DYNAMICAL ZETA FUNCTIONS FOR NON-UNIFORMLY HYPERBOLIC TRANSFORMATIONS

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Abstract. We consider a class of maps f of $[0,1]$ which are expanding everywhere but at a fixed point at 0, which we allow to be neutral. We follow two parallel approaches:

1) using an induced version g of the map f we are able to relate the analytic properties of the dynamical zeta functions associated to f and g and the spectral properties of the corresponding transfer operators;

2) using a suitable piecewise affine approximation $\hat{f}$ of the map f we obtain information on the behaviour of the corresponding zeta functions in the whole complex plane.

One result is that if f has a neutral fixed point then its zeta function extends meromorphically in the entire complex plane with a cross cut along the ray $(1, +\infty)$.

1. Introduction.

Weighted dynamical zeta functions as invariant objects associated to dynamical systems are receiving an increasing attention since the early work of Ruelle [R1]. For dynamical systems satisfying uniform hyperbolicity, like Axiom A systems, the zeta function usually turns out to have poles in some region of the complex plane, and these poles contain ergodic and geometric information about the dynamics.

On the other hand, counterexamples have been constructed (for suspensions of shifts) where the zeta function has a non-polar singularity arbitrarily close to its circle of convergence [G],[Po], even though a dynamical interpretation of this fact seems not to be available.

Consider now a transformation $f : M \to M$ of a compact metric space and set $\mathrm{Fix} f^n = \{x : f^n x = x\}$. Let $\phi : M \to \mathbb{C}$ be a weight function (in the sequel we shall consider only the case $\phi(x) = 1/|f'(x)|$). The formal weighted dynamical zeta function of f is then given by

$$\zeta(f, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} \prod_{k=0}^{n-1} \phi(f^k(x)) \tag{1.1}$$

When f satisfies uniform hyperbolicity and some smoothness assumptions, and ϕ is sufficently regular, one can prove that $\zeta(f, z)$ is analytic in some domain and extends meromorphically to a larger domain, where its poles are in bijection with the eigenvalues of a transfer operator acting on a suitable Banach space of functions. The most striking applications then concern the equidistribution of closed orbits and the determination of the decay rate of correlation functions. We refer to [B],[PP],[R2] for detailed expositions of the subject.

One expects that if the assumptions of uniform hyperbolicity is relaxed then the situation may change considerably.

In this paper we consider a class of smooth transformations $f : [0, 1] \to [0, 1]$ with a fixed point at the origin which we allow to be neutral or repelling. In the former case, we obtain the Pomeau-Manneville type 1 model at the intermittent transition point. The main result is that, when the fixed point at 0 is neutral, the zeta function is analytic in the unit disk and has a branch point in $z = 1$ whose order depends on the behaviour of the map in a neighborhood of the origin. The asymptotic behaviour of $\zeta(f, z)$ when $z \to 1_-$ is shown to be related with the mean return time in a given subset of $[0, 1]$. In addition $\zeta(f, z)$ extends meromorphically in the cut plane.

This result is acheived basically in two different ways: to the map f we associate an induced version g and a piecewise affine approximation $\hat{f}$, the former being uniformly hyperbolic and the latter allowing for exact calculations. Then, on one hand, we introduce a two-variables zeta function which relates $\zeta(f, z)$ to $\zeta(g, z)$ (Proposition 4.1), and a modified transfer operator which relates the transfer operator of f to that of g (Propositions 4.2 and 4.3), and study the analytic properties of both zeta functions through the spectral properties of this modified transfer operator (Theorems 4.1 and 4.2).

On the other hand, we compute an explicit power series expansion for $\zeta(\hat{f}, z)$ and then prove that $\zeta(f, z)$ and $\zeta(\hat{f}, z)$ have the same asymptotic behaviour in a neighborhood of $z = 1$ (Theorem 5.1).

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2. The model, its induced version and its piecewise affine approximation.

Our assumptions on the transformation $f : [0, 1] \to [0, 1]$ are the following:

(1) $f(0) = 0,\ f(1) = 1$;

(2) there is a number $p \in]0, 1[$ such that, setting $I_0 = [0, p[$ and $I_1 = [p, 1]$, we have $f(I_0) = [0, 1[,\ f(I_1) = [0, 1]$;

(3) the restriction $f_{|I_0}$ extends to a C^1-diffeomorphism f_0 on the closure of I_0 and $f_1 = f_{|I_1}$ is a C^1-diffeomorphism on I_1; the inverses are denoted as $\psi_i = f_i^{-1}$,

$i = 0, 1$;

(4) f_0 and f_1 have Lipschitz derivative;

(5) there are two numbers $0 < \eta < \alpha \leq 1$ such that $f' > 1/\alpha$ on $]0, p[$ and $f'(0) = 1/\alpha$; whereas $f' > 1/\eta$ on I_1;

(6) f has the following asymptotic behaviour when $x \to 0_+$

$$f(x) = \frac{x}{\alpha} + \gamma x^{1+s}(1 + u(x))$$

with constant $\gamma > 0$ and exponent $s + 1 > 1$, and where $u(x)$ is a C^1 function such that $u'(x) = \mathcal{O}(x^{t-1})$ as $x \to 0_+$, for some $t > 0$.

Remark 2.1. In what is to follow we shall consider the two situations corresponding to $\alpha = 1$ (neutral fixed point) and $\alpha < 1$ (uniform hyperbolicity) in a parallel way so as to emphasize the generality of the approach here considered and at the same time provide a direct comparison between the two cases.

We also introduce the sequence $\{c_k\}_{k \geq 0}$, given by

$$c_0 = 1, \quad c_k = \psi_0^k(c_0), \quad k \geq 1. \tag{2.1}$$

This sequence generates a countable partition of $[0, 1]$ into the intervals $A_k = [c_k, c_{k-1}]$, $k \geq 1$, which is a Markov partition. In particular, setting $A_0 = [0, 1]$, we have $f(A_k) = A_{k-1}$ for any $k \geq 1$.

In the sequel we shall need some information about the asymptotic behaviour of the c_k's.

Lemma 2.1. *Under the hypotheses (1)-(6) on the map f we have the following behaviour of the c_k when $k \to \infty$:*

i) if $\alpha < 1$ then $c_k = \mathcal{O}(1)\, \alpha^k$;

ii) if $\alpha = 1$ then $c_{k-1} = (s\gamma k)^{-1/s}(1 + \mathcal{O}(k^{-1}))$.

Proof. The last property of f gives for the inverse function:

$$\psi_0(x) = \alpha x - \gamma x^{1+s}(1 + v(x))$$

where $v(x)$ is some C^1 function such that $v'(x) = \mathcal{O}(x^{t-1})$ as $x \to 0_+$. We write this expression in a more manageable form, that is:

$$\psi_0(x) = \Big((\alpha x)^{-s} + \gamma s \alpha^{-1-s}(1+\tilde{v}(x))\Big)^{-1/s}$$

where $\tilde{v}(x)$ is another C^1 function such that $v'(x) - \tilde{v}'(x) = \mathcal{O}(x^{s-1})$. It is then easy to check that

$$\psi_0^k(x) = \Big((\alpha^k x)^{-s} + \gamma s \alpha^{-1-sk}\sum_{l=0}^{k-1} \alpha^{ls}(1+\tilde{v}(\psi_0^l(x)))\Big)^{-1/s}$$

On the other hand for $\alpha < 1$ we have

$$\sum_{l=0}^{k-1} \alpha^{ls}(1+\tilde{v}(\psi_0^l(x))) = \mathcal{O}(1)$$

and this concludes the proof of the first part. For $\alpha = 1$ we find

$$\psi_0^k(x) = \Big(x^{-s} + \gamma s k(1 + \frac{1}{k}\sum_{l=0}^{k-1} \tilde{v}(\psi_0^l(x)))\Big)^{-1/s}$$

whence

$$c_{k-1} = (s\gamma k)^{-1/s}(1+\frac{\mathcal{O}(1)}{k})^{-1/s}$$

and the assertion follows. q.e.d.

Remark 2.2. When $\alpha = 1$ the ergodic properties of the map f may have very different features according whether the series $\sum_k c_k$ is convergent or not. In the former case f has an absolutely continuous invariant probability measure which is ergodic and mixing, with polynomial decay of the correlations for regular (e.g. Hölder) observables. In the latter case, the only invariant non-singular measure is a σ-finite measure with anomalous ergodic properties. We shall see later how this difference is reflected on the analytic properties of the zeta function (see also Remarks 4.3 and 5.2). Concerning the ergodic properties of maps with a neutral fixed point we refer the reader to the following literature: [A],[AMU],[CF],[CI1],[CI2],[CI3],[FS],[LSV].

The induced version g.

The 'first passage' map (on the interval A_1), is the map $g : [0,1] \to [0,1]$ induced by f in the following way:

$$x \to g(x) = f^{n(x)}(x) \quad \text{where} \quad n(x) = 1 + \min\{n \geq 0 \ : \ f^n(x) \in A_1 \ \}$$

Equivalently, the map g is given by

$$x \to g(x) = g_k(x) = f^k(x) \quad \text{if} \quad x \in A_k, \ \ k \geq 1.$$

and defined arbitrarily on the set $\{c_k\}_{k\geq 0}$.

The map g is expanding and surjective, i.e.: $g_k'(x) \geq 1/\eta > 1$, for any $k \geq 1$, and $g_k(A_k) = [0,1]$. We denote by $\phi_k = g_k^{-1}$ the inverse branches of g. We shall see later on how the Lipschitz property of f' is reflected on g'.

Remark 2.3. Notice that the usual return time function $t(x)$ in the interval A_1 is given by

$$t(x) = \min\{n \geq 1 \ : \ f^n(x) \in A_1 \ \} = n \circ f(x).$$

The piecewise affine approximation $\hat{f}$.

For $k \geq 1$ we set (with $|A_0| = 1$),

$$\alpha_k = \frac{|A_k|}{|A_{k-1}|} \quad \text{and} \quad \beta_k = \prod_{j=1}^{k} \alpha_j = |A_k|. \tag{2.2}$$

Clearly we have

$$\overline{\lim}\, \alpha_k \ = \ \overline{\lim} (\beta_k)^{1/k} \ = \ \alpha. \tag{2.3}$$

We then define the map $\hat{f}$ as follows:

$$\hat{f}(x) = \begin{cases} (x - c_1)/\alpha_1, & \text{if } x \in A_1 \\ c_{k-1} + (x - c_k)/\alpha_k, & \text{if } x \in A_k, \ k \geq 2 \end{cases} \tag{2.4}$$

and its induced version $\hat{g}$ as

$$\hat{g}(x) = \frac{x - c_k}{\beta_k}, \qquad x \in A_k, \ k \geq 1. \tag{2.5}$$

We shall consider $\hat{f}$ and $\hat{g}$ as piecewise affine approximations of f and g, respectively.

3. The coding.

We now construct a coding. Let Ω be the set of one-sided sequences $\omega = (\omega_0\omega_1\ldots)$, $\omega_i \in \{1, 2, \ldots\}$ satisfying the compatibility condition: given ω_i then either $\omega_{i-1} = \omega_i + 1$ or $\omega_{i-1} = 1$. Then, the map $\xi : \Omega \to [0, 1]$ defined by

$$\xi(\omega) = x \quad \text{according to} \quad f^j(x) \in A_{\omega_j}, \;\; j \geq 0$$

is a bijection between Ω and the points of $[0, 1]$ which are not preimages of the origin. In other words, to any sequence $\omega \in \Omega$ corresponds, via the map ξ, a point $x \in [0, 1] \setminus \{c_k\}_{k\geq 0}$, and viceversa.

Moreover, ξ conjugates the map f with the shift τ on Ω.

Let us consider the infinite sequence $\{t_j\}_{j\geq 1}$ of successive entrance times in the state 1: $t_1(\omega) = \inf\{i \geq 0 \; : \; \omega_i = 1\}$ and, for $j \geq 2$, $t_j(\omega) = \inf\{i > t_{i-1} \; : \; \omega_i = 1\}$. Furthermore, we define a sequence of integer valued random variables by

$$\sigma_j(\omega) = t_{j+1} - t_j, \;\; j \geq 0$$

with the convention that $t_0 = -1$.

It is then easy to realize that for any $\omega \in \Omega$, we have $g^j(x) \in A_{\sigma_j}$, $j \geq 0$, where $x = \xi(\omega)$ and the integers $\sigma_j = \sigma_j(\omega)$ are defined above.

Let Σ be the set of *all* one-sided sequences σ of the form $\sigma = (\sigma_0\sigma_1\ldots)$, $\sigma_j \in \{1, 2, \ldots\}$. Then, the map $\pi : \Sigma \to [0, 1]$ defined by

$$\pi(\sigma) = x \quad \text{according to} \quad g^j(x) \in A_{\sigma_j}, \;\; j \geq 0$$

is a bijection between Σ and the points of $[0, 1] \setminus \{c_k\}_{k\geq 0}$. Moreover, π conjugates the map g with the shift τ on Σ. Observe that the map $\pi^{-1} \circ \xi$ determines a bijection between Ω and Σ.

All that can be repeated literally for the maps

$$\hat{\xi}(\omega) = x \quad \text{according to} \quad \hat{f}^j(x) \in A_{\omega_j}, \;\; j \geq 0$$

and

$$\hat{\pi}(\sigma) = x \quad \text{according to} \quad \hat{g}^j(x) \in A_{\sigma_j}, \;\; j \geq 0.$$

Remark 3.1. We can represent the dependence of the σ_j's on ω recursively as follows:

$$\sigma_0 = \omega_0 \quad \text{and} \quad \sigma_j = \omega_{s_j} \quad \text{for} \quad j > 0, \quad \text{where} \quad s_j = \sum_{i=0}^{j-1} \sigma_i.$$

Notice however that this rule may associate to a periodic sequence an eventually periodic one. More precisely, let $\omega \in \Omega$ be a periodic sequence of period n for the shift τ. We write it in the form $\omega = (\overline{\omega_0\omega_1 \ldots \omega_{n-1}})$. If $\omega_0 = 1$ then we may write, for some $k \geq 0$,

$$\omega_0\omega_1 \ldots \omega_{n-1} \; = \underbrace{1 \ldots 1}_{r_0 \geq 1} \underbrace{l_1 l_1 - 1 \ldots 1}_{l_1 \geq 1} \underbrace{1 \ldots 1}_{r_1 \geq 1} \;\cdots\; \underbrace{l_k l_k - 1 \ldots 1}_{l_k \geq 1} \underbrace{1 \ldots 1}_{r_k \geq 0}$$

where $r_0 + l_1 + \ldots + r_k = n$. Hence, the above rule gives

$$\sigma_0\sigma_1 \ldots \sigma_{k-1} \; = \underbrace{1 \ldots 1}_{r_0 \geq 1} \, l_1 \, \underbrace{1 \ldots 1}_{r_1 \geq 1} \;\cdots\; l_k \, \underbrace{1 \ldots 1}_{r_k \geq 0}$$

so that $\sigma(\omega) = (\overline{\sigma_0\sigma_1 \ldots \sigma_{m-1}})$ is periodic of period $m = k + r_0 + \ldots + r_k$ and satisfies $n = \sigma_0 + \ldots + \sigma_{m-1}$.

On the other hand, if $\omega_0 > 1$, it may happen that, for some $k \geq 1$,

$$\omega_0\omega_1 \ldots \omega_{n-1} \; = \underbrace{l_0 l_0 - 1 \ldots 1}_{l_0 \geq 1} \underbrace{1 \ldots 1}_{r_1 \geq 1} \underbrace{l_1 l_1 - 1 \ldots 1}_{l_1 \geq 1} \;\cdots\; \underbrace{1 \ldots 1}_{r_k \geq 1} \underbrace{l_k + l_0 \ldots l_0 + 1}_{l_k \geq 0}$$

and, according to the above rule, one would find the eventually periodic sequence $\sigma(\omega) = (\sigma_0\overline{\sigma_1\sigma_2 \ldots \sigma_m})$ where

$$\sigma_0 = l_0 \quad \text{and} \quad \sigma_1\sigma_2 \ldots \sigma_m \; = \underbrace{1 \ldots 1}_{r_1 \geq 1} \, l_1 \;\cdots\; \underbrace{1 \ldots 1}_{r_k \geq 1} \, l_k + l_0$$

whose ultimate period is $m = k+r_1+\ldots+r_k$ and which satisfies $n = \sigma_1+\ldots+\sigma_m$. Thus, in the latter case, in order to obtain a correspondence between periodic sequences it is necessary to apply the aformentioned rule to some iterate of the original sequence. By the way, this operation does not modify the weight associated to the periodic sequence (see below) and thus it has no influence in what follows.

Remark 3.2. For every integer $j \geq 0$, denote by x_j the projection on the j^{th} symbol, i.e. $x_j(\omega) = \omega_j$. Then the stochastic process on Ω given by $x_j(\omega) = \omega_j$, $j \geq 0$, is a Markov chain with transition probabilities $p_{ij} = |\psi_0(A_j) \bigcap A_i|/|A_i|$. This fact will be used in Section 5 to compute an explicit formula for $\zeta(\hat{f}, z)$. As easy consequence which can be noted here is that the random variables σ_j are independent and identically distributed. Their common law is given by: $\mathrm{Prob}(\sigma_j = k) = \beta_k$ for any $j \geq 0$ and $k \geq 1$.

4. Interactions, zeta functions and transfer operators.

We now use the maps ξ and π to project the functions $\log f'$ and $\log g'$, respectively, down to some 'interactions' on the symbol spaces. More precisely, given $\omega \in \Omega$ and $\sigma \in \Sigma$, we set

$$V(\omega) = \begin{cases} V_0(\omega) = \log[(\psi_0)'(\xi(\omega_1\omega_2\ldots))], & \text{if } \omega_0 > 1 \\ V_1(\omega) = \log[(\psi_1)'(\xi(\omega_1\omega_2\ldots))], & \text{if } \omega_0 = 1 \end{cases} \tag{4.1}$$

and

$$W(\sigma) = \log[(\phi_{\sigma_0})'(\pi(\sigma_1\sigma_2\ldots))]. \tag{4.2}$$

The corresponding 'linearized' functions are obtained through the maps $\hat{\xi}$ and $\hat{\pi}$:

$$\hat{V}(\omega) = \log \alpha_{\omega_0}, \quad \text{and} \quad \hat{W}(\sigma) = \log \beta_{\sigma_0}. \tag{4.3}$$

Given $0 < \theta < 1$ we define a metric on Σ by setting $d_\theta(\sigma, \sigma') = \theta^n$ where n is such that: $\sigma_j = \sigma'_j$ for $0 \leq j \leq n$. Moreover, for any continuous function

$\Phi : \Sigma \to \mathbb{C}$ and integer $n \geq 0$, set:

$$\mathrm{var}_n \Phi = \sup\{\, |\Phi(\sigma) - \Phi(\sigma')| \,:\, \sigma_j = \sigma'_j,\, 0 \leq j \leq n \,\}$$

and

$$|\Phi|_\theta = \sup\{\frac{\mathrm{var}_n \Phi}{\theta^n}, n \geq 0\}$$

Finally, we denote by $\mathcal{F}_\theta$ the space of all Lipschitz functions on Σ with respect to the metric d_θ, that is all continuous function Φ on Σ satisfying $\mathrm{var}_n \Phi \leq C\theta^n$ for some constant $C > 0$ (so $|\Phi|_\theta$ is the least Lipschitz constant). With the norm $\|\Phi\|_\theta = |\Phi|_\theta + |\Phi|_\infty$, one makes $\mathcal{F}_\theta$ a Banach space.

A direct consequence of the assumptions made above on the map f is the following result:

Lemma 4.1. *$W(\sigma) \in \mathcal{F}_\theta$ for any $\theta \geq \eta$.*

Proof. The proof is a trivial adaptation to the present context of the argument given in [CI1], Lemma 2.1. q.e.d.

We now write the dynamical zeta functions for the map f and g in the following way:

$$\zeta(f, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(f), \qquad \zeta(g, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(g) \tag{4.4}$$

where

$$Z_n(f) = \sum_{\substack{\omega \in \Omega \\ \tau^n \omega = \omega}} \exp \sum_{j=0}^{n-1} V(\tau^j \omega) + \alpha^n \tag{4.5}$$

and

$$Z_n(g) = \sum_{\substack{\sigma \in \Sigma \\ \tau^n \sigma = \sigma}} \exp \sum_{j=0}^{n-1} W(\tau^j \sigma) \tag{4.6}$$

The term α^n in (4.5) accounts for the contribution of the fixed point in 0. Using the functions $\hat{V}$ and $\hat{W}$ one obtains the corresponding quantities for the affine model.

Let us now examine how $\zeta(f, z)$ and $\zeta(g, z)$ are related to one another.

First, let $\omega \in \Omega$ be a periodic sequence of period n for the shift τ. Moreover, let $\sigma(\omega)$ be a periodic sequence of period m in the space Σ corresponding to ω or to some iterate of it (see Remark 3.1). Thus $s_m(\sigma) = \sum_{j=0}^{m-1} \sigma_j = n$. ¿From (4.1) and (4.2) we then have

$$\sum_{j=0}^{n-1} V(\tau^j \omega) = \sum_{j=0}^{m-1} W(\tau^j \sigma)$$

Using this fact we write $Z_n(f)$ as follows:

$$Z_n(f) = \alpha^n + \sum_{m=1}^{n} \frac{n}{m} \sum_{\substack{\sigma\in\Sigma,\, \tau^m\sigma=\sigma \\ s_m(\sigma)=n}} \exp \sum_{j=0}^{m-1} W(\tau^j \sigma) \tag{4.7}$$

The second sum ranges over the $\binom{n-1}{m-1}$ ways to write the integer n as a sum of m positive integers, counting all permutations.

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(f) &= \log(\frac{1}{1-\alpha z}) + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{m} \sum_{\substack{\sigma\in\Sigma,\, \tau^m\sigma=\sigma \\ s_m(\sigma)=n}} z^n \exp \sum_{j=0}^{m-1} W(\tau^j \sigma) \\ &= \log(\frac{1}{1-\alpha z}) + \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{\sigma\in\Sigma \\ \tau^m\sigma=\sigma}} z^{s_m(\sigma)} \exp \sum_{j=0}^{m-1} W(\tau^j \sigma)\end{aligned}$$

Putting together these observations we have the following

Proposition 4.1. *Consider the zeta function of two variables given by*

$$Z(f,w,z) = \exp \sum_{n=1}^{\infty} \frac{w^n}{n} \sum_{\substack{\sigma\in\Sigma \\ \tau^n\sigma=\sigma}} z^{s_n(\sigma)} \exp \sum_{j=0}^{n-1} W(\tau^j \sigma). \tag{4.8}$$

where $s_n(\sigma) = \sum_{j=0}^{n-1} \sigma_j$. *Then*

$$Z(f,1,z) = \zeta(f,z)(1-\alpha z) \quad \text{and} \quad Z(f,w,1) = \zeta(g,w). \tag{4.9}$$

We shall now study the relationship between the zeta functions we have introduced above, in particular the two-variables zeta function defined in Proposition 4.1, and suitable transfer operators acting on the space $\mathcal{F}_\theta(\Sigma)$.

For $W \in \mathcal{F}_\theta$ and $z \in \mathbb{C}$, let $\mathcal{L}_{W,z} : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ be defined by

$$(\mathcal{L}_{W,z} v)(\sigma) = \sum_{k=1}^{\infty} z^k \, e^{W(k\sigma)} \, v(k\sigma) \tag{4.10}$$

where $k\sigma$ denotes the sequence $(k\sigma_0\sigma_1 \ldots)$.

For the piecewise affine model one finds

$$(\mathcal{L}_{\hat{W},z} v)(\sigma) = \sum_{k=1}^{\infty} z^k \, \beta_k \, v(k\sigma).$$

Notice that for $z = 1$ one recovers the (symbolic version of the) Ruelle transfer operator $\mathcal{L}_W$ corresponding to the first passage map g (or $\hat{g}$).

The transfer operator associated to the map f is given by

$$\mathcal{L}_V u(\omega) = (\mathcal{L}_0 + \mathcal{L}_1) u(\omega)$$

with

$$\mathcal{L}_0 u(\omega) = e^{V_0((\omega_0+1)\omega)} u((\omega_0+1)\omega) \quad \text{and} \quad \mathcal{L}_1 u(\omega) = e^{V_1(1\,\omega)} u(1\,\omega)$$

We shall compare below the action of $\mathcal{L}_V$ and $\mathcal{L}_{W,z}$. So doing, it will be implicitly made use of the bijection $\vartheta = \xi^{-1} \circ \pi : \Sigma \to \Omega$. More precisely, let $\mathcal{I}_\vartheta : C(\Sigma) \to C(\Omega)$ be defined by $\mathcal{I}_\vartheta \psi = \psi \circ \vartheta^{-1}$. If $\psi \in \mathcal{F}_\theta(\Sigma)$ and z is small enough then it is easy to check that $\mathcal{L}_{W,z}\psi \in \mathcal{F}_\theta(\Sigma)$ and also $\mathcal{I}_\vartheta^{-1} \mathcal{L}_V \mathcal{I}_\vartheta \psi \in \mathcal{F}_\theta(\Sigma)$.

With this in mind, we first show that a simple algebraic relation exists between the operators $\mathcal{L}_{W,z}$, $\mathcal{L}_W$ and $\mathcal{L}_V$, which can be viewed as the counterpart of Proposition 4.1:

Proposition 4.2.

$$(1 - \mathcal{L}_{W,z})(1 - z\mathcal{L}_0) = 1 - z\mathcal{L}_V \quad \text{and} \quad (1 - \mathcal{L}_W)(1 - \mathcal{L}_0) = 1 - \mathcal{L}_V$$

Proof. We obtain the first identity, the second one will then follow by putting $z = 1$. First notice that if $\omega = \xi^{-1} \circ \pi(\sigma)$ we have

$$W(\sigma) = \sum_{j=0}^{\sigma_0 - 2} V_0(\tau^j \omega) + V_1(\tau^{\sigma_0 - 1}\omega). \tag{4.11}$$

Therefore, we have the representation

$$\mathcal{L}_{W,z} = \sum_{k=1}^{\infty} z^k \, \mathcal{L}_1 \mathcal{L}_0^{k-1} \tag{4.12}$$

and

$$\begin{aligned}(1 - \mathcal{L}_{W,z})(1 - z\mathcal{L}_0) &= 1 - \sum_{k=1}^{\infty} z^k \, \mathcal{L}_1 \mathcal{L}_0^{k-1} - z\mathcal{L}_0 + \sum_{k=1}^{\infty} z^{k+1} \, \mathcal{L}_1 \mathcal{L}_0^{k} \\ &= 1 - z\mathcal{L}_0 - z\mathcal{L}_1 = 1 - z\mathcal{L}_V. \qquad \text{q.e.d.}\end{aligned}$$

According to (4.12) $\mathcal{L}_{W,z}$ is an operator-valued power series whose radius of convergence is given by $\lim_{k\to\infty} \|\mathcal{L}_1\mathcal{L}_0^{k-1}\|_\theta^{-1/k}$. Using (4.11) we find

$$|\mathcal{L}_1\mathcal{L}_0^{k-1} v(\sigma)| \le e^{W(k\sigma)} |v|_\infty$$

and also

$$|e^{W(k\sigma)} v(k\sigma) - e^{W(k\sigma')} v(k\sigma')| \le \, e^{W(k\sigma)} \, (\, \theta \, |v|_\theta + C_1 \, \theta \, |v|_\infty \,)$$

for some constant $C_1 > 0$. Hence, for some $C_2 > 0$, we have

$$\|\mathcal{L}_1\mathcal{L}_0^{k-1}\|_\theta \le C_2 \, \sup_\sigma e^{W(k\sigma)}$$

Now, from Lemmas 2.1 and 4.1 it follows easily that for $\alpha < 1$ there is a constant $C_4 > 0$ such that $\sup_\sigma e^{W(k\sigma)} \le C_4 \alpha^k$; whereas, for $\alpha = 1$, one finds $\sup_\sigma e^{W(k\sigma)} \le C_5 k^{-(1+1/s)}$, for some $C_5 > 0$. On the other hand, if we denote by $r(T)$ the spectral radius of T, then it is a simple matter to show that $r(\mathcal{L}_0) = \alpha$. Thus we have proved the following

Lemma 4.2. *The radius of convergence of $\mathcal{L}_{W,z}$ is bounded below by $1/r(\mathcal{L}_0) = 1/\alpha$.*

Remark 4.1. In [CI1] we have constructed an absolutely continuous invariant probability measure $\rho(dx) = h(x)\, dx$ for the dynamical system $([0,1], g)$ as a Gibbs state on Σ for the function $W(\sigma)$, whose density $h(\sigma)$ is in $\mathcal{F}_\theta(\Sigma)$ and satisfies $d^{-1} < h < d$ for some $d > 0$ (with a slight abuse of notation we also denote by h the function $h\circ\pi^{-1}$). ¿From Proposition 4.2 it follows that if $\mathcal{L}_W h = h$ and $\mathcal{L}_V e = e$ then h and e are related by $h = (1-\mathcal{L}_0)\,e$ or else $e = \sum_{k=0}^{\infty}\mathcal{L}_0^k h$. This correspondence has been used in [CI2] and [CI3] to study some ergodic properties of the σ-finite absolutely continuous measure $\nu(dx) = e(x)\,dx$ invariant for the dynamical system $([0,1], f)$ with $\alpha = 1$ and $s = 1$.

We can actually say more.

Proposition 4.3. *Let $z \neq 0$ and $|z| \le 1/r(\mathcal{L}_0)$. Then $1 \in \mathrm{sp}\,(\mathcal{L}_{W,z})$ if and only if $1/z \in \mathrm{sp}\,(\mathcal{L}_V)$, and they have the same geometric multiplicity. Furthermore, the corresponding eigenfunctions e_z of $\mathcal{L}_V$ and h_z of $\mathcal{L}_{W,z}$ are related by $h_z = (1-z\mathcal{L}_0)e_z$ or else $e_z = \sum_{k=0}^{\infty} z^k\mathcal{L}_0^k h_z$.*

Proof. Assume that $\mathcal{L}_{W,z}h_z = h_z$. Then, from Proposition 4.2 it follows that $(1-z\mathcal{L}_V)\sum_{k=0}^{\infty} z^k\mathcal{L}_0^k h_z = 0$. Conversely, assume that $z\mathcal{L}_V e_z = e_z$, then $(1-\mathcal{L}_{W,z})(1-z\mathcal{L}_0)e_z = 0$. q.e.d.

We now get some more information about the spectrum of $\mathcal{L}_{W,z}$. Let us recall that the spectrum $\mathrm{sp}\,(T)$ of a bounded linear operator T can be decomposed into a discrete part, made up of isolated eigenvalues of finite multiplicity, and its complement, the essential spectrum, denoted by ess(T) (see, e.g., [DS]). The essential spectral radius is then defined as $r_{\mathrm{ess}}(T) = \sup\{\,|\lambda| \,:\, \lambda \in \mathrm{ess(T)}\,\}$.

Set moreover

$$R_n(W,z) = \sup_{\sigma\in\Sigma}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} z^{k_1+\ldots+k_n}\exp\sum_{i=1}^{n} W(k_i\ldots k_n\sigma)$$

and

$$P(W,z) = \lim_{n\to\infty}\frac{1}{n}\log R_n(W,z). \tag{4.13}$$

For the piecewise affine model one finds

$$R_n(\hat{W},z) = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} z^{k_1+\ldots+k_n}\beta_{k_1}\ldots\beta_{k_n} = \left(\sum_k \beta_k\, z^k\right)^n$$

so that

$$\exp P(\hat{W}, z) = \sum_{k=1}^{\infty} \beta_k \, z^k. \tag{4.14}$$

Theorem 4.1. *For any $|z| \leq 1/r(\mathcal{L}_0)$, the spectrum of $\mathcal{L}_{W,z} : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ consists of two disjoint parts:*

1) every point in the disk $\{ \lambda \, : \, |\lambda| \leq \theta \, \exp P(W, z) \}$;

2) isolated eigenvalues in the annulus $\{ \lambda \, : \, \theta \, \exp P(W, z) < |\lambda| \leq \exp P(W, z) \}$.

In particular, $\exp P(W, z)$ is a simple eigenvalue.

Remark 4.2. According to Lemma 4.1, the smallest essential spectral radius is attained by taking $\theta = \eta$.

Proof of Theorem 4.1. It is easy to check that for all $v \in \mathcal{F}_\theta(\Sigma)$

$$|\mathcal{L}^n_{W,z} v|_\infty \leq R_n(z) |v|_\infty \leq R_n(z) \|v\|_\theta$$

and

$$|\mathcal{L}^n_{W,z} v|_\theta \leq R_n(z)(C|v|_\infty + \theta^n |v|_\theta) \leq R_n(z)(C+1)\|v\|_\theta$$

for some constant $C > 0$. Therefore $\|\mathcal{L}^n_{W,z}\|_\theta \leq (C+2) R_n(z)$, where we have also denoted by $\| \; \|_\theta$ the operator norm. Thus, the spectral radius formula implies that

$$r(\mathcal{L}_{W,z}) \leq \exp P(W, z)$$

On the other hand, we can write

$$r(\mathcal{L}_{W,z}) = \lim_{n \to \infty} \left(\|\mathcal{L}^n_{W,z}\|_\theta \right)^{1/n} \geq \lim_{n \to \infty} \left(|\mathcal{L}^n_{W,z} 1|_\infty \right)^{1/n} = \exp P(W, z)$$

and therefore $r(\mathcal{L}_{W,z}) = \exp P(W, z)$.

Moreover, repeating literally the argument used in ([Po], p.151), one finds

$$r_{\text{ess}}(\mathcal{L}_{W,z}) \leq \theta \, \exp P(W, z)$$

and also shows that the disk of radius $\theta \, \exp P(W, z)$ is included in $\mathrm{sp}\,(\mathcal{L}_{W,z})$.

It remains to show that $\exp P(W,z)$ is a simple eigenvalue of $\mathcal{L}_{W,z}$. This is obvious for the operator $\mathcal{L}_{\hat{W},z}$ for which one finds:

$$\mathcal{L}_{\hat{W},z}1 = \exp P(\hat{W},z)1.$$

For the general case, let Ψ_z be such that $\mathcal{L}_{W,z}\Psi_z = \lambda_z\Psi_z$ where λ_z is the (simple) eigenvalue with largest modulus of $\mathcal{L}_{W,z}$. Reasoning as in ([CI1], Theorem 2.1) it is not difficult to show that $\Psi_z \in \mathcal{F}_\theta(\Sigma)$ and $d_z^{-1} \le \Psi_z \le d_z$ for some $d_z > 0$. Then,

$$\begin{aligned}\log\lambda_z &= \overline{\lim_{n\to\infty}}\, \frac{1}{n}\log \mathcal{L}^n_{W,z}\Psi_z \\ &= \overline{\lim_{n\to\infty}}\, \frac{1}{n}\log \mathcal{L}^n_{W,z}1 = P(W,z). \qquad \text{q.e.d.}\end{aligned}$$

Remark 4.3. For any $|z| \le 1/r(\mathcal{L}_0)$, the number $P(W,z)$ is called the pressure associated to the interaction $W(\sigma) + \sigma_0\log z$. In particular $P(W,1) = 1$.

It is worth noticing that the derivative

$$P'(W,1) = \lim_{z\to 1}\frac{P(W,z)-1}{z-1}$$

gives the mean return time $< t >$ in the state 1 with respect to the Gibbs measure on Σ for the function W (see [R3]).

On the other hand from Kac's formula we have (see Remark 2.3):

$$< t > = \int_{A_1} t(x)\,\nu(dx) = \int_0^1 n(x)\,\rho(dx) = \sum_{k=1}^{\infty} k\rho(A_k) \tag{4.14}$$

where the measures ρ and ν have been defined in Remark 4.1. Now, using Lemma 4.1 it is not difficult to show that $\rho(A_k) = \mathcal{O}(1)\beta_k$ for $k\to\infty$ (see [CI2]). Thus, formally,

$$P'(W,1) = \mathcal{O}(1)\sum_k c_k$$

So, we see that the pressure is analytic at $z=1$ in the two cases: $\alpha < 1$ and $\alpha = 1$, $s < 1$. If $\alpha = 1$ and $s \ge 1$ then $P(W,z)$ is no longer differentiable at $z = 1$ (see also Remark 5.2). This fact is related to the presence of a phase transition which can then be characterized by the coexistence, for $\alpha = 1$ and $s \geq 1$, of two equilibrium states for $(f, [0,1])$: the σ-finite measure ν and the Dirac delta measure concentrated at the neutral fixed point (see [PrS],[L] for related results).

Now, from the above spectral properties of $\mathcal{L}_{W,z}$ we get:

Theorem 4.2. *The two-variables zeta function defined in Proposition 4.1 has the following analytic structure:*

(i) for any $|z| \leq 1/r(\mathcal{L}_0)$, $1/Z(f,w,z)$, considered as a function of the variable w, is holomorphic in the disk of radius $1/\theta \exp P(W,z)$. Its zeroes in this disk, counted with multiplicity, are the inverses of the eigenvalues of $\mathcal{L}_{W,z} : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ in the corresponding annulus. Moreover, the zero of smallest modulus is simple and located at $1/\exp P(W,z)$;

(ii) for any $|w| \leq 1$, $1/Z(f,w,z)$, considered as a function of the variable z, is holomorphic in the disk of radius $1/|w|r(\mathcal{L}_0)$. Its zeroes in this disk are located at those values of z such that $\mathcal{L}_{W,z} : \mathcal{F}_\theta(\Sigma) \to \mathcal{F}_\theta(\Sigma)$ has $1/w$ as an eigenvalue.

Proof. We shall follow Haydn ([H], Theorem 4; see also [BK] and [R2]). We first choose $\Theta > \theta$ and write $\mathcal{P}_z$ for the projection corresponding to the part of the spectrum of $\mathcal{L}_{W,z}$ in the disk of radius $\Theta \exp P(W,z)$. Let moreover $\lambda_1(z), \ldots, \lambda_M(z)$ be the eigenvalues outside this disk, ordered with decreasing modulus, and $m_1(z), \ldots, m_M(z)$ their multiplicites. According to Theorem 4.1, $\lambda_1(z)$ is simple and equals $\exp P(W,z)$. We then use the decomposition

$$\mathcal{L}_{W,z} v = \sum_{i=1}^{M} \lambda_i(z) \cdot \Psi_{z,i} L_i(z) \Psi^*_{z,i}\, v + \mathcal{P}_z \mathcal{L}_{W,z} v$$

where the row vector $\Psi_{z,i}$ and the column vector $\Psi^*_{z,i}$ span the generalized eigenspaces of $\mathcal{L}_{W,z}$ and $\mathcal{L}^*_{W,z}$ corresponding to the eigenvalue $\lambda_i(z)$, and the matrices $L_i(z)$ can be assumed in Jordan normal form, so that $m_i(z) = \operatorname{tr} L_i(z)$. For any $n > 0$ we have

$$\mathcal{L}^n_{W,z} v = \sum_{i=1}^{M} \lambda^n_i(z) \cdot \Psi_{z,i} L^n_i(z) \Psi^*_{z,i}\, v + \mathcal{P}_z \mathcal{L}^n_{W,z}\, v$$

Now, fixing $n>0$ we let $\sum_\nu$ be the sum over words ν of length n, i.e. words of the form $\nu=(\sigma_0,\ldots,\sigma_{n-1})$, and denote by $\sigma^{(\nu)}$ the periodic concatenation $(\overline{\sigma_0\sigma_1\ldots\sigma_{n-1}})$. Let moreover $\chi_\nu\in\mathcal{F}_\theta(\Sigma)$ be such that $\chi_\nu(\sigma)=1$ if σ begins with the word ν, $\chi_\nu(\sigma)=0$ otherwise. Then we have the following key relation:

$$\Lambda_n(f,z):=\sum_{\substack{\sigma\in\Sigma\\ \tau^n\sigma=\sigma}} z^{s_n(\sigma)}\exp\sum_{j=0}^{n-1}W(\tau^j\sigma)=\sum_\nu(\mathcal{L}^n_{W,z}\chi_\nu)(\sigma^{(\nu)}) \tag{4.15}$$

Inserting (4.14) into (4.15) we get

$$\begin{aligned}\Lambda_n(f,z)&=\sum_\nu\sum_{i=1}^M\lambda_i^n(z)\cdot\Psi_{z,i}(\sigma^{(\nu)})L_i^n(z)\Psi^*_{z,i}\chi_\nu+\sum_\nu(\mathcal{P}_z\mathcal{L}^n_{W,z}\chi_\nu)(\sigma^{(\nu)})\\ &=\sum_{i=1}^M\lambda_i^n(z)\cdot(L_i^n(z)\Psi^*_{z,i})^{\mathrm{tr}}(\sum_\nu\Psi_{z,i}(\sigma^{(\nu)})\cdot\chi_\nu)^{\mathrm{tr}}+\sum_\nu(\mathcal{P}_z\mathcal{L}^n_{W,z}\chi_\nu)(\sigma^{(\nu)})\\ &=\Lambda_n^{(0)}(f,z)+\Lambda_n^{(1)}(f,z)+\Lambda_n^{(2)}(f,z)\end{aligned}$$

where

$$\begin{aligned}\Lambda_n^{(0)}(f,z)&=\sum_{i=1}^M m_i(z)\,\lambda_i(z)\\ \Lambda_n^{(1)}(f,z)&=\sum_{i=1}^M\lambda_i^n(z)\cdot(L_i^n(z)\Psi^*_{z,i})^{\mathrm{tr}}(\sum_\nu\Psi_{z,i}(\sigma^{(\nu)})\cdot\chi_\nu-\Psi_{z,i})^{\mathrm{tr}}\\ \Lambda_n^{(2)}(f,z)&=\sum_\nu(\mathcal{P}_z\mathcal{L}^n_{W,z}\chi_\nu)(\sigma^{(\nu)})\end{aligned}$$

and Ψ^{tr} denotes transposition. The first term gives the contribution:

$$\exp-\sum_{n=1}^\infty\frac{w^n}{n}\Lambda_n^{(0)}(f,z)=(1-w\exp P(W,z))\prod_{i=2}^M(1-w\lambda_i(z))^{m_i}. \tag{4.16}$$

Moreover, proceeding as in [H], one can show the existence of two constants $R_1,R_2>0$ such that

$$|\Lambda_n^{(1)}(f,z)|\le R_1\Theta^n e^{nP(W,z)}\quad\text{and}\quad|\Lambda_n^{(2)}(f,z)|\le R_2\Theta^n e^{nP(W,z)}$$

and the Theorem is proved. q.e.d.

Finally, putting together Proposition 4.1, Proposition 4.3, Theorem 4.2 and Remark 4.2 we obtain the following result:

Corollary 4.1.

(i) $1/\zeta(g,z)$ *is holomorphic in the disk of radius* $1/\eta$*. Its zeroes in this disk, counted with multiplicity, are the inverses of the eigenvalues of* $\mathcal{L}_W$ *in the annulus* $\eta < |\lambda| \leq 1$*.*

(ii) $1/\zeta(f,z)$ *is holomorphic in the disk of radius* $1/\alpha$*. In this disk,* $1/\zeta(f,z) = 0$ *if and only if* $1/z \in \mathrm{sp}(\mathcal{L}_V)$*.*

Remark 4.3. When $\alpha = 1$ the above result yields no zeroes of $1/\zeta(f,z)$ but the point $z = 1$. In this case one expects that the eigenvalue 1 of $\mathcal{L}_V$ is not isolated (i.e. there is no 'gap'). On the other hand, in order to understand the analytic properties of $1/\zeta(f,z)$ in the disk of radius $1/\alpha$, when $\alpha < 1$, it would be necessary to better specify a Banach space of functions where $\mathrm{sp}(\mathcal{L}_V)$ is non-trivial and then repeat the analysis made above for $\mathcal{L}_{W,z}$.

However, we will not insist any more in this direction. In the next Section we shall gain this information using a different method.

5. The zeta functions of $\hat{f}$ and f.

We now derive an explicit expression of $1/\zeta(\hat{f},z)$ as the limit of a sequence of suitable Fredholm determinants (see [PP], chap.10).

Proposition 5.1.

$$\frac{1}{\zeta(\hat{f},z)} = (1-z)(1-\alpha z)\sum_{n=0}^{\infty} c_n z^n$$

Proof.

For any integer N consider the matrix A_N given by

$$A_N = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_N & \alpha \\ \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha_N & \alpha \end{pmatrix}.$$

It is then easy to realize that

$$Z_n(\hat{f}) = \sum_{\substack{\omega\in\Omega \\ \tau^n\omega=\omega}} \alpha_{\omega_0}\dots\alpha_{\omega_{n-1}} + \alpha^n = \mathrm{tr}\, A_N^n$$

provided $N > n$. Therefore,

$$\frac{1}{\zeta(\hat{f}, z)} = \lim_{N\to\infty} \det(1 - zA_N),$$

On the other hand, one easily gets the expression

$$\begin{aligned} \det(1 - zA_N) &= (1 - \alpha z)(1 - \sum_{n=1}^{N} z^n \alpha_1 \dots \alpha_n) + z^{N+1}\alpha\alpha_1\dots\alpha_N \\ &= (1 - \alpha z)(1 - \sum_{n=1}^{N} z^n \beta_n) + z^{N+1}\alpha\beta_N \end{aligned}$$

Hence, taking $|z| < 1/\alpha$, so that $z^N\beta_N \to 0$ as $N \to \infty$, we get

$$\frac{1}{\zeta(\hat{f}, z)} = (1 - \alpha z)(1 - \sum_{n=1}^{\infty} \beta_n z^n)$$

We shall manipulate this expression a little further: set

$$1 - \sum_{n=1}^{\infty} \beta_n z^n = (1 - z)\sum_{n=0}^{\infty} c(n) z^n$$

so that

$$c(0) = 1 \quad \text{and} \quad c(n) = 1 - \sum_{i=1}^{n} \beta_i \quad \text{for} \quad n \geq 1,$$

namely $c(n) \equiv c_n = \psi_0^n(1)$ and the assertion follows. q.e.d.

Remark 5.1. It would have been not difficult to obtain $\zeta(\hat{f}, z)$ through a direct computation based on (4.7). So doing, one can also obtain the expression of the two-variables zeta function (4.8):

$$Z(\hat{f}, w, z) = \frac{1}{1 - w\sum_{k=1}^{\infty} \beta_k z^k}$$

so that from Proposition 4.1 we get at once Proposition 5.1 and also $\zeta(\hat{g}, z) = 1/(1-z)$.

Before discussing the analytic properties of $\zeta(\hat{f}, z)$ for the different situations that can be treated within this approach, we work out two simple and somehow opposite examples.

Example 5.1. For some $0 < p < 1$ set

$$f(x) = \hat{f}(x) = \begin{cases} x/p, & \text{if } x \le p; \\ (x-p)/(1-p), & \text{if } x > p. \end{cases}$$

Then $\alpha = p$ and $c_n = p^n$ so that

$$\zeta(f, z) = \frac{1}{1-z}$$

which is analytic in the entire z-plane except for a simple pole at $z = 1$.

Example 5.2. Set

$$f(x) = \begin{cases} x/(1-x), & \text{if } x \le 1/2; \\ 2x - 1, & \text{if } x > 1/2. \end{cases}$$

Then $\alpha = 1$ and $c_n = 1/(n+1)$ so that

$$\zeta(\hat{f}, z) = \frac{z}{(1-z)^2 \log 1/(1-z)}$$

which is analytic in the entire z-plane with a cross cut along the ray $(1, +\infty)$. Furthermore, in a neighborhood of the point $z = 1$ we have $(1-z)\zeta(\hat{f}, z) \to \infty$ but $(1-z)^2\zeta(\hat{f}, z) \to 0$.

More generally, according to Lemma 2.1 we have the following behaviour of $\zeta(\hat{f}, z)$:

Proposition 5.2. *Under the hypotheses (1)-(6) on the map f we have:*

a) if $\alpha < 1$ then $1/\zeta(\hat{f}, z)$ is holomorphic in the disk $|z| < 1/\alpha$ with only one zero; this zero is simple and located at $z = 1$;

b) if $\alpha = 1$ then $1/\zeta(\hat{f}, z)$ is holomorphic in the unit disk and can be continued analytically in the entire z-plane with a cross cut along the ray $(1, +\infty)$. The analytic continuation is given by the formula, valid for any $\delta \geq 0$,

$$1/\zeta(\hat{f}, z) = \frac{(1-z)^2}{2\pi i} \int_1^{+\infty} \int_{\delta - i\infty}^{\delta + i\infty} c(x) \frac{t^{-x}}{t - z} dx dt \tag{5.1}$$

where $c(x)$ is a regular function in the half-plane $\mathrm{Re}\, x \geq 0$ such that $c(n) = c_n$ for $n \geq 0$.

Proof. The first part is immediate. The proof of (b) follows by putting together Lemma 2.1 and standard techniques of analytic continuation of power series based on the use of the Mellin transform (see, e.g., [E], Theorem 6.1). q.e.d.

Relating $\zeta(\hat{f}, z)$ to $\zeta(f, z)$.

Theorem 5.1. *Under the hypotheses (1)-(6) on the map f we have:*

a) if $\alpha < 1$ then $1/\zeta(f, z)$ is holomorphic in the disk $|z| < 1/\alpha$ with only one zero; this zero is simple and located at $z = 1$;

b) if $\alpha = 1$ then $1/\zeta(f, z)$ is holomorphic in the unit disk and can be continued analytically in the entire z-plane with a cross cut along the ray $(1, +\infty)$. In addition, we have the asymptotic behaviour

$$1/\zeta(f, z) = \mathcal{O}(1)\, (1-z)^2 \sum_{n=0}^{\infty} c_n z^n \quad \text{as} \quad z \to 1_-.$$

Remark 5.2. If $\alpha = 1$ and $s < 1$, so that $\sum c_k < \infty$, then $\zeta(f, z)$ diverges as $(1-z)^{-2}$ when $z \to 1_-$. Instead, if $\alpha = 1$ and $s \geq 1$, so that $\sum c_k = \infty$, then $(1-z)\zeta(\hat{f}, z) \to \infty$ but $(1-z)^2 \zeta(\hat{f}, z) \to 0$ when $z \to 1_-$ (cf Remark 4.3). However, the asymptotic behaviour of $\zeta(\hat{f}, z)$ in a neighborhood of $z = 1$ and in particular the order of the branch point, may be inspected with the help of formula (5.1). We conclude by observing that from this behaviour one can obtain information on the behaviour of the partial sums $\sum_{n=1}^{N} c_n$, which may be useful for studying the mixing properties of the map f. This can be achieved by using Karamata's theorem in a suitable way [T]. However, an upper bound can be simply obtained as follows. Set

$$L(z) = \sum_{n=0}^{\infty} c_n\, z^n = \frac{1}{2\pi i} \int_1^{+\infty} \int_{\delta-i\infty}^{\delta+i\infty} c(x) \frac{t^{-x}}{t-z} dx dt.$$

Then, for any $N > 1$,

$$\begin{aligned} \sum_{n=1}^{N} c_n &\leq \left(1 - \frac{1}{N}\right)^{-N} \sum_{n=1}^{N} c_n \left(1 - \frac{1}{N}\right)^n \\ &\leq \mathcal{O}(1) \sum_{n=0}^{\infty} c_n \left(1 - \frac{1}{N}\right)^n \\ &= \mathcal{O}(1) L(1 - \frac{1}{N}). \end{aligned}$$

For instance, for the map in Example 5.2 (for which $\alpha = 1$ and $s = 1$) one finds $L(z) = \frac{1}{z} \log(\frac{1}{1-z})$ so that $\sum_{n=1}^{N} c_n \leq \mathcal{O}(1) \log N$.

Proof of Theorem 5.1. The first part is the analogous of Theorem 5.29 in [R3]. We then prove (b). Set

$$\begin{aligned} Q(W, z) &:= 1 - \exp P(W, z) \\ Q(\hat{W}, z) &:= 1 - \exp P(\hat{W}, z) = (1 - z) \sum_{k=0}^{\infty} c_k z^k \end{aligned}$$

It is easy to check that these functions satisfy

$$Q(W, 1) = Q(\hat{W}, 1) = 0 \quad \text{and} \quad Q(W, 0) = Q(\hat{W}, 0) = 1.$$

We then have

$$P'(W, 1) = \lim_{z \to 1_-} \frac{Q(W, z)}{1 - z}, \qquad P'(\hat{W}, 1) = \lim_{z \to 1_-} \frac{Q(\hat{W}, z)}{z - 1}.$$

Now, as we have already stressed, $P'(W,1)$ and $P'(\hat{W},1)$ give the mean return time in the state 1 with respect to the Gibbs measures on Σ for the functions W and $\hat{W}$, respectively. They are given formally by (cf (4.14)):

$$P'(W,1) = \sum_{k=1}^{\infty} k\rho(A_k) \quad \text{and} \quad P'(\hat{W},1) = \sum_{k=1}^{\infty} k\beta_k$$

On the other hand, under the assumptions (1)-(6) for the map f we have ([CI2], Lemma 2.4):

$$\sum_{k=1}^{\infty} k\rho(A_k) = \mathcal{O}(1) \sum_{k=1}^{\infty} k\beta_k$$

Therefore, the following limit exists:

$$\lim_{z\to 1_-} \frac{Q(W,z)}{Q(\hat{W},z)} = \frac{P'(W,1)}{P'(\hat{W},1)},$$

and moreover

$$Q(W,z) = \mathcal{O}(1)\, Q(\hat{W},z) \quad \text{as} \quad z \to 1_-.$$

Putting together Proposition 4.1, equation (4.16) with $w = 1$ and the above we get the announced asymptotic behaviour of $\zeta(f,z)$. The Theorem then follows in the same way as Proposition 5.2. q.e.d.

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