compatible sets.

Performing the Gaussian integral and after some combinatorics, (5.3) can be written as

$$Z_L^{ij} = \alpha^L\ \mathrm{Tr}\left(\overline{\mathrm{S}}\,\overline{\mathrm{T}}^{|\mathrm{j}-\mathrm{i}|+1}\overline{\mathrm{S}}\,\mathrm{T}^{\mathrm{L}-|\mathrm{j}-\mathrm{i}|-1}\right) \tag{5.4}$$

where T is given by (4.5),

$$\overline{T}(n,n') = T(n,n')\ \Delta(n,n') \tag{5.5}$$

with

$$\Delta(n,n') = \left(\frac{n(2(N-n)+1)}{N^2}\right)^{1/2} \frac{\zeta^2 N}{N-n-n'+1} \left(\frac{n'(2(N-n')+1)}{N^2}\right)^{1/2} \tag{5.6}$$

and

$$\overline{S}(n,n') = \zeta^2 \left(\frac{2n(2(N-n)+1)}{N^2}\right)^{1/2} \delta_{n,n'}$$

is a boundary term arising from the symmetrization of $\overline{T}$.

Notice that, by writing $\Delta(Nx,Ny) = 1 + (1/N)\delta(x,y)$ and using (4.12), we have

$$\delta(\overline{x},\overline{x}) = \frac{1}{2(1-\overline{x})(1-2\overline{x})} + O(1/N) \equiv \overline{\delta} + O(1/N) \tag{5.7}$$

The leading contribution to the equation (5.4) can be obtained using the transfer matrix method . Suppose that the eigenvalues $\{\overline{\lambda}_j\}_{j=0}^N$ of $\overline{T}$ is such that $\lambda_0 > \overline{\lambda}_0 > \overline{\lambda}_j$, $j = 1,\dots,N$. Denoting by $\overline{e}_0$ and e_0 the eigenvectors associated to the largest eigenvalue of $\overline{T}$ and T, respectively, let us assume that $\left(\overline{e}_0, \overline{S}\, e_0\right) \neq 0$. Then, the modified partition function (5.4) can be written as

$$Z_L^{ij} = (\alpha\lambda_0)^L \left|\left(\overline{e}_0, \overline{S}\, e_0\right)\right|^2 \left(\frac{\overline{\lambda}_0}{\lambda_0}\right)^{|j-i|+1} (1+r) \tag{5.8}$$

where

$$r = O\left(\left(\overline{\lambda}_1/\lambda_0\right)^{|j-i|}\right) + O\left((\lambda_1/\lambda_0)^{L-|j-i|}\right)$$

In the limit as $L \to \infty$, the behavior of the covariance is determined by

$$\frac{Z_L^{ij}}{Z_L} = \left|\left(\overline{e}_0, \overline{S}\, e_0\right)\right|^2 \left(\frac{\overline{\lambda}_0}{\lambda_0}\right)^{|j-i|+1} \left[1 + O\left((\overline{\lambda}_1/\lambda_0)^{|j-i|}\right)\right] \tag{5.9}$$

Equation (5.9) implies that $\ell^{-1} = \ln\left(\lambda_0/\overline{\lambda}_0\right)$ where ℓ denotes the correlation length given by

$$\ell^{-1} = \lim_{|j-i|\to\infty} \frac{-1}{|j-i|} \ln \int d\mu(\phi)\ \phi_i\phi_j \tag{5.10}$$

(recall $\mu(\phi) = \lim_{L\to\infty} \mu_L(\phi)$).

To conclude the proof of Theorem 1.5, we have:

Lemma 5.1 *For any $0 < \zeta < \infty$, there exist $N_0 = N_0(\zeta)$ and $c = c(\zeta) > 0$ such that, for $N > N_0$, the correlation length ℓ has the following form*

$$\ell^{-1} = \frac{c}{N} + \mathrm{O}\left(1/N^{3/2}\right).$$

□

Proof of Lemma 5.1: It follows from the Laplace's method developed in the previous section that, under the conditions of Lemma 4.1, the eigenvalues of $\overline{T}$ admit the following asymptotic expansion

$$\overline{\lambda}_j = c_0\, e^{N\overline{f}} \left(\overline{\lambda}_j^{(0)} + \overline{\lambda}_j^{(1)} \frac{1}{N} + O\left(1/N^{3/2}\right)\right) \;, \qquad j = 0, 1, \ldots, n_0 \tag{5.11}$$

with $\overline{\lambda}_0 > \left|\overline{\lambda}_1\right| > \ldots > \left|\overline{\lambda}_{n_0}\right| > \sup_{n>n_0} \left|\overline{\lambda}_n\right|$.

Following the procedure of section 4, the matrix $\overline{A} = c_0^{-1} e^{-N\overline{f}}\overline{T}$, in the eigenvalue equation $\overline{A}\varphi = \overline{\mu}\varphi$, can be approximate by an integral operator

$$\overline{\mathcal{T}}^{(1)}\psi = \left[\left(1 - \frac{1}{N}(\overline{l} + \overline{\delta})\right)\mathcal{T} + \frac{1}{N^{1/2}}\overline{\mathcal{V}}_1 + \frac{1}{N}\overline{\mathcal{V}}_2 + O\left(\frac{1}{N^{3/2}}\right)\right]\psi. \tag{5.12}$$

Here $\overline{\delta}$ is given by (5.7), $\overline{\mathcal{V}}_i = \mathcal{V}_i + \mathcal{W}_i$ where $\mathcal{T}$ and $\mathcal{V}_i$ are given by (4.16) and (4.25), respectively, and

$$\mathcal{W}_i u(z) = \frac{1}{2\sqrt{\pi} c_0} \int_{-\infty}^{\infty} \Sigma_i(z, z')\; e^{\frac{1}{2}q(z,z')}\; u(z')\; dz' \quad , \quad i = 1, 2 \tag{5.13}$$

are integral operators with $\Sigma_i(z, z')$ being the linear ($i = 1$) and quadratic ($i = 2$) term in the Taylor expansion of $\delta(z, z')\, L(z, z')$ about $(z, z') = (\overline{x}, \overline{x})$.

It is noteworthy that, due to (4.24) and (5.12), $\overline{\lambda}_0^{(0)} = \lambda_0^{(0)}$ and the first non vanishing term in the correlation length is $O\left(1/N\right)$, i.e.,

$$\ell^{-1} = \frac{1}{\lambda_0^{(0)}} \left(\lambda_0^{(1)} - \overline{\lambda}_0^{(1)}\right) \frac{1}{N} + O\left(1/N^{3/2}\right). \tag{5.14}$$

¿From equation (4.24) and standard perturbation theory, the $1/N$ correction to the largest eigenvalue of $\overline{A}$ is thus given by

$$\overline{\lambda}_0^{(1)} = \lambda_0^{(1)} - \frac{1}{2(1-\overline{x})(1-2\overline{x})} + \langle \psi_0, \mathcal{W}_2 \psi_0 \rangle + \sum_{j \neq 0} \frac{\langle \psi_j, \mathcal{W}_1 \psi_0 \rangle^2}{\lambda_0^{(0)} - \lambda_j^{(0)}} \tag{5.15}$$

with ψ_j given by (4.17).

The two last terms in the right hand side of (5.13) involve Gaussian integrals. Let M denote the 2×2 matrix such that $\Sigma_2(z, z') = (N/2)\, \mathbf{z}^T M \mathbf{z}$ with $\mathbf{z}^T = (z\ z')$ the transpose of $\mathbf{z}$. Then, we have

$$\langle \psi_0, \mathcal{W}_2 \psi_0 \rangle = \frac{1}{L(\overline{x}, \overline{x})} \frac{\rho M_{11} - (-\overline{\mathbf{h}}_{12}) M_{12}}{\rho^2 - \overline{\mathbf{h}}_{12}^2}$$

(recall $\Sigma_i(z, z') = \Sigma_i(z', z)$) which in view of (4.10) and (5.6) can be written as

$$-\left(\zeta^2 + \frac{1}{1-2\overline{x}}\right) \left[\frac{2\rho\zeta^2}{\rho^2 - \overline{\mathbf{h}}_{12}^2} + \frac{1}{\rho - \overline{\mathbf{h}}_{12}} \frac{1}{(1-\overline{x})(1-2\overline{x})} \right] - \frac{1}{\rho - \overline{\mathbf{h}}_{12}} \frac{1}{(1-2\overline{x})^2}. \tag{5.16}$$

For the last term of (5.15), all factors but $j = 1$ vanish and we have

$$\frac{\langle \psi_1, \mathcal{W}_1 \psi_0 \rangle^2}{\lambda_0^{(0)} - \lambda_1^{(0)}} = 4 \left(\zeta^2 + \frac{1}{1-2\overline{x}}\right) \frac{\rho + \overline{\mathbf{h}}_{12}}{\left(\rho - \overline{\mathbf{h}}_{12}\right)^2} \tag{5.17}$$

Substituting equations (5.16) and (5.17) into (5.15) and using (5.14), give $\ell^{-1} > 0$ for all $\zeta \in (0, \infty)$.

This concludes the proof of Lemma 5.1.

□

6 Proof of Theorem 1.6

We begin by noticing that Γ_L with L even is invertible and its inverse can be explicitly computed

$$\Gamma_\Lambda^{-1} = \begin{pmatrix} A & -B^t & \dots & -B^t \\ B & A & \dots & -B^t \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \dots & A \end{pmatrix} \tag{6.1}$$

where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

¿From (3.22), the two-point function can be written as

$$\left\langle \overline{\psi}_x \cdot \psi_y \right\rangle_L = 2N \frac{\widetilde{Z}_L^{kl}}{Z_L} \quad , \quad k = x/a \ , \mathrm{l} = \mathrm{y/a} \tag{6.2}$$

where

$$\begin{aligned} \widetilde{Z}_L^{kl} &= \int d\mu_{L,0}(\phi) \det \left(\Gamma_L + (\zeta/\sqrt{N})\Phi_L\right)^{2N} \left(\Gamma_L + (\zeta/\sqrt{N})\Phi_L\right)^{-1}_{kl} \\ &= \int d\mu_{L,0}(\phi) \det \left(\Gamma_L + (\zeta/\sqrt{N})\Phi_L\right)^{2N-1} \det \left(\Gamma_L + (\zeta/\sqrt{N})\Phi_L\right)^{(kl)} \end{aligned} \tag{6.3}$$

with $\det A^{(kl)}$ meaning the determinant of the cofactor matrix $A^{(kl)}$obtained by deleting the k-th row and l-th column of A.

In the free theory (ζ= 0), the two-point function (6.2) is given by $2N\,(\Gamma_L)^{-1}_{kl}$ which by (6.1) does not decay as $|x-y| \to \infty$. On the other hand, Theorem 1.6 states that the two-point function decays exponentially fast for any $\zeta \neq 0$ and N sufficiently large. Mass is thus generated even though the chiral symmetry $\overline{\psi}\psi \to -\overline{\psi}\psi$ remains unbroken.

Before replacing the determinants in (6.3) by the expansion (2.4) we notice that

$$\det\left(\Gamma_L + \Psi_L\right)^{(kl)} = \det\left(\Gamma_L + \Psi_L\right)^{<k} \det\left(\Gamma_L + \Psi_L\right)^{>l}$$

where $A^{<k}$ mean the $(k-1) \times (k-1)$ matrix formed by removing in A all rows and columns whose labels run from k to L. Similarly, $A^{>l}$ is a

$(L-l-1)\times(L-l-1)$ matrix with the rows and columns from 1 to l removed.

(6.3) can thus be written as

$$\widetilde{Z}_L^{kl} = \sum_{\mathbf{X}} \int d\mu_{L,0}(\phi) \prod_{i=1}^{2N} \left\{ \prod_{y\in X_i^c} (\zeta/\sqrt{N})\phi_y \right\} \tag{6.4}$$

where the summation is now over all collections $\mathbf{X}=(X_1,\ldots,X_{2N})$ of compatible sets with X_1 constrained to have the sites from k to l (ends included) covered by dimers. We notice that this constraint leads $\widetilde{Z}_L^{kl}$ to vanish for all k,l such that $|k-l|$ is even.

Integrating out the ϕ-variables and performing a resummation, (6.3) can be written (with periodic boundary conditions) as

$$\widetilde{Z}_L^{kl} = (-1)^{k+l}\alpha^L \ \mathrm{Tr}\left(\widetilde{\mathrm{S}}\ \left(\widetilde{\mathrm{T}}\,\widetilde{\mathrm{T}}^{\mathrm{t}}\right)^{|\mathrm{k}-\mathrm{l}|/2}\widetilde{\mathrm{S}}\ \mathrm{T}^{\mathrm{L}-|\mathrm{k}-\mathrm{l}|/2}\right) \tag{6.5}$$

where $\widetilde{T}$ is the $(N+1)\times(N+1)$ matrix whose entries are given by

$$\widetilde{T}(n,n') = T(n,n')\widetilde{\Delta}(n,n')$$

with T as in (4.5), $\widetilde{\Delta}(Nx,Ny)=\widetilde{\delta}(x,y)$ with

$$\widetilde{\delta}(x,y) = \left(\frac{x}{1-y}\right)^{1/2}$$

and $\widetilde{S}$ is a boundary term which will not be specified since they do not contribute to the mass gap expression.

Repeating the procedure in Section 5, we are able to write the eigenvalues $\{\widetilde{\lambda}_j\}_{j=0}^N$ of $\widetilde{T}$ in the following form

$$\widetilde{\lambda}_j = c_0 e^{N\overline{f}}\ \left(\widetilde{\lambda}_j^{(0)} + O\left(\frac{1}{\sqrt{N}}\right)\right)\ , \qquad j=0,1,\ldots,n_0 \tag{6.6}$$

The ratio between the partition function (6.2) can thus be written as

$$\frac{\widetilde{Z}_L^{kl}}{Z_L} = (-1)^{k+l}\left|\left(\widetilde{e}_0,\overline{S}\,e_0\right)\right|^2\left(\frac{\widetilde{\lambda}_0}{\lambda_0}\right)^{|k-l|} + O\left((\lambda_1/\lambda_0)^L\right)$$

which implies a mass gap $\widehat{m}$ given by

$$\widehat{m} = \frac{1}{a} \ln \frac{\lambda_0}{\widetilde{\lambda}_0} = \frac{1}{a} \ln \frac{\lambda_0^{(0)}}{\widetilde{\lambda}_0^{(0)}} + O\left(1/\sqrt{N}\right) \tag{6.7}$$

In order to prove Theorem 1.6 we need compute only the leading term in (6.7) and show that it coincides with the mean field mass gap.

Following the procedure in the Section 5, the matrix $\widetilde{A} = c_0^{-1} e{-}N\overline{f}\widetilde{T}$ in the eigenvalue equation is approximated by an integral operator given by

$$\widetilde{\mathcal{T}} = \widetilde{\delta}' \mathcal{T} + O(1/N^{1/2})$$

where $\mathcal{T}$ is given by (4.16) and

$$\widetilde{\delta}' = \widetilde{\delta}(\overline{x}, \overline{x}) = \left(\frac{\overline{x}}{1-\overline{x}}\right)^{1/2}$$

In view of (6.7), Lemma 4.1 and (1.18),

$$\begin{aligned} \widehat{m} &= \frac{1}{a} \ln \frac{1}{\widetilde{\delta}'} + O\left(1/\sqrt{N}\right) \\ &= \frac{1}{2a} \ln \left[\zeta^2 + (1+\zeta^4)^{1/2}\right] + O\left(1\sqrt{N}\right) \end{aligned}$$

which, by comparing to the mean field mass gap $\widehat{m}_{MF}$ in (1.15) , concludes the proof of Theorem 1.6.

□

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