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Relativistic Quantization and Improved Equation for a Free Relativistic Particle

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Dedicated to the memory of my grandfather M. M. Poggio (1907–1994)

Abstract

Usually the only difference between relativistic quantization and standard one is that the Lagrangian of the system under consideration should be Lorentz invariant. The standard approaches are logically incomplete and produce solutions with unpleasant properties: negative-energy, superluminal propagation etc.

We propose a two-projections scheme of (special) relativistic quantization. The first projection defines the quantization procedure (e.g. the Berezin-Toeplitz quantization). The second projection defines a casual structure of the relativistic system (e.g. the operator of multiplication by the characteristic function of the future cone). The two-projections quantization introduces in a natural way the existence of three types of relativistic particles (with 0, $\frac{1}{2}$, and 1 spins).

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Contents

1 Introduction

> One should keep the need for a sound mathematical basis dominating one's search for a new theory.
>
> P. A. M. Dirac [12].

The paper presents a new scheme of relativistic quantization. Our approach is different from other ones (for example, the Dirac construction). Usually the only difference between relativistic quantization and standard one is that the Lagrangian of the system under consideration should be Lorentz invariant. Quantization procedures themselves are the same for non-relativistic and relativistic cases. After solutions to the Schrödinger equation are obtained, one should consider additional restrictions: delete negative-energy solutions, forbid superluminal propagation or introduce second quantization to describe ensembles of bosons or fermions.

Another approach to relativistic quantization was used by Dirac [6, 12, 27] to introduce an equation for a free relativistic electron. This approach has the following features:

1. Its deduction contradicts all usual quantizations.

2. It gives a natural description of the spin of an electron and its magnetic moment.

3. It involves the inconvenient occurrence of states with negative energy with many unpleasant consequences (for example, *Zitterbewegung*).

One can suspect, that item 3 is a corollary of 1, but it was unknown how one can obtain item 2 without the false prerequisites and *negative* outcomes. We try to improve this situation by introduction a two-projections scheme of relativistic quantization.

The first projection defines quantization procedure (for example, the Berezin-Toeplitz quantization). The second projection defines a casual structure in relativistic system (for example, the operator of multiplication by the characteristic function of the future cone in the tangent space to space of events). We do *not* use in our construction the Lorentz (or another relativistic) group. The main idea of the presented quantization is an application of the causality constraint before the derivation of motion equations.

For the given energy operator H the two-projections quantization gives us a family of algebras of observables with the Heisenberg motion equation for them. This family is parametrized by a parameter $p \in [0, 1]$ such that:

1. For $p = 0$ the algebra of observables has the one-dimensional representation. The corresponding states are one-component (scalar fields).

2. For $p \in (0, 1)$, particularly for $p = \frac{1}{2}$, the algebra of observables has the two-dimensional spinor representation. The states are 2-spinors (spinor fields).

3. For $p = 1$ the algebra of observables has a reducible representation, which is the direct sum of two one-dimensional representations. The corresponding states are two-components (2-vector fields).

One can consider the parameter p as a value (without sign) of possible projection of the spin of particles. Then the two-projections quantization introduces in a natural way the existence of three main types of relativistic particles (corresponding to zero, one-half and unit spins). There is not an *elementary* particle with spin higher that 1 in our construction.

We make a very short (and a little bit skeptical) overview of the standard relativistic quantizations (quantum field theory) in Section 2. The background of our approach to this problem (quantum and relativistic projections) will be introduced in Section 3. In Section 4 we consider two models of the two-projections relativistic quantization. The first model is rather remote, nevertheless it demonstrates the appearance of spin-like structure in our setting. The second model describes a free relativistic particle with arbitrary spin in the Minkowski space-time.

The problem of a joining of quantum theory and general covariance is not a simple one. For example, in [10] it was suggested to make the notion of time even more relativistic than usual, i.e. the time flow should depend even from states of systems at hands. At the present paper another logical opportunity is studied: we admit a violence of relativistic invariance on the quantum level, i.e. the quantum and relativistic projections may not commute (see Remark 4.5). Such an assumption gives a description of observable spin effects.

There are some remarks on the paper style. The paper is addressed both to physicists and mathematicians. This explains why (sometimes) the explanation is too basic for someone. The bibliography on the subject is enormous. We usually refer only to the recent publication(s), which allow to reconstruct a wider set of references.

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2 Standard Quantum Field Theory and Relativistic Quantization

There are many (deeply intervening) ways to construct quantum field theory. Two important streams are:

- The path integral techniques: the Wiener functional and the Feynman path integrals [13, 14]. We will discuss this approach elsewhere [17].
- The second quantization technique: the Dirac-Fock-Jordan-Wigner approach [11].

The last one is a two step construction:

1. One should construct a relativistic equation for a single particle.
2. The second quantization procedure joins particles in the bosonic and/or fermionic Fock spaces [11, 25]. This provides a description of interactions, decays, creations, and annihilations of particles.

Herein we will focus on the first step—construction of relativistic equations of a single relativistic particle. The spin of a particle is the main characteristic of the equation. Other qualities of particles (mass, charge, etc.) are only parameters in the equation. But the type of equation itself sharply depends on particle's spin. Moreover, the type of second quantization is strictly predestined by the type of statistic (Fermi-Dirac or Bose-Einstein) and the statistic in its turn is also determined by the spin.

The spin is usually associated with representations of the inhomogeneous Lorentz group. Our consideration will show (Subsection 4.1) that connection between the notion of spin and the theory of special relativity is more deeper than only a representation of the inhomogeneous Lorentz group.

The equation for a single particle is usually constructed as Schrödinger type equation and the usual pre-requests[1] are:

- It should have a Lorentz invariant Hamiltonian. The fundamental relativistic metric form

  $$dt^2 - dx_1^2 - dx_2^2 - dx_3^2$$

  is the most natural Lorentz invariant object and it generates the wave equation (the Klein-Gordon equation without mass):

  $$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right) f(x) = 0.$$

  If one considers the Einstein expression for the energy of a free relativistic particle [20, (9,7)]

  $$E = \sqrt{p^2c^2 + m^2c^4}$$

  then the standard rules of quantization

  $$E \to i\hbar\frac{\partial}{\partial t}, \quad p_i = -i\hbar\frac{\partial}{\partial x_i}$$

  give the *square-root Klein-Gordon* equation for a free relativistic particle

  $$i\hbar\frac{\partial\phi(t)}{\partial t} = \sqrt{-c^2\hbar^2\Delta + m^2c^4}\,\phi(t). \tag{2.1}$$

  Here $\Delta = \sum_1^3 \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator.

[1] For the sake of simplicity all notions are illustrated by their simplest (original) form.

Spin	spin 0	spin $\frac{1}{2}$	spin 1
Particle	π-meson	electron	photon
Equation	Klein-Gordon	Dirac (factorized Klein-Gordon)	Klein-Gordon
States	scalar fields (1)	two-spinors fields (4)	4-vectors fields (4)
Correction	scalar fields (1)	spinors fields (2)	3-vectors fields (3)

Figure 1: Particles with different spins, their equations, types of states and their dimensionality (shown in braces) by the standard theory and results of correction by experiments.

- The equation under consideration should describe the number of freedom degrees corresponding to the spin of particle. The simple way for construction (2.1) does not meet this request.

Meanwhile the first condition is a very natural one, it is my impression, that the second condition was usually achieved by a hand-made work (if not to say artificial). For example [6, § 4], for a free particle with spin 1 it is usually *a priori* assumed that its states are described by a 4-vector field. Some weak motivation for this is the following: 4-vector field is a natural Lorentz invariant object. But immediate application of the Klein-Gordon equation shows that such states connected with negative energy solutions. To eliminate them one applies additional constraints and obtains states, which are described by 3 independent components only.

Another example is the Dirac equation for a free relativistic electron (see [12], [6, § 5] and [27, § 1.1]). It was mentioned in Introduction, its deduction contradicts all usual rules of quantization. Moreover, the Dirac "factorization procedure" was never used to any other problem in physics and thus may hardly be named a method[2]. Till now it was the only way to make the number of freedom degrees large enough for a description of the spin of the electron and its magnetic moment. But the Dirac equation provides us with too many degrees of freedom and half of them correspond again to negative energy solutions. This involves many unpleasant consequences (Zitterbewegung, superluminal propagation, etc. [27]). To eliminate them

[2] "What is the difference between method and device? A method is a device, which you use twice" [22, p. 208].

one should introduce, for example, the Dirac "holes in the sea of negative energy electron states".

We summarize information about different type of particle in Figure 1. We would like to avoid the question: *which types of elementary particles do exist, and which of them do are elementary*. At least photon and electron are often believed to exist and be rather elementary.

Our brief consideration justifies the following claim: *there is no any unified and natural procedure to obtain relativistic equations for different types of particles with right number of degrees of freedom.*

We will present in the next Sections a procedure of relativistic quantization based on the notion of casual structure. It is in an agreement with the "classical" non-relativistic quantizations and gives a simple description for the spin structure.

REMARK 2.1 If the reader is familiar with the Einstein-Podolsky-Rosen paradox then the question arises: *May standard quantum mechanics be combined with the notion of casualty at all*? I do not know the answer to this question, but would like to make two observations (see also Remark 4.5):

1. The Einstein-Podolsky-Rosen paradox is deeply connected with the theory of measurements and interpretation of quantum mechanics. We do not touch these topics herein.

2. The disagreement between the casualty structure and non-locality of standard quantum mechanics is not the only contradiction in quantum theory.

3 Origins of Two Projections

It this Section we explain how two projections arise in our approach. Indeed, two words from the paper title—*relativistic* and *quantization*—explain the existence of two projections.

3.1 Non-Relativistic Quantizations Defined by Projection

First, let us remind that the standard non-relativistic quantization may be obtained by application of a projection to the classical system under consideration. We give only a short summary of this topic, the relevant information may be found in [3, 4, 5, 7, 8, 9, 15] and their references.

Let $L_2(\mathbb{C}^n, d\mu_n)$ be a space of all square-integrable functions on $\mathbb{C}^n$ with respect to the Gaussian measure

$$d\mu_n(z) = \pi^{-n} e^{-z\cdot\overline{z}} dv(z),$$

where $dv(z) = dxdy$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. The Segal-Bargmann [1, 23] (or the Fock) space $F_2(\mathbb{C}^n)$ is the subspace of $L_2(\mathbb{C}^n, d\mu_n)$ consisting of all entire functions, i.e. such functions $f(z)$ that

$$\frac{\partial f}{\partial \bar{z}_j} = 0, 1 \leq j \leq n.$$

Denote by P_Q the orthogonal Bargmann projection [1] of $L_2(\mathbb{C}^n, d\mu_n)$ onto the Segal-Bargmann (Fock) space $F_2(\mathbb{C}^n)$. Then the formula

$$k(q,p) \rightarrow T_{k(q+ip)} = P_Q k(q+ip) I \tag{3.1}$$

defines Berezin-Toeplitz (anti-Wick) quantization, which maps a function $k(q,p) = k(q+ip)$ on $\mathbb{R}^{2n} = \mathbb{C}^n$ to the Toeplitz operator T_k with the pre-symbol $k(q+ip)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$. There is an identification between the Berezin quantization and the Weyl quantization [3, 7, 15]. The identification has the simplest form for observables depending only on p or q alone. We will use it in Subsection 4.3 for the construction of the Schrödinger type equation of a free particle at the Minkowski space.

Example 3.1 [5] At the Segal-Bargmann representation the operators of creation and annihilation of a particle at the j-th state are $a_j^+ = z_j I$ and $a_j^- = \frac{\partial}{\partial z_j}$ correspondingly. Let us consider harmonic oscillator with n degrees of freedom. Its Hamilton function is

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n}(q_j^2 + p_j^2).$$

Then the corresponding quantum Hamiltonian is the operator

$$T_{H(q,p)} = \frac{1}{2} P_Q \sum_{j=1}^{n}(q_j^2 + p_j^2) I = nI + \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}.$$

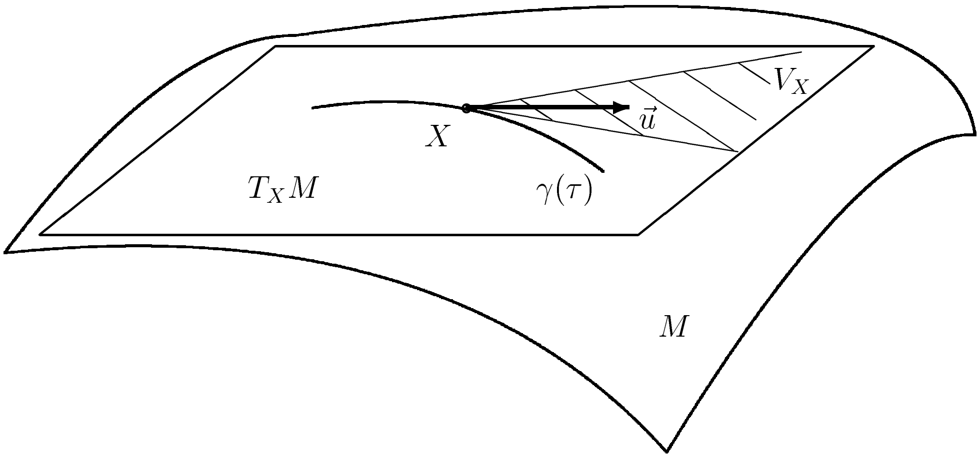

Figure 2: The casual structure for a dynamic system:
M —the space of events (space-time);
$\gamma(\tau)$ —a trajectory of the dynamical system parametrized by the proper “time” τ;
X —a point of the trajectory $\gamma(\tau)$ for the value $\tau = \tau_0$;
$T_X M$—the tangent space at X;
V_X —the casual cone in $T_X M$;
$\vec{u}$ —the velocity vector of the trajectory $\gamma(\tau)$ at X should belong to V_X.

REMARK 3.2 The Berezin-Toeplitz quantization is not the only quantization generated by a projection. Let us remind that *geometric quantization* procedure [16, 29] consists of two steps: prequantization and quantization. The second step is, in fact, restriction of operators achieved by prequantization to manifold defined by polarization (projection to functions depending only on “coordinates”, roughly speaking).

3.2 Casual Projection in Relativistic Mechanics

Now we introduce the second projection. Our consideration is based on the book [24, Chap. II]. This is an alternate approach to the theory of special relativity, which suggests that variations from the Lorentz group may be useful. The main idea is: one can introduce a general axiomatic relativistic structure not by means of the Lorentz group, but by the usage of the notion of casual structure. Namely, there is an "infinitesimal future cone" in the tangent space at each point to the space of events (space-time).

The axiomatic formulation and mathematical implementation may be found in [24, Chap. II]. We give only a short illustration here (see Figure 2). Let M be the space of events (space-time). We need not specify the dimensionality of M. Let $\gamma(\tau)$ be a trajectory of the dynamical system (point) under consideration parametrized by the proper "time" τ. Let X be a point of M lying on the trajectory $\gamma(\tau)$ for the value $\tau = \tau_0$. We denote by $T_X M$ the tangent space at the point X. The existence of the casual structure on M implies that in $T_X M$ there is the casual (or future, or light) cone, namely V_X. Thus the velocity vector $\vec{u}$ of trajectory $\gamma(\tau)$ at X should belong to the future cone V_X.

The casual constraint may be achieved by the multiplication of the Lagrangian function $L(q, \dot{q})$ by the characteristic function $\chi_R(q, \dot{q})$ of the future cone. The new Lagrangian $\chi_R(q, \dot{q})L(q, \dot{q})$ is non-zero only for admissible points of tangent bundle TM and thus allows only relativistic motion. We will denote the casual projection by $P_R = \chi_R I$. We would like to stress an analogy between the transition from classical mechanics to quantum by means of projection P_Q and the passing from non-relativistic mechanics to relativistic one by P_R.

Example 3.3 The "classical" example is the four-dimensional Minkowski space-time with the pseudo-Euclidean metric ([19, Chap. IX], [20, Chap. I, II])

$$ds^2 = c^2 dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2.$$

The casual structure is defined by the future part of the light cone:

$$c^2 dx_0 \geq \sqrt{dx_1^2 + dx_2^2 + dx_3^2}.$$

The relativistic projection P_R here is the operator of multiplication by the characteristic function $\chi_R(p)$ of the future cone. Note, that in this case the function χ_R does not depend on q and this will greatly simplify our consideration in Subsection 4.3.

Usually textbooks link the theory of special relativity with the Lorentz invariance of the theory. We will not touch the group of relativistic transformations herein. However, our theory (the relativistic projection P_R and the

Hamiltonian function) will be defined purely in terms of the light cone (see Section 4). Thus our theory will be invariant under all relativistic transformations, which (by their definition) preserve the light cone.

REMARK 3.4 It is interesting, that in classical mechanics we do not need such relativistic projection. Outside the light cone the Lagrangian of a free particle [19, (95.8)]

$$L = -c^2 m \sqrt{1 - \frac{1}{c^2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)}$$

is purely imaginary and thus is out classical theory. In quantum theory the complex numbers are on an equal footing with the real ones and separation of permitted and prohibited parts of phase space should be done explicitly.

REMARK 3.5 We have considered only the *future* part of the light cone. But it also has the *past* part. Let us remind that *anti*-particles may be considered as corresponding *particles* moving backward in time (CPT invariance [2, § 13]). To describe this one may wish to multiply the Lagrangian function by the characteristic function of the past part of the light cone. In both cases (the future and past part of light cone) (anti-)particles are moving the forward in proper time. Thus particles and corresponding anti-particles have appeared in our consideration on the equal symmetrical footing.

4 Models of the Two-Projections Relativistic Quantization

Previous consideration shows that a natural algebra of observables for a quantum relativistic particle should be generated by (at least) three operators: the quantum projection P_Q, the relativistic projection P_R, and by the Hamilton function $H(q,p)$. We will denote such C^*-algebra of observables by $\mathfrak{O}(P_Q, P_R, H(q,p))$. This Section is devoted to two concrete realizations of this algebra.

4.1 Relativistic Quantization: a Toy Model

First we will consider unrealistic case of the Hamiltonian function $H(q,p)$ identically equal to 1. Then algebra $\mathfrak{O}(P_Q, P_R, H(q,p))$ is generated only by two projections P_Q and P_R:

$$\mathfrak{O}(P_Q, P_R, H(q,p)) = \mathfrak{O}(P_Q, P_R)$$

The algebra generated by two projections is very well studded in mathematics (see, for example, [26, 28]) and have already appeared in quantum mechanics ("two questions generate infinitely many questions" [21, Appendix 3]). The following result is the basis of our construction.

Theorem 4.1 [28] *Let the points* 0 *and* 1 *be non-isolated points of the spectrum* $\sigma = \operatorname{sp}(P_Q - P_R)^2$. *Then the algebra* $\mathfrak{O}(P_Q, P_R)$ *generated by two projections* P_Q *and* P_R *is isometrically isomorphic to the algebra of all* 2×2 *continuous matrix-functions on* σ, *which are diagonal at the points of* $\{0,1\} \subset \sigma$. *This isomorphism* ϕ *is defined by the following mapping of the generators of* $\mathfrak{O}(P_Q, P_R)$

$$\begin{aligned}
\phi : P_Q &\longmapsto \begin{pmatrix} 1-p & \sqrt{p(1-p)} \\ \sqrt{p(1-p)} & p \end{pmatrix}, \\
\phi : P_R &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

where $p \in \sigma$.

The parameter p appeared at the previous Theorem has the following geometric meaning. The spectrum σ characterize the mutual disposition of $\operatorname{Im} P_Q$ and $\operatorname{Im} P_R$ and, in some sense, generalizes square of sine of the angle between two lines in the two-dimensional case. This geometric interpretation suggests to understand the parameter p as a possible value of projection (without sign) of spin of the particle under consideration.

Remark 4.2 In our approach values of projection of spin are not quantized in the quantitative sense and may fill whole interval $[0,1]$. But they do are "quantized" in the qualitative (phenomenon) sense. For $p=0$ and $p=1$ matrixes are diagonal (correspond to "vector" representations) and for $p \in (0,1)$ (particular $\frac{1}{2}$) they are general matrixes (correspond to "spinor" representations). Thus we will count all value $p \in (0,1)$ as corresponding to spin $\frac{1}{2}$. Probably, "hidden internal degree of freedom" p may be employed in future.

Let us assume, that $\sigma = [0,1]$. We introduce the subalgebra (even an ideal) $\mathfrak{O}'(P_Q, P_R)$ of the algebra $\mathfrak{O}(P_Q, P_R)$ such that

$$\mathfrak{O}'(P_Q, P_R) = \{A \in \mathfrak{O}(P_Q, P_R) \mid Af = 0 \text{ for all } f \in (\operatorname{Im} P_Q)^{\perp} \cap (\operatorname{Im} P_R)^{\perp}\}.$$

By other words, a non-quantum *and* non-relativistic behavior in $\mathfrak{O}'(P_Q, P_R)$ is unobservable. Then one obtains an evident

Corollary 4.3 *For parameter $p \in [0,1]$ the algebra $\mathfrak{O}'(P_Q, P_R)$ has:*

1. *For $p = 0$—the one-dimensional representation. The corresponding states are one-component (scalar fields);*

2. *For $p \in (0,1)$ (particularly for $p = \frac{1}{2}$)—the two-dimensional spinor representation. The states are 2-spinors (spinor fields).*

3. *For $p = 1$— a reducible representation, which is the direct sum of two one-dimensional representations. The corresponding states are two-components (2-vector fields);*

Example 4.4 For the observable $P_Q + P_R \in \mathfrak{O}'(P_Q, P_R)$ we have:

$$\begin{array}{ccc} p=0 & p=\frac{1}{2} & p=1 \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

In our framework *elementary* particles with spin 0, $\frac{1}{2}$, and 1 are only allowed. Particles with higher spin may be considered as composite ones.

There are some conclusions from such interpretation:

- *Particle of a spin s has $2s+1$ degree of freedom.* For spin 0, $\frac{1}{2}$, and 1 the possible values of its projection (without sign) are $\{0\}$, $\{\frac{1}{2}\}$, and $\{0,1\}$ correspondingly. Counting dimensionality of matching representations from Corollary 4.3 one can obtain the assertion.

- *If a particle has the luminal propagation it should have the spin with projection* 1. Indeed, for a particle with the luminal propagation the relativistic projections P_R is the operator of multiplication by the characteristic function of the boundary of light cone. But the boundary of a reasonable cone has the zero measure, thus $P_R = 0$. From here

$$\operatorname{sp}(P_Q - P_R)^2 = \operatorname{sp} P_Q^2 = \operatorname{sp} P_Q = \{0,1\}.$$

We should exclude the value 0, because it corresponds to a non-quantum and non-relativistic behavior. After that the only possibility is the projection with value 1. Note, that our conclusion is in the total agreement with the case of photons.

- *If a particle has an underluminal propagation then it may have the spin* $\frac{1}{2}$. For a particle with underluminal propagation the relativistic projection P_R is not zero and $\operatorname{sp}(P_Q - P_R)^2$ may contain more points than 0 and 1. It will depend on additional constrains, which an opportunity for the projection will realize.

REMARK 4.5 In the given model particles with the spin 0 or 1 are defined by commuting projections P_Q and P_R, i.e. their quantum and relativistic natures are in the agreement. In contrary, the spin $\frac{1}{2}$ may arise only for non-commuting projections P_Q and P_R. Thus *an existence of particles with spin* $\frac{1}{2}$ *may be explained by a non-compatibility of the relativity with the quantum world*.

It seems that first conclusions even from the very toy model are natural, thus we are going to construct more realistic model.

4.2 Realistic Hamiltonian and Improved Equation for a Free Relativistic Particle

We have seen in the previous Subsection that a spin-like structure has arisen in our approach from the very existence of two projections and does not depend on the properties of Hamiltonian function. Thus for particles with all types of spin we can select a *scalar* Hamiltonian function guiding by relativistic non-quantum mechanics.

Let us consider the Minkowski four-dimensional space-time (space of events) with the pseudo-Euclidean metric ([19, Chap. IX] and [20, Chap. I, II])

$$ds^2 = c^2 dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2.$$

The casual structure is defined by the future part of the light cone:

$$c^2 dx_0 \geq \sqrt{dx_1^2 + dx_2^2 + dx_3^2}. \tag{4.1}$$

Here the relativistic projection P_R is the operator of multiplication by the characteristic function $\chi_R(p)$ of the future cone (for anti-particles see Remark 3.5). The quantum projection P_Q is the Bargmann projection on the four-dimensional Segal-Bargmann space $F_2(\mathbb{C}^4)$. It was already calculated [18] that

Lemma 4.6 *The spectrum of the operator*

$$(P_Q - P_R)^2|_{F_2(\mathbb{C}^n)}$$

acting on $F_2(\mathbb{C}^n)$ *is equal to* $[0, 1]$.

Thus in this model particles of spin 0, $\frac{1}{2}$, and 1 are all permitted.

Let us find the Hamilton function. The variation of action integral for a free particle with mass m ([19, (95.8)], [20, (8,1)])

$$S = -mc \int_a^b ds$$

gives equation [20, (9,10)]

$$\frac{du_j}{ds} = 0, \text{ where } u_j = \frac{dx_j}{ds}.$$

The Hamilton formulation may be achieved by introduction of 4-momentums ([19, (95.15)], [20, (9,14)])

$$p_j = mc\frac{dx_j}{ds} = mcu_j$$

and Hamilton function of a free particle ([19, 96.8])

$$H(q,p) = \frac{1}{2mc}\sum_{j=0}^{3} p_i p^i = \frac{1}{2mc}(p_0^2 - p_1^2 - p_2^2 - p_3^2). \tag{4.2}$$

The Hamiltonian (4.2) is manifestly Lorentz invariant and positive inside the light cone.

Thus the algebra $\mathfrak{O}'(P_Q, P_R, H(q,p))$ of observables of a free particle in the Segal-Bargmann representation is the algebra generated by quantization projection P_Q (Subsection 3.1), relativistic projection P_R of multiplication by the characteristic function of the future cone (4.1) and the

Hamiltonian $H(q,p)$ (4.2). The Heisenberg equation for an observable $A \in \mathfrak{O}'(P_Q, P_R, H(q,p))$ may be set as follows

$$\frac{dA(\tau)}{d\tau} = \frac{i}{\hbar}[A(\tau), H_Q],$$

where $H_Q = P_Q P_R H(q,p) I$ is the quantum (operator) Hamiltonian and τ is the proper time in the Minkowski space.

The corresponding Schrödinger equation for a state $\phi(\tau)$ (which has a spin structure by the existence of two projections) has the form

$$\frac{\partial\phi(\tau)}{\partial\tau} = -\frac{i}{\hbar} H_Q \phi(\tau). \tag{4.3}$$

4.3 The Relativistic Equation in the Schrödinger Representation

Now we would like to write equation (4.3) in the Schrödinger representation. To this end we need transfer operator $H_Q = P_Q P_R H(q,p) I$ from the Segal-Bargmann representation to the Schrödinger one. It is easy to do because function $P_R H(q,p)$ for a free particle in a flat space-time depends on variables p only. We produce our calculations in the momentum representation.

We will use the following standard notations:

$$z = x + iy = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Let $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$ with the usual notion of the complex conjugation. For $z, w \in \mathbb{C}^n$ let

$$\begin{aligned} z \cdot w &= z_1 w_1 + \ldots + z_n w_n, \\ |z|^2 &= |z_1|^2 + \ldots + |z_n|^2 (= z \cdot \overline{z}), \\ (x,y) &= (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n. \end{aligned}$$

Denote by $d\mu_n(z)$ the following Gaussian measure over $\mathbb{C}^n$

$$d\mu_n(z) = \pi^{-n} e^{-z\cdot\overline{z}} dv(z),$$

where $dv(z) = dxdy$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

To do calculations let us introduce the following operators (see [18] for details of calculations). Introduce the unitary operator

$$U : L_2(\mathbb{C}^n, d\mu_n) \to L_2(\mathbb{R}^{2n}) = L_2(\mathbb{R}^{2n}, dxdy),$$

defined by

$$(U\varphi)(z) = \pi^{-\frac{n}{2}} e^{-\frac{z\cdot\overline{z}}{2}} \varphi(z),$$

or

$$(U\varphi)(x,y) = \pi^{-\frac{n}{2}} e^{-\frac{x^2+y^2}{2}} \varphi(x+iy).$$

The unitary operator $I \otimes F$, where

$$(Ff)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\eta\cdot y} f(\eta)\, d\eta$$

is the Fourier transformation, maps isometrically the space

$$L_2(\mathbb{R}^{2n}, dxdy) = L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy)$$

onto itself. Now introduce the isomorphism

$$W = W^* = W^{-1} : L_2(\mathbb{R}^{2n}) \to L_2(\mathbb{R}^{2n}),$$

where

$$(Wf)(x,y) = f(\frac{1}{\sqrt{2}}(x+y), \frac{1}{\sqrt{2}}(x-y)).$$

Introduce the isometrical imbedding

$$R_0 : L_2(\mathbb{R}^n, dx) \to L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy)$$

defined by

$$R_0 : g(x) \mapsto g(x) \cdot l(y).$$

Then the adjoint operator

$$R_0^* : L_2(\mathbb{R}^{2n}) \to L_2(\mathbb{R}^n)$$

is defined by

$$(R_0^* f)(x) = \pi^{-\frac{n}{4}} \int_{\mathbb{R}^n} f(x,y)\, e^{-\frac{1}{2}y^2}\, dy.$$

Now, the operator $R = R_0^* W(I \otimes F)U$ maps the space $L_2(\mathbb{C}^n, d\mu_n)$ onto $L_2(\mathbb{R}^n, dx)$ and the restriction

$$R|_{F_2(\mathbb{C}^n)} : F_2(\mathbb{C}^n) \to L_2(\mathbb{R}^n, dx)$$

is an isometrical isomorphism.

The adjoint operator

$$R^* = U^{-1}(I \otimes F^{-1})WR_0 : L_2(\mathbb{R}^n, dx) \to F_2(\mathbb{C}^n) \subset L_2(\mathbb{C}^n, d\mu_n)$$

maps isomorphically and isometrically the space $L_2(\mathbb{R}^n, dx)$ onto the Segal-Bargmann space $F_2(\mathbb{C}^n)$. We have the following representations

$$\begin{aligned} P_Q &= R^*R : L_2(\mathbb{C}^n, d\mu_n) \to F_2(\mathbb{C}^n), \\ I &= RR^* : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n). \end{aligned}$$

Now we calculate the image

$$H_S = RH_QR^* = RP_QP_RH(q,p)R^*$$

of operator H_Q under isometry R between the Segal-Bargmann and the Schrödinger representations.

$$\begin{aligned} H_S &= RP_QP_RH(q,p)R^* = \\ &= RR^* \cdot RP_RH(q,p)R^* = R\chi_R(p)H(q,p)R^* = \\ &= R_0^*W(I \otimes F)U\chi_R(p)H(q,p)U^{-1}(I \otimes F^{-1})WR_0 = \\ &= R_0^*W(I \otimes F)\chi_R(p)H(q,p)I \otimes F^{-1})WR_0 = \\ &= R_0^*W\chi_R(p)H(q,p)R_0 = \\ &= R_0^*\chi_R(\frac{\xi + x}{\sqrt{2}})H(q, \frac{\xi + x}{\sqrt{2}})R_0. \end{aligned}$$

Here we use that $H(q,p) = H(p) = \frac{1}{2mc}\sum_{j=0}^{3} p_ip^i$ does not depend on q. Thus

$$\begin{aligned} (H_Sf)(\xi) &= \pi^{-1}\int_{\mathbb{R}^4} e^{-\frac{1}{2}x^2}\chi_R(\frac{\xi + x}{\sqrt{2}})H(\frac{\xi + x}{\sqrt{2}})f(\xi)\pi^{-1}e^{-\frac{1}{2}x^2}\,dx = \\ &= \lambda(\xi)f(\xi), \end{aligned}$$

where

$$\begin{aligned} \lambda(\xi) &= \pi^{-2}\int_{\mathbb{R}^4} \chi_R(\frac{\xi + x}{\sqrt{2}})H(\frac{\xi + x}{\sqrt{2}})e^{-x^2}\,dx \\ &= \frac{1}{4mc\pi^2}\int_{V-\xi}\sum_{j=0}^{3}(\xi + x)_j(\xi + x)^j\,e^{-x^2}\,dx. \end{aligned} \tag{4.4}$$

Theorem 4.7 *The Hamilton operator H_S of a free relativistic particle in the momentum Schrödinger representation is the unbounded operator $H_S = \lambda(\xi)I$ of multiplication by the positive valued function $\lambda(\xi)$ (4.4). In the Schrödinger coordinate representation this operator is $H_S = F^{-1}\lambda(\xi)F$.*

Operator H_S have a positive spectrum $(0, +\infty)$ and is Lorentz invariant.

The Schrödinger equation in the coordinate representation takes the form:

$$\frac{\partial f(\tau)}{\partial \tau} = -\frac{i}{\hbar}F^{-1}\lambda(\xi)Ff(\tau). \tag{4.5}$$

We would like to give an insight on the function $\lambda(\xi)$ because it is definitely not elementary one.

Lemma 4.8 *One has decomposition*

$$\lambda(\xi) = \frac{1}{4mc}\chi_R(\xi)\sum_{j=0}^{3}\xi_j\xi^j + o(\sum_{j=0}^{3}\xi_j\xi^j).$$

Proof. Let us remind that $\chi_R(\xi)H(\xi)$ is only of polynomial growth at infinity on $\mathbb{R}^4$ and on the space of such function

$$\frac{t^4}{\pi^2}e^{-t^2y^2} \to \delta(y), \text{ where } t \to \infty$$

(in sense of generalized functions). Thus we have

$$\begin{aligned}
\lim_{t\to\infty} t^{-2}\lambda(t\xi) &= \lim_{t\to\infty}\frac{1}{4mc\pi^2t^{-2}}\int_{\mathbb{R}^4}\chi_R(t\xi+x)\sum_{j=0}^{3}(t\xi+x)_j(t\xi+x)^j\, e^{-x^2}\,dx \\
&= \lim_{t\to\infty}\frac{1}{4mc\pi^2t^{-2}}\int_{\mathbb{R}^4}\chi_R(\xi+\frac{x}{t})t^2\sum_{j=0}^{3}(\xi+\frac{x}{t})_j(\xi+\frac{x}{t})^j e^{-x^2}dx \\
&= \lim_{t\to\infty}\frac{1}{4mc}\int_{\mathbb{R}^4}\chi_R(\xi+\frac{x}{t})\sum_{j=0}^{3}(\xi+\frac{x}{t})_j(\xi+\frac{x}{t})^j\frac{t^4}{\pi^2}e^{-t^2(x/t)^2}d\frac{x}{t} \\
&= \lim_{t\to\infty}\frac{1}{4mc}\int_{\mathbb{R}^4}\chi_R(\xi+y)\sum_{j=0}^{3}(\xi+y)_j(\xi+y)^j\frac{t^4}{\pi^2}e^{-t^2y^2}dy \\
&= \lim_{t\to\infty}\frac{1}{4mc}\int_{\mathbb{R}^4}\chi_R(\xi+y)\sum_{j=0}^{3}(\xi+y)_j(\xi+y)^j\left(\frac{t^4}{\pi^2}e^{-t^2y^2}\right)dy
\end{aligned}$$

$$= \frac{1}{4mc}\int_{\mathbb{R}^4} \chi_R(\xi+y)\sum_{j=0}^{3}(\xi+y)_j(\xi+y)^j\,\delta(y)\,dy$$
$$= \frac{1}{4mc}\chi_R(\xi)\sum_{j=0}^{3}\xi_j\xi^j.$$

Granting homogeneity of the Hamiltonian one obtains the assertion. □

By this decomposition one can see, that the equation

$$\frac{\partial f(\tau)}{\partial \tau} = -\frac{i}{4\hbar mc}\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right) f(\tau)$$

may be considered as the first approximation to equation (4.5).

Remark 4.9 There are no principal difficulties to develop our formalism also for a particle in the external field (i.e. the Hamiltonian explicitly depending on coordinates x_i) and/or in the curved space-time (i.e. the characteristic function of the future cone depending on coordinates). But in this case one cannot expect the simplicity of equation (4.5).

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