# Effective Reducibility of Quasiperiodic Linear Equations close to Constant Coefficients

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**Abstract**

Let us consider the differential equation

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x, \quad |\varepsilon| \leq \varepsilon_0,$$

where $A$ is an elliptic constant matrix and $Q$ depends on time in a quasiperiodic (and analytic) way. It is also assumed that the eigenvalues of $A$ and the basic frequencies of $Q$ satisfy a diophantine condition. Then it is proved that this system can be reduced to

$$\dot{y} = (A^*(\varepsilon) + \varepsilon R^*(t, \varepsilon))y, \quad |\varepsilon| \leq \varepsilon_0,$$

where $R^*$ is exponentially small in $\varepsilon$, and the linear change of variables that performs such reduction is also quasiperiodic with the same basic frequencies than $Q$. The results are illustrated and discussed in a practical example.

## 1 Introduction

The well-known Floquet theorem states that any linear periodic system, $\dot{x} = A(t)x$, can be reduced to constant coefficients, $\dot{y} = By$, by means of a periodic change of variables. Moreover, this change of variables can be taken, over $\mathbb{C}$, with the same period than $A(t)$.

A natural extension is to consider the case in which the matrix $A(t)$ depends on time in a quasiperiodic way. Before starting the discussion of this issue, let us recall the definition and basic properties of quasiperiodic functions.

**Definition 1.1** *A function $f$ is a quasiperiodic function with vector of basic frequencies $\omega = (\omega_1, \ldots, \omega_r)$ if $f(t) = F(\theta_1, \ldots, \theta_r)$, where $F$ is $2\pi$ periodic in all its arguments and $\theta_j = \omega_j t$ for $j = 1, \ldots, r$. Moreover, $f$ is called analytic on a strip of width $\rho$ if $F$ is analytical on an open set containing $|\mathrm{Im}\ \theta_j| \leq \rho$ for $j = 1, \ldots, r$.*

It is also known that an analytic quasiperiodic function $f(t)$ on a strip of width $\rho$ has Fourier coefficients defined by

$$f_k = \frac{1}{(2\pi)^r} \int_{\mathbb{T}^r} F(\theta_1, \dots, \theta_r) e^{-(k,\theta)\sqrt{-1}}\, d\theta,$$

such that $f$ can be expanded as

$$f(t) = \sum_{k\in\mathbb{Z}^r} f_k e^{(k,\omega)\sqrt{-1}t},$$

for all $t$ such that $|\mathrm{Im}\ t| \le \rho/\|\omega\|_\infty$. We denote by $\|f\|_\rho$ the norm

$$\|f\|_\rho = \sum_{k\in\mathbb{Z}^r} |f_k| e^{|k|\rho},$$

and it is not difficult to check that it is well defined for any analytical quasiperiodic function defined on a strip of width $\rho$. Finally, to define an analytic quasiperiodic matrix, we note that all these definitions hold when $f$ is a matrix-valued function. In this case, to define $\|f\|_\rho$ we use the infinity norm (that will be denoted by $|\cdot|_\infty$) for the matrices $f_k$.

After those definitions and properties, let us return to the problem of the reducibility of a linear quasiperiodic equation, $\dot{x} = \widehat{A}(t)x$, to constant coefficients. The approach of this work is to assume that the system is close to constant coefficients, that is, $\widehat{A}(t) = A + \varepsilon Q(t,\varepsilon)$, where $\varepsilon$ is small. This case has already been considered in many papers (see [2], [8] and [9] among others), and the results can be summarized as follows: let $\lambda_i$ be the eigenvalues of $A$, and $\alpha_{ij} = \lambda_i - \lambda_j$, for $i \neq j$. Then, if all the values Re $\alpha_{ij}$ are different from zero, the reduction can be performed for $|\varepsilon| < \varepsilon_0$, $\varepsilon_0$ sufficiently small (see [2]). If some of the Re $\alpha_{ij}$ are zero (this happens, for instance, if $A$ is elliptic, that is, if all the $\lambda_i$ are on the imaginary axis) more hypothesis are needed. The usual one is a diophantine condition involving the $\alpha_{ij}$ and the basic frequencies of $Q(t,\varepsilon)$, and to assume a nondegeneracy condition with respect to $\varepsilon$ on the corresponding $\alpha_{ij}(\varepsilon)$ of the matrix $A+\varepsilon\overline{Q}(\varepsilon)$ ($\overline{Q}(\varepsilon)$ denotes the average of $Q(t,\varepsilon)$). This allows to prove (see [9] for the details) that there exists a Cantorian set $\mathcal{E}$ such that the reduction can be performed for all $\varepsilon \in \mathcal{E}$. Moreover, the relative measure of the set $[0,\,\varepsilon_0] \setminus \mathcal{E}$ in $[0,\,\varepsilon_0]$ is exponentially small in $\varepsilon_0$.

Our purpose here is a little bit different: instead of looking for a total reduction to constant coefficients (this seems to lead us to eliminate a dense set of values of $\varepsilon$, see [8] or [9]), we try to minimize the quasiperiodic part, without taking out any value of $\varepsilon$. The result obtained is that the quasiperiodic part can be made exponentially small. As all the proof is constructive (and it can be carried out with a finite number of steps), it can be applied to practical examples in order to do an "effective" reduction: if $\varepsilon$ is small enough, the remainder will be so small that, for practical purposes, it can be taken equal to zero. The error produced with this dropping can be bounded easily, by means of the Gronwall lemma. Finally, we want to stress that we have also eliminated the nondegeneracy hypothesis of previous papers ([8], [9]).

Before finishing this introduction, we want to mention some similar results obtained when the dynamics of the system is slow: $\dot{x} = \varepsilon(A + \varepsilon Q(t, \varepsilon))x$. This case is contained in [14], which is an extension of [12]. The result obtained is also that the quasiperiodic part can be made exponentially small in $\varepsilon$. Total reducibility has been also considered in this case: in [15] is stated that the reduction can be performed except for a set of values of $\varepsilon$ of measure exponentially small.

There are many other results for the reducibility problem. For instance, in the case of the Schrödinger equation with quasiperiodic potential we can mention [3], [4], [5], [10], [11] and [13]. Another classical and remarkable paper is [7], where the general case (that is, without asking to be close to constant coefficients) is considered. Finally, the classical results for quasiperiodic systems can be found in [6].

In order to simplify the reading, the paper has been divided in sections as follows: Section 2 contains the exposition (without technical details) of the main ideas and methodology, Section 3 contains the main theorem, Sections 4 and 5 are devoted to the proofs and, finally, Section 6 contains an example to show how these results can be applied to a concrete problem.

# 2 The method

The method used is based on the same inductive scheme that [8]. Let us write our equation as

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x, \tag{1}$$

where $A$ is an elliptic $d \times d$ matrix and $Q(t, \varepsilon)$ is quasiperiodic with $\omega = (\omega_1, \ldots, \omega_r)$ as vector of basic frequencies, and analytic on a strip of width $\rho$. First of all, let us rewrite this equation as

$$\dot{x} = (A_0(\varepsilon) + \varepsilon \widetilde{Q}(t, \varepsilon)),$$

where $A_0(\varepsilon) = A + \overline{Q}(\varepsilon)$ and $\widetilde{Q}(t, \varepsilon) = Q(t, \varepsilon) - \overline{Q}(\varepsilon)$. Now let us assume that we are able to find a quasiperiodic $d \times d$ matrix $P$ (with the same basic frequencies than $Q$) verifying

$$\dot{P} = A_0(\varepsilon)P - PA_0(\varepsilon) + \widetilde{Q}(t, \varepsilon), \tag{2}$$

such that $\|\varepsilon P(t, \varepsilon)\|_\sigma < 1$, for some $\sigma > 0$. In this case, it is not difficult to check that the change of variables $x = (I + \varepsilon P(t, \varepsilon))y$ transforms equation (1) into

$$\dot{y} = (A_0(\varepsilon) + \varepsilon^2 (I + \varepsilon P(t, \varepsilon))^{-1} \widetilde{Q}(t, \varepsilon) P(t, \varepsilon)). \tag{3}$$

As this equation is like (1) but with $\varepsilon^2$ instead of $\varepsilon$, the inductive scheme seems clear: to average the quasiperiodic part of (3) and to restart this process. The main difficulty that appear in this process comes from equation (2), because the solution contains the denominators $\lambda_i(\varepsilon) - \lambda_j(\varepsilon) + \sqrt{-1}(k, \omega)$, $1 \le i, j \le d$, where $\lambda_i(\varepsilon)$ are the eigenvalues of $A_0(\varepsilon)$ (this is shown inside the proof of Lemma 4.2). This divisor appears in the $k$th Fourier coefficient of $P$. Note that if the values $\lambda_i(\varepsilon) - \lambda_j(\varepsilon)$ are outside the imaginary axis, the (modulus of the) divisor can be bounded from below, being easy to prove the convergence. On the other hand, the value $\lambda_i(\varepsilon) - \lambda_j(\varepsilon) + \sqrt{-1}(k, \omega)$ can be arbitrarily small giving rise to convergence problems.

## 2.1 Avoiding the small divisors

Let us start assuming that the eigenvalues $\lambda_i$ of the original unperturbed matrix $A$ (see equation (1)) and the basic frequencies of $Q$ satisfy the diophantine condition

$$|\lambda_i - \lambda_j + \sqrt{-1}(k, \omega)| \ge \frac{c}{|k|^\gamma}, \quad \forall k \in \mathbb{Z}^r \setminus \{0\}. \tag{4}$$

where $|k| = |k_1| + \cdots + |k_r|$. Note that, in principle, we can not guarantee that in equation (2) this condition holds, because the eigenvalues of $A_0(\varepsilon)$ have been changed with respect to the ones of $A$ (in an amount of $\mathcal{O}(\varepsilon)$) and some of the divisors can be very small or even zero.

The key point is to realize that, as the eigenvalues of $A$ move in an amount of $\mathcal{O}(\varepsilon)$ at most, the quantities $\lambda_i(\varepsilon) - \lambda_j(\varepsilon)$ are contained in a (complex) ball $B_{i,j}(\varepsilon)$ centered in $\lambda_i - \lambda_j$ and with radius $\mathcal{O}(\varepsilon)$. As the centre of the ball satisfies condition (4), the values $(k, \omega)$ can not be inside that ball if $|k|$ is less than some value $M(\varepsilon)$. This implies that it is possible to cancel all the harmonics such that $0 < |k| < M(\varepsilon)$, because they do not produce small divisors (note that we can only have resonances when $(k, \omega)$ is inside $B_{i,j}(\varepsilon)$). The harmonics with $|k| \ge M(\varepsilon)$ are exponentially small in $M(\varepsilon)$ (when $M(\varepsilon) \to \infty$), this is, exponentially small in $\varepsilon$ (when $\varepsilon \to 0$), so we do not need to eliminate them.

The idea of considering only frequencies less than some threshold $M$ has already been applied before in other contexts (see, for instance, [1]).

## 2.2 The iterative scheme

To apply the considerations above we define, as before, $A_0(\varepsilon) = A + \varepsilon\overline{Q}(\varepsilon)$, $\tilde{Q}(t, \varepsilon) = Q(t, \varepsilon) - \overline{Q}(\varepsilon)$ and we split $\tilde{Q}(t, \varepsilon)$ in the sum of two matrices $Q_0(t, \varepsilon)$, $R_0(t, \varepsilon)$: $Q_0(t, \varepsilon)$ contains the harmonics $Q_k e^{(k,\omega)\sqrt{-1}t}$ with $|k| < M(\varepsilon)$ and $R_0(t, \varepsilon)$ the ones with $|k| \ge M(\varepsilon)$. So, (1) can be rewritted as

$$\dot{x} = (A_0(\varepsilon) + \varepsilon Q_0(t, \varepsilon) + \varepsilon R_0(t, \varepsilon))x, \tag{5}$$

Now the idea is to cancel $Q_0(t, \varepsilon)$ and to leave $R_0(t, \varepsilon)$ (it is already exponentially small with $\varepsilon$). So, we compute $P_0$ such that

$$\dot{P}_0 = A_0(\varepsilon)P_0 - P_0 A_0(\varepsilon) + Q_0(t, \varepsilon).$$

Then, the change $x = (I + \varepsilon P_0(t, \varepsilon))y$ gives

$$\dot{y} = \left[A_0 + \varepsilon^2 (I + \varepsilon P_0)^{-1} Q_0 P_0 + \varepsilon (I + \varepsilon P_0)^{-1} R_0 (I + \varepsilon P_0)\right] y.$$

This equation can be rewritten to be like (5) to repeat the process. Note that the size of the harmonics with $0 < |k| < M(\varepsilon)$ has been squared. As we will see in the proofs, this is enough to guarantee convergence of those terms to zero. Thus, the final equation has a purely quasiperiodic part exponentially small with $\varepsilon$.

## 2.3 Remarks

It is interesting to note that it is enough to apply a finite number of steps of the inductive process: we do not need to cancel completely the harmonics with $0 < |k| < M(\varepsilon)$ but we can stop the process when they are of the same size of the ones of $R$ (from the proof it can be seen that the number of steps needed to achieve this is of order $|\ln |\varepsilon||$). This allows to apply (with the help of a computer) this procedure on a practical example.

Another remarkable point is about the diophantine condition: note that we only need the condition up to a finite order ($M(\varepsilon)$, that is of order $(1/|\varepsilon|)^{1/\gamma}$, as we shall see in the proofs). This means that, in a practical example when the perturbing frequencies are known with finite precision, the diophantine condition can be checked easily.

# 3 The Theorem

In what follows, $\mathcal{Q}_d(\rho, \omega)$ states for the set of the analytic quasiperiodic $d \times d$ matrices on a strip of width $\rho$ and having $\omega$ as vector of basic frequencies. Moreover, $i$ will denote $\sqrt{-1}$.

**Theorem 3.1** *Consider the equation* $\dot{x} = (A + \varepsilon Q(t, \varepsilon))x$, $|\varepsilon| \le \varepsilon_0$, *and* $x \in \mathbb{R}^d$, *where*

1. *$A$ is a constant $d \times d$ matrix with different eigenvalues $\lambda_1, \ldots, \lambda_d$.*
2. *$Q(\cdot, \varepsilon) \in \mathcal{Q}_d(\rho, \omega)$ with $\|Q(\cdot, \varepsilon)\|_\rho \le q$, $\forall\, |\varepsilon| \le \varepsilon_0$, for some $\omega \in \mathbb{R}^r$, and $q, \rho > 0$.*
3. *The vector $\omega$ satisfies the diophantine conditions*

$$|\lambda_j - \lambda_\ell + i(k, \omega)| \ge \frac{c}{|k|^\gamma}, \quad \forall\, k \in \mathbb{Z}^r \setminus \{0\}, \quad \forall\, j,\, \ell \in \{1, \ldots, d\}, \tag{6}$$

*for some constants $c > 0$, $\gamma > r - 1$. As usual, $|k| = |k_1| + \cdots + |k_r|$.*

*Then there exist positive constants $\varepsilon^*$, $a^*$, $r^*$ and $m$ such that for all $\varepsilon$, $|\varepsilon| \le \varepsilon^*$, the initial equation can be transformed into*

$$\dot{y} = (A^*(\varepsilon) + \varepsilon R^*(t,\varepsilon))y\,, \tag{7}$$

*where:*

1. *$A^*$ is a constant matrix with $|A^*(\varepsilon) - A|_\infty \le a^*|\varepsilon|$.*
2. *$R^*(\cdot,\varepsilon) \in \mathcal{Q}_d(\rho,\omega)$ and $\|R^*(\cdot,\varepsilon)\|_{\rho-\delta} \le r^* \exp\left(-\left(\frac{m}{|\varepsilon|}\right)^{1/\gamma}\delta\right)$, $\forall\,\delta \in ]0,\rho]$.*

*Furthermore the quasiperiodic change of variables that performs this transformation is also an element of $\mathcal{Q}_d(\rho,\omega)$. Finally, a general explicit computation of $\varepsilon^*$, $a^*$, $r^*$ and $m$ is possible:*

$$\varepsilon^* = \min\left(\varepsilon_0,\, \frac{\alpha}{eq\beta(3d-1)}\right),\ a^* = \frac{eq\beta^2}{e-1},\ r^* = ea^*,\ m = \frac{c}{10eq\beta}$$

*where $e = \exp(1)$, $\alpha = \min_{j\neq\ell}(|\lambda_j - \lambda_\ell|)$ and $\beta$ is the condition number of a regular matrix $S$ such that $S^{-1}AS$ is diagonal, that is, $\beta = C(S) = |S^{-1}|_\infty |S|_\infty$.*

**Remark 3.1** *For fixed values of $\lambda_1,\ldots,\lambda_d$ and $\gamma$ hypothesis 3 is not satisfied for any $c > 0$ only for a set of values of $\omega$ of zero measure if $\gamma > r-1$.*

**Remark 3.2** *In case that the eigenvalues of the perturbed matrices move on balls of radius $\mathcal{O}(\varepsilon^p)$ (that is, if the nondegeneracy hypothesis needed in [8] or [9] is not satisfied), it is not difficult to show that the bound of the exponential can be improved: $\|R^*(\cdot,\varepsilon)\|_{\rho-\delta} \le r^* \exp(-(m/|\varepsilon|)^{p/\gamma}\delta)$. The proof is very similar, but using $M(\varepsilon) = (m/|\varepsilon|)^{p/\gamma}$ instead of $(m/|\varepsilon|)^{1/\gamma}$.*

This last remark seems to show that this nondegeneracy hypothesis is not necessary, and it is only used for technical reasons. In fact, the results seem to be better when this hypothesis is not satisfied.

**Remark 3.3** *If the unperturbed matrix $A$ has multiple eigenvalues (that is, if hypothesis 1 is not satisfied) the theorem is still true, but the exponent of $\varepsilon$ in the exponential of the remainder is slightly worse. This happens because the (small) divisors are now raised to a power that increases with the multiplicity of the eigenvalues. The proof is not included, since it does not introduce new ideas and the technical details are rather tedious.*

**Remark 3.4** *The values of $\varepsilon^*$, $a^*$, $r^*$ and $m$ given in the theorem are rather pessimistic. In the proof, we have preferred to use simple (but rough) bounds instead of cumbersome but more accurate ones. If one is interested in realistic bounds for a given problem, the best thing to do is to rewrite the proof for that particular case. We have done this in Section 6 where, with the help of a computer program, we have applied some steps of the method to an example. This allows not only to obtain better bounds, but also to obtain (numerically) the reduced matrix as well as the corresponding change of variables.*

# 4 Lemmas

We will use some lemmas to simplify the proof of the theorem.

## 4.1 Basic lemmas

**Lemma 4.1** *Let $Q(t) = \sum\limits_{k\in\mathbb{Z}^r} Q_k e^{i(k,\omega)t}$ be an element of $\mathcal{Q}_d(\rho,\omega)$ and $M > 0$. Let us define $\overline{Q} = Q_0$, $\widetilde{Q}(t) = Q(t) - Q_0$,*

$$Q_{\geq M}(t) = \sum_{\substack{k\in\mathbb{Z}^r \\ |k|\geq M}} Q_k e^{i(k,\omega)t},$$

*and $\widetilde{Q}_{<M} = \widetilde{Q} - Q_{\geq M}$. Then we have the bounds*

1. $|\overline{Q}|_\infty$, $\|\widetilde{Q}\|_\rho$, $\|\widetilde{Q}_{<M}\|_\rho \leq \|Q\|_\rho$.
2. $\|Q_{\geq M}\|_{\rho-\delta} \leq \|Q\|_\rho e^{-M\delta}$, $\forall\, \delta \in ]0,\rho]$.

**Proof**. It is an immediate check. ∎

The next lemma is used to control the variation of the eigenvalues of a perturbed diagonal matrix.

**Lemma 4.2** *Let $D$ be a $d\times d$ diagonal matrix with different eigenvalues $\lambda_1,\ldots,\lambda_d$ and $\alpha = \min\limits_{j\neq\ell}(|\lambda_j - \lambda_\ell|)$. Then if $A$ verifies $|A - D|_\infty \leq b \leq \frac{\alpha}{3d-1}$, the following conditions hold:*

1. *$A$ has different eigenvalues $\mu_1,\ldots,\mu_d$ and $|\lambda_j - \mu_j| \leq b$ if $j = 1,\ldots,d$.*
2. *There exists a regular matrix $S$ such that $S^{-1}AS = D^* = \mathrm{diag}\,(\mu_1,\ldots,\mu_d)$ satisfying $C(S) \leq 2$.*

**Proof**. It is contained in [8]. ∎

**Lemma 4.3** *Let $(q_n)_n$, $(a_n)_n$ and $(r_n)_n$ be sequences defined by*

$$q_{n+1} = q_n^2, \quad a_{n+1} = a_n + q_{n+1}, \quad r_{n+1} = \frac{2+q_n}{2-q_n} r_n + q_{n+1}\,.$$

*with initial values $q_0 = a_0 = r_0 = e^{-1}$. Then $(q_n)_n$ is decreasing to zero and $(a_n)_n$, $(r_n)_n$ are increasing and convergent to some values $a_\infty$ and $r_\infty$ respectively, with $a_\infty < \frac{1}{e-1}$, $r_\infty < \frac{e}{e-1}$.*

**Proof**. It is immediate that $q_n$ goes to zero quadratically and this implies that $a_n$ is convergent to the value $a_\infty$:

$$a_\infty = \sum_{j=0}^{\infty} q_j < \sum_{j=1}^{\infty} e^{-j} = \frac{1}{e-1}.$$

Then

$$r_n \le p\left(r_0 + \sum_{j=1}^{n} q_j\right) \le p a_\infty,$$

where $p = \prod_{j=0}^{\infty} \frac{2+q_j}{2-q_j}$. This product is convergent, in fact:

$$\ln p = \sum_{j=0}^{\infty} [\ln(1+q_j/2) - \ln(1-q_j/2)] \le \frac{3}{2} a_\infty \le \frac{3}{2(e-1)} < 1,$$

and so $p < e$, where we have used that $\ln(1+x) \le x$ and $-\ln(1-x) \le 2x$, for $x \in (0, 1/2)$. ∎

## 4.2 The inductive lemma

The next lemma is used to do a step of the inductive procedure.

Before stating the result, let us introduce some notation. Let $D$ and $\alpha$ be like in Lemma 4.2 and let $\varepsilon^*$, $q^*$, $L$ and $M(\varepsilon)$ be positive constants. We consider the equation at the step $n$ of the iterative process:

$$\dot{x}_n = (A_n(\varepsilon) + \varepsilon Q_n(t,\varepsilon) + \varepsilon R_n(t,\varepsilon)) x_n, \;\; |\varepsilon| \le \varepsilon^*, \tag{8}$$

where $Q_n(\cdot,\varepsilon)$, $R_n(\cdot,\varepsilon) \in \mathcal{Q}_d(\rho,\omega)$ and $\overline{Q}_n(\varepsilon) = Q_n(\cdot,\varepsilon)_{\ge M(\varepsilon)} = 0$. We assume that for some $a_n$, $q_n$, $r_n \ge 0$ and $|\varepsilon| < \varepsilon^*$ the following bounds hold:

$$|A_n(\varepsilon) - D| \le q^* a_n |\varepsilon|, \quad \|Q_n(\cdot,\varepsilon)\|_\rho \le q^* q_n, \quad \|R_n(\cdot,\varepsilon)\|_{\rho-\delta} \le q^* r_n e^{-M(\varepsilon)\delta},$$

where $\delta$ is such that $0 < \delta \le \rho$ (the constant $q^*$ has been introduced to simplify, later, the proof of the theorem). We want to see if it is possible to apply a step of the iterative process to equation (8) to obtain

$$\dot{x}_{n+1} = (A_{n+1}(\varepsilon) + \varepsilon Q_{n+1}(t,\varepsilon) + \varepsilon R_{n+1}(t,\varepsilon)) x_{n+1}, \quad |\varepsilon| \le \varepsilon^*, \tag{9}$$

such that $Q_{n+1}(\cdot,\varepsilon)$, $R_{n+1}(\cdot,\varepsilon) \in \mathcal{Q}_d(\rho,\omega)$, $\overline{Q}_{n+1}(\varepsilon) = Q_{n+1}(\cdot,\varepsilon)_{\ge M(\varepsilon)} = 0$. We also want to relate the bounds $a_{n+1}$, $q_{n+1}$ and $r_{n+1}$ of the terms of this equation with the corresponding bounds of equation (8).

**Lemma 4.4** *Let $\lambda_1^{(n)}(\varepsilon), \dots, \lambda_d^{(n)}(\varepsilon)$ be the eigenvalues of $A_n(\varepsilon)$. Under the previous notations, if*

1. *$L \geq 8q^*$, $\varepsilon^* \leq \frac{\alpha}{q^*(3d-1)}$,*

2. *$a_n \leq 1$, $q_n \leq e^{-1}$,*

3. *the condition*

$$|\lambda_j^{(n)}(\varepsilon) - \lambda_\ell^{(n)}(\varepsilon) + i(k,\omega)| \geq L|\varepsilon|, \quad |\varepsilon| \leq \varepsilon^*,$$

   *is satisfied for all $j$, $\ell$ and for all $k \in \mathbb{Z}^r$ such that $0 < |k| < M(\varepsilon)$,*

*then, equation (8) can be transformed into (9) and:*

$$q_{n+1} = q_n^2, \quad a_{n+1} = a_n + q_{n+1}, \quad r_{n+1} = \frac{2+q_n}{2-q_n} r_n + q_{n+1}\,.$$

*The quasiperiodic change of variables that performs this transformation is*

$$x_n = (I + \varepsilon P_n(t,\varepsilon))x_{n+1}, \tag{10}$$

*where $P_n(\cdot,\varepsilon)$ is the (only) solution of*

$$\dot{P}_n = A_n(\varepsilon)P_n - P_n A_n(\varepsilon) + Q_n(t,\varepsilon), \quad \overline{P}_n = 0, \tag{11}$$

*that belongs to $\mathcal{Q}_d(\rho,\omega)$. Moreover, $\|\varepsilon P_n(\cdot,\varepsilon)\|_\rho \leq q_n/2 < 1/2$.*

**Remark 4.1** *$A_n$, $Q_n$, $R_n$, $P_n$, $M$ and $\lambda_j^{(n)}$ depend on $\varepsilon$ but, for simplicity, we will not write this explicitely.*

**Proof**. Let us start studying the solutions of (11). Let $S_n$ be the matrix found in Lemma 4.2 with $S_n^{-1}A_nS_n = D_n = \text{diag}\,(\lambda_1^{(n)},\ldots,\lambda_d^{(n)})$, $C(S_n) \leq 2$. This lemma can be applied because

$$|A_n - D|_\infty \leq q^* a_n|\varepsilon| \leq q^*\varepsilon^* \leq \frac{\alpha}{3d-1} \quad \text{for all} \quad |\varepsilon| \leq \varepsilon^*.$$

Making the change of variables $P_n = S_nX_nS_n^{-1}$ and defining $Y_n = S_n^{-1}Q_nS_n$, equation (11) becomes

$$\dot{X}_n = D_nX_n - X_nD_n + Y_n, \quad \overline{Y}_n = 0.$$

As $D_n$ is a diagonal matrix we can handle this equation as $d^2$ unidimensional equations, that can be solved easily by expanding in Fourier series. If $X_n = (x_{\ell j,n})$, $Y_n = (y_{\ell j,n})$, with

$$x_{\ell j,n}(t) = \sum_{\substack{k\in\mathbb{Z}^r \\ 0<|k|<M}} x_{\ell j,n}^k e^{i(k,\omega)t}, \qquad y_{\ell j,n}(t) = \sum_{\substack{k\in\mathbb{Z}^r \\ 0<|k|<M}} y_{\ell j,n}^k e^{i(k,\omega)t},$$

the coefficients must be

$$x^k_{\ell j,n} = \frac{y^k_{\ell j,n}}{\lambda^{(n)}_j - \lambda^{(n)}_\ell + i(k,\omega)},$$

and, by hypothesis 3 they can be bounded by $|x^k_{\ell j,n}| \leq (L|\varepsilon|)^{-1}|y^k_{\ell j,n}|$, and this implies

$$\begin{aligned} \|P_n\|_\rho &\leq C(S_n)\|X_n\|_\rho \leq C(S_n)(L|\varepsilon|)^{-1}\|Y_n\|_\rho \leq C(S_n)^2(L|\varepsilon|)^{-1}\|Q_n\|_\rho \leq \\ &\leq 4(L|\varepsilon|)^{-1}q^*q_n \leq |\varepsilon|^{-1}\frac{q_n}{2}. \end{aligned}$$

Hence, $\|\varepsilon P_n\|_\rho \leq q_n/2 < 1/2$. Thus $I + \varepsilon P_n$ is invertible and

$$\|(I+\varepsilon P_n)^{-1}\|_\rho \leq \frac{1}{1 - \|\varepsilon P_n\|_\rho} < 2.$$

Now, applying the change (10) to (8) and defining $Q^*_n = \varepsilon(I+\varepsilon P_n)^{-1}Q_nP_n$, $A_{n+1} = A_n + \varepsilon\overline{Q^*_n}$, $Q_{n+1} = (\widetilde{Q^*_n})_{<M}$ and $R_{n+1} = (I+\varepsilon P_n)^{-1}R_n(I+\varepsilon P_n) + (Q^*_n)_{\geq M}$, it is easy to derive equation (9). Finally we use Lemma 4.1 to bound the terms of this equation:

$$\begin{aligned} \|Q^*_n\|_\rho &\leq \|(I+\varepsilon P_n)^{-1}\|_\rho\|Q_n\|_\rho\|\varepsilon P_n\|_\rho \leq \|Q_n\|_\rho q_n \leq q^*q_n^2 = q^*q_{n+1} \\ \|Q_{n+1}\|_\rho &\leq \|Q^*_n\|_\rho \leq q^*q_{n+1} \\ |A_{n+1} - D|_\infty &\leq |A_n - D|_\infty + |\varepsilon\overline{Q^*_n}|_\infty \leq q^*(a_n + q_{n+1})|\varepsilon| = q^*a_{n+1}|\varepsilon| \\ \|R_{n+1}\|_{\rho-\delta} &\leq \frac{1+\|\varepsilon P_n\|_\rho}{1-\|\varepsilon P_n\|_\rho}\|R_n\|_{\rho-\delta} + \|(Q^*_n)_{\geq M}\|_{\rho-\delta} \leq \\ &\leq \left(\frac{1+q_n/2}{1-q_n/2}r_n + q_{n+1}\right)q^*e^{-M\delta} = q^*r_{n+1}e^{-M\delta}, \quad \forall\, \delta \in ]0,\rho]. \qquad \blacksquare \end{aligned}$$

# 5 Proof of Theorem

Let $S$ be a regular matrix such that $S^{-1}AS = D = \operatorname{diag}(\lambda_1,\ldots,\lambda_d)$. We define $\varepsilon^*$, $\alpha$, $\beta$ and $m$ as in the statement of Theorem 3.1. We also define $q^* = e\beta q$, $M = M(\varepsilon) = \left(\frac{m}{|\varepsilon|}\right)^{1/\gamma}$ and $L = 8q^*$.

The (constant) change $x = Sx_0$ transforms the initial equation into

$$\dot{x}_0 = (D + \varepsilon Q^*(t,\varepsilon))x_0 \tag{12}$$

where $Q^* = S^{-1}QS$ and so $\|Q^*\|_\rho \leq e^{-1}q^*$ for $|\varepsilon| \leq \varepsilon^*$. We split equation (12) as

$$\dot{x} = (A_0 + \varepsilon Q_0(t) + \varepsilon R_0(t))x_0$$

where $A_0 = D + \varepsilon\overline{Q^*}$, $Q_0 = \widetilde{Q^*}_{<M}$ and $R_0 = Q^*_{\geq M}$. Using Lemma 4.1 it is easy to see that

$$|A_0 - D|_\infty \leq q^*a_0|\varepsilon|, \qquad \|Q_0\|_\rho \leq q^*q_0, \qquad \|R_0\|_{\rho-\delta} \leq q^*r_0e^{-M\delta},$$

$\forall\,\delta\in]0,\rho],\ |\varepsilon|\le\varepsilon^*$, if $a_0=q_0=r_0=e^{-1}$.

We are going to show that in all the steps the hypothesis of Lemma 4.4 are satisfied. As hypothesis 1 and 2 are easy to check, we focus on hypothesis 3.

Now since $a_n\le 1$ and $|\varepsilon|\le\varepsilon^*$, $|A_n-D|_\infty\le q^*|\varepsilon|\le\frac{\alpha}{3d-1}$, Lemma 4.2 gives that

$$|\alpha_{j\ell}^{(n)}-\alpha_{j\ell}|<2q^*|\varepsilon|\quad\text{for all } j,\ \ell,\ |\varepsilon|\le\varepsilon^*,$$

where $\alpha_{j\ell}=\lambda_j-\lambda_\ell$, $\alpha_{j\ell}^{(n)}=\lambda_j^{(n)}-\lambda_\ell^{(n)}$ being $\lambda_1^{(n)},\dots,\lambda_d^{(n)}$ the eigenvalues of $A_n(\varepsilon)$.

Using hypothesis 3 of the Theorem we obtain that, if $k\in\mathbb{Z}^r$ and $0<|k|<M(\varepsilon)$,

$$\begin{aligned}|\alpha_{j\ell}^{(n)}+i(k,\omega)| &\ge |\alpha_{j\ell}+i(k,\omega)|-|\alpha_{j\ell}^{(n)}-\alpha_{j\ell}|>\frac{c}{|k|^\gamma}-2q^*|\varepsilon|>\\ &> \left(\frac{c}{m}-2q^*\right)|\varepsilon|=L|\varepsilon|,\end{aligned}$$

and hypothesis 3 of Lemma 4.4 is verified.

In consequence the iterative process can be carried out and Lemma 4.3 ensures the convergence of the process. The composition of all the changes $I+\varepsilon P_n$ is convergent because $\|I+\varepsilon P_n\|_\rho\le 1+q_n/2$. Then the final equation is

$$\dot x_\infty=(A_\infty(\varepsilon)+\varepsilon R_\infty(t,\varepsilon))x_\infty,\quad |\varepsilon|\le\varepsilon^*, \tag{13}$$

where $|A_\infty(\varepsilon)-D|_\infty\le q^*a_\infty|\varepsilon|\le\frac{e\beta}{e-1}q|\varepsilon|$, and

$$\|R_\infty(\cdot,\varepsilon)\|_{\rho-\delta}\le q^*r_\infty e^{-M(\varepsilon)\delta}\le\frac{e^2\beta}{e-1}q\exp\left\{-\left(\frac{m}{|\varepsilon|}\right)^{1/\gamma}\delta\right\},\ \forall\,\delta\in]0,\rho].$$

To end up the proof, the change $x_\infty=S^{-1}y$ transforms equation (13) into equation (7) with the bounds that we were looking for. ■

# 6 An example

The results of this paper can be applied in many ways, according to the kind of problem we are interested in. Let us illustrate this with the help of an example.

Let us consider the equation

$$\ddot x+(1+\varepsilon q(t))x=0, \tag{14}$$

where $q(t)=\cos(\omega_1 t)+\cos(\omega_2 t)$, being $\omega_1=\sqrt{2}$ and $\omega_2=\sqrt{3}$. Defining $y$ as $\dot x$ we can rewrite (14) as

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\varepsilon\begin{pmatrix}0&0\\-q(t)&0\end{pmatrix}\right]\begin{pmatrix}x\\ y\end{pmatrix}. \tag{15}$$

As $\lambda_{1,2} = \pm i$, the diophantine condition (6) is satisfied for $\gamma = 1$ (because the frequencies are quadratic irrationals). The value of $c$ will be discussed later. For the sake of simplicity, let us take $\rho = 2$ and $\delta = 1$. This implies that $q = \|Q\|_\rho = 2e^2$. It is not difficult to derive $\beta = 2$ and, finally, $\varepsilon^* = 4.9787\ldots \times 10^{-3}$ and $r^* = 2.5419\ldots \times 10^2$.

The value of $c$ might be calculated for all $k = (k_1, k_2)$, but better (bigger) values can be used since we only need to consider $|k|$ up to a finite order. For instance, an easy computation shows that for $|k| \leq 125$ $c$ is 0.149. If $|k| = 126$, then $c$ must be 0.013 at most, due to the quasiresonance produced by $k = (70, -56)$. In the range $126 \leq |k| \leq 10^5$ there are no more relevant resonances, so the value $c = 0.013$ suffices.

To start the discussion, let us suppose that the value of $\varepsilon$ in (15) is $\varepsilon = 2 \times 10^{-6}$. If we take $c = 0.149$ we obtain that $m = 1.8545\ldots \times 10^{-4}$ and $M = 93$ (recall that the process cancels frequencies such that $|k| < M(\varepsilon)$). If the value of $M$ had been bigger than 125, we should have used the value $c = 0.013$ instead. So, we can reduce the system to constant coefficients with a remainder $R^*$ such that $\|R^*\|_{\rho-1} < 10^{-37}$.

If the given value of $\varepsilon$ is smaller, for instance $\varepsilon = 10^{-7}$, the computed value of $M$ if $c = 0.149$ is 1855, so $c = 0.013$ must be used. This produces $M = 162$ and $\|R^*\|_{\rho-1} < 10^{-67}$. A value of $\varepsilon = 5 \times 10^{-8}$ implies $M = 324$ and $\|R^*\|_{\rho-1} < 10^{-138}$. The computation of the reduced matrix, as well as the quasiperiodic change of variables will be discussed below.

Another interesting problem is to study the reducibility for a value of $\varepsilon$ bigger than the $\varepsilon^*$ given above. Let us continue working with the same equation but selecting, as an example, $\varepsilon = 0.1$.

To increase the value of $\varepsilon^*$, one may try to rewrite the proof, using optimal bounds at each step. This has not been done here in order to get an easy, clean and short proof. Instead of doing this, we think that it is much better to rewrite the proof for our example, using no bounds but exact values. This will produce the best results for this problem.

For that purpose, we have implemented the algorithm used in the proof of the theorem as a C program, for a (given) fixed value of $\varepsilon$. The program computes and performs a finite number of the changes of variables used to prove the theorem. As a result, the reduced system (including the remainder) as well as the final change of variables are written.

To simplify and make the program more efficient, all the coefficients have been stored as double precision variables. During all the operations, all the coefficients less than $10^{-20}$ have been dropped, in order to control the size of the Fourier series appearing during the process. Of course, this introduces some (small) numerical error in the results.[1]

[1] If one wants to control that error, it is possible to use intervalar arithmetic for the coefficients and to carry a bound of the remainder for each Fourier series.

After three changes of variables, (15) is transformed into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left[ \begin{pmatrix} 0.000000000000000 & 1.000000366251255 \\ -0.992421151834871 & 0.000000000000000 \end{pmatrix} + R(t) \right] \begin{pmatrix} x \\ y \end{pmatrix}. \qquad (16)$$

The remainder $R$ is very small: the biggest coefficient it contains is less than $10^{-16}$. Note that the accuracy (relative error) of this remainder is very poor, due to the use of double precision arithmetic (15–16 digits) for the coefficients. During the computations, $M$ has not been given a value. Instead, we have tried to cancel all the frequencies with amplitude bigger than $10^{-16}$ (it turns out from the computations that all these frequencies satisfy $|k| \leq 20$). It is also possible to obtain a better accuracy in the result, using a multiple-precision arithmetic.

Finally, to check the software, we have tabulated a solution of (16) for a time span of 10 time units. We have transformed this table by means of the (quasiperiodic) change of variables given by the program. Then, we have taken the first point of the transformed table as initial condition of (15), to produce (by means of numerical integration) a new table. The differences between these two tables are less than $10^{-13}$, as expected.

So, for practical purposes, this is an "effective" Floquet Theorem in the sense that it allows to compute the reduced matrix as well as the change of variables, with the usual accuracy used in numerical computations.

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