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Continued fraction transformations, Brjuno functions and BMO spaces

S. Marmi
Dipartimento di Matematica "U. Dini", Università di Firenze,
Viale Morgagni 67/A, 50134 Firenze, ITALY

P. Moussa
CEA, Service de Physique Théorique, CE-Saclay
F-91191 Gif-sur-Yvette Cedex, FRANCE

J.-C. Yoccoz
Université de Paris-Sud, Mathématiques
Bât.425, 91405 Orsay, FRANCE

ABSTRACT

The small divisors problem which is raised by stability questions in classical mechanics has an analog in holomorphic dynamical systems, namely the existence of Siegel disks. The size of these disks is well represented by the Brjuno functions. We analyse the relation between these functions and the various continued fraction transformations, and display the functional equation which is fulfilled by these highly singular functions. The analysis of this functional equation shows that the Brjuno function belong to the BMO space, and that a regular perturbation of this equation leads to a modification of the singular function which is 1/2-Hölder continuous. This leads us to believe that the most singular part of the size of the stability domains as function of the rotation number, is 'universal' up to a 1/2-Hölder continuous function.

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S. Marmi[1], P. Moussa [2], and J.-C. Yoccoz [3]

Abstract

The small divisors problem which is raised by stability questions in classical mechanics has an analog in holomorphic dynamical systems, namely the existence of Siegel disks. The size of these disks is well represented by the Brjuno functions. We analyse the relation between these functions and the various continued fraction transformations, and display the functional equation which is fulfilled by these highly singular functions. The analysis of this functional equation shows that the Brjuno function belong to the BMO space, and that a regular perturbation of this equation leads to a modification of the singular function which is 1/2-Hölder continuous. This leads us to believe that the most singular part of the size of the stability domains as function of the rotation number, is 'universal' up to a 1/2-Hölder continuous function.

[1] Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy

[2] Service de Physique Théorique, C.E. Saclay, 91191 Gif-Sur-Yvette, France

[3] Université de Paris-Sud, Mathématiques. Bât. 425, 91405-Orsay, France

0. Introduction

The Brjuno condition tells which invariant disks (called Siegel disks) persist when an irrational rotation is analytically perturbed. The Brjuno function tells more : it gives an estimate of minus the logarithm of the size of the Siegel disks as a function of the rotation number [Yo]. In this work, we first analyse the relation between the Brjuno function and the various kind of continued fractions. We establish the functional equation fulfilled by the Brjuno function, and show that its solution requires the inversion of an operator T. We show that T is a contracting operator for all L^p norms, and also for the BMO (Bounded Mean Oscillation) space. The Brjuno function is obtained as the action of $(1-T)^{-1}$ on a logarithmic function which belongs to the BMO space. Therefore the Brjuno function is also in this space. Noticing that the adjoint of T is nothing else than the Ruelle-Frobenius-Perron operator associated to the dynamical system which generate the continued fraction, the identification of the space adapted to T, seems to us promising for the dynamical properties. Finally, the action of T on continuous function is decribed according to Hölder's properties. We show that regular perturbations (at least $C^{1/2}$) of the logarithmic term do modify the solution only by a $C^{1/2}$ contribution, so that the most singular part remain unchanged. We anticipate that this result might be much more general : the geometric renormalisation for holomorphic dynamical systems will likely produce only C^1 perturbations to the renormalisation equation, and the most singular part of minus the logarithm of the size of the stability domains as function of the rotation number could be universally (that is modulo $C^{1/2}$) described by the Brjuno function.

1. On a family of continued fraction transformations and the related Brjuno functions

Let $\alpha \in [1/2, 1]$ and let $x \in \mathbb{R}$. We define

$$[x]_\alpha = \min\{p \in \mathbb{Z} \mid x < \alpha + p\} \tag{1.1}$$

that is

$$[x]_\alpha = p \text{ iff } \quad \alpha - 1 + p \le x < \alpha + p\,.$$

Note that

$$[x]_\alpha = [x - \alpha + 1]$$

where $[\ ] = [\ ]_1$ denotes the usual integer part of a real number. We will consider the iteration of

$$A_\alpha : (0, \alpha) \mapsto [0, \alpha] \tag{1.2}$$

defined by

$$A_\alpha(x) = \left| \frac{1}{x} - \left[\frac{1}{x} \right]_\alpha \right| . \tag{1.3}$$

Theorem 1.1. *The dynamical system defined by the iteration of (1.3) preserves an absolutely continuous (w.r.t. Lebesgue) probability measure m_α with density $c_\alpha \rho_\alpha(x)$: $m_\alpha(dx) = c_\alpha \rho_\alpha(x)dx$. The density is given by :*

(i) if $\frac{\sqrt{5}-1}{2} \le \alpha \le 1$

$$\begin{aligned} c_\alpha &= \frac{1}{\log(1+\alpha)}\,, \\ \rho_\alpha(x) &= \frac{1}{1+x}\chi_{(\frac{1-\alpha}{\alpha},\alpha)}(x) + \frac{1}{2+x}\chi_{(1-\alpha,\frac{1-\alpha}{\alpha}]}(x) \\ &\quad + \left(\frac{1}{x+2} + \frac{1}{2-x}\right)\chi_{(0,1-\alpha]}(x)\,, \end{aligned} \tag{1.4}$$

(ii) if $2 - \sqrt{2} \le \alpha < \frac{\sqrt{5}-1}{2}$

$$\begin{aligned} c_\alpha &= \frac{1}{\log G} \\ \rho_\alpha(x) &= \frac{1}{G+x}\chi_{(\frac{2\alpha-1}{1-\alpha},\alpha)}(x) + \frac{1}{2+x}\chi_{(1-\alpha,\frac{2\alpha-1}{1-\alpha}]}(x) \\ &\quad + \left(\frac{1}{x+2} + \frac{1}{G+1-x}\right)\chi_{(\frac{2\alpha-1}{\alpha},1-\alpha]}(x) \\ &\quad + \left(\frac{1}{x+2} + \frac{1}{2-x}\right)\chi_{(0,\frac{2\alpha-1}{\alpha}]}(x)\,, \end{aligned} \tag{1.5}$$

(iii) if $\frac{1}{2} \leq \alpha < 2-\sqrt{2}$

$$\begin{aligned} c_\alpha =& \frac{1}{\log G} \, , \\ \rho_\alpha(x) =& \frac{1}{G+x}\chi_{(1-\alpha,\alpha)}(x) + \left(\frac{1}{G+x} + \frac{1}{G+1-x}\right)\chi_{(\frac{2\alpha-1}{1-\alpha},1-\alpha]}(x) \\ + & \left(\frac{1}{x+2} + \frac{1}{G+1-x}\right)\chi_{(\frac{2\alpha-1}{\alpha},\frac{2\alpha-1}{1-\alpha}]}(x) + \left(\frac{1}{x+2} + \frac{1}{2-x}\right)\chi_{(0,\frac{2\alpha-1}{\alpha}]}(x) \, , \end{aligned} \tag{1.6}$$

where $G = \frac{\sqrt{5}+1}{2}$ *and* $\chi_{(a,b)}(x)$ *denotes the characteristic function of the interval* (a,b).

Remark 1.2. Note that if $\alpha = 1$ one finds Gauss' result : $c_1\rho_1(x) = \frac{1}{(1+x)\log 2}$.

Proof. By Theorem 4.1.1 of [LM] m_α will be an invariant probability measure for A_α if and only if its density ρ_α is a fixed point of the Perron-Frobenius operator P_α associated to A_α : if B denotes any measurable subset of $(0,\alpha)$, and f any summable function on $(0,\alpha)$, P_α is defined by

$$\int_B (P_\alpha f)(x)dx = \int_{A_\alpha^{-1}(B)} f(x)dx \, .$$

The Perron-Frobenius operator for all these maps has the following form :

$$(P_\alpha f)(x) = \sum_{m\geq m_+}^{+\infty} \frac{1}{(m+x)^2} f\left(\frac{1}{m+x}\right) + \sum_{m\geq m_-}^{+\infty} \frac{1}{(m-x)^2} f\left(\frac{1}{m-x}\right) \, ,$$

where m_+ and m_- depend both on α and on x.

If $\alpha = 1$ one has $m_+ = 1$ and $m_- = +\infty$ and it is immediate to check that $\frac{1}{1+x}$ is a fixed point of P_1.

If $\frac{\sqrt{5}-1}{2} < \alpha < 1$ one has three possible cases :

(a) $x \in (0, 1-\alpha]$ then $m_+ = 2$, $m_- = 2$ and $\rho_\alpha(x) = \frac{1}{x+2} + \frac{1}{2-x}$;

(b) $x \in (1-\alpha, \frac{1-\alpha}{\alpha}]$ then $m_+ = 2$, $m_- = +\infty$ and $\rho_\alpha(x) = \frac{1}{x+2}$;

(c) $x \in (\frac{1-\alpha}{\alpha}, \alpha)$ then $m_+ = 1$, $m_- = +\infty$ and $\rho_\alpha(x) = \frac{1}{1+x}$.

Let us consider case (c). By (1.4) it follows that the two sums in the definition of $\mathcal{P}_\alpha$ must be splitted into three parts according to which interval contains

$1/(m \pm x)$:

$$
\begin{aligned}
(P_\alpha \rho_\alpha)(x) = & \sum_{m=2}^{m_1} \frac{1}{(m+x)^2} \frac{1}{1+\dfrac{1}{m+x}} + \sum_{m=2}^{m_2} \frac{1}{(m-x)^2} \frac{1}{1+\dfrac{1}{m-x}} \\
& + \frac{1}{m_1+x+1} - \frac{1}{m_1+x+\frac{3}{2}} + \frac{1}{m_2-x+1} - \frac{1}{m_2-x+\frac{3}{2}} \\
& + \sum_{m=m_1+2}^{\infty} \frac{1}{(m+x)^2} \left(\frac{1}{2+\dfrac{1}{m+x}} + \frac{1}{2-\dfrac{1}{m+x}} \right) \\
& + \sum_{m=m_2+2}^{\infty} \frac{1}{(m-x)^2} \left(\frac{1}{2+\dfrac{1}{m-x}} + \frac{1}{2-\dfrac{1}{m-x}} \right) .
\end{aligned}
$$

Using

$$
\frac{1}{(m \pm x)^2} \frac{1}{1+\dfrac{1}{m \pm x}} = \frac{1}{m \pm x} - \frac{1}{m+1 \pm x}
$$

$$
\frac{1}{(m+x)^2} \left(\frac{1}{2+\dfrac{1}{m+x}} + \frac{1}{2-\dfrac{1}{m+x}} \right) = \frac{1}{m+x-\frac{1}{2}} - \frac{1}{m+x+\frac{1}{2}}
$$

$$
\frac{1}{(m-x)^2} \left(\frac{1}{2+\dfrac{1}{m-x}} + \frac{1}{2-\dfrac{1}{m-x}} \right) = \frac{1}{m-x-\frac{1}{2}} - \frac{1}{m-x+\frac{1}{2}}
$$

it is easy to show that all terms cancel except for $\frac{1}{x+2} + \frac{1}{2-x}$.

The remaining cases (a) and (b) are simpler, and the proof of (ii) and (iii) follows the same kinds of ideas (note that $G^2 = G+1$, $G-1 = 1/G$ and $2-G = 1/G^2$). □

Remark 1.3. It is well known that the Kolmogorov-Sinai entropy of these maps is given by

$$
h(\alpha) = -2 \int_0^\alpha c_\alpha \rho_\alpha(x) \log x \, dx .
$$

If $\alpha = 1$ one obtains

$$
h(1) = \frac{\pi^2}{6 \log 2} ,
$$

for $\alpha = 1/2$ one obtains,

$$
h(1/2) = \frac{\pi^2}{6 \log G} ,
$$

a result already given by Rieger [Ri]. For general α, we find that $c_\alpha h(\alpha)$ does not depend on α, thus

$$h(\alpha) = \begin{cases} \frac{\pi^2}{6\log(1+\alpha)} & \text{if } \frac{\sqrt{5}-1}{2} \le \alpha \le 1 \ , \\ \frac{\pi^2}{6\log G} & \text{if } \frac{1}{2} \le \alpha \le \frac{\sqrt{5}-1}{2} \ , \end{cases}$$

as already shown by Nakada [Na]. These results show the existence of a phase transition at $\alpha = \frac{\sqrt{5}-1}{2}$.

To each $x \in \mathbb{R} \setminus \mathbb{Q}$ we associate a continued fraction expansion by iterating A_α as follows. Let

$$\begin{aligned} x_0 &= |x - [x]_\alpha| \\ a_0 &= [x]_\alpha \end{aligned} \tag{1.7}$$

then one obviously has

$$x_0 = a_0 + \varepsilon_0 x_0 \tag{1.8}$$

where

$$\varepsilon_0 = \begin{cases} +1 \text{ iff } x \ge [x]_\alpha \\ -1 \text{ otherwise} \end{cases} \tag{1.9}$$

We now define inductively for all $n \ge 0$

$$\begin{aligned} x_{n+1} &= A_\alpha(x_n) \\ a_{n+1} &= \left[\frac{1}{x_n}\right]_\alpha \ge 1 \end{aligned} \tag{1.10}$$

thus

$$x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1} \tag{1.11}$$

where

$$\varepsilon_{n+1} = \begin{cases} +1 \text{ iff } \frac{1}{x_n} \ge a_{n+1} \\ -1 \text{ otherwise} \end{cases} \tag{1.12}$$

Therefore we have

$$x = a_0 + \varepsilon_0 x_0 = a_0 + \frac{\varepsilon_0}{a_1 + \varepsilon_1 x_1} = \ldots = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \ddots + \cfrac{\varepsilon_{n-1}}{a_n + \varepsilon_n x_n}}} \tag{1.13}$$

and we will write

$$x = [(a_0, \varepsilon_0), (a_1, \varepsilon_1), \ldots, (a_n, \varepsilon_n), \ldots] \ . \tag{1.14}$$

Note that for $\alpha = 1$ we recover the standard continued fraction expansion defined through the iteration of Gauss' map $x \mapsto x^{-1} \bmod 1$, and all $\varepsilon_n = +1$. When

$\alpha = 1/2$ one has the so-called nearest integer continued fraction, and $a_n \geq 2$ for all $n \geq 1$.

The nth-convergent is defined by

$$\frac{p_n}{q_n} = [(a_0, \varepsilon_0), (a_1, \varepsilon_1), \ldots, (a_n, \varepsilon_n)] = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \ddots + \cfrac{\varepsilon_{n-1}}{a_n}}} . \tag{1.15}$$

and it is immediate to check that the numerators p_n and denominators q_n are recursively determined by

$$p_{-1} = q_{-2} = 1 \ , \quad p_{-2} = q_{-1} = 0 \ , \tag{1.16}$$

and for all $n \geq 0$

$$\begin{aligned} p_n &= a_n p_{n-1} + \varepsilon_{n-1} p_{n-2} \ , \\ q_n &= a_n q_{n-1} + \varepsilon_{n-1} q_{n-2} \ . \end{aligned} \tag{1.17}$$

Moreover

$$x = \frac{p_n + p_{n-1}\varepsilon_n x_n}{q_n + q_{n-1}\varepsilon_n x_n} \tag{1.18}$$

$$x_n = -\varepsilon_n \frac{q_n x - p_n}{q_{n-1}x - p_{n-1}} \tag{1.19}$$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \varepsilon_0 \ldots \varepsilon_{n-1} \tag{1.20}$$

Let

$$\beta_n = \Pi_{i=0}^n x_i = (-1)^n \varepsilon_0 \ldots \varepsilon_n (q_n x - p_n) \tag{1.21}$$

Then

$$\begin{aligned} x_n &= \frac{\beta_n}{\beta_{n-1}} \\ \beta_{n-2} &= a_n \beta_{n-1} + \varepsilon_n \beta_n \end{aligned} \tag{1.22}$$

From the definitions given one easily proves by induction the following proposition

Proposition 1.4. *Given $\alpha \in [1/2, 1]$, for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and for all $n \geq 1$ one has*

(i) $q_{n+1} > q_n > 0$;

(ii) $p_n > 0$ *when* $x > 0$ *and* $p_n < 0$ *when* $x < 0$;

(iii) $|q_n x - p_n| = \dfrac{1}{q_{n+1} + \varepsilon_{n+1} q_n x_{n+1}}$, *so that* $\dfrac{1}{1+\alpha} < \beta_n q_{n+1} < \dfrac{1}{\alpha}$;

(iv) *if* $\alpha > \dfrac{\sqrt{5}-1}{2}$, $\beta_n \leq \alpha \left(\dfrac{\sqrt{5}-1}{2} \right)^n$;

(v) *if* $\alpha > \dfrac{\sqrt{5}-1}{2}$, $q_n \geq \dfrac{1}{\alpha(1+\alpha)} \left(\dfrac{\sqrt{5}+1}{2} \right)^{(n-1)}$;

(vi) *if* $\alpha \leq \dfrac{\sqrt{5}-1}{2}$, $\beta_n \leq \alpha \left(\sqrt{2} - 1) \right)^n$;

(vii) $\quad$ *if* $\alpha \le \frac{\sqrt{5}-1}{2}$, $q_n \ge \frac{1}{\alpha(1+\alpha)}\left(\sqrt{2}+1)\right)^{(n-1)}$.

Remark 1.5. We only quote here some aspects of the proof of the above theorem : one gets parts (i) and (ii) by recursion using (1.17). Part (iii) is easily obtained from (1.18), and is used to deduce part (v) from (iv) and part (vii) from (vi). Part iv) and vi) are easy to prove when α is close to one or one half, respectively. However the proof is much more intricate around $\alpha = (\sqrt{5}-1)/2$ (see [MMY]).

Remark 1.6. From (iii) one gets

$$\frac{1}{2q_nq_{n+1}} < \frac{1}{q_n(q_n+q_{n+1})} \le \frac{1}{q_n(\alpha q_n+q_{n+1})} < \left|x-\frac{p_n}{q_n}\right| < \frac{1}{q_nq_{n+1}} \tag{1.23}$$

if $\varepsilon_{n+1} = +1$, whereas

$$\frac{1}{q_nq_{n+1}} < \left|x-\frac{p_n}{q_n}\right| < \frac{1}{q_n(q_{n+1}-(1-\alpha)q_n)} < \frac{1}{\alpha q_n^2} \tag{1.24}$$

if $\varepsilon_{n+1} = -1$.

Remark 1.7. By (v) and (vii), Proposition 1.4, there exists two positive constants c_1 and c_2 such that

$$\sum_{k=0}^{\infty}\frac{\log q_k}{q_k} \le c_1 \ ,$$
$$\sum_{k=0}^{\infty}\frac{\log 2}{q_k} \le c_2 \ ,$$

for all $\alpha \in [1/2, 1]$ and for all $x \in (0, \alpha)$.

Following Yoccoz [Yo] we define a (generalized) Brjuno function :

Definition 1.8. *The α-Brjuno function $B_\alpha : \mathbb{R}\setminus\mathbb{Q} \to \bar{\mathbb{R}}$ is defined by the formula*

$$B_\alpha(x) = -\sum_{i=0}^{\infty}\beta_{i-1}\log x_i \tag{1.25}$$

where we have posed $\beta_{-1} = 1$.

Remark 1.9. The Brjuno function defined in [Yo] corresponds to $B_{1/2}$, the one defined by the nearest integer continued fraction map $A_{1/2}$.

Proposition 1.10. *Given $\alpha \in [1/2, 1]$ one has*
(i) $B_\alpha(x) = B_\alpha(x+1)$ for all $x \in \mathbb{R}\setminus\mathbb{Q}$;
(ii) For all $x \in (0,\alpha)\cap\mathbb{R}\setminus\mathbb{Q}$

$$B_\alpha(x) = -\log x + xB_\alpha\left(\frac{1}{x}\right)$$

(iii) if $x \in [\alpha - 1, 0) \cap \mathbb{R} \setminus \mathbb{Q}$ *then* $B_\alpha(-x) = B_\alpha(x)$.
(iv) There exists a constant $C_\alpha > 0$ *such that for all* $x \in \mathbb{R} \setminus \mathbb{Q}$ *one has*

$$\left| B_\alpha(x) - \sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j} \right| \le C_\alpha$$

where $\{q_j\}_{j\ge 0}$ *denotes the sequence of the denominators of the convergents to* x *of the* α*-continued fraction expansion.*

Proof. Given $x \in \mathbb{R}\setminus\mathbb{Q}$, the sequences $(x_i)_{i\ge 0}$ and $(\beta_i)_{i\ge 0}$ associated to x and $x+1$ are the same, which proves (i). The same is true for x and $-x$ if $x \in (\alpha - 1, 0)$, which proves (iii).

If $x \in (0, \alpha)$, let $y = 1/x$ and denote by y_i, $a_i(y)$, $\beta_i(y)$, and x_i, $a_i(x)$, $\beta_i(x)$ the sequences (1.10) and (1.21) associated to y and to x respectively. From (1.7) and (1.8) it follows that $x_0 = x$, $a_0(y) = a_1(x)$, $y_0 = x_1$ and by induction for all $n \ge 0$ $y_n = x_{n+1}$ and $\beta_n(y) = \frac{\beta_{n+1}(x)}{x}$. Thus

$$\begin{aligned} B_\alpha(y) &= -\sum_{i=0}^{\infty} \beta_{i-1}(y) \log y_i = -\log y_0 - \sum_{i=1}^{\infty} \frac{1}{x} \beta_i(x) \log x_{i+1} \\ &= -\frac{1}{x} \sum_{i=1}^{\infty} \beta_{i-1}(x) \log x_i = \frac{1}{x} [B_\alpha(x) + \log x] \ , \end{aligned}$$

which proves (ii).

To prove (iv) we first remark that

$$q_i \beta_{i-1} + \varepsilon_i q_{i-1} \beta_i = 1$$

for all $i \ge 0$. Then

$$\begin{aligned} -B_\alpha(x) + \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_i} &= \sum_{i=0}^{\infty} \beta_{i-1} \log \frac{\beta_i}{\beta_{i-1}} + \sum_{i=0}^{\infty} \left(\beta_{i-1} + \varepsilon_i \frac{q_{i-1}}{q_i} \beta_i \right) \log q_{i+1} \\ &= \sum_{i=0}^{\infty} \beta_{i-1} \log \beta_i q_{i+1} - \sum_{i=0}^{\infty} \beta_{i-1} \log \beta_{i-1} + \sum_{i=0}^{\infty} \varepsilon_i \frac{q_{i-1}}{q_i} \beta_i \log q_{i+1} \ , \end{aligned}$$

but by (1.21), (1.23) and (1.24) one has

$$\left| \sum_{i=0}^{\infty} \beta_{i-1} \log \beta_i q_{i+1} \right| \le 2 \sum_{i=0}^{\infty} \frac{\log 2}{q_i} \le 2c_2 \ ,$$

$$\left| \sum_{i=0}^{\infty} \beta_{i-1} \log \beta_{i-1} \right| \le 2 \sum_{i=0}^{\infty} \frac{\log 2 + \log q_i}{q_i} \le 2(c_1 + c_2) \ ,$$

$$\left| \sum_{i=0}^{\infty} \varepsilon_i \frac{q_{i-1}}{q_i} \beta_i \log q_{i+1} \right| \le 2 \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_{i+1}} \le 2c_1 \ ,$$

from which it follows that

$$\left| B_\alpha(x) - \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_i} \right| \le 4(c_1 + c_2)\ .$$

$\square$

Let P_n/Q_n denote the n-th convergent to x according to the standard continued fraction expansions (i.e. obtained by the iteration of the Gauss map A_1). Using the results of [Bo] one relates the n-th convergents of the α–continued fractions to P_n/Q_n :

Lemma 1.11. *Let $k^\alpha : \mathbb{N} \to \mathbb{N}$ be the arithmetic function inductively defined by k(-1)=-1 and*

$$k(n+1) = \begin{cases} k(n)+1 & \text{if } \varepsilon_{n+1} = +1, \\ k(n)+2 & \text{if } \varepsilon_{n+1} = -1, \end{cases}$$

where ε_k is defined as in (1.10) and (1.11). Then k is strictly increasing and for all $n \in \mathbb{N}$

$$\frac{p_n}{q_n} = \frac{P_{k(n)}}{Q_{k(n)}}\ .$$

Moreover, when $k(n+1) = k(n)+2$, we have for the denominators of the convergent of Gauss'continued fraction $Q_{k(n+1)} = Q_{k(n)+2} = Q_{k(n)+1} + Q_{k(n)}$

By means of Lemma 1.11 one can prove the following

Theorem 1.12. *There exists a positive constant $C > 0$ such that for all $\alpha \in [1/2, 1]$ and for all $x \in \mathbb{R} \setminus \mathbb{Q}$ one has*

$$\left| B_\alpha(x) - \sum_{j=0}^{\infty} \frac{\log Q_{j+1}}{Q_j} \right| \le C \tag{1.26}$$

Proof. Thanks to (iv), Proposition 1.10, it suffices to compare $\sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j}$ with $\sum_{j=0}^{\infty} \frac{\log Q_{j+1}}{Q_j}$. By Lemma 1.11, one has $q_j = Q_{k(j)}$ for all j thus

$$\sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j} = \sum_{k(j+1)=k(j)+1} \frac{\log Q_{k(j+1)}}{Q_{k(j)}} + \sum_{k(j+1)=k(j)+2} \frac{\log Q_{k(j+1)}}{Q_{k(j)}}\ .$$

Using the fact that $Q_{k(j+1)} = Q_{k(j)+2} = Q_{k(j)+1} + Q_{k(j)}$ we have

$$\frac{\log Q_{k+2}}{Q_k} = \frac{\log(Q_{k+1} + Q_k)}{Q_k} = \frac{\log Q_{k+1}}{Q_k} + \frac{\log\left(1 + \frac{Q_k}{Q_{k+1}}\right)}{Q_k}$$

but

$$0 \leq \frac{\log\left(1+\frac{Q_k}{Q_{k+1}}\right)}{Q_k} \leq \frac{\log 2}{Q_k} \ ,$$

By applying the estimates of Remark 1.7 one gets the result :

$$\left| \sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j} - \sum_{j=0}^{\infty} \frac{\log Q_{j+1}}{Q_j} \right| \leq 2c_2 + c_1 \ .$$

□

Remark 1.13. The *Brjuno numbers* [Br] are usually defined by the condition

$$\sum_{i=0}^{\infty} \frac{\log Q_{i+1}}{Q_i} < +\infty \ .$$

Theorem 1.12 shows that the α-Brjuno functions B_α are finite at x if and only if x is a Brjuno number and that all the generalized Brjuno functions differ one from the other for a L^∞ function.

On the other hand, the advantage of the functions B_α with respect to the Brjuno condition is that they verify a nice functional equation under the action of the modular group $\mathrm{SL}\,(2,\mathbb{Z})$.

Another important characterization of the generalized Brjuno functions comes from their "uniqueness", as it is stated by Corollary 1.15 below.

Let us consider the operator

$$(Tf)(x) = xf\left(\frac{1}{x}\right) \ , \tag{1.27}$$

if $x \in (0,\alpha)$, defined for the moment on measurable functions of $\mathbb{R}$ which verify

$$f(x) = f(x+1) \text{ for almost every } x \in \mathbb{R} \ , \quad f(-x) = f(x) \ \text{ for a.e. } x \in (0, 1-\alpha) \ . \tag{1.28}$$

It is understood that the function Tf is completed outside $(0,\alpha)$ by imposing on Tf the same parity and periodicity conditions which are expressed for f in (1.28).

The functional equation for the α-Brjuno function can be written in the form

$$[(1-T)B_\alpha](x) = -\log x \ , \tag{1.29}$$

for all $x \in (0,\alpha)$, complemented with the periodicity and symmetry conditions (1.28). This suggest to study the operator T on the Banach spaces

$$X_{\alpha,p} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ verifies } (1.28) \ , \ \ f \in L^p(0,\alpha), \} \tag{1.30}$$

endowed with the norm of $L^p(0,\alpha)$, as α varies in $(1/2,1)$ and $p\in[1,\infty]$. Note that if $p<p'$ one has the obvious inclusion $X_{\alpha,p'}\subset X_{\alpha,p}$ and let

$$X_\alpha = \cap_{p\geq 1} X_{\alpha,p}\ . \tag{1.31}$$

Clearly

$$X_{\alpha,\infty} \subsetneq X_\alpha\ ,$$

since, for example, $\log|x-[x]_\alpha|\in X_\alpha\setminus X_{\alpha,\infty}$.

Theorem 1.14. *T is a linear bounded operator from $X_{\alpha,p}$ into itself for all $\alpha\in[\frac{1}{2},1]$ and for all $p\in[1,\infty]$. Indeed we have*

$$||T||_{\mathcal{L}(X_{\alpha,\infty})} = \alpha\ , \tag{1.32}$$

moreover, if $\frac{1}{2}\leq\alpha\leq\frac{\sqrt{5}-1}{2}$, then

$$||T||_{\mathcal{L}(X_{\alpha,p})} \leq [\zeta(2+p)-1+\zeta(2+p,\alpha)-\alpha^{-(p+2)}]^{\frac{1}{p}}\ , \tag{1.33}$$

and if $\frac{\sqrt{5}-1}{2}<\alpha\leq 1$, then

$$||T||_{\mathcal{L}(X_{\alpha,p})} \leq [\zeta(2+p)-1+\zeta(2+p,\alpha)-\alpha^{-(p+2)}+\alpha^{p+2}]^{\frac{1}{p}}\ , \tag{1.34}$$

where $\zeta(s)=\sum_{k=1}^\infty k^{-s}$ and $\zeta(s,\alpha)=\sum_{k=0}^\infty(k+\alpha)^{-s}$ are Riemann's and Hurwitz's zeta functions.

Proof. (1.32) is immediately established by remarking that the constant function 1 on $(0,\alpha)$, extended to $\mathbb{R}$ by means of (1.28), belongs to $X_{\alpha,\infty}$, has norm equal to 1 whereas $(T1)(x)=x-[x]_\alpha$, the norm of which is evidently equal to α.

If $f\in X_{\alpha,p}$ and $\frac{1}{2}\leq\alpha\leq\frac{\sqrt{5}-1}{2}$, since $\frac{1}{\alpha}\geq\alpha+1$

$$\begin{aligned}\int_0^\alpha |(Tf)(x)|^p dx &= \int_{\frac{1}{\alpha}}^\infty \frac{|f(y)|^p}{y^{p+2}}dy \\ &\leq \sum_{k=2}^\infty\left[\int_0^\alpha \frac{|f(y)|^p}{(k+y)^{p+2}}dy + \int_0^{1-\alpha}\frac{|f(y)|^p}{(k-y)^{p+2}}dy\right] \\ &\leq ||f||^p_{L^p(0,\alpha)}\left(\sum_{k=2}^\infty k^{-(p+2)}+\sum_{k=1}^\infty(k+\alpha)^{-(p+2)}\right)\ ,\end{aligned} \tag{1.35}$$

which proves (1.33).

If $\frac{\sqrt{5}-1}{2}<\alpha\leq 1$, $\frac{1}{\alpha}<\alpha+1$ and by the same argument of above one gets

$$\begin{aligned}\int_0^\alpha |(Tf)(x)|^p dx \leq\ & ||f||^p_{L^p(0,\alpha)}\left(\sum_{k=2}^\infty k^{-(p+2)}+\sum_{k=1}^\infty(k+\alpha)^{-(p+2)}\right) \\ &+\int_{\frac{1}{\alpha}}^{\alpha+1}\frac{|f(y)|^p}{y^{p+2}}dy \\ &\leq ||f||^p_{L^p(0,\alpha)}\left(\sum_{k=2}^\infty k^{-(p+2)}+\sum_{k=1}^\infty(k+\alpha)^{-(p+2)}+\alpha^{p+2}\right)\ .\end{aligned} \tag{1.36}$$

□

Corollary 1.15. *For all $\alpha \in [\frac{1}{2}, 1)$, the α-Brjuno function B_α is the unique solution of (1.29) which belongs to X_α.*

Proof. The idea of the proof is to show that T is a contraction on $X_{\alpha,p}$ for p sufficiently large. Then B_α is uniquely defined by $B_\alpha(x) = \sum_{n=0}^{\infty} T^n(\log x)$. If $\alpha \le \frac{\sqrt{5}-1}{2}$ one can show that T is a contraction on $X_{\alpha,p}$ for all $p \ge 1$: it suffices to remark that

$$\zeta(p+2) - 1 + \zeta(p+2,\alpha) - \alpha^{-(p+2)} \le \zeta(3) - 1 + \zeta(3, 1/2) - 8 ,$$

but

$$\zeta(s, 1/2) = (2^s - 1)\zeta(s) ,$$

thus for the norm of the linear operator T we have

$$||T||_{\mathcal{L}(X_{\alpha,p})} \le 8\zeta(3) - 9 \le .6164553 ,$$

where we have used the inequality (see [AS]) $\zeta(3) \le 1.2020569032$.

If $\frac{\sqrt{5}-1}{2} < \alpha < 1$ one needs a slighlty more refined argument to prove that T is a contraction on $X_{\alpha,p}$ if p is large enough. First of all we remark that

$$\begin{aligned}\zeta(s) &= \sum_{k=1}^{n} k^{-s} + \frac{n^{-s+1}}{s-1} - s\int_n^\infty \frac{x-[x]}{x^{s+1}}dx ,\\ \zeta(s,v) &= \sum_{k=0}^{n}(k+v)^{-s} + \frac{(n+v)^{-s+1}}{s-1} - s\int_n^\infty \frac{x-[x]}{(x+v)^{s+1}}dx ,\end{aligned} \tag{1.37}$$

as it is immediate to show integrating by parts. Since

$$-s\int_n^\infty \frac{x-[x]}{x^{s+1}}dx < 0 , \quad -s\int_n^\infty \frac{x-[x]}{(x+v)^{s+1}}dx < 0 ,$$

by applying (1.37) with $v = \alpha$ and $n = 2$ one gets

$$\begin{aligned}\zeta(2+p) - 1 &\le \frac{2^{-p-1}}{p+1} + 2^{-p-2} ,\\ \zeta(2+p,\alpha) - \alpha^{-p-2} &\le \frac{(2+\alpha)^{-p-1}}{p+1} + (2+\alpha)^{-p-2} + (1+\alpha)^{-p-2} ,\end{aligned}$$

thus the condition for T being a contraction reads

$$\frac{2^{-p-1}}{p+1} + 2^{-p-2} + \frac{(2+\alpha)^{-p-1}}{p+1} + (2+\alpha)^{-p-2} + (1+\alpha)^{-p-2} + \alpha^{p+2} < 1 ,$$

which is evidently satisfied for any fixed $\alpha < 1$ by choosing p large enough. $\square$

Remark 1.16. By slightly refining the proof one can very easily show that T is a contraction on $X_{\alpha,p}$ if :

1) $p \geq 1$ and $\alpha < .726$;
2) $p \geq 2$ and $\alpha < .909$;
3) $p \geq 3$ and $\alpha < .960$;

and so on, and our bounds are still far from being optimal. On the other hand it is easy to convince oneself by considering some examples that T cannot be a contraction on $X_{1,p}$.

Remark 1.17. One can easily compute the adjoint $T^* \in \mathcal{L}(X_{\alpha,q})$ of $T \in \mathcal{L}(X_{\alpha,p})$, where $p^{-1} + q^{-1} = 1$: let $f \in X_{\alpha,p}$, $g \in X_{\alpha,q}$

$$\begin{aligned}\int_0^\alpha (Tf)(x)g(x)dx = \int_0^\alpha xf\left(\frac{1}{x}\right) g(x)dx = \int_{\frac{1}{\alpha}}^\infty f(y)g\left(\frac{1}{y}\right)\frac{dy}{y^3} \\ = \int_0^\alpha f(y)(T^*g)(y)dy \ ,\end{aligned}$$

where

$$(T^*g)(y) = \sum_{m\geq m_+} \frac{1}{(m+y)^3} g\left(\frac{1}{m+y}\right) + \sum_{m\geq m_-} \frac{1}{(m-y)^3} g\left(\frac{1}{m-y}\right) ,$$

and m_+, m_- depend both on α and on y, as explained above in the proof of Theorem 1.1. Note that the adjoint of T belongs to the class of the so-called Perron-Frobenius-Ruelle operators [Me1,Me2] associated to A_α

$$(\mathcal{P}_{\alpha,\phi} g)(y) = \sum_{m\geq m_+} \phi\left(\frac{1}{(m+y)}\right) g\left(\frac{1}{m+y}\right) + \sum_{m\geq m_-} \phi\left(\frac{1}{(m-y)}\right) g\left(\frac{1}{m-y}\right) .$$

If $\phi(y) = |A'_\alpha(y)|^{-1}$ then $\mathcal{P}_{\alpha,\phi} = \mathcal{P}_\alpha$; if $\phi(y) = |A'_\alpha(y)|^{-3/2}$ then $\mathcal{P}_{\alpha,\phi} = T^*$.

Remark 1.18. When $\alpha = 1/2$, we have $1/\alpha = 2$. Therefore in (1.35), the second sum in the right hand side needs only to run over $k = 2$ to ∞. As a consequence, we have

$$||T||_{\mathcal{L}(X_{1/2,p})} \leq \left(\sum_{k=2}^\infty \left(k^{-(p+2)} + (k+1/2)^{-(p+2)}\right)\right)^{\frac{1}{p}} \tag{1.38}$$

$$= \left(\zeta(2+p) - 1 + \zeta(2+p,1/2) - 2^{p+2} - \left(\frac{2}{3}\right)^{p+2}\right)^{\frac{1}{p}} . \tag{1.39}$$

This can be rewritten as

$$||T||_{\mathcal{L}(X_{1/2,p})} \leq \left(2^{p+2}(\zeta(p+2) - 1) - 1 - \left(\frac{2}{3}\right)^{p+2}\right)^{\frac{1}{p}} . \tag{1.40}$$

The bound in the above equation is sharper than the bound in (1.33). The explicit values for $p = 2$ are

$$||T||_{\mathcal{L}(X_{1/2,2})} \leq \left(\frac{8\pi^4}{45} - \frac{1393}{81} \right)^{\frac{1}{2}} \sim 0.34589 \; . \tag{1.41}$$

By comparison, the numerical value obtained from Equation (1.33) would be 0.56318.

Remark 1.19. The results obtained for the norm of T can easily be extended to the operators $T_{(\nu)}$ defined as follows for $\nu > 0$

$$(T_{(\nu)} f)(x) = x^{\nu} f\left(\frac{1}{x}\right) \; , \tag{1.42}$$

if $x \in (0, \alpha)$, and completed as in (1.28) outside $(0, \alpha)$. The adjoint operators $T^*_{(\nu)}$ also belong to the class of the Perron-Frobenius-Ruelle operators associated to A_α (see remark 1.17 above).

Similarly as equations (1.33), (1.34), and (1.40), we get the following results : if $\frac{1}{2} < \alpha \leq \frac{\sqrt{5}-1}{2}$, then

$$||T_{(\nu)}||_{\mathcal{L}(X_{\alpha,p})} \leq [\zeta(2 + p\nu) - 1 + \zeta(2 + p\nu, \alpha) - \alpha^{-(p\nu+2)}]^{\frac{1}{p}} \; , \tag{1.43}$$

if $\frac{\sqrt{5}-1}{2} < \alpha \leq 1$, then

$$||T_{(\nu)}||_{\mathcal{L}(X_{\alpha,p})} \leq [\zeta(2 + p\nu) - 1 + \zeta(2 + p\nu, \alpha) - \alpha^{-(p\nu+2)} + \alpha^{p\nu+2}]^{\frac{1}{p}} \; , \tag{1.44}$$

and finally if $\alpha = \frac{1}{2}$

$$||T_{(\nu)}||_{\mathcal{L}(X_{1/2,p})} \leq \left(2^{p\nu+2}(\zeta(p\nu + 2) - 1) - 1 - \left(\frac{2}{3}\right)^{p\nu+2} \right)^{\frac{1}{p}} \; . \tag{1.45}$$

For $p = 2$, this last estimate can be rewritten as

$$||T_{(\nu)}||_{\mathcal{L}(X_{1/2,2})} \leq \left(2^{2\nu+2}(\zeta(2\nu + 2) - 1) - 1 - \left(\frac{2}{3}\right)^{2\nu+2} \right)^{\frac{1}{2}} \; . \tag{1.46}$$

Proceeding as in the proof of corollary 1.15, one easily checks that $T_{(\nu)}$ is a contraction on $X_{\alpha,p}$ for p sufficiently large.

2. BMO and the Brjuno function

In this section we want to prove that the Brjuno functions B_α belong to the space BMO (see [Gr], and also Appendix A for a short account on some fundamental results concerning BMO). Since they all differ by a L^∞ function, and $L^\infty \subset$ BMO, it will be enough to prove that B_α is BMO for a fixed value of α.

In this section we fix $\alpha = 1/2$ and denote $B_{1/2}$ simply by B.

Let $I \subset [0,1/2]$ be any interval, and $f \in L^1([0,1/2])$. We denote by

$$f_I = \frac{1}{|I|}\int_I f(x)dx \tag{2.1}$$

the mean of f over I. Using Lemma A.10, we also define the quadratic oscillation $\mathcal{O}_I(f)$ as follows

$$\mathcal{O}_I(f) = \left(\frac{1}{2|I|^2}\int_I\int_I (f(s)-f(t))^2 dsdt\right)^{\frac{1}{2}} . \tag{2.2}$$

As in the previous Section, we define for $1 \le p \le \infty$ the space $X_p \equiv X_{1/2,p}$:

$$X_p = \{f \in L^p([0,1/2]) \mid f(x+1) = f(x)\ \forall x \in \mathbb{R}\ ,\ f(-x) = f(x)\ \forall x \in [0,1/2]\}\ , \tag{2.3}$$

with the usual L^p-norm on $[0,1/2]$. In the above definition, f is extended from $[0,1/2]$ to $\mathbb{R}$, in such a way that it is even and periodic with period 1.

We now consider the space

$$X_* = \{f \in \mathrm{BMO}(\mathbb{R})\ \mid f(x+1) = f(x)\ \forall x \in \mathbb{R}\ ,\ f(-x) = f(x)\ \forall x \in [0,1/2]\}\ , \tag{2.4}$$

with the norm

$$||f||_* = A|f|_* + B||f||_{2,[0,1/2]}\ , \tag{2.5}$$

where $A > 0$ and $B > 0$ are fixed, $||f||_{2,[0,1/2]} = ||f||_{L^2(0,1/2)}$, and

$$|f|_* = \sup_{I \subset [0,1/2]} \mathcal{O}_I(f)\ . \tag{2.6}$$

Therefore we have

$$||f||_* = A \sup_{I\subset[0,1/2]}\left(\frac{1}{2|I|^2}\int_I\int_I (f(s)-f(t))^2\, dsdt\right)^{\frac{1}{2}} + B\left(\int_0^{1/2}(f(x))^2\, dx\right)^{\frac{1}{2}} . \tag{2.7}$$

Remark 2.1. Using John-Nirenberg's theorem and its corollary, the norm we use is equivalent to the usual BMO norm (see Appendix A, Propositions A.4, A.6, A.8, and A.10).

Remark 2.2. One has the obvious inclusions $X_\infty \subsetneqq X_* \subsetneqq X_1$. By elementary (but tedious) calculations, one checks that the even and periodic function wich coincides with $\log x$ on $(0,1/2]$ is in X_*, which can be written as $\log|x-[x]_{1/2}| \in X_*$.

We now recall the definition of the operator T. For $x \in \mathbb{R}$, let $[x]_{1/2}$ be the distance from x to the nearest integer. For $f \in L^1([0,1/2])$, we define Tf as in (1.27 and 28)

$$(Tf)(x) = xf\left(\left|\frac{1}{x} - \left[\frac{1}{x}\right]_{1/2}\right|\right) \quad \text{for} \quad x \in [0,1/2]\ . \tag{2.8}$$

We continue f , and Tf, to the full real axis so that they are even and periodic, and (2.8) can be rewritten as $(Tf)(x) = xf(1/x)$, for $0 \le x \le 1/2$. As already mentioned in Theorem 1.14, Tf is a linear operator from X_p to X_p for all $p \ge 1$ including ∞. More precisely, the norm of the operator T in the various X_p can be estimated. We have

$$||T||_{\mathcal{L}(X_\infty)} = \frac{1}{2}\ . \tag{2.9}$$

For finite $p \ge 1$, we use the bound described in Remark 1.18 above when $\alpha = 1/2$, namely

$$||T||_{\mathcal{L}(X_p)} \le C_p\ , \tag{2.10}$$

with

$$C_p = \left(\sum_{n=2}^{\infty}\left(\frac{1}{n^{p+2}} + \frac{1}{(n+1/2)^{p+2}}\right)\right)^{\frac{1}{p}}\ . \tag{2.11}$$

By comparing the sums in (2.11) to an integral over the interval $(3/2,\infty)$, which we cut at every integer and half-integer points, we get an easy estimate

$$C_p \le \left(2\int_{3/2}^{\infty}\frac{dx}{x^{p+2}}\right)^{\frac{1}{p}} = \left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{3}\right)^{1+\frac{1}{p}} \le \frac{2}{3} \quad \text{for } p \ge 1\ . \tag{2.12}$$

Therefore, as already stated in Theorem 2.14, the operator T is a contraction on all X_p, for $1 \le p \le \infty$. We shall now consider the action of T on X_*. We have the following theorem.

Theorem 2.3. *T is a bounded linear operator $T : X_* \mapsto X_*$. Moreover it is a contraction for the norm (2.7) with $A = 1$ and $B \ge 7.405$ (as in (2.41) below), that is*

$$||T||_{\mathcal{L}(X_*)} \le c_* < 1 \quad , \quad \text{with} \ \ c_* = \frac{40\sqrt{3}}{81} \sim 0.855. \tag{2.13}$$

In fact, if we allow to increase B to a sufficiently large value, one can reduce c_* to be as close as one wants to $(2/5)\sqrt{3} \sim 0.6928$.

Proof.

a) Let $f \in X_*$, we shall show that for the seminorm (2.6) we have $|Tf|_* < \infty$, which will prove that T is a linear operator $T : X_* \mapsto X_*$. For $I = [a,b] \subset [0,1/2]$, with $b > a$, we have from (2.6)

$$
\begin{aligned}
(\mathcal{O}_I(Tf))^2 &= \frac{1}{2|I|^2} \int_{I\times I} \left((Tf)(s) - (Tf)(t)\right)^2 \, dsdt \\
&= \frac{1}{2|I|^2}\left[2|I| \int_I ((Tf)(s))^2 \, ds - 2\left(\int_I (Tf)(s)\, ds\right)^2\right] \leq \frac{1}{|I|}\int_I ((Tf)(s))^2 \, ds \\
&\leq \frac{1}{|I|}\int_J (f(s))^2 \frac{ds}{s^4} \ , \qquad (2.14)
\end{aligned}
$$

where $J = [1/b, 1/a] \subset [2, \infty]$. Let now $[m/2, n/2]$ be the smallest interval with half integer edges, which contains J. We have

$$
1/b = \beta + m/2 \quad , \quad 1/a = n/2 - \alpha \quad , \quad 0 \leq \alpha < 1/2 \quad , \quad 0 \leq \beta < 1/2 \ . \qquad (2.15)
$$

Since $b - a > 0$, we have for the integers m and n, $4 \leq m < m+1 \leq n$. We get

$$
\begin{aligned}
(\mathcal{O}_I(Tf))^2 &\leq \frac{(m+2\beta)(n-2\alpha)}{2(n-m-2\beta-2\alpha)} \int_{m/2}^{n/2} (f(s))^2 \frac{ds}{s^4} \qquad (2.16) \\
&\leq ||f||^2_{2,[0,1/2]} \frac{(m+2\beta)(n-2\alpha)}{2(n-m-2\beta-2\alpha)} \sum_{k=0}^{n-m-1} \frac{2^4}{(m+k)^4} \ . \qquad (2.17)
\end{aligned}
$$

Comparing the last sum with the corresponding integral gives

$$
\begin{aligned}
(\mathcal{O}_I(Tf))^2 &\leq ||f||^2_{2,[0,1/2]} \frac{(m+2\beta)(n-2\alpha)}{2(n-m-2\beta-2\alpha)} \int_m^n \frac{2^4 ds}{(s-1)^4} \\
&\leq ||f||^2_{2,[0,1/2]} \frac{8(m+2\beta)(n-2\alpha)}{3(n-m-2\beta-2\alpha)} \left(\frac{1}{(m-1)^3} - \frac{1}{(n-1)^3}\right) \qquad (2.18)
\end{aligned}
$$

However, since $n > m$,

$$
\begin{aligned}
\frac{1}{(m-1)^3} - \frac{1}{(n-1)^3} &= \frac{(n-m)((n-1)^2 + (n-1)(m-1) + (m-1)^2)}{(n-1)^3(m-1)^3} \\
&\leq \frac{3(n-m)}{(n-1)(m-1)^3} \ ,
\end{aligned}
$$

moreover, $m + 2\beta \leq m+1$, and $n - 2\alpha \leq n$, which gives

$$
(\mathcal{O}_I(Tf))^2 \leq 8||f||^2_{2,[0,1/2]} \left(\frac{n-m}{n-m-2\alpha-2\beta}\right)\left(\frac{n}{n-1}\right)\left(\frac{m+1}{(m-1)^3}\right) \ .
$$

We have $m \geq 4$, and $n \geq 5$, therefore

$$\left(\frac{m+1}{(m-1)^3}\right) \leq \frac{5}{27} \quad , \quad \text{and,} \quad \left(\frac{n}{n-1}\right) \leq \frac{5}{4} \; .$$

In addition, we have $2|J| = n - m - 2\alpha - 2\beta$, and $0 \leq \alpha + \beta < 1$, therefore

$$(\mathcal{O}_I(Tf))^2 \leq \frac{50}{27}\left(1 + \frac{1}{|J|}\right) ||f||^2_{2,[0,1/2]} \; .$$

For intervals I such that the inverse interval J is not too small, that is $|J| \geq \delta$, we have a uniform bound for $\mathcal{O}_I(Tf)$ in terms of the L^2-norm of f. For $I = [a, b]$,

$$\text{if } \left(\frac{1}{a} - \frac{1}{b}\right) \geq \delta \; , \text{ we have } \quad \mathcal{O}_I(Tf) \leq \frac{5}{3}\sqrt{\frac{2}{3}}\left(1 + \frac{1}{\delta}\right)^{\frac{1}{2}} ||f||_{2,[0,1/2]} \; . \tag{2.19}$$

b) Now we must bound $\mathcal{O}_I(Tf)$ when $|J| < \delta$. For convenience, we will assume $\delta \leq 1/2$. From (2.6), we have

$$\mathcal{O}_I(Tf) = \left(\frac{1}{2|I|^2}\int_{I\times I} \left[(s-t)f(1/s) + t(f(1/s) - f(1/t))\right]^2 \, dsdt\right)^{\frac{1}{2}} \; .$$

Now, using the triangular inequality for the norm $L^2(I \times I)$, one gets

$$\mathcal{O}_I(Tf) \leq \Gamma_1 + \Gamma_2 \; , \tag{2.20}$$

where

$$\Gamma_1 = \left(\frac{1}{2|I|^2}\int_{I\times I} (s-t)^2 \left[f(1/s)\right]^2 \, dsdt\right)^{\frac{1}{2}} \; , \tag{2.21}$$

$$\Gamma_2 = \left(\frac{1}{2|I|^2}\int_{I\times I} t^2 \left[f(1/s) - f(1/t)\right]^2 \, dsdt\right)^{\frac{1}{2}} \; . \tag{2.22}$$

c) We first find an upper bound for Γ_1. In the integral in formula (2.21), we have $|s - t| < |I|$, and $|I| \leq 1/2$, therefore one gets

$$\begin{aligned}
\Gamma_1^2 \; &\leq \frac{1}{4}\int_I \left(f\left(\frac{1}{s}\right)\right)^2 \, ds = \int_J \frac{(f(s))^2}{4s^2} \, ds \\
&\leq \int_2^\infty \frac{(f(s))^2}{4s^2} \, ds \leq \sum_{n=0}^{\infty} \frac{1}{4(2+n/2)^2}\int_{2+n/2}^{3/2+n/2} (f(s))^2 \, ds \\
&\leq \frac{||f||^2_{2,[0,1/2]}}{4}\sum_{n=0}^{\infty} \frac{1}{(2+n/2)^2} \leq \frac{||f||^2_{2,[0,1/2]}}{2}\int_{-1}^{\infty} \frac{1}{(2+s/2)^2} \, ds \; ,
\end{aligned}$$

where we have used the comparison of the sum with an integral. Finally, one gets

$$\Gamma_1 \le \sqrt{\frac{2}{3}}||f||_{2,[0,1/2]}\ . \tag{2.23}$$

The contribution Γ_1 to $\mathcal{O}_I(Tf)$ is estimated in terms of The L^2-norm of f.

d) Now we will estimate Γ_2. From (2.22), one gets

$$\Gamma_2 = \left(\frac{1}{2|I|^2}\int_{J\times J}[(f(s)-f(t))]^2\ \frac{dsdt}{s^2t^4}\right)^{\frac{1}{2}}\ , \tag{2.24}$$

where, once more, $J=[1/b,1/a]$ is the image of I under inversion. We assume $|J|<\delta\le 1/2$, and we still define m, n, α, β, as in (2.15). Then we have $n-m-2\alpha-2\beta<2\delta\le 1$. We then have either $n=m+1$, or $n=m+2$.

If $n=m+1$, there is no points with half-integer coordinates in $J\times J$, and therefore $J\times J$ is contained in the interior of one cell of the lattice of half integer points. We bound from below s^2t^4 by $(m/2+\beta)^6$ in (2.24), and using the parity and periodicity of f, in order to shift the integration over $[0,1/2]\times[0,1/2]$, we get from (2.6)

$$\Gamma_2 \le \frac{1}{(m/2+\beta)^3}\left(\frac{|J|}{|I|}\right)|f|_* = \frac{m/2+1/2-\alpha}{(m/2+\beta)^2}|f|_*\ , \tag{2.25}$$

and

$$\Gamma_2 \le \left(\frac{1}{m+2\beta}+\frac{2\delta}{(m+2\beta)^2}\right)2|f|_* \le \left(\frac{1}{2}+\frac{\delta}{4}\right)|f|_* \le \frac{5}{8}|f|_*\ . \tag{2.26}$$

If $n=m+2$, there is one, and only one point with half-integer coordinates (s,t), which is interior to $J\times J$. This point has cordinates (q,q), with $q=p/2$ and $p=m+1=n-1$. We have $q\ge 5/2$, and we set $\alpha'=1/2-\alpha$ and $\beta'=1/2-\beta$.

The square $J\times J$ has non empty intersection with four cells of the lattice of half integer points. The lower left edge of these four cells are $(q-1/2,q-1/2)$ and (q,q) for the 'diagonal' ones, and $(q-1/2,q)$ and $(q,q-1/2)$ for the 'off diagonal' ones. The integral representation for Γ_2^2 deduced by squaring (2.24) can be split into four contributions corresponding to each cell.

The intersection of $J\times J$ with each diagonal cell is a square, and in the corresponding integral, we first bound from below s and t by the coordinates of its lower left edge. The remaining integrals are nothing else than $(|K|/|I|)^2(\mathcal{O}_K(f))^2$, with $K=[q-\beta',q]$ or $K=[q,q+\alpha']$ respectively. Then, due to the parity and periodicity property of f, these integrals are bounded by $(|K|/|I|)^2|f|_*^2$, with $|K|=\beta'$ or α' respectively.

The intersection of $J\times J$ with the off diagonal cells are rectangular but we still can bound from below s and t in the integrals by the coordinates of the lower left edge. Then, we enlarge each rectangle up to the smallest square in which it is contained, and having with the rectangle the common edge (q,q). Due to the parity and periodicity property of f, the two remaining integrals are both equal to $(|K|/|I|)^2(\mathcal{O}_K(f))^2$, with $K = [q, q+\gamma']$ and $\gamma' = \sup(\alpha', \beta')$. They can be bounded by $(|K|/|I|)^2|f|_*^2$, with $|K| = \gamma'$. Putting everything together, one gets

$$\Gamma_2 \leq C_2|f|_* \ , \tag{2.27}$$

with

$$|I|^2C_2^2 \leq \frac{\beta'^2}{(q-\beta')^6} + \frac{\alpha'^2}{q^6} + \frac{\gamma'^2}{q^2(q-\beta')^4} + \frac{\gamma'^2}{q^4(q-\beta')^2} \ . \tag{2.28}$$

From (2.15), we have

$$|I| = b - a = \frac{1}{\beta + m/2} - \frac{1}{n/2 - \alpha} = \frac{\alpha' + \beta'}{(q-\beta')(q+\alpha')} \ .$$

Therefore

$$C_2^2 = \frac{(q+\alpha')^2}{(q-\beta')^4}\left(\frac{\beta'^2 + \alpha'^2(1-\beta'/q)^6 + \gamma'^2(1-\beta'/q)^4 + \gamma'^2(1-\beta'/q)^2}{(\alpha'+\beta')^2}\right) \ ,$$

which implies, since $q \geq 5/2$, and $\beta' < 1/2$,

$$C_2^2 = \frac{(q+\alpha')^2}{(q-\beta')^4}\left(\frac{\beta'^2 + \alpha'^2 + 2\gamma'^2}{(\alpha'+\beta')^2}\right) \ . \tag{2.29}$$

However, $\beta'^2+\alpha'^2+2\gamma'^2 = 3\beta'^2+\alpha'^2$ or $\beta'^2+3\alpha'^2$, depending on whether $\beta' \geq \alpha'$, or $\beta' \leq \alpha'$. In both cases, it is easy to check that $\beta'^2+\alpha'^2+2\gamma'^2 \leq 3(\alpha'+\beta')^2$, since α', β' (and γ') are non negative. Then

$$C_2^2 = 3\left(\frac{(q+\alpha')^2}{(q-\beta')^4}\right) \ , \tag{2.30}$$

so that,

$$\Gamma_2 \leq \sqrt{3}\left(\frac{(q+\alpha')}{(q-\beta')^2}\right)|f|_* \ , \tag{2.31}$$

which is valid for $q \geq 5/2$, $0 \leq \alpha' < 1/2$, $0 \leq \beta' < 1/2$, and, since $n - m = 2$, for $\alpha' + \beta' < \delta$. Now (2.31) implies

$$\Gamma_2 \leq \sqrt{3}\left(\frac{1}{(q-\beta')} + \frac{\alpha'+\beta'}{(q-\beta')^2}\right)|f|_* \leq \sqrt{3}\left(\frac{1}{(q-\delta)} + \frac{\delta}{(q-\delta)^2}\right)|f|_* \ ,$$

that is

$$\Gamma_2 \leq \sqrt{3}\left(\frac{q}{(q-\delta)^2}\right)|f|_* \leq \sqrt{3}\left(\frac{10}{(5-2\delta)^2}\right)|f|_* \ . \tag{2.32}$$

e) Putting together (2.20), (2.23), (2.26) and (2.32), we get the following statement.

For $I=[a,b]$, if $\left(\frac{1}{a}-\frac{1}{b}\right)<\delta$ we have

$$\mathcal{O}_I(Tf)\leq\sqrt{\frac{2}{3}}||f||_{2,[0,1/2]}+\sup\left(\frac{1}{2}+\frac{\delta}{4},\frac{10\sqrt{3}}{(5-2\delta)^2}\right)|f|_*\ .$$

A bit of analysis shows that the supremum is always reached by the second argument when $0\leq\delta\leq 1/2$. Therefore

$$\mathcal{O}_I(Tf)\leq\sqrt{\frac{2}{3}}||f||_{2,[0,1/2]}+\left(\frac{10\sqrt{3}}{(5-2\delta)^2}\right)|f|_*\ . \tag{2.33}$$

The Equations (2.19) and (2.33) show that $|Tf|_*$ is finite, when $|f|_*$ and $||f||_{2,[0,1/2]}$ are finite, which completes the proof that $f\in X_*$ implies $Tf\in X_*$.

f) We will now show that T is a bounded linear operator $X_*\to X_*$ for a suitable choice of the coefficients A and B in the norm (2.7).

First we note that (2.10) and (2.12) imply

$$||Tf||_{2,[0,1/2]}\leq c_1||f||_{2,[0,1/2]}\ , \tag{2.34}$$

with $c_1=4/9$. Then, collecting results of (2.19) and (2.33) we get for some choice of the parameter δ,

$$|Tf|_*\leq c_*|f|_*+c_2||Tf||_{2,[0,1/2]}\ . \tag{2.35}$$

Therefore,

$$A|Tf|_*+B||Tf||_{2,[0,1/2]}\leq c_*\left(A|f|_*+\left(\frac{Ac_2}{c_*}+\frac{Bc_1}{c_*}\right)||f||_{2,[0,1/2]}\right)\ .$$

However can choose $A>0$ and $B>0$ such that

$$Bc_*\geq Ac_2+Bc_1\ , \tag{2.36}$$

because $c_*>c_1=4/9$. In fact the latter inequality results from (2.33) which shows that $c_*>1/2$. With such a choice for A and B, we would get

$$A|Tf|_*+B||Tf||_{2,[0,1/2]}\leq c_*\left(A|f|_*+B||f||_{2,[0,1/2]}\right)\ ,$$

that is, in view of (2.7),

$$||Tf||_*\leq c_*||f||_*\ . \tag{2.37}$$

The latter equation shows that T is a bounded operator in X_*, for the norms $||f||_*$ with

$$A=1\quad,\quad\text{and}\qquad B\geq\frac{c_2}{c_*-c_1}\ . \tag{2.38}$$

With this norm on X_*, the norm of the operator T fulfils $||T||_{\mathcal{L}(X_*)}\leq c_*$.

g) We now show that by a suitable choice of δ, we will get $c_* < 1$, which will show that T is a contraction on X_*. we have from (2.33)

$$c_* = \left(\frac{10\sqrt{3}}{(5-2\delta)^2} \right) \quad . \tag{2.39}$$

Comparing (2.19) and (2.33) shows that

$$c_2 = \frac{5}{3}\sqrt{\frac{2}{3}}\left(1+\frac{1}{\delta}\right)^{\frac{1}{2}} \quad . \tag{2.40}$$

However, as seen from (2.39), c_* is a monotone function of δ, which increases from 0.6928 to 1.0825 when δ increases from 0 to 1/2. It reaches one for δ around 0.4. Taking δ smaller than 0.4, proves that T is a contraction. In order to give numbers, we take $\delta = 1/4$, wich gives

$$c_2 = \frac{5}{3}\sqrt{\frac{10}{3}} \qquad , \qquad c_* = \frac{40\sqrt{3}}{81} \sim 0.855$$

The corresponding values of B are

$$B \geq \left(\frac{5}{3}\sqrt{\frac{2}{3}}\left(1+\frac{1}{\delta}\right)^{\frac{1}{2}} \right) \left(\frac{10\sqrt{3}}{(5-2\delta)^2} - \frac{4}{9} \right)^{-1} = \frac{45}{2}\sqrt{\frac{5}{2}}\frac{1}{10-3\sqrt{3}} \sim 7.405. \tag{2.41}$$

If now one takes δ very small, one can reduce c_* as close as one wants to its value for $\delta = 0$, which is $(2/5)\sqrt{3} = 0.6928$, but meanwhile, B must become very large, and increase proportionnaly with $\delta^{-(1/2)}$. This completes the proof of Theorem 2.3. □

An immediate consequence of Theorem 2.3 is the following theorem

Theorem 2.4 *The Brjuno function $B(x) \equiv B_{1/2}(x)$ belongs to X_*, and therefore to* BMO($\mathbb{R}$)*. The Brjuno functions $B_\alpha(x)$ belong to* BMO($\mathbb{R}$) *for $1/2 \leq \alpha \leq 1$.*

Proof. $B(x)$ is the unique solution of (1.29) for $\alpha = 1/2$, that is

$$((1-T)B)(x) = -\log x \quad , \quad \text{when} \quad 0 \leq x \leq 1/2 \ .$$

However T is a contraction in X_*, and $\log x$ is in BMO([0, 1/2]). Therefore, $1 - T$ is invertible in BMO([0, 1/2]), and $B \in$ BMO([0, 1/2]). From Corollary A.9, we deduce $B \in$ BMO($\mathbb{R}$). Now, from Theorem 1.12, for $1/2 \leq \alpha \leq 1$ we have $B_\alpha - B_{1/2} \in L^\infty(\mathbb{R})$, which implies $B_\alpha \in$ BMO($\mathbb{R}$), since $L^\infty(\mathbb{R}) \subset$ BMO($\mathbb{R}$). □

Let us now go back to Theorem 2.3. The contraction property of T depends on the norm we have taken on X_*. Indeed it is not valid in all possible equivalent norms. However, the spectral radius of T in X_* does not depend on the choice of the norm in the equivalence class. This makes the following Theorem 2.5 relevant.

Theorem 2.5. *The spectral radius of T in X_* is at most equal to $\sqrt{2}-1$.*

Proof.

a) We already know from Theorem 2.3 that the spectral radius of T, which is defined as $\rho(T) = \limsup ||T^n||^{1/n}$, fulfils

$$\rho(T) \le \frac{2\sqrt{3}}{5} \sim 0.6928 \ . \tag{2.42}$$

b) We first show that $||T||_{\mathcal{L}(X_2)} < \sqrt{2}-1$. But that is just where we need the estimate of Equation (1.40). Indeed we have for $f \in X_2$

$$||Tf||_{2,[0,1/2]} \le c_1 ||f||_{2,[0,1/2]} \ , \quad \text{so that} \quad ||T||_{\mathcal{L}(X_2)} \le c_1 \ , \tag{2.43}$$

with

$$c_1 = \left(\frac{8\pi^4}{45} - \frac{1393}{81}\right)^{\frac{1}{2}} \sim 0.34589... \le \sqrt{2}-1 \sim 0.4142 \ . \tag{2.44}$$

Obviously we have

$$||T^n f||_{2,[0,1/2]} \le c_1^n ||f||_{2,[0,1/2]} \ . \tag{2.45}$$

c) Assume now $f \in X_*$, and let δ be a positive number such that $\delta \le 1/2$. We consider intervals $I = [a,b] \subset [0,1/2]$. Let $J = [1/b, 1/a]$ be the image of I under the change $x \to 1/x$. There are two possibilities :

c1) J has length greater or equal to δ.

c2) J has length smaller than δ.

We will now estimate in case c1) the quadratic oscillation $\mathcal{O}_I(Tf)$ as defined in (2.6). For this purpose, the argument in part a) of the proof of Theorem 2.3 which leads to (2.19) may be used without changes. Therefore we have for $I = [a,b]$ and $\delta \le 1/2$,

$$\text{if} \quad \left(\frac{1}{a} - \frac{1}{b}\right) \ge \delta \ , \text{ we have } \quad \mathcal{O}_I(Tf) \le \frac{5}{3}\sqrt{\frac{2}{3}}\left(1+\frac{1}{\delta}\right)^{\frac{1}{2}} ||f||_{2,[0,1/2]} \ . \tag{2.46}$$

Obviously from (2.43), we have (still in case c1) above) for $n \ge 1$,

$$\mathcal{O}_I(T^n f) \le \frac{5}{3}\sqrt{\frac{2}{3}}\left(1+\frac{1}{\delta}\right)^{\frac{1}{2}} c_1^{n-1} ||f||_{2,[0,1/2]} \ . \tag{2.47}$$

We must now deal with case c2), that is we assume $(1/a - 1/b) < \delta \le 1/2$. The argument of part b) of the proof of Theorem 2.3 applies without changes and we get as in (2.20)

$$\mathcal{O}_I(Tf) \le \Gamma_1 + \Gamma_2 \ , \tag{2.48}$$

where Γ_1 and Γ_2 are given by (2.21) and (2.22) respectively. We get a bound for Γ_1 as in part c) of the proof of Theorem 2.3, namely

$$\Gamma_1 \le \sqrt{\frac{2}{3}}||f||_{2,[0,1/2]} \ . \tag{2.49}$$

which is nothing else than Equation (2.23).

Now we use (2.24) to estimate Γ_2, and we will introduce the transformation $\tau : I \to \tau I$, defined on intervals I which are subintervals of $[0, 1/2]$, and such that the inverse interval J has length less than $\delta \le 1/2$. When these conditions are not fulfilled (that is in case c1) above), we shall say that τI is *undefined*. Let now $[r/2, s/2]$ be the smallest interval with integer or half integer edges, which contains $J = [1/b, 1/a] \subset [2, \infty]$. We have

$$1/b = r/2 + \beta \quad , \quad 1/a = s/2 - \alpha \quad , \quad 0 \le \alpha < 1/2 \quad , \quad 0 \le \beta < 1/2 \ . \tag{2.50}$$

Since $b - a > 0$, we have for the integers r and s, $4 \le r < r+1 \le s$. Since we assume $|J| < \delta \le 1/2$, we have $s - r - 2\alpha - 2\beta < 2\delta \le 1$. We then have either $s = r+1$, or $s = r+2$.

If $s = r+1$, there is no integer or half-integer point in the interior of J, and we can define τI as the interval contained in $[0, 1/2]$ deduced from J by translation or by translation followed by a symmetry around zero. We get $\tau I = [\beta, 1/2 - \alpha]$ when r is even, and $\tau I = [\alpha, 1/2 - \beta]$ when r is odd. We have $|\tau I| = |J| = 1/2 - \alpha - \beta < \delta \le 1/2$. It is easy to check that, in the present case, τI is nothing else than the image of I under the map $A_{1/2}$ defined in Section 1, Equation (1.3). Then, following the argument which leaded us to (2.25) we get

$$\Gamma_2 \le \frac{1}{(r/2+\beta)^3}\left(\frac{|\tau I|}{|I|}\right)\mathcal{O}_{\tau I}(f) = \frac{r/2 + 1/2 - \alpha}{(r/2+\beta)^2}\mathcal{O}_{\tau I}(f) = \frac{b^2}{a}\mathcal{O}_{\tau I}(f) \ . \tag{2.51}$$

If $s = r+2$, we have $J = [(r+1)/2 - (1/2 - \beta), (r+1)/2 + (1/2 - \alpha)]$, so that the point $(r+1)/2$ is in J. When r is odd, by translation, we can shift J to $J' = [\beta - 1/2, 1/2 - \alpha]$ which contains 0. In this case, we define $\tau I = [0, \sup(1/2 - \alpha, 1/2 - \beta)]$. When r is even, by translation, we can shift J to $J' = [1/2 - (1/2 - \beta), 1/2 + (1/2 - \alpha)]$ which contains $1/2$. In this case we define $\tau I = [1/2 - \sup(1/2 - \alpha, 1/2 - \beta), 1/2]$. In both cases, we have $|\tau I| = \sup(1/2 - \alpha, 1/2 - \beta)$, and

$$|J|/2 \le (1 - \alpha - \beta)/2 \le |\tau I| \le 1 - \alpha - \beta = |J| < \delta \ . \tag{2.52}$$

In the present case τI is contained in the image of I through the map $A_{1/2}$. Once more, following the argument which leaded us to (2.25) we get, since $|J| = |J'|$

$$\Gamma_2 \le \frac{1}{(r/2+\beta)^3}\left(\frac{|J|}{|I|}\right)\mathcal{O}_{J'}(f) = \frac{r/2 + 1 - \alpha}{(r/2+\beta)^2}\mathcal{O}_{J'}(f) \ . \tag{2.53}$$

By Lemma A.11, we have $\mathcal{O}_{J'}(f) \le 2\mathcal{O}_{\tau I}(f)$, and

$$\Gamma_2 \le \frac{2}{(r/2+\beta)^3}\left(\frac{|J|}{|I|}\right)\mathcal{O}_{\tau I}(f) = \frac{2(r/2+1-\alpha)}{(r/2+\beta)^2}\mathcal{O}_{\tau I}(f) = \frac{2b^2}{a}\mathcal{O}_{\tau I}(f) \ . \quad (2.54)$$

Note than when $s = r+2$, one of the edges of τI is either 0 or 1/2. We summarize these results :

$$\text{if } \tau I \text{ is undefined} \quad \mathcal{O}_I(Tf) \le \frac{5}{3}\sqrt{\frac{2}{3}}\left(1+\frac{1}{\delta}\right)^{\frac{1}{2}} ||f||_{2,[0,1/2]} \quad (2.55)$$

$$\text{if } \tau I \text{ is defined} \quad \mathcal{O}_I(Tf) \le \sqrt{\frac{2}{3}}||f||_{2,[0,1/2]} + \frac{\lambda b^2}{a}\mathcal{O}_{\tau I}(f) \ , \quad (2.56)$$

where $\lambda = 1$ or 2, depending on whether $s = r+1$, in which case (2.51) holds, or $s = r+2$, in which case (2.54) holds. When τI is defined, a is strictly positive, and conversely when $a = 0$, τI is undefined, because $|J|$ is then infinite. When $\lambda = 2$, either 0 or 1/2 belongs to τI. Setting

$$C = \frac{5}{3}\sqrt{\frac{2}{3}}\left(1+\frac{1}{\delta}\right)^{\frac{1}{2}} > \sqrt{\frac{2}{3}} \ , \quad (2.57)$$

we rewrite (2.55) and (2.56) in one single equation

$$\mathcal{O}_I(Tf) \le C||f||_{2,[0,1/2]} + \frac{\lambda b^2}{a}\mathcal{O}_{\tau I}(f) \ , \quad (2.58)$$

with the convention that $\lambda = 0$ if τI is undefined, so that the second term in the above equation is absent (this happens in particular when $a = 0$).

Note that $1/a - 1/b < \delta$ implies $b/a < 1 + b\delta$, and since $b \le 1/2$,

$$\frac{2b^2}{a} < \gamma = 1 + \frac{\delta}{2} \ . \quad (2.59)$$

We have $|\tau I| = |J|$ when $\lambda = 1$, and from (2.52), $|J|/2 \le |\tau I| \le |J|$ when $\lambda = 2$. Also $|I|/|J| = ab \le 1/4$, therefore we have

$$\left.\begin{array}{ll} \text{when } \lambda = 1 & \dfrac{|\tau I|}{|I|} = \dfrac{1}{ab} \ge 4 \\ \text{when } \lambda = 2 & \dfrac{|\tau I|}{|I|} \ge \dfrac{1}{2ab} \ge 2 \end{array}\right\} \ . \quad (2.60)$$

d) We are now ready to iterate the procedure. We start from

$$\begin{aligned} \mathcal{O}_I(T^n f) &\le C||T^{n-1}f||_{2,[0,1/2]} + \frac{\lambda b^2}{a}\mathcal{O}_{\tau I}(T^{n-1}f) \\ &\le Cc_1^{n-1}||f||_{2,[0,1/2]} + \frac{\lambda b^2}{a}\mathcal{O}_{\tau I}(T^{n-1}f) \ , \end{aligned}$$

and use once more the equation when τI is defined, shifting n to $n-1$, and so one. We get after k steps (for $k \leq n-1$)

$$\mathcal{O}_I(T^n f) \leq C_k ||f||_{2,[0,1/2]} + \left(\prod_{i=0}^{k} \frac{\lambda_i b_i^2}{a_i} \right) \mathcal{O}_{\tau^{k+1} I}(T^{n-k-1} f) \ . \qquad (2.61)$$

The definition of $\tau^k I$ in the above equation, is recursive. Obviously, if $\tau^k I$ is undefined, $\tau^{k+1} I$ is also undefined. We sart from $\tau^0 I = I$, then we set $\tau^{k+1} I = \tau(\tau^k I)$ if it is defined, and the value of λ_k is 1 or 2 as explained above after (2.56). If $\tau(\tau^k I)$ is undefined, we set $\lambda_k = 0$. We also have set $\tau^k I = [a_k, b_k]$, and $I = [a_0, b_0]$. The recursion starts with

$$C_0 = c_1^{n-1} C \ ,$$

and proceeds as

$$C_{k+1} = C_k + c_1^{n-k-2} C \left(\prod_{i=0}^{k} \frac{\lambda_i b_i^2}{a_i} \right) \ ,$$

so that

$$C_{k+1} = C_0 + C \sum_{j=0}^{k} c_1^{n-j-2} \left(\prod_{i=0}^{j} \frac{\lambda_i b_i^2}{a_i} \right) \leq C_0 + C \sum_{j=0}^{k} c_1^{n-j-2} \gamma^{j+1} \ ,$$

that is

$$C_{k+1} = C c_1^{n-k-2} \frac{\gamma^{k+2} - c_1^{k+2}}{\gamma - c_1} \ .$$

Setting $k = n-2$, we get

$$C_{n-1} = C c_1^{-2} \frac{\gamma^n - c_1^n}{\gamma - c_1} \ , \qquad (2.62)$$

where γ is given by (2.59) and c_1 by (2.44). From (2.61) we get

$$\mathcal{O}_I(T^n f) \leq C_{n-1} ||f||_{2,[0,1/2]} + \left(\prod_{i=0}^{n-1} \frac{\lambda_i b_i^2}{a_i} \right) \mathcal{O}_{\tau^n I}(f) \ . \qquad (2.63)$$

The important feature of the above equation is that the coefficient of $||f||_{2,[0,1/2]}$ does not depend on I. We will now bound the coefficient of $\mathcal{O}_{\tau^n I}(f)$. For this purpose, we observe that in the sequence $\lambda_0, \lambda_1, \ldots \lambda_{n-1}$, there is at most one element equal to 2. Indeed, assume that $\lambda_r = 2$ for some r. We have already noticed after (2.56) that if $\lambda_r = 2$, then $\tau^{r+1} I$ contains either 0 or 1/2. If it contains zero, its inverse is infinite, then $\lambda_{r+1} = 0$ since then $\tau^{r+1} I$ is undefined and all following $\lambda' s$ vanish. If $\tau^{r+1} I$ contains 1/2, then its inverse contains 2, and either $\tau^{r+1}I$ is undefined in which case $\lambda_{r+1} = 0$, or it has 0 as lower edge, and then $\lambda_{r+1} = 1$. By the same argument $\tau^{r+2}I$ is then undefined, $\lambda_{r+2} = 0$ and all following $\lambda's$ vanish. If all $\lambda_i \neq 0$ for $0 \le i \le n-1$, then all $\lambda_i = 1$ except may be λ_{n-2} or λ_{n-1}. Therefore we have

$$\mathcal{O}_I(T^n f) \le C_{n-1}||f||_{2,[0,1/2]} + \left(2 \prod_{i=0}^{n-1} \frac{b_i^2}{a_i} \right) \mathcal{O}_{\tau^n I}(f) \ . \tag{2.64}$$

Taking as in (2.6) the supremum over all possible intervals I, we get

$$|T^n f|_* \le C_{n-1}||f||_{2,[0,1/2]} + C_{*,n}|f|_* \ , \tag{2.65}$$

where $C_{*,n}$ is given by

$$C_{*,n} = \sup_I \left(2 \prod_{i=0}^{n-1} \frac{b_i^2}{a_i} \right) \ , \tag{2.66}$$

where the supremum is taken over intervals I such that the intervals $\tau^k I = [a_k, b_k]$ are defined for $1 \le k \le n$.

e) We shall now prove there exists a constant $K > 1$ independant on n such that

$$C_{*,n} \le 2K(\sqrt{2} - 1)^n \ . \tag{2.67}$$

From the definition of τ given in part c) above, it is easy to see that one can find for $0 \le i \le n-1$ a sequence of numbers x_i, such that $x_i \in \tau^i I = [a_i, b_i]$, and $x_{i+1} = A_{1/2}(x_i)$, for $0 \le i \le n-2$, where

$$A_{1/2}(x) = \left| \frac{1}{x} - \left[\frac{1}{x} \right]_{1/2} \right| \ , \tag{2.68}$$

as in (1.3). For that purpose, take x_{n-1} arbitrarily in the interior of $\tau^{n-1}I$, and compute backward x_k from x_{k+1}, adding at each step the suitable integer part. We choose x_{n-1} irrational to avoid the discontinuity points of the integer part $[x]_{1/2}$ defined in (1.1). We obviously have

$$|b_i - a_i| \le |\tau^i I| \quad , \quad |x_i - a_i| \le |\tau^i I| \quad , \quad |x_i - b_i| \le |\tau^i I| \ . \tag{2.69}$$

From (2.60), we get for $0 \le k \le n-1$,

$$|\tau^k I| \le 2^{-(n-k)}|\tau^n I| \le 2^{-(n-k)}\delta \le 2^{-(n-k+1)} \ , \tag{2.70}$$

since $\delta \le 1/2$. In fact we also get a better bound

$$|\tau^k I| \le 2a_k b_k |\tau^{k+1} I| \le a_k b_k 2^{-(n-k-2)}\delta \le a_k b_k 2^{-(n-k-1)} \ . \tag{2.71}$$

Then we get

$$\frac{b_i^2}{a_i} = x_i \left(1 + \frac{b_i - a_i}{a_i}\right) \left(1 + \frac{b_i - x_i}{x_i}\right) \le x_i \left(1 + \frac{b_i - a_i}{a_i}\right)^2 ,$$

so that using (2.69) and (2.71), we have

$$\frac{b_i^2}{a_i} \le x_i \left(1 + \frac{b_i}{2^{n-1-i}}\right)^2 \le x_i \left(1 + \frac{1}{2^{n-i}}\right)^2 .$$

Therefore

$$\prod_{i=0}^{n-1} \frac{b_i^2}{a_i} \le \exp\left(2\sum_{i=0}^{n-1} \log(1 + 2^{-(n-i)})\right) \prod_{i=0}^{n-1} x_i \le \exp\left(2\sum_{i=0}^{n-1} 2^{-(n-i)}\right) \prod_{i=0}^{n-1} x_i$$
$$\le \exp\left(\sum_{i=0}^{n-1} 2^{-i}\right) \prod_{i=0}^{n-1} x_i \le \exp\left(\sum_{i=0}^{\infty} 2^{-i}\right) \prod_{i=0}^{n-1} x_i \le e^2 \prod_{i=0}^{n-1} x_i .$$

Finally,

$$C_{*,n} \le 2e^2 \sup_{x_0} \left(\prod_{i=0}^{n-1} x_i\right) = 2e^2 \sup_{x_0} \left(\prod_{i=0}^{n-1} A_{1/2}^k(x_0)\right) , \tag{2.72}$$

where now the supremum is taken over x_0, that is over all possible trajectories of the map $x \to A_{1/2}(x)$. The estimate (2.72) deserves some comments. Although the map τ which transforms the interval $[a_i, b_i]$ into $[a_{i+1}, b_{i+1}]$ give the same result as the action of $A_{1/2}$ when λ_i is equal to 1, the sequences $a_0, a_1, \ldots, a_n$, and $b_0, b_1, \ldots, b_n$ are not in general trajectories for $A_{1/2}$, because at each step a_{i+1} is defined as the lower edge of $\tau[a_i, b_i]$. Moreover, when $\lambda = 2$, there is a further restriction of the interval. Nevertheless these sequences can be approximated in a suitable way by a true trajectory of $A_{1/2}$, because this map is expansive. In particular, this approximation prevents the a_i's to become too small. The determination of the trajectory of $A_{1/2}$ is in fact an appliction of the so-called Shadowing Lemma, which is a classical trick in expansive dynamical systems.

The estimate (2.67) then resuts from Proposition 1.4. which is an almost classical result in the theory of continued fractions described in Section 1.

f) From (2.45) we have

$$||T^n f||_{2,[0,1/2]} \le c_1^n ||f||_{2,[0,1/2]} , \tag{2.73}$$

and from (2.64) and (2.67) we get

$$|T^n f|_* \le C_{n-1} ||f||_{2,[0,1/2]} + 2K(\sqrt{2} - 1)^n |f|_* , \tag{2.74}$$

By an argument similar to part f) of the proof of Theorem 2.3, we use a norm of the family (2.7) with $A = 1$ and

$$B > \frac{C_{n-1}}{2K(\sqrt{2} - 1)^n - c_1^n} ,$$

which is positive in view of (2.44). With this norm,

$$||T^n f||_* \le 2K(\sqrt{2}-1)^n ||f||_* \ . \tag{2.75}$$

Since $\limsup\ (2K)^{1/n} = 1$, the spectral radius of T in X_* fulfils

$$||T||_{\mathcal{L}(X_*)} \le \sqrt{2}-1 \ . \tag{2.76}$$

which completes the proof of Theorem 2.5. $\square$

Remark 2.6. The same argument can be applied to the operator $T_{(\nu)}$ introduced in Remark 1.18. The main change in the argument is the replacement of (b^2/a) in the equations (2.51) and (2.56) by $(b^{1+\nu}/a)$, so that (after recomputing all constants !) one finally gets instead of (2.74) and estimate of the kind.

$$|T^n_{(\nu)} f|_* \le C_{n-1,\nu} ||f||_{2,[0,1/2]} + K_\nu(\sqrt{2}-1)^{n\nu} |f|_* \ , \tag{2.77}$$

A careful examination of (1.35) for $\alpha = 1/2$ shows that one can reduce the L_2-norm estimate up to

$$\begin{aligned} ||T_{(\nu)}||_{\mathcal{L}(X_{1/2,2})} &\le \left(2 \sum_{n=2}^{\infty} \frac{1}{(m+1/2)^{2+2\nu}} \right)^{\frac{1}{2}} \\ &\le \left(2(2^{2+2\nu}-1)\zeta(2+2\nu) - 2^{3+2\nu} - 2\left(\frac{2}{3}\right)^{2+2\nu} \right))^{\frac{1}{2}} \end{aligned} \tag{2.78}$$

Therefore, the spectral radius of $T_{(\nu)}$ is less or equal to the largest of the two following numbers : the BMO-norm estimate $(\sqrt{2}-1)^\nu$ and the above L_2-norm estimate (2.78), which is better than (1.46). We already know that the BMO-norm estimate is smaller when $\nu = 1$, we also see that the L_2-norm will be smaller for large ν since it behaves as $(2/5)^\nu$.

A more concise way toward the spectral radius of T can be obtained using the invariant measure described in Section 1 above. This can be found in [MMY].

3. Regularity properties for the functional equation of the Brjuno function

The functional equation (1.29) for the Brjuno function for $\alpha = 1/2$ is

$$[(1-T)B_{1/2}](x) = -\log x \ , \tag{3.1}$$

for all $x \in (0, 1/2)$, complemented with the condition that $B = B_{1/2}$ is even and periodic. In this Section we will suppose that the right hand side of this equation is pertubed, by an additional term f, which is less singular than the logarithmic function. We want to study the singular properties of the perturbed solution. Since the equation is linear, we only need to consider the action on f of T and $(1-T)^{-1}$, which will be conveniently called the Brjuno operator $\mathbf{B}$. We will consider even and periodic functions f which are *continuous.* It is sufficient to know the value of f on $[0, 1/2]$, so we assume $f \in C^0_{[0,1/2]}$. One can check that Tf is also continuous provided we set $Tf(0) = 0$. We need now the usual Hölder's type semi-norms for continuous functions.

Definition 3.1. *Let $f \in C^0_{[0,1/2]}$, Then we define the Hölder's γ-norm as*

$$|f|_\gamma = \sup_{0 \le x < y \le 1/2} \frac{|f(x) - f(y)|}{|x-y|^\gamma} \ , \tag{3.2}$$

with $0 < \gamma \le 1$. This is a seminorm since it vanishes on constant functions, so that we introduce the norm :

$$||f||_\gamma = A|f|_\gamma + B|f|_\infty \ , \tag{3.3}$$

with A and B positive constants. We say that $f \in C^\gamma$, if $f \in C^0_{[0,1/2]}$ and $|f|_\gamma$ is finite.

We now have :

Proposition 3.2.
i) T is a bounded operator in C^γ, for the norm $||f||_\gamma$, provided $0 < \gamma \le 1/2$, for B/A large enough : if $B/A > (2^\gamma - 2^{-\gamma})^{-1}$, the norm of T corresponding to the norm (3.3) satisfies $||T||_\gamma \le 2^{(2\gamma-1)} \le 1$.
ii) For $0 < \gamma < 1/2$, T is a contraction.

We need the following Lemma

Lemma 3.3 *Let $0 < y < x \le 1/2$, and define x_1 and y_1 by the following conditions*

$$y = \frac{1}{n + y_1} \quad , \quad x = \frac{1}{m + x_1} \ , \tag{3.4}$$

with $n \ge 2$ and $m \ge 2$, and $-1/2 \le x_1 < 1/2$ and $-1/2 \le y_1 < 1/2$, then we have

$$||x_1| - |y_1|| \le \frac{|x-y|}{|x||y|} \ . \tag{3.5}$$

Proof of Lemma 3.3. Since $y < x$, we have $n - m > x_1 - y_1 > -1$. therefore $n \geq m$. Let $n - m = p \geq 0$. We have $x - y = xy(p + y_1 - x_1)$. So that we need to prove $||x_1| - |y_1|| \leq |p + y_1 - x_1|$. This is obvious when $p = 0$. We always have $||x_1| - |y_1|| \leq 1/2$, so the required inequality also holds when $p \geq 2$. In the remaining case $p = 1$, we set $\eta = \mathrm{sign}(y_1)$ and $\epsilon = \mathrm{sign}(x_1)$, and we need to check that $||x_1| - |y_1|| \leq |1 + \eta|y_1| - \epsilon|x_1||$. Still because the left hand side is smaller or equal to $1/2$, this last inequality is not obvious only when $\eta = -1$ and $\epsilon = +1$. It therefore remains to show that $||x_1| - |y_1|| \leq |1 - |y_1| - |x_1||$. Setting $u = 1/2 - |x_1|$ and $v = 1/2 - |y_1|$, the last inequality is equivalent to $|1 - v/u| \leq |1 + v/u|$, which is readily checked since u/v is real and non-negative. □

Proof of Proposition 3.2 Let $0 < y < x \leq 1/2$, and x_1 and y_1 as in the preceding lemma. We have

$$\begin{aligned}
|Tf(x) - Tf(y)| &= |xf(1/x) - yf(1/y)| = |xf(|x_1|) - yf(|y_1|)| \\
&\leq |x - y||f(|x_1|)| + |y||f(|x_1|) - f(|y_1|)| \\
&\leq |x - y||f|_\infty + |y||f|_\gamma ||x_1| - |y_1||^\gamma \\
&\leq |x - y||f|_\infty + |f|_\gamma \frac{|x - y|^\gamma}{|x|^\gamma |y|^{\gamma - 1}} ,
\end{aligned}$$

where we have used $f \in C^\gamma$, and Lemma 3.3. Therefore

$$\begin{aligned}
\frac{|Tf(x) - Tf(y)|}{|x - y|^\gamma} &\leq |x - y|^{1-\gamma}|f|_\infty + \left(\frac{|y|}{|x|}\right)^\gamma |y|^{1-2\gamma}|f|_\gamma \\
&\leq (1/2)^{1-\gamma}|f|_\infty + (1/2)^{1-2\gamma}|f|_\gamma ,
\end{aligned}$$

since $0 < y < x \leq 1/2$, and $\gamma \leq 1/2$. For $y = 0$, the right hand side can be replaced by its first term $(1/2)^{1-\gamma}|f|_\infty$, and the above inequality extends to the case where y vanishes, so that

$$|Tf|_\gamma \leq K_\gamma(f) = 2^{\gamma-1}|f|_\infty + 2^{2\gamma-1}|f|_\gamma . \tag{3.6}$$

For the norm, we get

$$\begin{aligned}
||Tf||_\gamma =& A|Tf|_\gamma + B|Tf|_\infty \\
&\leq 2^{2\gamma-1}\left[A|f|_\gamma + (2^{-2\gamma}B + 2^{-\gamma}A)|f|_\infty\right] , \\
&\leq 2^{(2\gamma-1)}||f||_\gamma ,
\end{aligned}$$

provided $2^{-2\gamma}B + 2^{-\gamma}A \leq B$, that is

$$A/B \leq 2^\gamma - 2^{-\gamma}$$

which completes the proof. □

The above proposition has two obvious consequences

i) Since $C^\gamma \subset C^{\gamma'}$ whenever $\gamma' \leq \gamma$, we have

$$f \in C^\gamma \quad \text{and} \quad \gamma \geq 1/2 \qquad \Longrightarrow \qquad Tf \in C^{1/2}$$
$$f \in C^\gamma \quad \text{and} \quad \gamma \geq \gamma_0 \quad , \quad \gamma_0 \leq 1/2 \qquad \Longrightarrow \qquad Tf \in C^{\gamma_0} \ .$$

ii) When $\gamma < 1/2$, T is a contraction on C^γ. Therefore $1 - T$ is invertible and $\mathbf{B} = \sum_0^\infty T^n = (1-T)^{-1}$ preserves C^γ, and we have.

$$f \in C^\gamma \quad \text{and} \quad 0 < \gamma < 1/2 \qquad \Longrightarrow \qquad \mathbf{B}f \in C^\gamma$$
$$f \in C^{1/2} \qquad \Longrightarrow \qquad \mathbf{B}f \in C^\gamma \quad , \quad \forall \gamma \quad \text{such that} \quad 0 < \gamma < 1/2 \ .$$

The Brjuno function which we have studied in the previous section is nothing else than $\mathbf{B}\ell$, where ℓ is equal to minus the logarithmic fonction restricted to $[0, 1/2]$. When made even and periodic, this function is not continuous. Suppose that we perturb ℓ by a function with enough regularity properties (for example C^1 or $C^{1/2}$, the change in $\mathbf{B}\ell$ will be continuous and even $C^{1/2-\epsilon}$ for any small ϵ. In this sense, the 'most singular ' part of the Brjuno function is stable or 'universal', roughly speaking modulo Hölder one-half continuous contributions.

In fact we have a slightly better result for $\mathbf{B}$ than for T, as shown in the next proposition, which shows that the $C^{1/2}$ property is effectively reached.

Proposition 3.4. *If $f \in C^\gamma$, and $\gamma > 1/2$, then $\mathbf{B}f \in C^{1/2}$.*

Proof. We use $\mathbf{B}f(x) = \sum_{n=0}^\infty (T^n f)(x) = \sum_{n=0}^\infty \beta_{n-1}(x) f(x_n)$, which is analogous to (1.25), using convention $\beta_{-1}(x) = 1$. Therefore we have

$$|\mathbf{B}f(x) - \mathbf{B}f(y)| = \left| \sum_{n=0}^\infty \beta_{n-1}(x) f(x_n) - \beta_{n-1}(y) f(y_n) \right| \ , \tag{3.7}$$

where x_i and y_i are the respective trajectories of x and y under $A_{1/2}$.

Now, for each pair of numbers (x, y), such that $x \neq y$, we define the number N which tells how far the continued fraction expansions of x and y do coincide, more precisely : we have $x_k^{-1} = a_{k+1}(x) + \varepsilon_{k+1}(x) x_{k+1}$, and $y_k^{-1} = a_{k+1}(y) + \varepsilon_{k+1}(y) y_{k+1}$. We assume that $a_k(x) = a_k(y)$, $\varepsilon_k(x) = \varepsilon_k(y)$ as long as $k \leq N$. At least one of these equalities fail for $k > N$. In order to avoid ambiguities in the rational case, where we have $x_n = 0$ for some n, we take $\varepsilon_n(x) = +1$ and $x_k = 0$ whenever $k > n$, and let for $k > n$, a_k and ε_k be undefined (one could let a_k be infinite). Therefore, if x_n or y_n vanish for some n, we get $N \leq n$. We then have

$$\frac{1}{x_k} - \frac{1}{y_k} = \varepsilon_{k+1}(x_{k+1} - y_{k+1}) \quad \text{when} \quad k + 1 \leq N \ . \tag{3.8}$$

Therefore we get, still for $k+1 \le N$,

$$|x_{k+1} - y_{k+1}| = \frac{|x_k - y_k|}{|x_k||y_k|} = \frac{|x-y|}{\beta_k(x)\beta_k(y)} \ . \tag{3.9}$$

We also see that the numerators p_k, and the denominators q_k of the truncated continued fraction expansion of x and y coincide as long as $k \le N$, therefore, we first have

$$\text{for} \quad k \le N \quad , \quad |\beta_k(x) - \beta_k(y)| = q_k |x-y| \ , \tag{3.10}$$

and also, due to Proposition (1.4, iii),

$$\text{for} \quad k \le N-1 \quad , \quad \frac{1}{3} \ \le \ \frac{\beta_k(x)}{\beta_k(y)} \ \le \ 3 \ . \tag{3.11}$$

We separate (3.7) into four contributions and divide by $|x-y|^{1/2}$, that is,

$$\frac{|\mathbf{B}f(x) - \mathbf{B}f(y)|}{|x-y|^{1/2}} \le E_1 + E_2 + E_3 + E_4 \tag{3.12}$$

with

$$E_1 = |x-y|^{-1/2} \sum_{n=0}^{N} |\beta_{n-1}(x) - \beta_{n-1}(y)||f(x_n)| \ , \tag{3.13}$$

$$E_2 = |x-y|^{-1/2} \sum_{n=0}^{N} |\beta_{n-1}(y)||f(x_n) - f(y_n)| \ , \tag{3.14}$$

$$E_3 = |x-y|^{-1/2} \sum_{n=N+1}^{N+2} |\beta_{n-1}(x) f(x_n) - \beta_{n-1}(y) f(y_n)| \ . \tag{3.15}$$

$$E_4 = |x-y|^{-1/2} \sum_{n=N+3}^{\infty} |\beta_{n-1}(x) f(x_n) - \beta_{n-1}(y) f(y_n)| \ . \tag{3.16}$$

Our proof will be completed if we can bound E_1 to E_4 in terms of $|f|_\infty$ and $|f|_\gamma$ for $f \in C^\gamma$ and $\gamma > 1/2$.

1) Bound for E_1. From (3.10) we get $E_1 \le |x-y|^{1/2} |f|_\infty \sum_{n=1}^{N} q_{n-1}$, which using (3.8), gives $E_1 \le |x_N - y_N|^{1/2} |f|_\infty (\beta_{N-1}(x)\beta_{N-1}(y))^{1/2} \sum_{n=0}^{N-1} q_n$, but from Proposition 1.4., we have for $n \le N$, $q_n < 2(\beta_{n-1}(x))^{-1}$ and $q_n < 2(\beta_{n-1}(y))^{-1}$ from which we get $q_n < 2(\beta_{n-1}(x)\beta_{n-1}(y))^{-1/2}$. Therefore,

$$E_1 \le 2|x_N - y_N|^{1/2} |f|_\infty \sum_{n=0}^{N-1} \left(\frac{\beta_{N-1}(x)}{\beta_{n-1}(x)} \right)^{1/2} \left(\frac{\beta_{N-1}(y)}{\beta_{n-1}(y)} \right)^{1/2} \ , \tag{3.17}$$

However, using Proposition 1.4.,

$$\left(\frac{\beta_{N-1}(x)}{\beta_{n-1}(x)}\right) = \beta_{N-n-1}(x_n) \le (\sqrt{2}-1)^{N-n-1} ,$$

and the sum in (3.14) is bounded by the converging series $\sum_{n=0}^{\infty}(\sqrt{2}-1)^n = (2-\sqrt{2})^{-1}$. Furthermore, $|x_N - y_N| \le 1/2$ and E_1 is bounded by a constant time $|f|_\infty$.

2) Bound for E_2. We have $E_2 \le |x-y|^{-1/2}|f|_\gamma \sum_{n=0}^{N} |\beta_{n-1}(y)||x_n - y_n|^\gamma$, so that, using (3.9),

$$E_2 \le |f|_\gamma \sum_{n=0}^{N} \left(\frac{\beta_{n-1}(y)}{\beta_{n-1}(x)}\right)^{1/2} |x_n - y_n|^{\gamma-1/2} . \tag{3.18}$$

Using (3.11), we get $E_2 \le 3^{1/2}|f|_\gamma \sum_{n=0}^{N} |x_n - y_n|^{\gamma-1/2}$. However for $n \le N$, we see from (3.9) that

$$|x_n - y_n| = |x_N - y_N| \left(\frac{\beta_{N-1}(x)\beta_{N-1}(y)}{\beta_{n-1}(x)\beta_{n-1}(y)}\right) \le (\sqrt{2}-1)^{2N-2n-2}|x_N - y_N| .$$

Then

$$|x_n - y_n| \le \frac{(\sqrt{2}-1)^{2N-2n-2}}{2} ,$$

since $x_N - y_N \le 1/2$, and the sum from 0 to N is bounded since the geometric series $\sum_{n=0}^{\infty}(\sqrt{2}-1)^{2n(\gamma-1/2)}$ converges. As a consequence, E_2 is bounded by a constant times $|f|_\gamma$. Note that we have explicitely used here the hypothesis $\gamma > 1/2$.

3) Bound for E_3. We observe that since $\gamma > 1/2$, $f \in C_\gamma \Rightarrow f \in C_{1/2}$, and in fact, we have $|f|_{1/2} \le 2^{(1/2)-\gamma}|f|_\gamma$. We first bound the term of order $N+1$.

$$\begin{aligned}
|\beta_N(x)f(x_{N+1}) - \beta_N(y)f(y_{N+1})| &= |\beta_{N-1}(x)(Tf)(x_N) - \beta_{N-1}(y)(Tf)(y_N)| \\
&\le |(\beta_{N-1}(x) - \beta_{N-1}(y))(Tf)(x_N)| + |\beta_{N-1}(y)((Tf)(x_N) - (Tf)(y_N))| \\
&\le (1/2)q_{N-1}|f|_\infty|x-y| + K_{1/2}(f)|x_N - y_N|^{1/2}\beta_{N-1}(y) ,
\end{aligned}$$

where we have used (3.10) and (3.6) for the exponent 1/2. Therefore, using (3.9), one gets

$$\begin{aligned}
&\frac{|\beta_N(x)f(x_{N+1}) - \beta_N(y)f(y_{N+1})|}{|x-y|^{1/2}} \\
&\qquad \le (1/2)|f|_\infty q_{N-1}|x-y|^{1/2} + \frac{K_{1/2}(f)\beta_{N-1}(y)}{|\beta_{N-1}(x)\beta_{N-1}(y)|^{1/2}} .
\end{aligned}$$

However, from (3.9), one gets $|x-y| = |x_N - y_N|x_{N-1}y_{N-1}\beta_{N-2}(x)\beta_{N-2}(y)$, and from Proposition (1.4), $q_{N-1}\beta_{N-2} \le 2$. Then (3.11) shows that both terms of the right hand side in the above inequality are bounded (in terms of $|f|_\infty$ and $|f|_{1/2}$).

We now bound the term of order $N+2$ in E_3. We have

$$\begin{aligned}|\beta_{N+1}(x)f(x_{N+2})-\beta_{N+1}(y)f(y_{N+2})| =&|\beta_N(x)((Tf)(x_{N+1})-(Tf)(y_{N+1}))|\\
+|(\beta_{N-1}(x)-\beta_{N-1}(y))y_N(Tf)(y_{N+1})|+&|\beta_{N-1}(x)(Tf)(y_{N+1})(x_N-y_N)|\ ,\\
\le&\beta_N(x)K_{1/2}(f)|x_{N+1}-y_{N+1}|^{1/2}\\
+y_Ny_{N+1}|f|_\infty q_{N-1}|x-y|+&y_Ny_{N+1}|f|_\infty\beta_{N-1}(x)|x_N-y_N|\ .\end{aligned}$$

We now use Lemma 3.3 to get $x_Ny_N|x_{N+1}-y_{N+1}|\le|x_N-y_N|$, and we get

$$\begin{aligned}&\frac{|\beta_{N+1}(x)f(x_{N+2})-\beta_{N+1}(y)f(y_{N+2})|}{|x-y|^{1/2}}\\
\le&K_{1/2}(f)\left(\frac{y_N}{x_N}\right)^{1/2}\beta_{N-1}(x)\left|\frac{x_N-y_N}{x-y}\right|^{1/2}\\
+&(1/4)|f|_\infty\left(q_{N-1}|x-y|^{1/2}+\beta_{N-1}(x)\frac{|x_N-y_N|}{|x-y|^{1/2}}\right)\ .\end{aligned}$$

In the above expression, we have got three terms to bound. In the first we have the factor $(y_N/x_N)^{1/2}$ which is bounded when $y_N\le x_N$. In the other case it is sufficient to permute x and y. It then remains the factor $\beta_{N-1}(x)(|x_N-y_N|/|x-y|)^{1/2}$, which is easily bounded using (3.8) and (3.10). The same combination occurs in the third term, with an additional bounded factor $|x_N-y_N|^{1/2}$. The middle term $q_{N-1}|x-y|^{1/2}$ has been already treated for the previous term (of order $N+1$).

4) Bound for E_4. We have

$$E_4=|x_N-y_N|^{-1/2}(\beta_{N-1}(x)\beta_{N-1}(y))^{-1/2}\sum_{n=N+3}^{\infty}|\beta_{n-1}(x)f(x_n)-\beta_{n-1}(y)f(y_n)|\ . \tag{3.19}$$

From the above definition of N, we know that $2x_N^{-1}$ and $2y_N^{-1}$ do not both belong to the same interval in the family of intervals including for all $M\ge2$, $[2M,2M+1)$ and $[2M+1,2M+2)$. Therefore the interval $[2x_N^{-1},2y_N^{-1}]$ contains at least one integer point (we assume $x_N>y_N$, the other possibility being treated by permuting x and y). Let now r and R, be the rationals between x and y having, up to order N included, the same continued fraction expansion coefficient as x and y, and such that $2r_N^{-1}$ and $2R_N^{-1}$ are respectively the smallest and the largest integer such that $2x_N^{-1}\le 2r_N^{-1}\le 2R_N^{-1}\le y_N^{-1}$. We now have

$$|\mathbf{B}f(x)-\mathbf{B}f(y)|\le|\mathbf{B}f(x)-\mathbf{B}f(r)|+|\mathbf{B}f(r)-\mathbf{B}f(R)|+|\mathbf{B}f(R)-\mathbf{B}f(y)|$$

and since we have either $x\le r\le R\le y$ or $y\le R\le r\le x$, we see that we need only to get a bound for E_4 in the two following cases : either $x=r$ and $y=R$ are both rational and $2x_N^{-1}$ and $2y_N^{-1}$ are different integers, or one of these (say $y=r$)

has these properties, and $|x_N^{-1} - y_N^{-1}| < 1/2$. Indeed the bound already obtained for E_1, E_2, and E_3, apply to the present situations.

Now we observe that when $2r_N^{-1}$ is even, we have $r_{N+1} = 0$, and when $2r_N^{-1}$ is odd, we have $r_{N+1} = 1/2$ and $r_{N+2} = 0$, so that the sum in (3.19) contains no terms coming from r or R. More precisely, there is no E_4 type contribution in $|r - R|^{-1/2}|\mathbf{B}f(r) - \mathbf{B}f(R)|$, which was the motivation to split the sum for $n > N$ in (3.7) between E_3 and E_4. For the same reason, in the other relevant case—where $y = r$, and $-1/2 < x_N^{-1} - r_N^{-1} < 1/2$—Equation (3.19) reduces to

$$
\begin{aligned}
E_4 =& |x_N - r_N|^{-1/2}(\beta_{N-1}(x)\beta_{N-1}(r))^{-1/2} \sum_{n=N+3}^{\infty} \beta_{n-1}(x)|f(x_n)| \\
\leq& \frac{\sqrt{3}|f|_\infty x_N x_{N+1} x_{N+2}}{|x_N - r_N|^{1/2}} \sum_{n=N+3}^{\infty} \beta_{n-N+3}(x_{N+3}) \\
\leq& \left(\frac{\sqrt{3}|f|_\infty}{2-\sqrt{2}}\right)\left(\frac{x_N x_{N+1} x_{N+2}}{|x_N - r_N|^{1/2}}\right) \ , \qquad (3.20)
\end{aligned}
$$

where we have used (3.11), proposition (1.4), and the sum of the converging series $\sum_{n=0}^{\infty}(\sqrt{2}-1)^n$. We now define θ such that

$$
\frac{1}{x_N} = \frac{1}{r_N} + \theta \quad , \quad -1/2 < \theta < 1/2 \ , \qquad (3.21)
$$

so that $|x_N - r_N| = r_N x_N |\theta|$, and we have now to get a bound for the remaining factor

$$
\frac{x_N x_{N+1} x_{N+2}}{|x_N - r_N|^{1/2}} = \frac{x_N x_{N+1} x_{N+2}}{x_N^{1/2} r_N^{1/2} |\theta|^{1/2}} = \frac{x_{N+1} x_{N+2}}{|\theta|^{1/2}} \left(\frac{x_N}{r_N}\right)^{1/2} .
$$

Notice that since $2x_N^{-1}$ and $2r_N^{-1}$ belong to the same interval $[M, M+1]$, with $M \geq 2$, their ratio remain bounded, that is $2/3 \leq x_N/r_N \leq 3/2$, so we need therefore only to get a bound for $x_{N+1}x_{N+2}|\theta|^{-1/2}$.

In the case where $2r_N^{-1}$ is even, we have $x_{N+1} = |\theta|$, so that $x_{N+1}x_{N+2}\theta^{-1/2} < 2^{-3/2}$. In the case where $2r_N^{-1} = 2P+1$ is odd, we have $x_{N+1} = 1/2 - |\theta|$. Either $|\theta| \geq 1/10$ which implies $x_{N+1} \leq 2/5$ and $x_{N+1}x_{N+2}|\theta|^{-1/2} \leq (2/5)^{1/2}$ or $|\theta| < 1/10$ and $2/5 < x_{N+1} \leq 1/2$, in which case $x_{N+2} = 2 - x_{N+1}^{-1} = 2|\theta|/(1/2 - |\theta|)$,and $x_{N+1}x_{N+2}|\theta|^{-1/2} = 2\sqrt{\theta} \leq (2/5)^{1/2}$. This concludes the proof of the proposition. □

The regularity results of the present section might provide an explanation for the numerical results [Ma], which suggested a relation between the Brjuno function and the size of the stability domain as function of the rotation number in some holomorphic area preserving maps. Note that some rigorous results have been obtained in the case of the semistandard map [Ma,Da]. We anticipate that this result might be much more general : the geometric renormalisation for holomorphic dynamical systems will likely produce only C^1 perturbations to the renormalisation equation, and the most singular part of minus the logarithm of the size of the stability domains as function of the rotation number could be universally (that is modulo $C^{1/2}$) described by the Brjuno function.

Appendix A. Norms on B.M.O. spaces

1. B. M. O. Norms

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. We define the mean value f_I of f on the interval I as :

$$f_I = \frac{1}{|I|}\int_I f\,dx \ , \tag{A.1}$$

where $|I|$ is the length of the interval I. Then we define for any interval U

$$||f||_{*,U} = \sup_{I\subset U}\frac{1}{|I|}\int_I |f - f_I|\,dx \ . \tag{A.2}$$

We then say f belongs to the space $\mathrm{BMO}(U)$ if $||f||_{*,U} < \infty$, *i.e.* is finite. BMO is an abbreviation for 'bounded mean oscillation'. $||f||_{*,U}$ is a seminorm on $\mathrm{BMO}(U)$, since for any constant c, we have $||f+c||_{*,U} = ||f||_{*,U}$. In particular $||f||_{*,U} = 0$ if f is constant on U. This applies to $U = \mathbb{R}$ and leads to the space $\mathrm{BMO}(\mathbb{R})$, abbreviated as BMO and the seminorm $||f||_{*,\mathbb{R}}$ on $\mathrm{BMO}(\mathbb{R})$ will simply be written $||f||_{*,}$. In fact this seminorm is a norm on the quotient space of function in $L^1_{\mathrm{loc}}(\mathbb{R})$ modulo the constant functions. With this norm, the quotient space is complete.

We list now some classical results and lemmas [Gr,GRCF].

Proposition A.1. *If there exists a positive number M such that for any interval $I \subset U$, we can find a number α_I with $\frac{1}{|I|}\int_I |f-\alpha_I|\,dx \le M$, then $|f_I - \alpha_I| \le M$ and $||f||_{*,U} \le 2M$.*

Proof. We have $(f_I - \alpha_I) = \frac{1}{|I|}\int_I (f-\alpha_I)\,dx$, thus $|f_I-\alpha_I| \le \frac{1}{|I|}\int_I |f-\alpha_I|\,dx \le M$, and $\frac{1}{|I|}\int_I |f-f_I|\,dx = \frac{1}{|I|}\int_I |f-\alpha_I+\alpha_I-f_I|\,dx \le \frac{1}{|I|}\int_I |f-\alpha_I|\,dx + |\alpha_I - f_I| \le 2M$. Therefore $||f||_{*,U} \le 2M$. □

Proposition A.2. *The space $L^\infty(U)$ is a subspace of* $\mathrm{BMO}(U)$, *and* $||f||_{*,U} \le \inf_c ||f-c||_{\infty,U}$.

Proof. From Hölder's inequality, we get for any interval $I \subset U$, $\frac{1}{|I|}\int_I |f-f_I|\,dx \le \left(\frac{1}{|I|}\int_I |f-f_I|^2\,dx\right)^{1/2}$. But $\frac{1}{|I|}\int_I (f-f_I)^2\,dx = \frac{1}{|I|}\int_I f^2\,dx - (f_I)^2 \le \frac{1}{|I|}\int_I f^2\,dx \le (||f||_{\infty,U})^2$. As a result, $||f||_{*,U} \le ||f||_{\infty,U}$, the L^∞ norm of f on U. Since $||f||_{*,U}$ is equal for any c to $||f+c||_{*,U}$, we immediately deduce that $||f||_{*,U} \le ||f-c||_{\infty,U}$ on U, for any c. □

Proposition A.3. *Assume $f \in \mathrm{BMO}(U)$, with seminorm $||f||_{*,U}$. Let I and J be two intervals in U . Let m be an arbitrary integer such that $m > 1$, we have*

a) If $I \subset J$ and $1 \le \frac{|J|}{|I|} \le m$, then $|f_I - f_J| \le m||f||_{*,U}$.

b) If $I \subset J$ and $m < \frac{|J|}{|I|}$, then

$$|f_I - f_J| \le \left(1 + \frac{1}{\log m} \log\left(\frac{|J|}{|I|}\right)\right) m||f||_{*,U} \le \frac{2m}{\log m} \log\left(\frac{|J|}{|I|}\right) ||f||_{*,U} \ .$$

c) If $|I| = |J|$, let K be the smallest interval containing both I and J. Assume first $|K| \le 2|I| = 2|J|$, which means that I and J overlap or are consecutive intervals, then $|f_I - f_J| \le 4||f||_{*,U}$. On the contrary, if $|K| > 2|I| = 2|J|$, which means that I and J are disjoint intervals, then $|f_I - f_J| \le \frac{8}{\log 2}||f||_{*,U} \log\left(\frac{|K|}{|I|}\right) = \frac{8}{\log 2}||f||_{*,U} \log\left(2 + \frac{\delta(I,J)}{|I|}\right)$ where $\delta(I,J)$ is the length of the interval between I and J.

Proof. a) $|f_I - f_J| \le \frac{1}{|I|}\int_I |f - f_J|\, dx = \frac{|J|}{|I|}\frac{1}{|J|}\int_I |f - f_J|\, dx \le \frac{|J|}{|I|}\frac{1}{|J|}\int_J |f - f_J|\, dx \le \frac{|J|}{|I|}||f||_{*,U} \le m||f||_{*,U}$.

b) Let $t = \frac{|J|}{|I|}$, and for $n > 1$, let $\lambda = t^{1/n}$. It then easy to find intervals $I_2, I_3, \ldots, I_n$ chosen in an arbitrary way, but subjected to the conditions $I = I_1 \subset I_2 \subset I_3 \ldots \subset I_n \subset I_{n+1} = J$, $|I_{k+1}|/|I_k| = \lambda$ for $1 \le k \le n$, and

$$\frac{|J|}{|I|} = \frac{|I_{n+1}|}{|I_n|}\frac{|I_n|}{|I_{n-1}|} \cdots \frac{|I_2|}{|I_1|} = \lambda^n = t \ .$$

Using $|f_I - f_J| = |f_{I_1} - f_{I_2}| + |f_{I_2} - f_{I_3}| + \ldots + |f_{I_n} - f_{I_{n+1}}|$, we get

$$|f_I - f_J| \le \left(\sum_{i=k}^{n} \frac{|I_{k+1}|}{|I_k|}\right) ||f||_{*,U} = nt^{\frac{1}{n}}||f||_{*,U} \quad \text{as in the proof of a).}$$

For $t > m$, let now $n > 1$ be the integer such that $m^{n-1} < t \le m^n$, so that $t^{1/n} \le m$, and

$$n < 1 + \frac{1}{\log m}\log t \le \frac{2}{\log m}\log t \ .$$

Therefore, from $|f_I - f_J| \le nt^{1/n}||f||_{*,U}$, we get

$$|f_I - f_J| \le \left(1 + \frac{1}{\log m} \log\left(\frac{|J|}{|I|}\right)\right) m||f||_{*,U} \le \frac{2m}{\log m} \log\left(\frac{|J|}{|I|}\right) ||f||_{*,U} \ .$$

In fact we have got much better than statements a) and b) : for $I \subset J \subset U$, we have $|f_I - f_J| \le S\left(\frac{|J|}{|I|}\right)||f||_{*,U}$, where the function $S(t)$ is defined for $t \ge 1$ as $S(t) = \inf_{n\ge 1} nt^{1/n}$. After some calculations, we find $S(t) = t$ for $1 \le t \le 4$, and $S(t) \le (2/\log 2)\log t$ for $t \ge 4$. For t large, we get $S(t) \sim e\log t$.

c) We have $|f_I - f_J| \le |f_K - f_I| + |f_K - f_J|$ and the statement follows from parts a) or b) with $m = 2$. □

The preceding proposition shows that when $|I|$ goes to zero, the mean f_I cannot grow faster than $|\log(|I|)|$ times some constant.

Proposition A.4. *We have the following 'magic reverse Hölder's inequality' : let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and suppose that for some interval $U \subset \mathbb{R}$, the seminorm $||f||_{*,U}$ is finite, then for any bounded real $p \geq 1$, there exists a constant A_p such that*

$$\sup_{I\subset U} \left(\frac{1}{|I|}\int_I |f-f_I|^p \, dx\right)^{\frac{1}{p}} \leq A_p ||f||_{*,U} \ .$$

Proof. In fact it is a corollary of the John-Nirenberg theorem, see Garnett [Gr]. The proof there is for $U = \mathbb{R}$, but a careful reading show that it works for any U. The constant A_p *does not depend on* U, and may be shown to be smaller than pC with an explicit constant C (our evaluation gives $C < 10$). Note that the inequality does not work in the limit $p \to \infty$. □

The preceding proposition shows that replacing the L^1 norm in the definition of the BMO norm $||f||_{*,U}$, by the analogous L^p norm (with p finite), leads to the same BMO space. More precisely using the usual L^p norm

$$||f||_{p,U} = \left(\int_U |f|^p \, dx\right)^{\frac{1}{p}} \ ,$$

we define

$$||f||_{*,p,U} = \sup_{I\subset U} \left(\frac{1}{|I|}\int_I |f-f_I|^p \, dx\right)^{\frac{1}{p}} = \sup_{I\subset U} |I|^{-\frac{1}{p}} ||f-f_I||_{p,I} \ . \tag{A.3}$$

We then have

Proposition A.5. *The space* $\mathrm{BMO}(U)$, *is a subspace of $L^p(U)$ when U is a bounded interval.*

Proof. Let $f \in \mathrm{BMO}(U)$, then Proposition A.4 shows that $||f-f_U||_{p,U} \leq |U|^{(1/p)} A_p ||f||_{*,U}$. But $||f||_{p,U} \leq ||f-f_U||_{p,U} + ||f_U||_{p,U} \leq |U|^{(1/p)}(A_p||f||_{*,U} + |f_U|)$, and since $f \in L^1(U)$, $|f_U| \leq |U|^{-1} \times ||f||_{1,U}$. Therefore, $||f||_{p,U} \leq |U|^{(1/p)} A_p ||f||_{*,U} + |U|^{(1/p)-1}||f||_{1,U}$, and $||f||_{p,U}$ is bounded. □

Therefore, $\mathrm{BMO}(U)$ is a subspace of $\cap_{p=1}^{\infty} L^p(U)$, but *not* a subspace of $L^\infty(U)$. In fact, on $\mathrm{BMO}(U)$, we have a family of equivalent norms, as shown in the following proposition.

Proposition A.6. *On* $\mathrm{BMO}(U)$, *where U a bounded interval, define for any real $\alpha > 0$, and $\beta > 0$, and for any integer $p \geq 1$ (finite), the following family of norms*

$$N(f,\alpha,\beta,p) = \alpha ||f||_{*,p,U} + \beta ||f||_{p,U} \ , \tag{A.4}$$

then

a) these norms are all equivalent for various α and β and p fixed.
b) these norms are all equivalent for any p.

Proof. a) is a consequence of the identities

$$(\alpha x+\beta y)=\frac{\alpha}{\alpha'}(\alpha' x+\beta' y)+\left(\frac{\beta}{\beta'}-\frac{\alpha}{\alpha'}\right)\beta' y=\frac{\beta}{\beta'}(\alpha' x+\beta' y)-\left(\frac{\beta}{\beta'}-\frac{\alpha}{\alpha'}\right)\alpha' x\ , \tag{A.5}$$

which show that when all numbers $\alpha,\alpha',\beta,\beta'$ are positive, and x,y nonnegative, then $\inf(\alpha/\alpha',\beta/\beta')(\alpha' x+\beta' y)\le \alpha x+\beta y\le \sup(\alpha/\alpha',\beta/\beta')(\alpha' x+\beta' y)$.

b) Using Proposition A.4 and the argument in the proof of Proposition A.5, we have $N(f,\alpha,\beta,p)\le \alpha A_p||f||_{*,U}+\beta(|U|^{(1/p)}A_p||f||_{*,U}+|U|^{(1/p)-1}||f||_{1,U})$. Therefore, $N(f,\alpha,\beta,p)\le(\alpha+\beta|U|^{(1/p)})A_p||f||_{*,U}+\beta|U|^{(1/p)-1}||f||_{1,U}$. On the other hand Hölder's inequality shows that $N(f,\alpha,\beta,p)\ge \alpha||f||_{*,U}+\beta|U|^{(1/p)-1}||f||_{1,U}$. So that we have :

$$N(f,\alpha,\beta|U|^{(1/p)-1},1)\le N(f,\alpha,\beta,p)\le N(f,\alpha+\beta|U|^{(1/p)},\beta|U|^{(1/p)-1},1)\ .$$

Using Eq. (A.5), one easily finds positive constants C_- and C_+ such that

$$N(f,\alpha+\beta|U|^{(1/p)},\beta|U|^{(1/p)-1},1)\le C_+N(f,\alpha,\beta,1)$$

and

$$N(f,\alpha,\beta|U|^{(1/p)-1},1)\ge C_-N(f,\alpha,\beta,1)\ .$$

Therefore

$$C_-N(f,\alpha,\beta,1)\le N(f,\alpha,\beta,p)\le C_+N(f,\alpha,\beta,1)\ .$$

and the norms $N(f,\alpha,\beta,p)$ and $N(f,\alpha,\beta,1)$ are equivalent. □

Remark that these quantities $N(f,\alpha,\beta,p)$ are norms and not only seminorms when $\beta\ne 0$. With respect to all these norms $N(f,\alpha,\beta,p)$ (with $\alpha>0$ and $\beta>0$), the space BMO(U) is complete, since any Cauchy sequence with respect to $N(f,\alpha,\beta,p)$ is also a Cauchy sequence in $L^p(U)$ and therefore in $L^1(U)$ for U bounded. Moreover the L^1-limit of a sequence of functions in BMO(U) is obviously also in BMO(U).

We now define the oscillation $O_I(f)$ of f on the interval I as

$$O_I(f)=\frac{1}{|I|}\int_I|f-f_I|\,dx\ , \tag{A.6}$$

such that $||f||_{*,U}=\sup_{I\subset U}O_I(f)$. We then have

Proposition A.7. *Let I_1 and I_2 be two consecutive intervals, which mean that the closure $\overline{I_1}$ and $\overline{I_2}$ have one and only one common point, and let $I=\overline{I_1}\cup\overline{I_2}$. Let $a_1=|I_1|$, $a_2=|I_2|$ so that $|I|=a_1+a_2$. Then*

$$f_I=\frac{a_1f_{I_1}+a_2f_{I_2}}{a_1+a_2}\ , \tag{A.7}$$

and

$$O_I(f) = \frac{a_1}{a_1+a_2} O_{I_1}(f) + \frac{a_2}{a_1+a_2} O_{I_2}(f) + \frac{2a_1a_2}{(a_1+a_2)^2} |f_{I_1} - f_{I_2}| \ . \qquad \text{(A.8)}$$

Proof. An elementary calculation gives Eq. (A.7) as well as

$$(f_I - f_{I_1}) = \frac{a_2}{a_1+a_2}(f_{I_2} - f_{I_1}) \quad \text{and,} \quad (f_I - f_{I_2}) = \frac{a_1}{a_1+a_2}(f_{I_1} - f_{I_2}) \ . \quad \text{(A.9)}$$

Now

$$O_I(f) = \frac{1}{|I|}\int_I |f - f_I|\, dx = \frac{|I_1|}{|I|}\frac{1}{|I_1|}\int_{I_1} |f - f_I|\, dx + \frac{|I_2|}{|I|}\frac{1}{|I_2|}\int_{I_2} |f - f_I|\, dx \ ,$$

so that

$$\begin{aligned} O_I(f) \quad &\leq \quad \frac{|I_1|}{|I|}\left(|f_{I_1} - f_I| + \frac{1}{|I_1|}\int_{I_1} |f - f_{I_1}|\, dx\right) \\ &\quad + \frac{|I_2|}{|I|}\left(|f_{I_2} - f_I| + \frac{1}{|I_2|}\int_{I_2} |f - f_{I_2}|\, dx\right) \\ &= \frac{a_1}{a_1+a_2} O_{I_1}(f) + \frac{a_2}{a_1+a_2} O_{I_2}(f) + \frac{a_1}{a_1+a_2}|f_I - f_{I_1}| + \frac{a_2}{a_1+a_2}|f_I - f_{I_2}| \ , \end{aligned}$$

and the proposition follows from Eq. (A.9). □

The above Proposition shows that in order to bound the oscillation on the union of two consecutive intervals, one needs to control the difference between the mean values on the two intervals.

2. B. M. O. Norms for periodic functions

In BMO($\mathbb{R}$) we can consider the various seminorms $||f||_{*,U}$ defined in Eq. (A.2) above, for any bounded interval U. We will also define the seminorm $||f||_{*,\mathbb{T}}$ as follows

$$||f||_{*,\mathbb{T}} = \sup_{|I|\leq 1} \frac{1}{|I|}\int_I |f - f_I|\, dx \ , \tag{A.10}$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ with length less or equal to 1. The seminorm $||f||_{*,\mathbb{T}}$ is convenient for periodic functions with period 1. For any $f \in$ BMO($\mathbb{R}$) we obviously have :

$$||f||_{*,[-(1/2),+(1/2)]} \leq ||f||_{*,\mathbb{T}} \leq ||f||_{*,\mathbb{R}} \ . \tag{A.11}$$

This observation will be useful if we now consider functions $f \in$ BMO($\mathbb{R}$) *which are even and periodic with period 1.* For such periodic functions, we have the following result

Proposition A.8. *There exist constant $C_1 > 1$, $C_2 > 1$, $C_3 > 1$ such that for any $f \in$ BMO($\mathbb{R}$), which is even and periodic with period 1, we have*

$$\begin{array}{ll} a) & ||f||_{*,\mathbb{R}} \leq C_1 ||f||_{*,\mathbb{T}} \\ b) & ||f||_{*,\mathbb{T}} \leq C_2 ||f||_{*,[0,1]} \\ b) & ||f||_{*,[0,1]} \leq C_3 ||f||_{*,[0,1/2]} \end{array} \quad .$$

Proof. a) We observe first that the mean value and the oscillation of f on any interval K with length 1 does not depend on K. Let us call these quantities $\overline{f}$ and $\overline{O}(f)$ respectively. Now consider an arbitrary interval I. It is always possible to cut this interval into two consecutive subintervals I_1 and I_2, such that $I = I_1 \cup I_2$, $|I_1| = n \geq 0$ is the integer part of $|I|$, and $|I_2| = a < 1$. If I_1 is not empty, it is made of n complete periods of f, so that $f_{I_1} = \overline{f}$ and $O_{I_1}(f) = \overline{O}(f) < ||f||_{*,\mathbb{T}}$. If $n = 0$, we have $O_I(f) \leq ||f||_{*,\mathbb{T}}$. From Proposition A.7 we get for $n > 0$

$$O_I(f) = \frac{n}{n+a}\overline{O}(f) + \frac{a}{n+a}O_{I_2}(f) + \frac{2na}{(n+a)^2}|f_{I_2} - \overline{f}| \ .$$

but $O_{I_2}(f) \leq ||f||_{*,\mathbb{T}}$ and also $\overline{O}(f) \leq ||f||_{*,\mathbb{T}}$. On the other hand, I_2 is contained in some interval with length equal to 1, and one can bound $|f_{I_2} - \overline{f}|$ using Proposition A.3 a) or b). We get $|f_{I_2} - \overline{f}| \leq 2||f||_{*,\mathbb{T}}$ for $a \geq (1/2)$, and for $a < (1/2)$,

$$|f_{I_2} - \overline{f}| \leq \left(1 + \frac{|\log a|}{\log 2}\right) 2||f||_{*,\mathbb{T}} \ .$$

Therefore for $a \geq (1/2)$

$$O_I(f) \leq \left(1 + \frac{4na}{(n+a)^2}\right) ||f||_{*,\mathbb{T}} \ ,$$

and for $a < (1/2)$

$$O_I(f) \le \left(1 + \frac{4na}{(n+a)^2}\left(1 + \frac{|\log a|}{\log 2}\right)\right) ||f||_{*,\mathbb{T}} .$$

An elementary study of the function $-x\log(x)$ on $[0,1]$ shows that it is bounded by e^{-1},and for $n \ge 1$ we have :

$$\text{for } a \ge 1/2 \ , \ \frac{n}{(n+a)^2} \le \frac{4}{9} \le 1 \ , \text{ and for } a < 1/2 \ , \ \frac{n}{(n+a)^2} \le \frac{1}{n} \le 1 \ .$$

Therefore in both cases,

$$O_I(f) \le \left(3 + \frac{4}{e\log 2}\right) ||f||_{*,\mathbb{T}} = C_1||f||_{*,\mathbb{T}} \ ,$$

and part a) holds with $C_1 \sim 5.13$. Using the improved bounds mentioned at the end of the proof of part b) of Proposition A.3, it is possible to get $C_1 \le 3.13$. Note that part a) of the Proposition does not require the hypothesis that f is even.

b) Consider an arbitrary interval $I = (a,b)$ with length $(b-a) \le 1$. If $b-a = 1$, then $O_I(f) = \overline{O}(f) \le ||f||_{*,[0,1]}$. If $(b-a) < 1$ and if the integer parts of $[a]$ of a and $[b]$ of b are equal, then the interval I corresponds by periodicity to $([a],[b]) \subset [0,1]$ and $O_I(f) = O_{([a],[b])}(f) \le ||f||_{*,[0,1]}$. If $(b-a) < 1$, and if $[b] = [a]+1$, then $I = I_1 \cup I_2$, where $I_1 = (a,[a]+1]$ and $I_2 = [[a]+1,b)$. Let $a' = [a]+1-a = |I_1|$, $b' = b-[b] = |I_2|$. Then Eq. (A.8), shows that

$$O_I(f) = \frac{a'}{a'+b'}O_{I_1}(f) + \frac{b'}{a'+b'}O_{I_2}(f) + \frac{2a'b'}{(a'+b')^2}|f_{I_1} - f_{I_2}| \ . \tag{A.12}$$

By periodicity both $O_{I_1}(f)$ and $O_{I_2}(f)$ are bounded by $||f||_{*,[0,1]}$, then we get

$$O_I(f) = ||f||_{*,[0,1]} + \frac{2a'b'}{(a'+b')^2}|f_{I_1} - f_{I_2}| = ||f||_{*,[0,1]} + \frac{2a'b'}{(a'+b')^2}|f_{(1-a',1]} - f_{[0,b')}| \ .$$

However, since f is even, $f_{(1-a',1]} = f_{[0,a')}$. By symmetry, we can assume $b' \ge a'$. Let $x = a'/b' < 1$, then from Proposition A.3, if $x^{-1} \le 2$, we get

$$|f_{[0,a')} - f_{[0,b')}| \le 2||f||_{*,[0,1]} \ ,$$

and

$$O_I(f) = \left(1 + \frac{4x}{(1+x)^2}\right) ||f||_{*,[0,1]} \le 2||f||_{*,[0,1]} \ . \tag{A.13}$$

If $x^{-1} > 2$

$$|f_{[0,a')} - f_{[0,b')}| \le \left(1 + \frac{|\log x|}{\log 2}\right) 2||f||_{*,[0,1]} \ .$$

From this we get

$$O_I(f) = \left(1 + \frac{4x}{(1+x)^2} + \frac{4x|\log x|}{(1+x)^2 \log 2}\right) ||f||_{*,[0,1]} \ . \tag{A.14}$$

Now $4x/((1+x)^2) < 8/9$ and $1/((1+x)^2) < 1$. Therefore we get the expected result with

$$C_2 \leq \left(1 + \frac{8}{9} + \frac{4}{e \log 2}\right) \leq 4.012 \ , \tag{A.15}$$

by an argument similar to the one given in part a). Once more, improved bounds would reduce C_2 up to 3.13.

c) Let $I \subset [0,1]$. If either $I \subset [0,1/2]$ or $I \subset [1/2,1]$, we get $O_I(f) \leq ||f||_{*,[0,1/2]}$, using the parity property in the second case. On the other hand, if I contains $1/2$ in its interior, one divides I into two parts I_1 and I_2 contained in $[0,1/2]$ or $[1/2,1]$ repectively. Using the same argument and setting again $a' = |I_1|$ and $b' = |I_2|$, we still have Eq. (A.12). Now, we have

$$|f_{I_1} - f_{I_2}| = |f_{(1/2-a',1/2]} - f_{[1/2,1/2+b')}| = |f_{(1/2-a',1/2]} - f_{[1/2,1/2-b')}| \ ,$$

since f is even and periodic. Using Proposition A.3, we get here the same estimates as Eqs. (A.13) and (A.14), which completes the proof with $C_3 = C_2$. □

Note that parts b) and c) are not true if the periodic function f is not even. A non trivial, but immediate consequence of these results is the following corollary.

Corollary A.9. *Let f be a function defined in $[0,1/2]$, which belongs to* BMO$([0,1/2])$. *The function g which is even and periodic with period 1, and which coincides with f on $[0,1/2]$ is in* BMO$(\mathbb{R})$.

Using the above Proposition A.8, we see that a natural choice for a norm on the space of functions $f \in \text{BMO}(\mathbb{R})$ which are even and periodic with period 1 is $N(f,\alpha,\beta,p)$ defined in (A.4), with $p = 2$ and $U = [0,1/2]$, for some fixed $\alpha > 0$ and $\beta > 0$. Changing the values of p, α, and β will lead to equivalent norms. Changing U into $[0,1]$, or $\mathbb{T}$ as in Eq. (A.10), will again give an equivalent norm. Also equivalent would be a norm like $\alpha||g||_{*,\mathbb{R}} + \beta||g||_{1,[0,1]}$ for the function g which is even and periodic with period 1, and which coincides with f on $[0,1/2]$. The L^1-norm $||g||_{1,[0,1]}$ of g on $[0,1]$, can obviously replaced by the similar norm on $[0,1/2]$ that is $||g||_{1,[0,1/2]}$.

We will now quote two lemmas which are useful when using the L^2-norm. Let $I \subset [0,1/2]$ be any interval, and $f \in L^1([0,1/2])$. We denote by

$$f_I = \frac{1}{|I|}\int_I f(x)dx \tag{$A.16$)}$$

the mean of f over I.

Lemma A.10. *If $f \in L^2([0,1/2])$ and $I \subset [0,1/2]$ is any interval*

$$\int_I (f - f_I)^2 ds = \frac{1}{2|I|} \int_I \int_I (f(s) - f(t))^2 ds dt \,. \tag{A.17}$$

Proof. It is a simple calculation : first one observes that

$$\int_I (f - f_I)^2 ds = \int_I f(s)^2 ds - \frac{1}{|I|} \left(\int_I f(s) ds \right)^2 = \frac{1}{|I|} \int_I \int_I (f(s)^2 - f(s)f(t)) ds dt \,,$$

and then since one obviously has

$$\frac{1}{|I|} \int_I \int_I (f(s)^2 - f(s)f(t)) ds dt = \frac{1}{|I|} \int_I \int_I (f(t)^2 - f(s)f(t)) ds dt$$

one immediately gets (A.17). □

The following lemma deals with the control of the oscillation in L_2-norm.

Lemma A.11. *Let f be an even function in $L^1_{\text{loc}}(\mathbb{R})$. Let $0 < a < b$. For any finite interval $I \in \mathbb{R}$ define*

$$\mathcal{O}_I(f) = \left(\frac{1}{2|I|^2} \int_{I \times I} (f(s) - f(t))^2 ds dt \right)^{\frac{1}{2}} \,.$$

Then we have

$$\mathcal{O}_{[-a,b]}(f) \le 2\mathcal{O}_{[0,b]}(f) \,.$$

Proof. Set $I = [-a, b]$, $I_+ = [0, b]$, $I_- = [-a, 0]$. Then we have

$$2(a+b)^2 (\mathcal{O}_{[-a,b]}(f))^2 = \int_{I \times I} (f(s) - f(t))^2 ds dt \,.$$

The integral over $I \times I$ can be splitted as a sum over four integrals over $I_+ \times I_+$, $I_- \times I_-$, $I_+ \times I_-$, $I_- \times I_+$ respectively. However, due to the parity property of f, each of these four integrals is less than (or equal to) the integral over $I_+ \times I_+$. Therefore,

$$2(a+b)^2 (\mathcal{O}_{[-a,b]}(f))^2 \le 8b^2 (\mathcal{O}_{[0,b]}(f))^2 \,,$$

and

$$\mathcal{O}_{[-a,b]}(f) \le \frac{2b}{a+b} \mathcal{O}_{[0,b]}(f) \le 2\mathcal{O}_{[0,b]}(f) \,.$$

□

The preceding Lemma shows that the quadratic oscillation of an even function f on an interval I containing 0 in its interior, is at most twice the quadratic oscillation over the largest among the positive or negative subintervals of I.

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