Invariant sections and finitely smooth conjugacy of dynamical systems

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Abstract

New versions of the smooth invariant section theorem are established and used to prove some results on finitely smooth conjugacy of two flows in the neighbourhood of their common smooth invariant submanifold.

In this paper, we establish new versions of the smooth invariant section theorem (see, for example, [1], Theorem A.33) and employ them to prove some results on finitely smooth conjugacy of two flows in the vicinity of their common smooth invariant submanifold. To this end, we also make use of the deformation method (see [2]).

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At first, let us recall some definitions and state the standing assumptions. We keep the terminology and notation adopted in [1].

Let $p : (E, \mathbb{R}, \pi) \to (B, \mathbb{R}, \rho)$ be a linear extension. The numbers

$$\Omega(\pi, b) = \limsup_{t\to+\infty} \tfrac{1}{t} \ln \|\pi^t(b)\|,$$
$$\omega(\pi, b) = -\limsup_{t\to+\infty} \tfrac{1}{t} \ln \|\pi^{-t}(\rho^t(b))\| \quad (b \in B)$$

are called *the upper and the lower Lyapunov exponents*. They do not depend on the choice of the Riemannian metric on (E, ρ, B).

Let $\mathcal{M}$ be a compact smooth manifold, $\xi : \mathcal{M} \to T\mathcal{M}$ be a C^{k+1} vector field and $(\mathcal{M}, \mathbb{R}, f)$ be the phase flow determined by ξ. Let Δ be a submanifold of $\mathcal{M}$ of class C^1 invariant under the flow f and $T_\Delta\mathcal{M}$ denote the restriction of the tangent bundle $T\mathcal{M}$ on Δ.

In what follows, we shall assume that there exist Tf–invariant vector subbundles X and Y of $T_\Delta\mathcal{M}$ such that

(a) $T_\Delta\mathcal{M} = T\Delta \oplus X \oplus Y$;

(b) $\Omega(Tf|Y, b) < 0, \quad -\omega(Tf|X, b) < 0,$
$\Omega(Tf|Y, b) - k\omega(Tf|T\Delta, b) < 0,$
$-\omega(Tf|X, b) + k\Omega(Tf|T\Delta, b) < 0 \quad (b \in B).$

These conditions mean that the submanifold Δ is *normally k–hyperbolic*. According to the well–known results (see, for example, [1], Theorem V.2.2), there exist the stable manifold $W^s(\Delta)$ and the unstable manifold $W^u(\Delta)$, both of class C^k, such that

$$W^s(\Delta) \cap W^u(\Delta) = \Delta, \quad T_b W^s(\Delta) = T_b\Delta \oplus Y_b, \quad T_b W^u(\Delta) = T_b\Delta \oplus X_b \quad (b \in B).$$

Consequently, Δ is a C^k submanifold.

Assume, in addition, that

$$\begin{aligned}
&\text{(c)} \quad \Omega(Tf|Y, b) - \omega(Tf|T\Delta, b) + k \cdot \Omega(Tf|T\Delta, b) < 0, \\
&\qquad -\omega(Tf|X, b) + \Omega(Tf|T\Delta, b) - k \cdot \omega(Tf|T\Delta, b) < 0 \quad (b \in B).
\end{aligned}$$

By [1], Theorem V.4.6, there exist f–invariant foliations of class $C^{k,k+1}$

$$W^s(\Delta) = \cup_{b\in\Delta} W^{ss}(b), \quad W^u(\Delta) = \cup_{b\in\Delta} W^{uu}(b)$$

such that $T_b W^{ss}(b) = Y_b$, $T_b W^{uu}(b) = X_b$ $(b \in \Delta)$. Smoothness of class $C^{k,k+1}$ means that the leaves $W^{ss}(b)$ and $W^{uu}(b)$ are C^{k+1} submanifolds which C^k–smoothly depend on $b \in \Delta$. Hence it follows that X and Y are C^k vector subbundles of $T_\Delta\mathcal{M}$.

Let $U = U(\Delta)$ denote a sufficiently small neighbourhood of Δ in $\mathcal{M}$ and $H : X \oplus Y \to U(\Delta)$ be some tubular neighbourhood of class $C^{k,k+1}$. By using the map H^{-1} transfer the local flow $f|U$ to the neighbourhood of the zero section $Z(\Delta)$ of the vector bundle $X \oplus Y$. In other words, henceforth we shall assume that in the vicinity U of $Z(\Delta) \approx \Delta$ there is defined a local flow of class $C^{k,k+1}$ which preserves $Z(\Delta)$. Taking into account what has been said about the foliations $\{W^{ss}(b)\}$ and $\{W^{uu}(b)\}$, we can straighten out these foliations by using an appropriate $C^{k,k+1}$ local diffeomorphism. So we shall assume that $W^{ss}(b) = Y_b$, $W^{uu}(b) = X_b$ $(b \in B)$.

Let $p : X \oplus Y \to \Delta$, $p_X : X \oplus Y \to X$ and $p_Y : X \oplus Y \to Y$ be the natural projections. Given a point $z \in X_b \oplus Y_b$, we shall write $z = (b, x, y)$ $(b \in \Delta,\ x \in X_b,\ y \in Y_b)$ and

$$f^t(z) = (f_1^t(b,x,y), f_2^t(b,x,y), f_3^t(b,x,y)).$$

Recall that f_i^t $(i = 1, 2, 3)$ are $C^{k,k+1}$ smooth and

$$f_1^t(b, 0, 0) = f^t(b), \quad f_2^t(b, 0, y) = 0, \quad f_3^t(b, x, 0) = 0.$$

Now let us state the last but one standing assumption. Let $(E, \pi, X \oplus Y)$ be a vector bundle of class C^k and $F^t : E|U(\Delta) \to E|U(\Delta)$ be an affine extension of the local flow $f^t|U(\Delta)$, *i.e.*,

$$F^t(w) = A^t(b,x,y)w + \Phi^t(b,x,y) \quad (w \in E_z,\ z = (b,x,y)), \tag{1}$$

where A^t is a linear f^t–morphism. Moreover, assume that

$$\text{(d)} \quad \Phi^t(b,x,y) = \varphi^t(b,x,y)y^Q + \psi^t(b,x,y)x^S,$$

where $\varphi^t : X \oplus Y \to P_Q(Y, E)$ is a C^k section of the bundle $P_Q(Y, E)$ (the space $P_Q(Y_b, E_z)$ of homogeneous polynomials of degree Q serves as a fiber of $P_Q(Y, E)$ over $z = (b, x, y) \in X \oplus Y$) and $\psi^t : X \oplus Y \to P_S(X, E)$ is a C^k section of the bundle $P_S(X, E)$.

Let E_Δ be the restriction of the vector bundle E to Δ and $A_0^t = A^t|E_\Delta$.

Theorem 1. *If the standing assumption* (a), (b), (c), (d) *and*

$$\begin{aligned}&\text{(e)}\quad -\omega(A_o,b)+Q\cdot\Omega(Tf|Y,b)+k\cdot\Omega(Tf|X,b)<0,\\ &\qquad\ \Omega(A_o,b)-S\cdot\omega(Tf|X,b)-k\cdot\omega(Tf|Y,b)<0\quad(b\in\Delta)\end{aligned}$$

are fulfilled, then there exists a C^k smooth local section $\sigma : U(\Delta)\to E$ invariant under the local flow F^t.

Proof. Because A^t is a linear morphism, it suffices to prove the statement in the case where $\psi^t\equiv 0$.

According to [1], Lemma III.2.3, the conditions

$$\begin{aligned}&\sup_{b\in\Delta}\|Tf^t|Y_b\|<1,\quad \sup_{b\in\Delta}\|Tf^{-t}|X_{f^t(b)}\|<1,\\ &\sup_{b\in\Delta}\|Tf^t|Y_b\|\|Tf^{-t}|T_{f^t(b)}\Delta\|^k<1,\\ &\sup_{b\in\Delta}\|Tf^{-t}|X_{f^t(b)}\|\|Tf^t|T_b\Delta\|^k<1,\\ &\sup_{b\in\Delta}\|A^{-t}|E_{(f^t(b),0,0)}\|\|Tf^t|Y_b\|^Q\|Tf^t|X_b\|^k<1\end{aligned}\tag{2}$$

hold for all sufficiently large numbers $t>0$. Fix such a number t. Let us prove the existence of a C^k smooth section $\sigma : U\to E$ invariant under the local diffeomorphism F^t. For simplicity of notation we shall drop the upper index t.

At first, extend the map $f|U$ to $X\oplus Y$ and the map F to E in the following way. Denote $L|X_b\oplus Y_b = Df(b,0,0)$ $(b\in\Delta)$. Clearly, $L : X\oplus Y\to X\oplus Y$ is a linear morphism of class C^k and covers the diffeomorphism $f|\Delta$. By shrinking the neighbourhood U of the zero section $Z(\Delta)\subset X\oplus Y$ we can make the map $L^{-1}\circ f|U$ as near to id as we like. Therefore there exists a vector field $v : U\to TU$ of class $C^{k,k+1}$ such that $L^{-1}\circ f = \exp\circ v$ on U. Let $\tilde v : X\oplus Y\to T(X\oplus Y)$ be a vector field of class $C^{k,k+1}$ which agrees with v on U and vanishes out of some larger neighbourhood U_0. Define $\tilde f = L\circ\exp\circ\tilde v$. Clearly, $\tilde f|U = f|U$ and $\tilde f = L$ on $(X\oplus Y)\backslash U_0$. Henceforth, we shall omit the tilda over f. Analogously, extend the linear morphism A to E so that $A(b,x,y)=A(b,0,0)$ outside of some neighbourhood U_0. At last, extend Φ to E by the cut–off procedure.

Because $f_3(b,x,0)=0$ and $f_3\in C^{k,k+1}$, there exists a mapping $\delta : X\oplus Y\to L(Y,Y)$ of class C^k such that $f_3(b,x,y)=\delta(b,x,y)y$. Moreover, $\delta = L|Y$ outside of some neighbourhood of the zero section.

That the section $\sigma : X\oplus Y\to E$ is F–invariant can be written in the form

$$A(b,x,y)\sigma(b,x,y)+\Phi(f(b,x,y))=\sigma(f(b,x,y)),$$

or

$$\sigma(b,x,y)=[A(b,x,y)]^{-1}\sigma(f(b,x,y))-[A(b,x,y)]^{-1}\Phi(f(b,x,y)),$$

which is equivalent to

$$\sigma(b,x,y)=A^{-1}(f(b,x,y))\sigma(f(b,x,y))-A^{-1}(f(b,x,y))\Phi(f(b,x,y)).\tag{3}$$

At first, let us prove that equation (3) has a continuous solution. Denote by $\Gamma^0(E)$ the set of all continuous sections of the vector bundle E and by $\Gamma^0_Q(E)$ the set of all $\sigma\in\Gamma^0(E)$ such that

$$|||\sigma|||_0\equiv\sup\left\{\frac{\|\sigma(b,x,y)\|}{\|y\|^Q} : b\in\Delta, x\in X_b, y\in Y_b, \|y\|\neq 0\right\}<\infty.$$

Clearly, $\Gamma_Q^0(E)$ provided with the norm $|||\cdot|||_0$ is a Banach space. Consider the affine operator $F_\# : \Gamma^0(E) \to \Gamma^0(E)$ defined by

$$F_\#\sigma(b,x,y) = A^{-1}(f(b,x,y))\sigma(f(b,x,y)) - A^{-1}(f(b,x,y))\Phi(f(b,x,y)).$$

Let us show that $\Gamma_Q^0(E)$ is invariant under $F_\#$. In fact, let $\sigma \in \Gamma_Q^0(E)$. Note that $\Phi \in \Gamma_Q^0(E)$ according to hypothesis (d). Therefore

$$|||F_\#\sigma|||_0 = \sup\left\{\frac{\|A^{-1}(f(b,x,y))\sigma(f(b,x,y)) - A^{-1}(f(b,x,y))\Phi(f(b,x,y))\|}{\|y\|^Q}\right\}$$

$$\leq \sup\frac{\|A^{-1}(f(b,x,y))\sigma(f(b,x,y))\|}{\|y\|^Q} + \sup\frac{\|A^{-1}(f(b,x,y))\Phi(f(b,x,y))\|}{\|y\|^Q}$$

$$\leq \sup_{b\in\Delta}(\|A^{-1}|E_{f(b,0,0)}\| + \varepsilon)\cdot\sup\frac{\|\Phi(f(b,x,y))\|}{\|f_3(b,x,y)\|^Q}\cdot\sup\frac{\|\delta(b,x,y)\|^Q\|y\|^Q}{\|y\|^Q}$$

$$+\sup_{b\in\Delta}(\|A^{-1}|E_{f(b,0,0)}\| + \varepsilon)\cdot\sup\frac{\|\sigma(f(b,x,y))\|}{\|f_3(b,x,y)\|^Q}\cdot\sup\frac{\|\delta(b,x,y)\|^Q\|y\|^Q}{\|y\|^Q}$$

$$\leq \sup_{b\in\Delta}(\|A^{-1}|E_{f(b,0,0)}\| + \varepsilon)\cdot\sup_{b\in\Delta}(\|Tf|Y_b\| + \varepsilon)^Q\{|||\sigma|||_0 + |||\Phi|||_0\},$$

where ε is a positive number which depends on the neighbourhood U of $Z(\Delta)$ and can be made arbitrarily small by shrinking U. Thus, $F_\#\sigma \in \Gamma_Q^0(E)$.

Next let us show that $F_\# : \Gamma_Q^0(E) \to \Gamma_Q^0(E)$ is a contracting operator. It is easily seen from the above estimates that for any $\sigma_1, \sigma_2 \in \Gamma_Q^0(E)$ we have

$$|||F_\#\sigma_1 - F_\#\sigma_2|||_0 \leq (\sup_{b\in\Delta}\|A^{-1}|E_{f(b,0,0)}\| + \varepsilon)^Q\cdot(\sup_{b\in\Delta}\|Tf|Y_b\| + \varepsilon)^Q|||\sigma_1 - \sigma_2|||_0.$$

Recalling (2) and assuming that $\varepsilon > 0$ is sufficiently small we conclude that $F_\#|\Gamma_Q^0(E)$ is a contraction. Let $\bar\sigma \in \Gamma_Q^0(E)$ be its fixed point. Then $\bar\sigma$ is a continuous F–invariant section of the vector bundle E.

Now let us pass to the proof that $\bar\sigma$ is C^k smooth. Denote $B = X \oplus Y$. Recall that $(E, \pi, X \oplus Y) \equiv (E, \pi, B)$ is a C^k vector bundle. Let us briefly present some notions and facts from [3] which we shall use later on. For $m = 0, 1, \ldots, k$, denote by $P^m(E)$ the set of all m–jets of sections of the bundle (E, π, B) endowed with the natural structure of a C^{k-m} vector bundle over B. There exists a canonical filtration

$$E \xleftarrow{\rho^1} P^1(E) \xleftarrow{\rho^2} \ldots \xleftarrow{\rho^m} P^m(E),$$

where ρ^i are surjective linear morphisms of class C^{k-i} $(i = 1, \ldots, m)$. Let $N^m(E)$ denote the kernel of ρ^m. Then $P^m(E) = P^{m-1}(E) \oplus N^m(E)$ is a C^{k-m} vector bundle over $P^{m-1}(E)$. The kernel $N^m(E)$ is naturally isomorphic to $L_m(TB, E)$. There exists an isomorphism

$$\mu_m : P^m(E) \to E \oplus L(TB, E) \oplus \ldots \oplus L_m(TB, E) \equiv L^m(E),$$

but μ_m is not natural. Every C^k automorphism (F,f) of the bundle (E,π,B) induces a morphism $P^m(F): P^m(E)\to P^m(E)$. By using the isomorphism μ_m, $P^m(F)$ can be written as follows:

$$P^m(F)(\eta_{m-1},\xi^m)=(P^{m-1}(F)(\eta_{m-1}),D_VF(b)\circ\xi^m\circ(Tf^{-1})^m+\psi_m(\eta_{m-1})),$$

where $\eta_m\equiv(\eta_{m-1},\xi^m)=(v,\xi^1,\dots,\xi^{m-1},\xi^m)$, $v\in E$, $\xi^i\in L_i(TB,E)\quad(i=1,\dots,m)$, $\psi_m\in L(L^{m-1}(E),L_m(TB,E))$ and D_V denotes the derivative along the fiber.

Let $\Gamma^0(P^m(E))$ denote the space of all continuous bounded sections of the vector bundle $P^m(E)$. Clearly,

$$\Gamma^0(P^m(E))=\Gamma^0(P^{m-1}(E))\oplus\Gamma^0(L_m(TB,E)).$$

Let $\rho_*^m:\Gamma^0(P^m(E))\to\Gamma^0(P^{m-1}(E))$ denote the natural projection. The automorphism (F,f) induces $F_\#:\Gamma^0(E)\to\Gamma^0(E)$, where

$$F_\#(\sigma)(d)=F(\sigma(f^{-1}(d)))\quad(d\in B),$$

and $P^m_\#(F):\Gamma^0(P^m(E))\to\Gamma^0(P^m(E))$, which can be written as

$$P^m_\#(F)(\lambda,\mu)(d)=\Big(P^{m-1}_\#(F)(\lambda)(d),D_VF(\sigma(f^{-1}(d)))\circ\mu\circ(Tf^{-1}(d))^m+\psi^*_m(\lambda)(d)\Big),\tag{4}$$

where $\lambda\in\Gamma^0(P^{m-1}(E))$, $\mu\in\Gamma^0(L_m(TB,E))$, $\sigma=\rho^1_*\circ\ldots\circ\ \rho^{m-1}_*(\lambda)$, and $\psi^*_m(\lambda)\in\Gamma^0(L_m(TB,E))$ is a section which depends only on λ.

Let us return to the proof that $\bar\sigma$ is C^k smooth. Apply formula (4) to the morphism (F^{-1},f^{-1}), where F is defined by (1). Because $N^m(E)$ can be decomposed as a direct sum of $L_{q,r,s}(T\Delta,X,Y;E)$ for all non–negative integers q,r,s such that $q+r+s=m$, we get

$$\begin{aligned}P^m_\#(F^{-1})(\lambda,\mu)(b,x,y)\ &=(P^{m-1}_\#(F^{-1})(\lambda)(b,x,y),\psi^*_m(\lambda)(b,x,y)\\ &+\sum_{q+r+s=m}A^{-1}(f(b,x,y))\circ\mu_{q,r,s}\circ(Tf|T_b\Delta)^q(Tf|X_b)^r(Tf|Y_b)^s\\ &+\sum_{q+r+s=m}A^{-1}(f(b,x,y))\circ\Phi_{q,r,s}\circ(Tf|T_b\Delta)^q(Tf|X_b)^r(Tf|Y_b)^s),\end{aligned}$$

where $\mu_{q,r,s}$ denotes the projection of μ onto $L_{q,r,s}(T\Delta,X,Y;E)$ and $\Phi_{q,r,s}$ is the same projection of the main part of $j^m(\Phi)$.

Let $\Gamma^0_Q(P^m(E))$ denote the subspace of $\Gamma^0(P^m(E))$ which consists of sections $\xi\in\Gamma^0(P^m(E))$ such that the projection $\xi_{q,r,s}$ of ξ onto $\Gamma^0(L_{q,r,s}(T\Delta,X,Y;E))$ satisfies the condition

$$|||\xi|||_m\ \equiv\ \sup\Big\{\frac{\|\xi_{q,r,s}(b,x,y)\|}{\|y\|^{Q-s}}:\ (b,x,y)\in X\oplus Y,\ \|y\|\neq0,\ q+r+s\le m\Big\}<\infty.$$

Provide $\Gamma^0_Q(P^m(E))$ with the norm $|||\cdot|||_m$.

To prove that $\bar\sigma$ is C^k smooth, we proceed by induction.

Induction hypothesis. For $1\le m\le k$ and all l, $l=0,1,\dots,m-1$, $\bar\sigma$ is a C^l smooth section and, moreover, $j^l(\bar\sigma)$ is the attracting fixed point of the operator $P^l_\#(F^{-1}):\Gamma^0_Q(P^l(E))\to\Gamma^0_Q(P^l(E))$.

Consider the morphism $P^m_\#(F^{-1}) : \Gamma^0_Q(P^m(E)) \to \Gamma^0_Q(P^m(E))$. Let us show that $P^m_\#(F^{-1})$ transforms $\Gamma^0_Q(P^m(E))$ into itself. Let $\eta_{m-1} \in \Gamma^0_Q(P^{m-1}(E))$ and $(\eta_{m-1}, \xi^m) \in \Gamma^0_Q(P^m(E))$. Since $\Phi(b,x,y) = \varphi(b,x,y)y^Q$ where φ is bounded in the C^k norm on $X \oplus Y$, it is not hard to show that $j^l(\Phi) \in \Gamma^0_Q(P^l(E))$ for all $l = 0,1,\ldots,k$ (see [1], section A.8). Hence

$$
\begin{aligned}
&|||P^m_\#(F^{-1})(\eta_{m-1},\xi^m)|||_m \\
&= \sup_{q+r+s\le m} \frac{\|\mathrm{pr}_{q,r,s}\circ P^m_\#(F^{-1})(\eta_{m-1},\xi^m)(b,x,y)\|}{\|y\|^{Q-s}} \\
&\le \sup_{q+r+s\le m} \frac{\|\mathrm{pr}_{q,r,s}\circ \psi^*_m(\eta_{m-1})(b,x,y)\|}{\|y\|^{Q-s}} \\
&+ \sum_{q+r+s=m} \sup \frac{\|A^{-1}(f(b,x,y))\circ \xi_{q,r,s}(f(b,x,y))\circ (Tf|T_b\Lambda)^q (Tf|X_b)^r (Tf|Y_b)^s\|}{\|y\|^{Q-s}} \\
&+ \sum_{q+r+s=m} \sup \frac{\|A^{-1}(f(b,x,y))\circ \Phi_{q,r,s}(f(b,x,y))\circ (Tf|T_b\Lambda)^q (Tf|X_b)^r (Tf|Y_b)^s\|}{\|y\|^{Q-s}} \\
&< \infty.
\end{aligned}
$$

In fact, the first term is bounded because $\eta_{m-1} \in \Gamma^0_Q(P^{m-1}(E))$, the second and the third terms are bounded since $j^m(\Phi) \in \Gamma^0_Q(P^m(E))$, $(\eta_{m-1},\xi^m) \in \Gamma^0_Q(P^m(E))$ and $f_3(b,x,y) = \delta(b,x,y)y$.

Next let us show that $P^m_\#(F^{-1})$ is fiberwise contracting on $\Gamma^0_Q(P^m(E))$ over $\Gamma^0_Q(P^{m-1}(E))$. If (η_{m-1},ξ^m) and $(\eta_{m-1},\bar{\xi}^m)$ belong to $\Gamma^0_Q(P^m(E))$, then

$$
\begin{aligned}
&|||P^m_\#(F^{-1})(\eta_{m-1},\xi^m) - P^m_\#(F^{-1})(\eta_{m-1},\bar{\xi}^m)|||_m \\
&\le \sum_{q+r+s=m} \sup \frac{\|A^{-1}(f(b,x,y))\circ (\xi^m-\bar{\xi}^m)(f(b,x,y))\circ (Tf|T_b\Delta)^q (Tf|X_b)^r (Tf|Y_b)^s\|}{\|y\|^{Q-s}} \\
&\le \sum_{q+r+s=m} \sup_{b\in\Delta} (\|A^{-1}|E_{f(b,0,0)}\|+\varepsilon)(\|Tf|T_b\Delta\|+\varepsilon)^q(\|Tf|X_b\|+\varepsilon)^r \times \\
&\qquad \times (\|Tf|Y_b\|+\varepsilon)^s \cdot \sup \frac{\|(\xi^m-\bar{\xi}^m)(f(b,x,y))\|}{\|f_3(b,x,y)\|^{Q-s}} \cdot \sup \frac{\|\delta(b,x,y)y\|^{Q-s}}{\|y\|^{Q-s}} \\
&\le \sum_{q+r+s=m} \sup_{b\in\Delta} (\|A^{-1}|E_{f(b,0,0)}\|+\varepsilon)(\|Tf|Y_b+\varepsilon)^Q(\|Tf|T_b\Delta\|+\varepsilon)^q \times \\
&\qquad \times (\|Tf|X_b\|+\varepsilon)^r \cdot \sup \frac{\|(\xi^m-\bar{\xi}^m)(f(b,x,y))\|}{\|y\|^{Q-s}} \\
&\qquad \le \sup_{b\in\Delta} (\|A^{-1}|E_{f(b,0,0)}\|+\varepsilon)(\|Tf|Y_b\|+\varepsilon)^Q(\|Tf|X_b\|+\varepsilon)^k \cdot |||\xi^m-\bar{\xi}^m|||_m,
\end{aligned}
$$

because $\Omega(Tf|T\Delta, b) < \omega(Tf|X, b) \le \Omega(Tf|X, b)$. Taking into account (2) and assuming $\varepsilon > 0$ to be small enough, we derive that $P^m_\#(F^{-1})$ is fiberwise contracting on $\Gamma^0_Q(P^m(E))$ over $\Gamma^0_Q(P^{m-1}(E))$. According to the induction hypothesis, $\bar\sigma$ is C^{m-1} and $j^{m-1}(\bar\sigma)$ is the attracting fixed point of $P^{m-1}_\#(F^{-1}) : \Gamma^0_Q(P^{m-1}(E)) \to \Gamma^0_Q(P^{m-1}(E))$. By Theorem A.25 from [1], there exists a section $\eta_m \in \Gamma^0_Q(N^m(E))$ such that $\eta_m \equiv (j^{m-1}(\bar\sigma), \xi^m)$ is the attracting fixed point of $P^m_\#(F^{-1})$.

Let us prove that $\bar\sigma \in C^m$ and $\eta_m = j^m(\bar\sigma)$. To this end, consider the sequence $\{j^m(F_\#^{-n}(\Phi))\}_n = \{[P^m_\#(F^{-1})]^n(j^m(\Phi))\}$ $(n = 1, 2, \ldots)$. Because η_m is the attracting fixed point of $P^m_\#(F^{-1})$, the sequence $\{j^m(F_\#^{-n}(\Phi))\}_{n=1,2,\ldots}$ uniformly converges to η_m and, moreover, $\{F_\#^{-n}(\Phi)\} \to \bar\sigma$ in the C^0 topology. Hence, $\bar\sigma$ is C^m smooth and $j^m(\bar\sigma) = \eta_m$. This means that $j^m(\bar\sigma)$ is the attracting fixed point of the morphism $P^m_\#(F^{-1}) : \Gamma^0_Q(P^m(E)) \to \Gamma^0_Q(P^m(E))$. Because $\bar\sigma$ is the unique F–invariant section of $\Gamma^0_Q(E)$, it is necessarily invariant under the flow $\{F^t\}$. The proof is complete.

Remark 1. The conclusion of Theorem 1 remains true if we replace conditions (d) and (e) by

(d′) $\Phi^t = \Phi^t_1 + \Phi^t_2$ and there exists a number $C > 0$ such that

$$\|j^p(\Phi^t_1)(b,x,y)\| \le C\|y\|^{Q-p}, \ \|j^p(\Phi^t_2)(b,x,y)\| \le C\|x\|^{S-p}$$

for all $p = 0, 1, \ldots, k;$ $(b, x, y) \in U(\Delta),$ $t \in \mathbb{R};$

(e′) $-\omega(A_0, b) + (Q-k)\cdot\Omega(Tf|Y, b) + k\cdot\Omega(Tf|X, b) < 0,$
$\Omega(A_0, b) - (S-k)\cdot\omega(Tf|X, b) - k\cdot\omega(Tf|Y, b) < 0 \quad (b \in \Delta).$

The proof is left to the reader.

2

Let $\dot z = \xi(z)$ $(z \in \mathbb{R}^d)$ be a C^k smooth vector field. Assume that the origin $z = 0$ is a singular point of ξ. Denote $L = D\xi(0)$. Decompose $\mathbb{R}^d$ into a direct sum of L–invariant linear subspaces W, X and Y so that the spectrum of $L|W$ is pure imaginary and the real parts of the eigenvalues of $L|X$ $(L|Y)$ are positive (negative, respectively). Let $\{f^t\}$ denote the phase flow of ξ. Without loss of generality, we may suppose that W, X and Y are f^t–invariant. Besides, consider the vector field

$$\dot u = A(z)u + \Phi(z), \quad \dot z = \xi(z) \ \ (z \in \mathbb{R}^d, u \in \mathbb{R}^c), \tag{5}$$

where $A : \mathbb{R}^d \to L(\mathbb{R}^c, \mathbb{R}^c)$ and $\Phi : \mathbb{R}^d \to \mathbb{R}^c$ are C^k smooth functions.

Let us introduce some more notation. Given a linear operator $\mathfrak{L}$, denote by $\sigma(\mathfrak{L})$ its spectrum and define

$$\omega(\mathfrak{L}) = \min\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathfrak{L})\}, \quad \Omega(\mathfrak{L}) = \max\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathfrak{L})\}.$$

Put

$$\Theta = \Omega(A(0)), \quad \theta = -\omega(A(0)), \quad \mathrm{M} = \Omega(L|X), \quad \mu = \omega(L|X),$$
$$\Lambda = -\omega(L|Y), \quad \lambda = -\Omega(L|Y).$$

Observe that M, μ, Λ and λ are positive numbers.

Theorem 2. *Assume that $\Phi(b,x,y) = \varphi(b,x,y) \cdot y^Q + \psi(b,x,y) \cdot x^S$, where $b \in W, x \in X, y \in Y, \varphi, \psi \in C^k, \theta - Q\lambda + k\mathrm{M} < 0, \Theta - S\mu + k\Lambda < 0$. Then there exists a C^k smooth function $u = \sigma(z)$ defined in the neighbourhood of $z = 0$ such that its graph is invariant under the vector field* (5).

The proof of this statement is similar to that of Theorem 1 and will be omitted. We only note that there is no need in assuming that the vector field (5) is of class C^{k+1} (as we did in Theorem 1).

Remark 2. The conclusion of Theorem 2 remains true if $\Phi = \Phi_1 + \Phi_2$ and there exists a number $C > 0$ such that

$$\|D^p\Phi_1(b,x,y)\| \leq C\|y\|^{Q-p}, \quad \|D^p\Phi_2(b,x,y)\| \leq C\|x\|^{S-p}$$

for all $p = 0, 1, \dots, k$; $\theta - Q\lambda + k\mathrm{M} < 0$, $\Theta - S\mu + k\Lambda < 0$.

Let us state another variant of Theorem 2.

Theorem 2$'$. *Assume that the fuction $\Phi : \mathbb{R}^d \to \mathbb{R}^c$ is of class C^K and $\|\Phi(b,x,y)\| \leq C\|(x,y)\|^Q$, where*

$$K > \theta/\lambda + \Theta/\mu + k(\mathrm{M}/\lambda + \Lambda/\mu + 1) + 2,$$
$$Q > \theta/\lambda + \Theta/\mu + k(\mathrm{M}/\lambda + \Lambda/\mu) + 2.$$

Then there exists a (local) invariant C^k smooth section of the affine extension determined by (5).

Theorem 2$'$ follows immediately from Theorem 2 by virtue of Taylor's formula.

3

Let k and K be positive integers, $k < K$; $\mathcal{M}$ be a compact smooth manifold, $\xi : \mathcal{M} \to T\mathcal{M}$ be a vector field of class C^k. Let Δ be a C^k smooth submanifold invariant under the flow $(\mathcal{M}, \mathbb{R}, f)$ of ξ. Assume that f satisfies the conditions (a), (b) and (c). Suppose $\eta : \mathcal{M} \to T\mathcal{M}$ is another vector field of class C^k leaving Δ invariant, as well. We shall assume that the vector field ξ and η have contact of order Q, $Q \leq K$, at each point $b \in \Delta$, *i.e.*, $j_b^Q(\xi) = j_b^Q(\eta)$. Let g denote the phase flow of η.

Our aim is to establish conditions ensuring that the flows f and g are C^k conjugate near the submanifold Δ. To this purpose we shall use the *deformation method* [2]. Let us outline this very useful approach to solving conjugacy problems.

Let $\{\xi_\varepsilon\}$ denote the family of vector fields $\xi_\varepsilon = \varepsilon\eta + (1-\varepsilon)\xi$. Clearly, $\xi_0 = \xi$ and $\xi_1 = \eta$. We seek a family $\{h_\varepsilon\}$ of C^k diffeomorphisms $h_\varepsilon : \mathcal{M} \to \mathcal{M}$ such that h_ε transfers ξ_0 to ξ_ε, i.e.

$$(h_\varepsilon)_*(\xi_0) \equiv Th_\varepsilon \circ \xi_0 \circ h_\varepsilon^{-1} = \xi_\varepsilon. \tag{6}$$

In reality, instead of $\{h_\varepsilon\}$, we are looking for a vector field Φ on $\mathcal{M} \times \mathbb{R}$ of the form $\dot{x} = \varphi_\varepsilon(x)$, $\dot{\varepsilon} = 1$, which commutes with the vector field $\Xi : \dot{x} = \xi_\varepsilon(x)$, $\dot{\varepsilon} = 0$. Suppose we have found such a vector field Φ, *i.e.*,

$$[\Phi, \Xi] = 0. \tag{7}$$

Let $F(x,\varepsilon,t)\quad (x\in\mathcal{M},\ \varepsilon\in\mathbb{R},\ t\in\mathbb{R})$ denote the phase flow of Φ. Put $h_\varepsilon=\mathrm{pr}_1\circ F(x,0,\varepsilon)$. It is easy to show that h_ε satisfies (6). Observe that (7) is equivalent to

$$D\varphi_\varepsilon(x)\cdot\xi_\varepsilon(x)=D\cdot\xi_\varepsilon(x)\cdot\varphi_\varepsilon(x)+\eta(x)-\xi(x), \tag{8}$$

which is called the *cohomology equation*. The corresponding characteristic system is

$$\dot y=D\xi_\varepsilon(x)\cdot y+\eta(x)-\xi(x),\quad \dot x=\xi_\varepsilon(x)\quad (x,y\in\mathcal{M}). \tag{9}$$

The solutions of (8) are nothing else than invariant sections $y=\varphi_\varepsilon(x)$ of (9). Summing up, the deformation method reduces the conjugacy problem to finding smooth invariant sections of the affine extension (9). In this way, we obtain the following results.

Theorem 3. *Suppose $\mathcal{M},\Delta,\xi,\eta,k,K$ and Q satisfy the conditions formulated in the first paragraph of this section. Let P and R be positive integers such that*

$$\begin{gathered}R\cdot\Omega(Tf|Y,b)-\omega(Tf|Y,b)+k\cdot\Omega(Tf|X,b)<0,\\ P\cdot\omega(Tf|X,b)+\Omega(Tf|X,b)-k\cdot\omega(Tf|Y,b)<0\end{gathered}$$

for all $b\in\Delta$. If $Q\ge P+R$ and $K\ge Q+k+1$, then the vector fields ξ and η are C^k conjugate in the vicinity of Δ.

Proof. By using Taylor's formula, one easily derives from $j_b^Q(\xi)=j_b^Q(\eta)\quad(b\in\Delta)$ and $K\ge P+R+k$ that $\eta-\xi$ can be expressed in the form

$$\eta(b,x,y)-\xi(b,x,y)=\phi(b,x,y)\cdot y^R+\psi(b,x,y)\cdot x^P\quad ((b,x,y)\in X\oplus Y),$$

where φ and ψ are functions of class C^k. Now apply Theorem 1 to the affine extension (9) and observe that

$$\Omega(A_0,b)\le\Omega(Tf|X,b),\quad -\omega(A_0,b)\le-\omega(Tf|Y,b)\quad (b\in\Delta).$$

The proof is complete.

Now let ξ and η be vector fields on $\mathbb{R}^d=W\oplus X\oplus Y$ of class C^K, $\xi(0)=\eta(0)=0$. Assume that W is their common central manifold of the singular point $z=0$ and $j_z^Q(\xi)=j_z^Q(\eta)\quad(z\in W)$.

Theorem 4. *Let P and R be positive integers such that*

$$\Lambda-R\lambda+k\mathrm{M}<0,\quad \mathrm{M}-P\mu+k\Lambda<0,\quad Q\ge P+R,\quad K\ge P+R+k.$$

Then ξ and η are C^k conjugate with one another near the point $z=0$.

Remark 3. The conditions of Theorem 4 hold whenever

$$\begin{gathered}Q>\Lambda/\lambda+\mathrm{M}/\mu+k(\Lambda/\mu+\mathrm{M}/\lambda)+2,\\ K>\Lambda/\lambda+\mathrm{M}/\mu+k(\Lambda/\mu+\mathrm{M}/\lambda+1)+2.\end{gathered}$$

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