Faddeev's anomaly and bundle gerbes.

A.L. Carey[1], M.K. Murray[2]

Department of Pure Mathematics,
University of Adelaide,
South Australia 5005,
AUSTRALIA.

Abstract: We show that transgression of the Dixmier-Douady class coming from the natural gerbe arising in the discussion of Faddeev's anomaly is the Faddeev-Mickelsson two cocycle.

[1]acarey@maths.adelaide.edu.au
[2]mmurray@maths.adelaide.edu.au

1 Introduction

In [1] we indicated how Segal's discussion of Faddeev's anomaly [7] could be phrased in the language of bundle gerbes and conjectured that the Faddeev anomaly was the transgression of the Dixmier-Douady class of the corresponding bundle gerbe. In this note we prove this conjecture. We utilise the results of [5] on the Faddeev-Mickelsson 2-cocycle on the gauge group to determine the Dixmier-Douady class via our notion of transgression (defined in section 2).

This proof is of interest for several reasons. First, Segal's work in [7] remains unpublished. Second, the discussion of this issue in [4] gives the impression that there is no connection between Segal's approach and that of [5]. The reason for this is that the notion of bundle gerbe was invented recently and is essential for the method we adopt here of relating the two. Third, the notion of bundle gerbe provides a way of realising degree three integral cohomology of a manifold geometrically that is naturally adapted to the classes arising in the study of anomalies (see [2]). The example discussed in this paper lends further weight to this last assertion.

2 Transgression

If M is a compact three-manifold with Riemannian structure and spin-structure, denote by $\mathcal{A}$ the affine space of connections on a complex vector bundle over M and by $\mathcal{G}$ the group of gauge transformations of this bundle. We define the *transgression map* [1] as follows. If ω is a three form on $\mathcal{A}/\mathcal{G}$ representing a cohomology class pull it back to a closed three form $\pi^*(\omega)$ on $\mathcal{A}$. If $A \in \mathcal{A}$ and X and Y are elements of the Lie algebra of $\mathcal{G}$ they define vectors X and Y in the tangent space at A and we define

$$c(X, Y)(A) = \mu_A(X, Y)$$

a co-cycle which defines a class in $H^2(L\mathcal{G}, \mathrm{Map}(\mathcal{A}, \mathbf{C}))$ the Lie algebra cohomology of the Lie algebra of the gauge group with values in the module $\mathrm{Map}(\mathcal{A}, \mathbf{C})$. The resulting map

$$H^3(\mathcal{A}/\mathcal{G}, \mathbf{C}) \to H^2(L\mathcal{G}, \mathrm{Map}(\mathcal{A}, \mathbf{C}))$$

we call the transgression.

3 Bundle gerbes

Let $\pi: Y \to P$ be a submersion. That is Y and P are manifolds and π is onto and has onto derivative. It follows from the submersion theorem that $Y \to P$ admits local sections. This will be obvious in the example we consider below. Define the fibre product of Y with itself to be

$$Y^{[2]} = \{(x, y)|\pi(x) = \pi(y)\} \subset Y \times Y.$$

Notice that the diagonal $\triangle = \{(y, y)|y \in Y\}$ is a subset of $Y^{[2]}$.

A bundle gerbe over P is a pair (J, Y) where Y is as above and J is a complex line bundle over $Y^{[2]}$ satisfying three conditions:

1. restricted to the diagonal J is trivial,
2. for any x, y in the same fibre of $\pi: Y \to P$ there is an isomorphism $J_{(x,y)} = J^*_{(y,x)}$,
3. for any x, y and z in the same fibre there is an isomorphism

$$J_{(x,y)} \otimes J_{(y,z)} \to J_{(x,z)}$$

where $J_{(x,y)}$ denotes the fibre of J over (x, y). Moreover all these isomorphisms are globally defined and commute with each other. In the example we are interested in below it will be straightforward to check that these conditions hold.

An easy way to construct a bundle gerbe J is to choose a line bundle $L \to Y$ and define $J_{(x,y)} = L^*_x \otimes L_y$ for all x and y where L_x denotes the fibre of L over x. We say that a bundle gerbe is *trivial* if can be shown to arise in this way. A bundle gerbe has a characteristic class called its Dixmier-Douady class which is a three class in the integral cohomology of P and is the obstruction to the bundle gerbe being trivial. More details are in [6].

The definition we have given here differs from that in [6] in two respects. The first is that we consider here complex rank one vector bundles instead of $\mathbf{C}^\times$ principal bundles. This is just the usual relationship between vector bundles and their frame bundles and gives equivalent definitions. More significant is that in [6] the map $Y \to X$ is required to be a fibration. The requirement that it be a submersion is weaker but necesary for the example below and still sufficient to recover the theory developed in [6].

4 Faddeev's anomaly

Following Segal [7] we consider the Dirac operator D_A coupled to a connection $A \in \mathcal{A}$. This is an operator on the Hilbert space H of spinors. Define a subset $\mathcal{A}_0 \subset \mathcal{A} \times \mathbf{R}$ to be all pairs (A, s) where s is not in the spectrum of the Dirac operator D_A. Given such a pair (A, s) we can decompose the Hilbert space of spinors into the direct sum of $H^+_{(A,s)}$ and $H^-_{(A,s)}$; the sums of the eigenspaces for eigenvalues greater and less than s respectively. We can then form the Fock bundle

$$F_{(A,s)} = \bigwedge H^+_{(A,s)} \otimes \bigwedge (H^-_{(A,s)})^*.$$

If we leave A fixed and consider another $t < s$ not in the spectrum of D_A then we have

$$H = H^-_{(A,t)} \oplus V_{(A,t,s)} \oplus H^+_{(A,s)}$$

where $V_{(A,t,s)}$ is the sum of all the eigenspaces for eigenvalues between t and s. Moreover

$$H^+_{(A,t)} = V_{(A,t,s)} \oplus H^+_{(A,s)} \quad \text{and} \quad H^-_{(A,s)} = H^-_{(A,t)} \oplus V_{(A,t,s)}.$$

As $\bigwedge V^*_{(A,s,t)} \otimes \det V_{(A,s,t)}$ is canonically isomorphic to $\bigwedge V_{(A,s,t)}$ it follows that

$$F_{(A,s)} = F_{(A,t)} \otimes \det V_{(A,s,t)}.$$

Hence the projective spaces $P(F_{(A,s)})$ and $P(F_{(A,t)})$ can be identified and descend to a projective bundle $\mathcal{P}$ on $\mathcal{A}$.

We are interested in whether or not there is a Hilbert bundle $\mathcal{H} \to \mathcal{A}/\mathcal{G}$ whose projective bundle is isomorphic to $\mathcal{P}/\mathcal{G}$. This question can be phrased in two equivalent ways. Firstly we note that $\mathcal{P} \to \mathcal{A}$ is always the projective bundle of a Hilbert bundle $\mathcal{H}$ over $\mathcal{A}$ because $\mathcal{A}$ is contractible. However to make $\mathcal{H}$ a bundle on $\mathcal{A}/\mathcal{G}$ we need to lift the group action of $\mathcal{G}$ to $\mathcal{H}$ and the obstruction to that is a two-cocycle with values in $\mathrm{Map}(\mathcal{A}, \mathbf{C}^\times)$. Equivalently we can tackle the problem directly on $\mathcal{A}/\mathcal{G}$. There it is well-known that the obstruction to a projective bundle over $\mathcal{A}/\mathcal{G}$ being the projectivisation of a Hilbert bundle is a three class in the cohomology of $\mathcal{A}/\mathcal{G}$.

It was shown in [1] that these two approaches are related by the transgression defined in Sec. 2. More precisely if we transgress the three class in question we get a $L\mathcal{G}$ cocycle which is the derivative of the group cocycle. We will show that there is a naturally defined bundle gerbe whose Dixmier-Douady class is the three cocycle and such that the transgressed Lie algebra two-cocycle is that of Faddeev and Mickelsson [4]. (Somewhat confusingly, the occurrence of the latter in chiral gauge theories is often referred to as Faddeev's anomaly).

5 The bundle gerbe

The space $\mathcal{A}_0$ introduced in Sec. 4 projects onto the space $\mathcal{A}$ of connections. This projection is clearly onto and a submersion because locally $\mathcal{A}_0$ is a product of an open set in $\mathcal{A}$ and an open set in $\mathbf{R}$. However the map $\mathcal{A}_0 \to \mathcal{A}$ is not a fibration because it fails to be locally trivial near points of $\mathcal{A}$ for which D_A has degenerate eigenvalues.

The fibre product $\mathcal{A}_0^{[2]}$ is just the set of all triples (A, t, s) where neither of the real numbers s and t are in the spectrum of D_A. We define a line bundle J over $\mathcal{A}_0^{[2]}$ by defining its fibre at (A, t, s) to be $\det V_{(A,t,s)}$ if $t \leq s$ and $\det V^*_{(A,t,s)}$ if $t \geq s$. The first two conditions for a bundle gerbe follow naturally. For the third consider $t < s < r$ then

$$V_{(A,t,s)} \oplus V_{(A,s,r)} = V_{(A,t,r)}$$

so that

$$\det(V_{(A,t,s)}) \otimes \det(V_{(A,s,r)}) = \det(V_{(A,t,r)})$$

as required.

In the notation of Sec. 4 we have

$$F_{(A,s)} = F_{(A,t)} \otimes J_{(A,s,t)}.$$

Consider what happens if J is trivial. That means there is a line bundle L over $\mathcal{A}_0$ such that

$$J_{(A,s,t)} = L^*_{(A,s)} \otimes L_{(A,t)}.$$

Then we have

$$F_{(A,s)} \otimes L_{(A,s)} = F_{(A,t)} \otimes L_{(A,t)}$$

and we can define a Hilbert bundle $\mathcal{H}$ on $\mathcal{A}$ by taking its fibre over A to be any of the $F_{(A,s)} \otimes L_{(A,s)}$ which are now all identified. Clearly we have $P(\mathcal{H}) \simeq \mathcal{P}$.

So if the bundle gerbe J is trivial then the projective bundle $\mathcal{P}$ is the projectivisation of a Hilbert bundle. The converse is also true. Assume that we have identified $\mathcal{P} = P(H)$ for some $H \to \mathcal{A}$. Then define $L_{(A,s)}$ to be all linear isomorphisms from $F_{(A,s)}$ to H_A whose projectivisation is the identification map $\mathcal{P}_A \to P(H_A)$. Notice that elements of $J_{(A,s,t)}$ define linear isomorphisms from $F_{(A,s)}$ to $F_{(A,t)}$ which induce the identity map $P(F_{(A,s)}) = \mathcal{P}_A$ to $P(F_{(A,t)}) = \mathcal{P}_A$ because that is how $\mathcal{P}$ is defined. It follows that composing these maps with elements of $L_{(A,t)}$ defines an isomorphism

$$J_{(A,s,t)} = L^*_{(A,s)} \otimes L_{(A,t)} \tag{1}$$

and hence the bundle gerbe is trivial.

So far everything we have said is happening on $\mathcal{A}$ where the bundle gerbe is trivial. However $\mathcal{G}$ clearly acts on J and it descends to a bundle gerbe on $\mathcal{A}/\mathcal{G}$ where the same discussion holds. A trivialisation of the bundle gerbe on $\mathcal{A}/\mathcal{G}$ is therefore equivalent to a $\mathcal{G}$ equivariant trivialisation of the bundle gerbe over $\mathcal{A}$.

Notice that if J has two trivialisations L and $\hat{L}$ then we must have

$$L^*_{(A,s)} \otimes \hat{L}_{(A,s)} = L^*_{(A,t)} \otimes \hat{L}_{(A,t)}$$

and hence $L^* \otimes \hat{L}$ descends to a line bundle on $\mathcal{A}$. As $\mathcal{A}$ is contractible any line bundle is trivial so we conclude that any two trivialisations of J over $\mathcal{A}$ are isomorphic as line bundles. Moreover because $\mathcal{A}$ is contractible J does have a trivialisation which is therefore essentially unique. We shall see how to construct it in the next section.

We conclude that the bundle $\mathcal{P}/\mathcal{G}$ is a Hilbert bundle if the trivialisation L of J admits a lift of the gauge group action, compatible with (1).

Given that there exists a bundle L and a trivialisation as in (1) consider what it means for L to admit a lift of the gauge group action compatible with (1). The group of all automorphisms of L covering the gauge group action, $\mathrm{Aut}_{\mathcal{G}}(L)$ is an extension of $\mathcal{G}$ by the group $\mathrm{Map}(\mathcal{A}_0, \mathbf{C}^\times)$ where the latter is the subgroup of automorphisms that cover the identity in $\mathcal{G}$. We are interested in the subgroup $\hat{\mathcal{G}}$ of automorphisms compatible with (1). The intersection of this with $\mathrm{Map}(\mathcal{A}_0, \mathbf{C}^\times)$ is easily seen to be the space all functions $f(A, s)$ such that

$$f(A, s)^{-1} f(A, t) = 1$$

that is $f(A, s)$ is independent of s. Hence we have an extension

$$0 \to \mathrm{Map}(\mathcal{A}, \mathbf{C}^\times) \to \hat{\mathcal{G}} \to \mathcal{G} \to 0 \tag{2}$$

of groups. A lift of the $\mathcal{G}$ action to an action on L compatible with (1) is a splitting of (2).

6 The Grassmanian

To construct the trivialisation of J we use the results of [3] and [5]. There it is shown that if $(A, s) \in \mathcal{A}_0$ then $H^+_{(A,s)}$ defines an element of a Grassmanian Gr_2 of subspaces satisfying a Schatten class condition. This defines a map

$$p\colon \mathcal{A}_0 \to \mathrm{Gr}_2. \tag{3}$$

Notice that the embedding $\mathcal{A} \to \mathrm{Gr}_2$ defined in [5] is actually singular.

In [5] [4] a determinant line bundle $\mathcal{L}$ is defined over Gr_2. If we denote by L the pull-back of this under p, that is

$$L_{(A,s)} = \mathcal{L}_{p(A,s)},$$

we shall show in the Appendix that this trivialises the gerbe J. It is also shown in [5] and [3] that the obstruction to lifting the gauge group action to L is a cocycle cohomologous to the Faddeev-Mickelsson 2-cocycle. Moreover that cocycle has values in $\mathrm{Map}(\mathcal{A}, \mathbf{C}^\times)$ not just $\mathrm{Map}(\mathcal{A}_0, \mathbf{C}^\times)$ so that it must be the obstruction to a lift compatible with (1).

We conclude that the obstruction to the bundle $\mathcal{P}/\mathcal{G}$ being the projectivisation of a bundle of Hilbert spaces over $\mathcal{A}/\mathcal{G}$ is the Fadeev cocycle. It is a straightforward calculation [2] to check that this is the transgression of the three form

$$\omega_A(X, Y, Z) = \int_M \mathrm{tr}(X \wedge Y \wedge Z)$$

on $\mathcal{A}/\mathcal{G}$. This is therefore the Dixmier-Douady class of the gerbe $J/\mathcal{G}$.

7 Appendix

This calculation relies on definitions and techniques from [4]. We want to construct an isomorphism

$$L^*_{(A,s)} \otimes L_{(A,t)} \to J_{(A,s,t)} \tag{4}$$

Recall from [5] that we begin by choosing a reference subspace H_0 with an orthonormal basis $e_i, i \geq 0$, of the Hilbert space of sections of the spin bundle. We assume that $e_i, i \in \mathbf{Z}$ forms a basis of all L^2 sections. Then the spaces $H_k = \text{span}\{e_i \mid i = k, k+1, \ldots\}$ are in the Grassmanian Gr_2 of [5]. In fact each H_k will be in a different connected component of Gr_2. If our reference subspace H_0 is one of the spaces $H^+_{(A,s)}$ then $H^+_{(A,s)} \in Gr_2$ for all A and s.

The element $H^+_{(A,s)}$ in the Grassmanian and its orthogonal complement define a splitting of H as does H_0 and its orthogonal complement. There is therefore a map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

mapping the former splitting to the latter. The index of $H^+_{(A,s)}$ is defined to be the index of the operator a. If $H^+_{(A,s)}$ has index d then an admissible basis is a basis

$$w_{-d}, w_{-d+1}, \ldots$$

of $H^+_{(A,s)}$ which can be reached from an orthonormal basis by a linear isomorphism and such that the matrix w_+ defined by

$$pr_{-d}(w_j) = \sum_{k=-d}^{\infty} (w_+)_{kj} e_k,$$

where pr_{-d} is the orthogonal projection onto H_{-d}, is a map in GL^2.

Assume that we are at a point (A, s) where $H^+_{(A,s)}$ has index n and that at another point (A, t) with $t < s$ we have

$$H^+_{(A,t)} = V_{(A,t,s)} \oplus H^+_{(A,s)} \tag{5}$$

with $\dim(V_{(A,t,s)}) = d$ so that $H^+_{(A,t)}$ has index $d + n$. Then we can restrict attention to admissible bases $\widetilde{w}$ for $H^+_{(A,t)}$ that respect the orthogonal decomposition in (5). That is $\widetilde{w} = (u, w)$ for a u a basis of $V_{(A,s,t)}$ and w an admissible basis for $H^+_{(A,s)}$. We then have that

$$\widetilde{w}_+ = \begin{pmatrix} \alpha & \beta \\ \gamma & w_+ \end{pmatrix}.$$

The space of admissible bases is acted on upon the right by elements of the group GL^2. Again in the case of $\tilde{w}$ we restrict attention to elements $\tilde{t}$ that preserve the decomposition (5) and hence we have

$$\tilde{t} = \begin{pmatrix} T & 0 \\ 0 & t \end{pmatrix}$$

where T is a d by d matrix and t an element of GL^2 acting on w.

The fibre of the pull-back line bundle L over the point (A, s) is the set of all equivence classes consisting of an admissible basis w and a complex number λ subject to the equivalence relation

$$(w, \lambda) \simeq (wt, \lambda\omega_2(w_+, t)^{-1})$$

where $\omega_2(w_+, t)$ is defined in [4]. Using square brackets to denote equivalence classes we define the isomorphism in (4) by

$$[w, \lambda] \otimes [\tilde{w}, \mu] \mapsto \frac{\lambda}{\mu} \exp(\mathrm{tr}(\alpha - 1)) \det(u)$$

where $\det(u)$ is the wedge product of the elements of u. We need to check that this is well-defined. That reduces to proving that

$$\omega_2(\tilde{w}_+, \tilde{t}) = \omega_2(w_+, t) \det(T) \exp(-\mathrm{tr}(\alpha T - 1)) \exp(\mathrm{tr}(T - 1)).$$

For this kind of identity we can follow the standard technique and use the fact that both sides are continuous and that the various groups GL^p are dense in each other. Hence it suffices to prove it for elements in GL^1. Applying the results of [4], namely

$$\omega_2(A, B) = \det{}_2(B) \exp(-\mathrm{tr}(AB - 1) + \mathrm{tr}(A - 1) + \mathrm{tr}(B - 1))$$

and

$$\det{}_2(A) = \det(A) \exp(-\mathrm{tr}(A - 1))$$

yields

$$\omega_2(A, B) = \det(B) \exp(-\mathrm{tr}(AB - 1) + \mathrm{tr}(A - 1)).$$

The result then follows by a calculation.

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