UNIVERSITÉ DE GENÈVE

SCHOLA GENEVENSIS MDLIX

On a Model for Quantum Friction II
Fermi's Golden Rule and Dynamics at Positive Temperature

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UGVA–DPT 1994/12–874

Abstract. We investigate the dynamics of an N-level system linearly coupled to a field of mass-less bosons at positive temperature. Using complex deformation techniques, we develop time-dependent perturbation theory and study spectral properties of the total Hamiltonian. We also calculate the lifetime of resonances to second order in the coupling.

1. Introduction

Let $\mathcal{A}$ be a quantum mechanical N-level system with energy operator H_A on the Hilbert space $\mathcal{H}_A = \mathbf{C}^N$. We denote by $E_1 < E_2 < \cdots < E_M$ the eigenvalues of H_A listed in increasing order. We will colloquially refer to $\mathcal{A}$ as an *atom* or *small system.* Even though we formulate our results for the N-level system $\mathcal{A}$ most of them will, in some sense, extend to situations where $\mathcal{H}_A$ is infinite dimensional and H_A unbounded — see Remark 4 at the end of Section 2 for more details.

Let $\mathcal{B}$ be an infinite heat bath. In this paper $\mathcal{B}$ will be an infinite free Bose gas at inverse temperature $\beta = 1/kT$, without Bose-Einstein condensate. This system is described (see e.g. [BR], [D1], [D2], [LP]) by a triple $\{\mathcal{H}_B, \Omega_B, H_B\}$ where $\mathcal{H}_B$ is a Hilbert space, H_B a self-adjoint operator on $\mathcal{H}_B$, and Ω_B a unit vector in $\mathcal{H}_B$. Let us denote by $\omega(\mathbf{k})$ the energy of a boson with momentum $\mathbf{k} \in \mathbf{R}^3$. Then the equilibrium momentum distribution of bosons at inverse temperature β is given by Planck's law

$$\rho(\mathbf{k}) = \frac{1}{\exp(\beta\omega(\mathbf{k})) - 1}.$$

The space $\mathcal{H}_B$ carries a representation of Weyl's algebra (CCR),

$$W_B(f) = \exp(i\varphi_B(f)), \tag{1.1}$$

where the field operators $\varphi_B(f)$ satisfy, for $(1 + \omega^{-1/2})f \in L^2(\mathbf{R}^3)$, the relation

$$(\Omega_B, W_B(f)\Omega_B) = \exp\left[-\frac{\|f\|^2}{4} - \frac{1}{2}\int_{\mathbf{R}^3} |f(\mathbf{k})|^2 \rho(\mathbf{k}) d^3k\right]. \tag{1.2}$$

The action of H_B is determined by the formula

$$\exp(itH_B)\, W_B(f)\, \exp(-itH_B) = W_B(\exp(it\omega)f). \tag{1.3}$$

We are interested in the physically realistic case of mass-less bosons: $\omega(\mathbf{k}) = |\mathbf{k}|$.

Let us suppose that the systems $\mathcal{A}$ and $\mathcal{B}$, isolated at time $t = 0$, start interacting. One expects the temperature of the small system to change. Since the heat reservoir is an infinite system its temperature will remain constant, and thermal equilibrium is achieved when both systems reach the same temperature $1/\beta$. Roughly speaking, this series of papers is devoted to study this approach to thermal equilibrium.

A representation of CCR satisfying Properties (1.1)–(1.3) is usually constructed using the abstract GNS construction. We prefer to work in an explicit representation due to Araki and Woods [AW]. This representation is central in our approach.

The configuration space of a single boson is $\mathbf{R}^3$ and its energy is $\omega(\mathbf{k}) = |\mathbf{k}|$ (we will always work in the momentum representation). The single particle Hilbert space is $L^2(\mathbf{R}^3)$. Let $\mathcal{H}_b$ be the symmetric Fock space constructed from $L^2(\mathbf{R}^3)$, and denote by Ω_b its vacuum. Let $a_b(\mathbf{k})$ and $a_b^*(\mathbf{k})$ be the usual annihilation and creation operators on $\mathcal{H}_b$ (see [RS2] for definitions, note that $a_b^*(f) = \int d^3k \, a_b^*(\mathbf{k}) f(\mathbf{k})$ is linear in f, while $a_b(f) = [a_b^*(f)]^*$ is anti-linear). Define the energy operator by

$$H_b = \int_{\mathbf{R}^3} d^3k \, \omega(\mathbf{k}) \, a_b^*(\mathbf{k}) a_b(\mathbf{k}),$$

and the field operators by

$$\varphi_b(f) = \frac{1}{\sqrt{2}}(a_b(f) + a_b^*(f)).$$

In the Araki-Woods representation the triple $\{\mathcal{H}_B, \Omega_B, H_B\}$ is given by

$$\mathcal{H}_B = \mathcal{H}_b \otimes \mathcal{H}_b, \qquad \Omega_B = \Omega_b \otimes \Omega_b, \qquad H_B = H_b \otimes I - I \otimes H_b.$$

The annihilation and creation operators are

$$a_B(f) = a_b((1+\rho)^{1/2} f) \otimes I + I \otimes a_b^*(\rho^{1/2} \overline{f}),$$
$$a_B^*(f) = a_b^*((1+\rho)^{1/2} f) \otimes I + I \otimes a_b(\rho^{1/2} \overline{f}),$$

and the field operators are given by

$$\varphi_B(f) = \frac{1}{\sqrt{2}}(a_B(f) + a_B^*(f)).$$

Notation. We write A instead of $A \otimes I$ or $I \otimes A$, whenever the meaning is clear within the context.

When the thermal bath is at zero-temperature, the following formalism is used to describe the system $\mathcal{A} + \mathcal{B}$: The Hilbert space of the system is $\mathcal{H}_A \otimes \mathcal{H}_b$ and its Hamiltonian is given by

$$\widetilde{H}_\lambda = H_A \otimes I + I \otimes H_b + \lambda Q \otimes \varphi_b(\alpha) = H_A + H_b + \lambda \widetilde{H}_I. \tag{1.4}$$

There Q is a self-adjoint operator on $\mathcal{H}_A$, $\alpha \in L^2(\mathbf{R}^3)$ and $\lambda \in \mathbf{R}$. In the sequel we will refer to α as the *form factor* and λ as the *friction constant*. If $\omega^{-1/2}\alpha \in L^2(\mathbf{R}^3)$ then $\widetilde{H}_I$ is infinitesimally small with respect to $\widetilde{H}_0$ and the operator $\widetilde{H}_\lambda$ is essentially self-adjoint on $\mathcal{H}_A \otimes D(H_b)$. The particular choice of the interaction Hamiltonian $\widetilde{H}_I$ is motivated by the dipole approximation in non-relativistic QED. The extensively studied spin-boson Hamiltonian also has the form (1.4).

When the heat bath is at positive temperature, the Hilbert space of the joint system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and the generator of the dynamics is formally given by

$$H_\lambda = H_A \otimes I + I \otimes H_B + \lambda Q \otimes \varphi_B(\alpha) = H_A + H_B + \lambda H_I, \tag{1.5}$$

see [D1], [D2], [PU], [H] and [BR]. In Section 3 we will prove that, if $\omega^{-1}\alpha$ and $\omega\alpha$ both belong to $L^2(\mathbf{R}^3)$, the operator H_λ is essentially self-adjoint on $\mathcal{H}_A \otimes D(H_b) \otimes D(H_b)$. However, H_I is not a relatively bounded perturbation of H_0. Note that, at zero-temperature ($\beta = \infty$), the operator H_λ decouples and acts trivially on the second Fock space. One then recovers an effective Hamiltonian on $\mathcal{H}_A \otimes \mathcal{H}_b$ which has the form (1.4). Thus the zero-temperature model can be realized as a (strong resolvent) limit of positive temperature models, as expected.

The goal of this paper is to develop time-dependent perturbation theory for the model (1.5). In the remaining part of this section we briefly outline the physical content of the theory. It will be further discussed in the third and fourth paper in the series.

Time-dependent perturbation theory was developed by Dirac in 1920's [DI], and further refined by Weisskopf and Wigner in [W]. For the other developments we refer the reader to [HEI] and [SC]. Dirac used the theory to study emission and absorption of light by matter, and to derive Einstein's A–B law from the first principles of quantum mechanics. Weisskopf and Wigner gave an improved solution of the equations of perturbation theory, computed atomic radiative lifetimes, and showed how the theory accounts for the observed width of the spectral lines.

The Hamiltonian H_0 has the following spectrum:

$$\begin{aligned} \sigma_{ac}(H_0) &= \mathbf{R}, \\ \sigma_{sc}(H_0) &= \emptyset, \\ \sigma_{pp}(H_0) &= \{E_1, \cdots, E_M\}. \end{aligned} \tag{1.6}$$

To simplify the discussion, suppose that the spectrum of H_A is simple, and denote by $\psi_1, \cdots, \psi_N$ its eigenvectors. Clearly $\Psi_j = \psi_j \otimes \Omega_B$ is the eigenfunction of H_0 corresponding to the eigenvalue E_j, and

$$b_j(t) = |(\Psi_j, \exp(-itH_\lambda)\Psi_j)|^2, \tag{1.7}$$

is the survival probability of the state Ψ_j. The usual "textbook" derivation of radiative lifetimes starts with the relation

$$b_j(t) = \exp(-\Gamma_j(\lambda)t). \tag{1.8}$$

The inverse radiative lifetime $\Gamma_j(\lambda)$ of the state Ψ_j is related to the width of the spectral lines by the uncertainty relation for time and energy. Formal perturbation theory yields

$$\Gamma_j(\lambda) = \lambda^2\Gamma_j + O(\lambda^3), \tag{1.9}$$

where the coefficient Γ_j is given by the expression

$$\Gamma_j = \sum_{\substack{k=1 \\ k \neq j}}^{N} \Gamma_{jk}, \tag{1.10}$$

with

$$\Gamma_{jk} = \pi \left| \left(\psi_k, Q\psi_j \right) \right|^2 g_\beta(E_j - E_k). \tag{1.11}$$

Here the weight g_β is given, in term of the form factor α, by the following formula

$$g_\beta(s) = \frac{s^2}{|1 - \exp(-\beta s)|} \int_{S^2} |\alpha(|s|\hat{k})|^2 \, d\sigma(\hat{k}), \tag{1.12}$$

where the integral is over the unit sphere S^2 in $\mathbf{R}^3$.

Second order perturbation theory accounts only for processes in which a single quanta of radiation is either emitted or absorbed. It follows from Dirac's theory that if $E_k < E_j$ then $\lambda^2\Gamma_{jk}$ is the probability per unit time that an atom will emit a photon of frequency $\nu = (E_j - E_k)/2\pi$, and make a transition $j \to k$. If $E_k > E_j$ then $\lambda^2\Gamma_{jk}$ is the probability per unit time that an atom will absorb a photon of frequency $\nu = (E_k - E_j)/2\pi$, and make a transition $k \to j$. For historical reasons (see e.g. [H], page 52) the Γ_j are often referred to as *Fermi's Golden Rule*. Note that, at zero-temperature, $\Gamma_{jk} = 0$ if $E_j < E_k$. The coefficient $\lambda^2\Gamma_j$ is the total transition probability per unit time from the level j. Let now p_j be the probability that the small system is in the pure state $|\psi_j\rangle\langle\psi_j|$. If the system $\mathcal{A} + \mathcal{B}$ is in thermal equilibrium, detailed balance requires:

$$p_j\Gamma_j = \sum_{k \neq j} p_k\Gamma_{kj},$$

to hold for all j. A solution of the above system is

$$p_j = \frac{\exp(-\beta E_j)}{\sum_k \exp(-\beta E_k)}.$$

Moreover this solution is unique, provided all Γ_{jk} are positive [D2]. Therefore, an atom in thermal equilibrium with the blackbody radiation is in its Gibbs state, as expected.

Time-dependent perturbation theory, as used in the above formal argument, resisted a general mathematical formulation for over forty years. Among the partly successful work on the subject, the most notable involve the master equation techniques [D1], [D2], [D3], [HA] and [PR]. This method has been discussed in [JP] and will be further discussed in the latter papers in this series. Concerning the "usual" derivation of (1.8)–(1.12), note that Relation (1.8) cannot hold at zero-temperature for all times since the spectrum of $\widetilde{H}_\lambda$ is bounded from below. Even at positive temperature it can hold only as an approximation and, to quote [SI], "it is often discussed fact in the physics literature that the usual "textbook derivation" of the time-dependent series is internally inconsistent and there is not universal agreement among physicists concerning either the higher order terms in the series or the precise quantity which is being approximated".

The foundations of time-dependent perturbation theory for N-body, non-relativistic quantum systems, as well as the precise mathematical definition of resonance, were given in [SI]. We refer the reader to [SI] and [RS3] for a list of references concerning earlier work on the subject. The notions introduced in [SI] have a natural extension to non-relativistic QED. The time-dependent perturbation series is supposed to describe the fate of the eigenvalues of H_0 (which are embedded in the continuous spectrum), after the perturbation H_I is "turned on". It is expected that these eigenvalues will "dissolve": There are $\varepsilon > 0$ and $\eta > 0$ such that, for $0 < |\lambda| < \varepsilon$, the operator H_λ has no eigenvalues in $]E_j - \eta, E_j + \eta[$. Let γ be a contour enclosing the spectrum of H_λ. The formula

$$(\Psi, \exp(-itH_\lambda)\Psi) = \oint_\gamma \exp(-itz) \left(\Psi, (z - H_\lambda)^{-1}\Psi\right) \frac{dz}{2\pi i}, \qquad (1.13)$$

relates the radiative lifetime of the state Ψ to the poles of the function

$$R_\Psi(z) = \left(\Psi, (z - H_\lambda)^{-1}\Psi\right). \qquad (1.14)$$

Following [SI], we now formulate the strategy for the analysis of the spectrum in the interval $]E_j - \eta, E_j + \eta[$, and the rigorous derivation of Relation (1.8): If the form factor α is sufficiently regular, there exists a dense subspace, $\mathcal{E} \subset \mathcal{H}$, on which the matrix elements $R_\Psi(z)$ have a meromorphic continuation from the upper half-plane onto the region $\mathcal{O} = \{z : |z - E_j| < \eta\}$. In $\mathcal{O}$ the functions R_Ψ are regular, except for a simple pole at a point $E_j(\lambda)$, independent of the choice of $\Psi \in \mathcal{E}$. If $\Gamma_j(\lambda) = -2\mathrm{Im}(E_j(\lambda)) > 0$, then H_λ has purely absolutely continuous spectrum on $]E_j - \eta, E_j + \eta[$. The resonance $E_j(\lambda)$ is expected to be an analytic function of λ for $|\lambda| < \varepsilon$. Finally, the first non-trivial coefficient in the Taylor expansion of $E_j(\lambda)$ should have an imaginary part given by Equations (1.9)–(1.12). One then can attempt to derive a formula for the decay of $b_j(t)$ using Relation (1.13). In first approximation one should get Equation (1.8).

For the zero-temperature model with massive bosons, this program was carried in part in [JP] and [OY]. However, the physically important case of mass-less bosons was beyond reach, except in some special cases [A1], [A2]. The difficulty, usually called *infrared catastrophe*, is related to the fact that there are vectors Ψ in the domain of $\widetilde{H}_\lambda$ which contain infinitely many soft photons, i.e., $(\Psi, N\Psi) = \infty$. For many years no method could be designed to avoid this difficulty. Recently, V. Bach, J. Fröhlich and I.M. Sigal [BFS] have developed a sophisticated renormalization algorithm to address this problem. We refer reader to [HS] for an exposition of their results.

In this paper, the program presented above is carried out for the positive temperature model defined by Equation (1.5).

Finally, we note that formal scattering theory relates (1.8)–(1.12) to experimental results [M]. It is therefore important to develop a scattering theory for the model (1.5). The method exposed here yields some partial understanding of the scattering processes: We plan to do a perturbative analysis of the resonance scattering and to calculate the energy distribution of photons emitted and absorbed in transitions. This will be the subject of the fourth paper in the series [JP2]. The investigation of the long time behavior of the interacting system $\mathcal{A}+\mathcal{B}$, and in particular the study of the stability of the equilibrium states, is based on the fusion of algebraic and spectral methods. This will be the content of a third paper[JP1].

Acknowledgments. We are grateful to I. M. Sigal for suggesting the problem to us, and to V. Bach, R. Seiler, I. M. Sigal and H. Spohn for useful discussions. Part of this work has been done while the first author was post-doctoral fellow at University of Toronto and visitor at University of Kentucky, Université de Paris VII and Technische Universität Berlin. At various stages of this work, C.- A. P. was visiting the University of Toronto, and the Institute for Mathematics and its Applications at the University of Minnesota. V. J. is grateful to V. Ivrii and J. Edward for much friendly support, and to P. Hislop, S. de Bièvre, J. P. Gazeau, R. Seiler and V. Bach for their hospitality. The research of the first author was supported in part by NSERC under grant OGP 0007901 and by Deutsche Forschungsgemeinschaft SFB 288 Differentielgeometrie und Quantenphysik. The second author was supported by the Fond National Suisse under grant 21–30607.

2. Statement of Results

We will need the following condition on the form factor α.

(H1) $(\omega + \omega^{-1})\alpha \in L^2(\mathbf{R}^3)$.

We begin with a self-adjointness statement for the generator of the dynamics (1.5).

Proposition 2.1. *If Hypothesis (H1) is satisfied, then H_λ is essentially self-adjoint on $\mathcal{H}_A \otimes D(H_b) \otimes D(H_b)$ for any $\lambda \in \mathbf{R}$.*

To state our results, we need some additional notation. If $\mathfrak{H}$ is a Hilbert space, we denote by $H^2(\delta, \mathfrak{H})$ the Hardy class of $\mathfrak{H}$-valued functions on the strip

$$\mathfrak{S}(\delta) \equiv \{z \,:\, |\mathrm{Im}(z)| < \delta\}.$$

The Hilbert space $H^2(\delta, \mathfrak{H})$ consists of all functions, $f\colon \mathfrak{S}(\delta) \to \mathfrak{H}$, which are analytic in $\mathfrak{S}(\delta)$ and satisfy

$$\|f\|^2_{H^2(\delta,\mathfrak{H})} \equiv \sup_{|a|<\delta} \int_{-\infty}^{\infty} \|f(x + ia)\|^2_{\mathfrak{H}}\, dx < \infty. \tag{2.1}$$

Given a function f on $\mathbf{R}^3$, we define a new function $\widetilde{f}$ on $\mathbf{R} \times \mathrm{S}^2$ by the formula

$$\widetilde{f}(s, \hat{k}) \equiv \begin{cases} -|s|^{1/2}\,\overline{f}(|s|\hat{k}) & \text{if } s < 0, \\ s^{1/2} f(s\hat{k}) & \text{if } s \geq 0. \end{cases} \tag{2.2}$$

With this notation, we can now formulate our central technical hypothesis:

(H2) *There exists $\delta > 0$ such that $\widetilde{\alpha} \in H^2(\delta, L^2(\mathrm{S}^2))$*

The hypotheses (H1)–(H2) is satisfied, for example, by the function $\alpha(\mathbf{k}) = \sqrt{|\mathbf{k}|}\exp(-|\mathbf{k}|^2)$. We may assume, without loss of generality, that $\delta < 2\pi/\beta$ (see Section 3 for details).

Here is our main result.

Theorem 2.2. *Suppose that (H1)–(H2) are satisfied. Then there exist a dense subspace $\mathcal{E} \subset \mathcal{H}$ and, for each $\eta \in]0, \delta[$, a constant $\Lambda(\eta) > 0$ such that for $\lambda \in]-\Lambda(\eta), \Lambda(\eta)[$ and $\Phi, \Psi \in \mathcal{E}$, the functions*

$$z \mapsto \left(\Phi, (H_\lambda - z)^{-1}\Psi\right), \tag{2.3}$$

have a meromorphic continuation from the upper half-plane onto the region

$$\mathcal{O} \equiv \{z : \mathrm{Im}(z) > -\eta\}.$$

The poles of the matrix elements (2.3) in $\mathcal{O}$ are independent of Φ and Ψ. They are identical to the eigenvalues of a **quasi-energy** *operator Σ_λ on $\mathcal{H}_A$. This operator is analytic for $|\lambda| < \Lambda(\eta)$, and has a power series representation of the form*

$$\Sigma_\lambda = H_A + \sum_{n=1}^{\infty} \lambda^{2n} \Sigma^{(2n)}.$$

The first non-trivial coefficient in this expansion satisfies

$$P_j \mathrm{Im}(\Sigma^{(2)}) P_j = P_j Q\, g_\beta(H_A - E_j)\, Q P_j, \tag{2.4}$$

where P_j is the orthogonal projection on the eigenspace of H_A corresponding to the eigenvalue E_j, and g_β is given in (1.12).

Remark 1. For any $\psi \in \mathcal{H}_A$, one has $\psi \otimes \Omega_B \in \mathcal{E}$.

Remark 2. Formula (2.4) is an obvious generalization of Equations (1.9)–(1.11) to degenerate eigenvalues. By first order perturbation theory, the eigenvalues of the operator $-2P_j \mathrm{Im}(\Sigma^{(2)}) P_j$ yield the coefficients of λ^2 in the expansion of the inverse *eigenlifetimes* of the eigenstates of energy E_j. In particular, if E_j non-degenerate, one easily gets the following corollary.

Corollary 2.3. *Suppose that (H1)–(H2) are satisfied, and let E_j be a simple eigenvalue of H_A. Then, for small λ, the quasi-energy operator Σ_λ has a unique simple eigenvalue $E_j(\lambda)$ near E_j. This eigenvalue is analytic and satisfies*

$$E_j(\lambda) = E_j + \lambda^2 a_j^{(2)} + O(\lambda^4),$$
$$\mathrm{Im}(a_j^{(2)}) = -\Gamma_j/2,$$

where Γ_j is given by Equations (1.9)–(1.12).

Theorem 2.2 and Proposition 4.1 in [CFKS] immediately yield the following

Corollary 2.4. *Suppose that (H1)–(H2) are satisfied, and that the operators $P_j \mathrm{Im}(\Sigma^{(2)}) P_j$ are non-singular for $1 \le j \le M$. Then there exists a constant $\Lambda > 0$ such that, for $\lambda \in]-\Lambda, \Lambda[$ and $\lambda \neq 0$, the operator H_λ has purely absolutely continuous spectrum filling the real axis.*

We now turn to the dynamical aspects of the system.

Theorem 2.5. *Suppose that (H1)–(H2) are satisfied. Then there exist a dense subspace $\mathcal{E} \subset \mathcal{H}$ and, for each $\eta \in]0, \delta[$, a constant $\Lambda(\eta) > 0$ with the following property: For*

$|\lambda| < \Lambda(\eta)$ there are two maps $W_\lambda^{\pm}: \mathcal{E} \to \mathcal{H}_A$ such that, for any $\Phi, \Psi \in \mathcal{E}$, one has $(W_\lambda^- \Phi, W_\lambda^+ \Psi) = (\Phi, \Psi)$ and

$$\big(\Phi, \exp(-itH_\lambda)\Psi\big) = \big(W_\lambda^- \Phi, \exp(-i\Sigma_\lambda t) W_\lambda^+ \Psi\big) + O(\exp(-\eta t)),$$

as $t \to +\infty$.

Finally let the survival probabilities $b_j(t)$ be given by Equation (1.7).

Corollary 2.6. *Assume that the hypotheses of Corollary 2.3 and Corollary 2.4 are satisfied, and set $\Gamma_j(\lambda) = -2\mathrm{Im}(E_j(\lambda))$. Then there exist positive constants Λ, a and C such that, for $|\lambda| < \Lambda$,*

$$\big|b_j(t) - \exp(-\Gamma_j(\lambda)t)\big| \le C\,\lambda^2 \exp(-a\lambda^2 t),$$

holds for $t > 0$.

Remark 1. It follows from our arguments that the constant $\Lambda(\eta)$ in Theorem 2.2 behaves like $\beta^{-3/2}$ as $\beta \uparrow \infty$. This forbids the use a limiting argument to analyze the zero-temperature case.

Remark 2. Hypotheses (H1)–(H2) covers physically important examples in which $\alpha(\mathbf{k}) \sim \sqrt{|\mathbf{k}|}$ for small $\mathbf{k}$. From the discussion in Section 3 one can deduce variants of (H2). For example: If the measurable function $h: \mathbf{R} \to \mathbf{C}$ satisfies $|h(s)| = 1$, and if $h(s)\widetilde{\alpha}(s, \hat{k}) \in H^2(\delta, L^2(\mathrm{S}^2))$, then all results hold. The configuration space of the bosons can be any $\mathbf{R}^d$, and the fact that $\omega(\mathbf{k}) = |\mathbf{k}|$ is of no particular importance. Let $\omega(\mathbf{k}) = g(|\mathbf{k}|)$ be a rotationally invariant function. Assume that $g(0) = 0$ and that $g(s)$ is a strictly increasing, unbounded, differentiable function on $\mathbf{R}^+$. Denote by h its inverse. If the form factor α is real-valued, and if

$$\alpha^\sharp(s, \hat{k}) \equiv \frac{s}{|s|^{3/2}} h'(|s|)\alpha(h(|s|)\hat{k}),$$

belongs to $H^2(\delta, L^2(\mathrm{S}^2))$ for some $\delta > 0$, then all results hold.

Remark 3. All the results hold if the system $\mathcal{B}$ is an infinite free Fermi gas.

Remark 4. Our results have simple extensions to the case of infinite dimensional $\mathcal{H}_A$. In fact, if we assume that

(i) H_A is positive.
(ii) Q is bounded.
(iii) $|\mathrm{Im}\,(H_A\psi, Q\psi)| \le C(\psi, (H_A+1)^{1/2}\psi)$ for some constant C and all $\psi \in D(H_A)$.

Then Proposition 2.1 holds with $\mathcal{H}_A$ replaced by $D(H_A)$. Theorem 2.2 and Theorem 2.5 also hold in this case, except that Σ_λ is now an analytic family of type A, and may have non-discrete spectrum. This means that the matrix elements of the resolvent of H_λ may have essential singularities in $\mathcal{O}$. However, for any bounded region $\mathcal{R}$, there exists a constant $\Lambda(\eta, \mathcal{R})$ such that Σ_λ has purely discrete spectrum in $\mathcal{R} \cap \mathcal{O}$. In particular Corollary 2.3 holds too. Corollary 2.4 also holds locally, i.e., H_λ has purely absolutely continuous spectrum in $\mathbf{R} \cap \mathcal{R}$ for $|\lambda| < \Lambda(\mathcal{R})$. But we can assert that H_λ has no singular continuous spectrum for small λ. Finally if we make the following assumptions on the spectrum of H_A,

(iv) The eigenvalues of H_A have bounded multiplicity.

(v) $d_0 \equiv \liminf_{j \to \infty} (E_{j+1} - E_j) > 0$.

Then one can choose the constant $\Lambda(\eta)$ in Theorem 2.2 in such a way that Σ_λ has purely discrete spectrum. The reader will find a few remarks scattered in the remaining parts of this work to support these claims.

3. Preliminaries

The primary purpose of this section is the construction of a new representation of the bath Hilbert space. As a first application of this representation, we will then prove Proposition 2.1.

We denote by $\mathfrak{F}(\mathfrak{H})$ the symmetric Fock space constructed on the Hilbert space $\mathfrak{H}$. For the proof of the following well-known theorem we refer the reader to [BSZ].

Theorem 3.1. *For any two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, there exists a unitary mapping*

$$U: \mathfrak{F}(\mathfrak{H}_1) \otimes \mathfrak{F}(\mathfrak{H}_2) \to \mathfrak{F}(\mathfrak{H}_1 \oplus \mathfrak{H}_2),$$

so that, for any two unitaries U_1, U_2, and any two vectors f, g, one has

$$U\Big(\Gamma(U_1) \otimes \Gamma(U_2)\Big)U^{-1} = \Gamma(U_1 \oplus U_2),$$

$$U\Big(\exp(i\varphi(f)) \otimes \exp(i\varphi(g))\Big)U^{-1} = \exp(i\varphi(f \oplus g)).$$

Furthermore, if Ω is the vacuum on $\mathfrak{F}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$, and Ω_1, Ω_2 are the vacua on $\mathfrak{F}(\mathfrak{H}_1)$, $\mathfrak{F}(\mathfrak{H}_2)$, then

$$U(\Omega_1 \otimes \Omega_2) = \Omega.$$

It follows from this theorem that a unitary transformation

$$U: \mathcal{H}_B \to \mathfrak{F}\left(L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3)\right), \tag{3.1}$$

exists, so that

$$
\begin{aligned}
U \exp(itH_B)U^{-1} &= \Gamma\left(\exp(it\omega) \oplus \exp(-it\omega)\right), \\
UW_B(f)U^{-1} &= \exp\left(i\varphi\left((1+\rho)^{1/2} f \oplus \rho^{1/2}\bar{f}\right)\right).
\end{aligned} \tag{3.2}
$$

We now define a unitary map

$$
V\colon L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3) \to L^2(\mathbf{R} \times \mathrm{S}^2, ds \otimes d\sigma),
$$

by the formula

$$
\Big(V(f \oplus g)\Big)(s, \hat{k}) \equiv \begin{cases} s\, g(|s|\hat{k}) & \text{if } s < 0, \\ s\, f(s\hat{k}) & \text{if } s \geq 0. \end{cases} \tag{3.3}
$$

It is easy to show that

$$
\begin{aligned}
V\Big(\exp(it\omega) \oplus \exp(-it\omega)\Big)V^{-1} &= \exp(its), \\
V\Big((1+\rho)^{1/2} f \oplus \rho^{1/2}\bar{f}\Big) &= f_\beta,
\end{aligned} \tag{3.4}
$$

where

$$
f_\beta(s, \hat{k}) \equiv \left(\frac{s}{1 - \exp(-\beta s)}\right)^{1/2} \widetilde{f}(s, \hat{k}), \tag{3.5}
$$

with $\widetilde{f}$ defined by Equation (2.2).

Remark. If $\widetilde{f} \in H^2(\delta, L^2(\mathrm{S}^2))$ for some $\delta > 2\pi/\beta$ then $f_\beta \in H^2(2\pi/\beta - \varepsilon, L^2(\mathrm{S}^2))$ for any $0 < \varepsilon < 2\pi/\beta$, but $f_\beta \notin H^2(2\pi/\beta + \varepsilon, L^2(\mathrm{S}^2))$ for any $\varepsilon > 0$. Thus, without loss of generality, we may assume in Hypothesis (H1) that $\delta < 2\pi/\beta$.

Notation. In the sequel we will identify the spaces $L^2(\mathbf{R} \times \mathrm{S}^2)$, $L^2(\mathbf{R}) \otimes L^2(\mathrm{S}^2)$ and $L^2(\mathbf{R}; L^2(\mathrm{S}^2))$, denoting all of them by $\mathcal{H}_s$.

We now come to the central point of the construction. With U and V given by Equations (3.1) and (3.3), we define the unitary map

$$
\widehat{U} = I \otimes \Gamma(V)U : \mathcal{H} \to \widehat{\mathcal{H}} \equiv \mathcal{H}_A \otimes \mathfrak{F}(\mathcal{H}_s).
$$

From Equations (3.2) (3.4), one easily infers that the following relations are satisfied

$$
\begin{aligned}
\widehat{H}_0 &\equiv H_A + \mathrm{d}\Gamma(s) = \widehat{U} H_0 \widehat{U}^{-1}, \\
\widehat{H}_I &\equiv Q \otimes \varphi(\alpha_\beta) = \widehat{U} H_I \widehat{U}^{-1}.
\end{aligned} \tag{3.6}
$$

Here α_β is obtained from the original form factor α by the transformations (2.2) and (3.5). Furthermore, if we denote by Ω the vacuum in $\mathfrak{F}(\mathcal{H}_s)$, then

$$\psi \otimes \Omega = \widehat{U}\left(\psi \otimes \Omega_B\right),$$

holds for any $\psi \in \mathcal{H}_A$. To complete our new picture, we shall now construct a self-adjoint generator for the dynamics of the coupled system. This is the purpose of the following lemma.

Lemma 3.2. *If* $\lambda \in \mathbf{R}$ *and* $(|s| + |s|^{-1/2})\alpha_\beta \in \mathcal{H}_s$, *then the operator*

$$\widehat{H}_\lambda \equiv \widehat{H}_0 + \lambda \widehat{H}_I, \tag{3.7}$$

is essentially self-adjoint on any core of $\mathrm{d}\Gamma(|s|)$.

For the proof, we need the following well-known results [GJ, Proposition 1.2.3],

Proposition 3.3. *Let* $\mu(k)$ *be a positive, measurable function on some measure space* M. *Denote by* F *the subspace of finite particle vectors of the Fock space* $\mathfrak{F}\left(L^2(M)\right)$, *and by* $N \equiv \mathrm{d}\Gamma(1)$ *the number operator.*

(i) *Assume* $f \in L^2(M)$, *then for any* $\Psi \in F$,

$$\|a^\#(f)\Psi\| \le \|f\|\,\|(N + I)^{1/2}\Psi\|,$$

where $a^\#(f)$ *represents either* $a(f)$ *or* $a^*(f)$.

(ii) *Assume* $(1 + \mu^{-1/2})f \in L^2(M)$, *then for any* $\Psi \in F$,

$$\|a^\#(f)\Psi\| \le \|(1 + \mu^{-1/2})f\|\,\|(\mathrm{d}\Gamma(\mu) + I)^{1/2}\Psi\|.$$

In particular the field operator $\varphi(f)$ *is infinitesimally small with respect to* $\mathrm{d}\Gamma(\mu)$.

Proof of Lemma 3.2. We invoke Nelson's commutator theorem (in the form of Theorem X.37 in [RS2]). Let $\widehat{N} = I + \mathrm{d}\Gamma(|s|)$. We must show that there is a constant $d > 0$, such that the following estimates hold for any $\Psi \in D(\mathrm{d}\Gamma(|s|))$:

$$\begin{gathered} \|\widehat{H}_\lambda \Psi\| \le d\,\|\widehat{N}\Psi\|, \\ |(\widehat{H}_\lambda \Psi, \widehat{N}\Psi) - (\widehat{N}\Psi, \widehat{H}_\lambda \Psi)| \le d\,\|\widehat{N}^{1/2}\Psi\|^2. \end{gathered} \tag{3.8}$$

Since $i[\widehat{N}, \varphi(\alpha_\beta)] = \varphi(i|s|\alpha_\beta)$, Inequalities (3.8) follow from Proposition 3.3, and the obvious fact that $\mathrm{d}\Gamma(s)$ is bounded with respect to $\mathrm{d}\Gamma(|s|)$. ∎

Proof of Proposition 2.1. We start by observing that Hypothesis (H1) implies that $(|s| + |s|^{-1/2})\alpha_\beta \in \mathcal{H}_s$. Therefore the conclusion of Lemma 3.2 holds. Let us define an auxiliary self-adjoint operator M on $\mathcal{H}$ by the formula

$$\exp(iMt) \equiv \Gamma(\exp(i\omega t)) \otimes \Gamma(\exp(i\omega t)).$$

Using the fundamental property of U (Theorem 3.1) and Definition (3.3) of V, one shows that

$$M = \widehat{U}^{-1}\mathrm{d}\Gamma(|s|)\widehat{U}.$$

It follows from Equation (3.6) and Lemma 3.2 that $H_\lambda = \widehat{U}^{-1}\widehat{H}_\lambda\widehat{U}$ is essentially self-adjoint on any core of M. The fact that $\mathcal{H}_A \otimes D(H_b) \otimes D(H_b)$ is such a core is well known (see [RS1], Theorem VIII.33). ■

Remark. Setting $\widehat{N} = I + H_A + \mathrm{d}\Gamma(|s|)$, the proofs of Lemma 3.2 and Proposition 2.1 extend to the situation where H_A is unbounded, provided one makes the following assumptions:

(i) $H_A \geq 0$.

(ii) Q is bounded with respect to $H_A^{1/2}$.

(iii) $|\mathrm{Im}\,(H_A\psi, Q\psi)| \leq C(\psi, (H_A+1)^{1/2}\psi)$ for some constant C and all $\psi \in D(H_A)$.

Of course one also has to replace $\mathcal{H}_A$ with $D(H_A)$ in Proposition 2.1, and $\mathrm{d}\Gamma(|s|)$ with $H_A + \mathrm{d}\Gamma(|s|)$ in Lemma 3.2.

Let us summarize the results of this section in

Theorem 3.4. *There exists a unitary mapping $\widehat{U}: \mathcal{H} \to \widehat{\mathcal{H}}$, such that*

$$\widehat{U}H_\lambda\widehat{U}^{-1} = \widehat{H}_\lambda.$$

In the sequel we shall identify $\mathcal{H}$ with $\widehat{\mathcal{H}}$ and H_λ with $\widehat{H}_\lambda$, and always work in the new representation. We would like to add a few comments concerning the above construction.

The simplest and most widely used complex-deformation technique is based on the Aguilar-Combes theory and the group of dilation operators: [AC], [BC], [RS3] and [SI]. The investigation of the zero-temperature model has been, so far, based on the second-quantization of the dilation group. This approach has been used in [OY], [JP], as well as in a recent work of Bach, Fröhlich and Sigal [BFS]. In the mass-less case the infrared problem reflects itself in the fact that the eigenvalues $\{E_j\}$ are not uncovered by a dilation of the Hamiltonian $\widetilde{H}_0$: Regular perturbation theory does not apply directly. Since $\widetilde{H}_I$ is not a relatively compact perturbation of $\widetilde{H}_0$, it is difficult to analyze the spectrum of $\widetilde{H}_\lambda$ near E_j, and to show that the matrix elements (1.14) have a meromorphic continuation across the continuous spectrum.

The resolution of the problem in the positive temperature case is based on the replacement of dilation analyticity with translation analyticity. The latter one originated in the study of resonances of an atom in a homogeneous electric field (see [AH] and [HE] for an example). The formal connection between the two problems becomes transparent in Equation (3.6). The complex deformation shifts the essential spectrum into the lower half-plane, and uncovers the eigenvalues $\{E_j\}$. However, the domain of the Hamiltonian is modified by the deformation, and the bulk of the technical work below will center around the resolution of this difficulty.

4. Spectral Deformations and Fermi's Golden Rule

Throughout this section we assume that Hypotheses (H1)–(H2) hold. For $a \in \mathbf{R}$, let $u(a)$ be the unitary translation group on $\mathcal{H}_s$,

$$\Big(u(a)f\Big)(s) = f^a(s) \equiv f(s+a). \tag{4.1}$$

Denote by $U(a) = \Gamma(u(a))$ the second quantization of $u(a)$. One easily shows that

$$U(a)\varphi(f)U(-a) = \varphi(f^a),$$
$$U(a)\mathrm{d}\Gamma(s)U(-a) = \mathrm{d}\Gamma(s) + aN,$$

where $N \equiv \mathrm{d}\Gamma(1)$ is the number operator on $\mathfrak{F}(\mathcal{H}_s)$. Thus, under a second quantized translation, the operator H_λ transforms according to

$$H_\lambda(a) \equiv U(a)H_\lambda U(-a) = H_A + \mathrm{d}\Gamma(s) + \lambda Q \otimes \varphi(\alpha_\beta^a) + aN.$$

Remark that if $f \in H^2(\delta, \mathfrak{H})$, Equation (4.1) defines a map from $\mathfrak{S}(\delta)$ to $L^2(\mathbf{R}) \otimes \mathfrak{H}$. The first lemma in this section states some basic properties of such *complex translations.*

Lemma 4.1. *Let $0 < \delta' < \delta$, then the following holds:*

(i) *If f belongs to $H^2(\delta, \mathfrak{H})$ then its derivative f' belongs to $H^2(\delta', \mathfrak{H})$. Furthermore, one has the bound*

$$\|f'\|_{H^2(\delta',\mathfrak{H})} \leq \frac{1}{\delta - \delta'} \|f\|_{H^2(\delta,\mathfrak{H})}.$$

(ii) *If f belongs to $H^2(\delta, \mathfrak{H})$ then the map*

$$\mathfrak{S}(\delta) \to L^2(\mathbf{R}) \otimes \mathfrak{H}$$
$$a \mapsto f^a$$

is analytic, and $\frac{df^a}{da} = f'^a$.

(iii) *If $a, b \in \mathfrak{S}(\delta')$ then, for any $f \in H^2(\delta, \mathfrak{H})$, one has the bound*

$$\|f^a - f^b\|_{L^2(\mathbf{R})\otimes\mathfrak{H}} \le \frac{|a-b|}{\delta-\delta'}\,\|f\|_{H^2(\delta,\mathfrak{H})}.$$

Proof. Unless explicitly mentioned, all norms will refer to the space $L^2(\mathbf{R}) \otimes \mathfrak{H}$. We first prove (i). Since, by definition, $f \in H^2(\delta, \mathfrak{H})$ implies that the function $f\colon \mathfrak{S}(\delta) \to \mathfrak{H}$ is analytic, we only have to prove the bound on the derivative f'. Denote by $\hat{f} \in L^2(\mathbf{R}, dr) \otimes \mathfrak{H}$ the Fourier transform of $f \in L^2(\mathbf{R}, ds) \otimes \mathfrak{H}$, then the norm (2.1) can be expressed as

$$\|f\|_{H^2(\delta,\mathfrak{H})} = \sup_{|a|<\delta} \left\|\exp(ar)\hat{f}\right\|.$$

Therefore we have

$$\begin{aligned}\|f'\|_{H^2(\delta',\mathfrak{H})} &= \sup_{|a|<\delta'} \left\|r\exp(ar)\hat{f}\right\| \\ &\le \sup_{r\in\mathbf{R}} \left|r\exp\left(-(\delta-\delta')|r|\right)\right| \sup_{|a|<\delta'} \left\|\exp\left(ar+(\delta-\delta')|r|\right)\hat{f}\right\|,\end{aligned}$$

and an explicit calculation leads to the desired inequality:

$$\|f'\|_{H^2(\delta',\mathfrak{H})} \le \frac{1}{\mathrm{e}(\delta-\delta')} \sup_{|a|<\delta} \left\|\exp(|ar|)\hat{f}\right\| \le \frac{\sqrt{2}}{\mathrm{e}}\,\frac{1}{\delta-\delta'}\,\|f\|_{H^2(\delta,\mathfrak{H})}.$$

Using the same notation, we now prove (ii). Assume that $|\mathrm{Im}(a)| < \delta' < \delta$. Then, for small $h \in \mathbf{C}$,

$$\begin{aligned}\left\|f^{a+h} - f^a - hf'^a\right\| &= \left\|\exp(iar)\left(\exp(ihr) - 1 - ihr\right)\hat{f}\right\| \\ &\le \sup_{r\in\mathbf{R}} \left|\exp\left(-\mathrm{Im}(a)r - \delta'|r|\right)\left(\exp(ihr)-1-ihr\right)\right| \left\|\exp(\delta'|r|)\hat{f}\right\|,\end{aligned}$$

and another simple calculation gives

$$\left\|f^{a+h} - f^a - hf'^a\right\| \le \mathrm{o}(h)\,\|f\|_{H^2(\delta,\mathfrak{H})}, \tag{4.2}$$

as $h \to 0$, which is the desired estimate. To prove (iii) remark that, as a consequence of (ii), we have

$$f^a - f^b = (b-a)\int_0^1 f'^{a+t(b-a)}\,dt.$$

Therefore (i) gives

$$\left\|f^a - f^b\right\| \le |b-a| \sup_{0\le t\le 1} \left\|f'^{a+t(b-a)}\right\| \le |b-a|\,\|f'\|_{H^2(\delta',\mathfrak{H})} \le \frac{|b-a|}{\delta-\delta'}\|f\|_{H^2(\delta,\mathfrak{H})},$$

as required. ■

Let now $a \in \mathfrak{S}(\delta)$ be complex, and define

$$\begin{aligned} H_I(a) &\equiv Q \otimes \frac{1}{\sqrt{2}}\left(a(\alpha_\beta^{\overline{a}}) + a^*(\alpha_\beta^a)\right), \\ H_\lambda(a) &\equiv H_A + \mathrm{d}\Gamma(s) + \lambda H_I(a) + aN. \end{aligned} \tag{4.3}$$

These operators are well defined on the dense subspace

$$\mathcal{D} = D(N) \cap D(\mathrm{d}\Gamma(s)),$$

One easily checks that, as an operator on $\mathcal{D}$, $H_\lambda(a)$ satisfies the relation

$$H_\lambda(a)^* \supset H_{\bar{\lambda}}(\bar{a}). \tag{4.4}$$

Therefore, $H_\lambda(a)$ is closable for each $(\lambda, a) \in \mathbf{C} \times \mathfrak{S}(\delta)$. We use the same symbol to denote its closure. The following proposition summarizes some simple facts about the family of closed operators $\{H_0(a) : a \in \mathbf{C}\}$.

Proposition 4.2. *Assume that $a \in \mathbf{C}$, then the following holds:*

(i) *For any $\Psi \in \mathcal{D}$ one has*

$$\|H_0(a)\Psi\|^2 = \|H_0(Re(a))\Psi\|^2 + |\mathrm{Im}(a)|^2\|N\Psi\|^2.$$

(ii) *If $\mathrm{Im} a \neq 0$, then $H_0(a)$ is a normal operator satisfying*

$$\begin{aligned} D(H_0(a)) &= \mathcal{D}, \\ H_0(a)^* &= H_0(\bar{a}). \end{aligned}$$

(iii) *The spectrum of $H_0(a)$ is given by*

$$\sigma(H_0(a)) = \{na + t : n = 1, 2, \cdots; t \in \mathbf{R}\} \cup \sigma(H_A).$$

Proof. Remark that, on the sector $N = n$, the operator $H_0(a)$ reduces to the normal operator

$$H_0^{(n)}(a) \equiv H_A + s_1 + \cdots + s_n + na.$$

A simple calculation immediately yields Identity (i). From this identity one easily shows that, if $\mathrm{Im}(a) \neq 0$,

$$\mathcal{D} = \left\{\Psi = \{\Psi^{(n)}\} : \Psi^{(n)} \in D(H_0^{(n)}(a));\ \textstyle\sum_n \|H_0^{(n)}(a)\Psi^{(n)}\|^2 < \infty\right\},$$

and it follows that $H_0(a)$ is a closed normal operator on $\mathcal{D}$. The last assertion in (ii) , and (iii) both follow from corresponding statements about $H_0^{(n)}(a)$. ■

The next result provides us with the necessary control of the interaction $H_I(a)$.

Lemma 4.3. *Let $a \in \mathfrak{S}(\delta)$, and* $\mathrm{Im}(a) \neq 0$. *Then the interaction $H_I(a)$ is infinitesimally small with respect to $H_0(a)$.*

Proof. By Cauchy-Schwarz inequality,

$$\|Q \otimes a^{\#}(f)\Psi\|^2 \leq \|Q^2\Psi\| \, \|a^{\#}(f)^* a^{\#}(f)\Psi\|.$$

Applying a well-known trick we obtain, for any $\varepsilon > 0$,

$$\|Q \otimes a^{\#}(f)\Psi\| \leq \frac{1}{2\varepsilon}\|Q^2\Psi\| + \frac{\varepsilon}{2}\|a^{\#}(f)^* a^{\#}(f)\Psi\|.$$

By Proposition 3.3, we further get

$$\|Q \otimes a^{\#}(f)\Psi\| \leq \frac{1}{2\varepsilon}\|Q^2\Psi\| + \frac{\varepsilon}{2}\|f\|^2\|(N+2)\Psi\|.$$

Finally, since $\mathrm{Im}(a) \neq 0$, the first statement of Proposition 4.2 and Equation (4.3) lead to

$$\|H_I(a)\Psi\| \leq \varepsilon\|H_0(a)\Psi\| + C_\varepsilon\|\Psi\|,$$

for appropriate $C_\varepsilon > 0$. ■

Let us introduce the strips

$$\mathfrak{S}^{\pm}(\delta) = \{z : 0 < \pm\mathrm{Im}(z) < \delta\}.$$

We are now ready to prove some basic properties of the deformed operator $H_\lambda(a)$.

Proposition 4.4. *Assume that $(\lambda, a) \in \mathbf{C} \times \mathfrak{S}^-(\delta)$, then:*

(i) *The following identities hold,*

$$D(H_\lambda(a)) = \mathcal{D},$$
$$H_\lambda(a)^* = H_{\overline{\lambda}}(\overline{a}).$$

(ii) *The spectrum of $H_\lambda(a)$ satisfies*

$$\sigma(H_\lambda(a)) \subset \{z : \mathrm{Im}(z) \leq D(\lambda, a)\},$$

where $D(\lambda, a)$ is given by

$$D(\lambda, a) \equiv \frac{1}{2}\left(\frac{|\mathrm{Re}(\lambda)|}{\delta - |\mathrm{Im}(a)|}\,|\mathrm{Im}(a)|^{1/2} + |\mathrm{Im}(\lambda)|\,|\mathrm{Im}(a)|^{-1/2}\right)^2 \|Q\|^2\,\|\alpha_\beta\|^2_{H^2(\delta)}.$$

Furthermore, if $\mathrm{Im}(z) > D(\lambda, a)$*, one has the bound*

$$\left\|\left(H_\lambda(a) - z\right)^{-1}\right\| \le \frac{1}{\mathrm{Im}(z) - D(\lambda, a)}.$$

(iii) *The map*

$$(\lambda, a) \mapsto H_\lambda(a)$$

from $\mathbf{C} \times \mathfrak{S}^-(\delta)$ *to the closed operators on* $\mathcal{H}$*, is an analytic family of type A in each variable separately.*

Remark. A similar statement holds for $(\lambda, a) \in \mathbf{C} \times \mathfrak{S}^+(\delta)$.

To prove Proposition 4.4, we need the following simple facts (see e.g., [K] Chapter V, Section 3.2): If T is a closed operator on a Hilbert space $\mathfrak{H}$, the convex set

$$\Theta(T) = \{(\phi, T\phi) : \phi \in D(T),\ \|\phi\| = 1\},$$

is called *numerical range* of T. Let us denote by $\mathfrak{N}(T)$ the closure of this set.

Lemma 4.5. *Let T be a closed operator on a Hilbert space $\mathfrak{H}$, such that $D(T) = D(T^*)$. Then*

$$\sigma(T) \subset \mathfrak{N}(T),$$

and, for $z \in \mathbf{C} \setminus \mathfrak{N}(T)$, one has the bound

$$\left\|(T - z)^{-1}\right\| \le \frac{1}{\mathrm{dist}\,(z, \mathfrak{N}(T))}. \tag{4.5}$$

Proof. Let $\phi \in D(T)$ be a unit vector, then Cauchy-Schwarz inequality implies

$$\|(T - z)\phi\| \ge |z - (\phi, T\phi)|\,. \tag{4.6}$$

It follows that $T - z$ is one-to-one if $z \notin \mathfrak{N}(T)$. Arguing similarly we see that $(T - z)^*$ is one-to-one if $\overline{z} \notin \mathfrak{N}(T^*)$. Since $\mathfrak{N}(T^*) = \overline{\mathfrak{N}(T)}$, we conclude that $T - z$ has a bounded, everywhere defined inverse for $z \notin \mathfrak{N}(T)$. Estimate (4.5) follows from Inequality (4.6).■

Proof of Proposition 4.4. The first assertion is a simple consequence of Lemma 4.3. To establish the second assertion, we set

$$\widehat{D}(\lambda, a) \equiv \sup \operatorname{Im}\left(\mathfrak{N}(H_\lambda(a))\right). \tag{4.7}$$

The assertion will follow from Lemma 4.5, provided we can show that $\widehat{D}(\lambda, a) \leq D(\lambda, a)$. To this end we first observe that, since real translations are unitary, we can choose $a = -i\mu$ with $0 < \mu < \delta$. Then a simple calculation shows that

$$\operatorname{Im}\left(H_\lambda(a)\right) = -\mu N + \gamma Q \otimes \varphi(g), \tag{4.8}$$

where

$$\gamma g \equiv \frac{1}{2i}\left(\lambda \alpha_\beta^a - \bar{\lambda}\alpha_\beta^{\bar{a}}\right), \qquad \|g\| = 1.$$

Denote by P the orthogonal projection on g, and set $P^\perp = 1 - P$. By construction, we have $N = \mathrm{d}\Gamma(P) \oplus \mathrm{d}\Gamma(P^\perp) = a^*(g)a(g) \oplus N^\perp$, and (4.8) splits into a direct sum

$$\operatorname{Im}\left(H_\lambda(a)\right) = -\mu\left\{\left(a^*(g)a(g) - \frac{\gamma}{\mu}\,Q \otimes \varphi(g)\right) \oplus N^\perp\right\}.$$

In the above formula, we recognize the sum of a (shifted) harmonic oscillator and a number operator. Completing the square in the first term, and performing a unitary transformation, we can rewrite

$$\operatorname{Im}\left(H_\lambda(a)\right) = -\mu\, I \otimes \left(N_0 \oplus N^\perp\right) + \frac{\gamma^2}{2\mu}Q^2 \otimes I,$$

where N_0 is a simple harmonic oscillator. Therefore we have

$$\sigma\left(\operatorname{Im}\left(H_\lambda(a)\right)\right) = \left\{-\mu n + \frac{\gamma^2}{2\mu}q^2 : n = 0, 1, \cdots;\ q \in \sigma(Q)\right\},$$

which, by Definition (4.7), means

$$\widehat{D}(\lambda, a) \leq \frac{\gamma^2}{2\mu}\|Q\|^2.$$

We conclude by estimating γ with the help of Lemma 4.1. To prove the last assertion, we first claim that, for fixed $\lambda \in \mathbf{C}$ and $\Psi \in \mathcal{D}$, the vector valued function $a \mapsto H_\lambda(a)\Psi$ is analytic. In fact, with

$$\frac{\partial H_\lambda(a)}{\partial a} \equiv N + \lambda Q \otimes \frac{1}{\sqrt{2}}\left(a\left({\alpha'_\beta}^{\bar{a}}\right) + a^*\left({\alpha'_\beta}^{a}\right)\right),$$

Proposition 3.3 (i) implies

$$\left\| H_\lambda(a+h)\Psi - H_\lambda(a)\Psi - h\frac{\partial H_\lambda(a)}{\partial a}\Psi \right\| = \mathrm{O}\left(\|\alpha_\beta^{a+h} - \alpha_\beta^a - h{\alpha'}_\beta^a\| \right).$$

By Lemma 4.1 (ii), the right hand side of the last inequality is $\mathrm{o}(h)$, proving the claim. Since Strong analyticity in λ for fixed a is obvious, type A analyticity now follows from the first two assertions of the proposition. ■

Remark. If we replace $\mathcal{D}$ by $D(N) \cap D(H_A + \mathrm{d}\Gamma(s))$, the proof of Proposition 4.2 extends to unbounded H_A. The same remark holds for the proof of Lemma 4.3 and Proposition 4.4 provided Q is bounded.

We now further investigate the spectrum of $H_\lambda(a)$. We will denote by $\mathrm{P}(\eta)$ the open half-plane $\{z : \mathrm{Im}(z) > \eta\}$.

Theorem 4.6. *There exists a constant $\Lambda > 0$ such that, for $(\lambda, a) \in \mathbf{C} \times \mathfrak{S}^-(\delta)$, the following statements hold:*

(i) If

$$|\lambda| < \Lambda\, |\mathrm{Im}(a)|, \tag{4.9}$$

then the spectrum of the operator $H_\lambda(a)$ in the half-plane $\mathrm{P}(\mathrm{Im}(a) + \frac{|\lambda|}{\Lambda})$ is purely discrete and independent of a.

(ii) If

$$|\lambda| < \frac{1}{4}\Lambda\, |\mathrm{Im}(a)|,$$

then the spectral projection $P_\lambda(a)$ associated to the spectrum of $H_\lambda(a)$ in $\mathrm{P}(\mathrm{Im}(a) + \frac{|\lambda|}{\Lambda})$ is analytic in λ and satisfies the bound

$$\|P_\lambda(a) - P_0(a)\| < \frac{3\lambda}{\Lambda |\mathrm{Im}(a)|}.$$

Proof. Remark that, by Proposition 4.2 and Lemma 4.3, the resolvent formula

$$\left(H_\lambda(a) - z\right)^{-1} = \left(H_0(a) - z\right)^{-1} \left(1 + \lambda H_I(a) \left(H_0(a) - z\right)^{-1}\right)^{-1}, \tag{4.10}$$

holds for small λ, as long as z belongs to a *cone* of the form $\{z : 0 < c_1 < |z| < c_2\, \mathrm{Im}(z)\}$. We organize the proof of Theorem 4.6 in two steps: First we will extend the domain of validity of Formula (4.10) by refining our estimate on the product $H_I(a)(H_0(a)-z)^{-1}$. Then we will invoke analytic perturbation theory to control the spectrum.

Applying Proposition 3.3 (i), we get

$$\left\| H_I(a)\left(H_0(a)-z\right)^{-1}\right\| \leq \sqrt{2}\,\|Q\|\,\|\alpha_\beta\|_{H^2(\delta)}\left\|(N+1)^{1/2}\left(H_0(a)-z\right)^{-1}\right\|.$$

Since N and $H_0(a)$ are commuting normal operators, it is rather easy to compute the norm of $T=(N+1)^{1/2}(H_0(a)-z)^{-1}$. On the sector $N=0$, the operator T reduces to

$$T^{(0)}=\left(H_A-z\right)^{-1},$$

and therefore,

$$\|T^{(0)}\|=\frac{1}{\mathrm{dist}(z,\sigma(H_A))}. \tag{4.11}$$

On the other hand if $z=E+i\eta$ and $a=\xi-i\mu$, the sector $N=n>0$ reduces T to

$$T^{(n)}=\frac{\sqrt{n+1}}{(H_A+s_1+\cdots+s_n+n\xi-E)-i\,(\mu n+\eta)}.$$

It follows that

$$\|T^{(n)}\|=\frac{\sqrt{n+1}}{|\mu n+\eta|}. \tag{4.12}$$

Since $\|T\|=\sup_{n\geq 0}\|T^{(n)}\|$, Equations (4.11) and (4.12) lead, after an elementary analysis, to the bound

$$\|T\|\leq\begin{cases}\frac{\sqrt{2}}{\mathrm{dist}(z,\sigma(H_0(a)))} & \text{if } -\mu<\eta<3\mu;\\ \frac{1}{2\sqrt{\mu(\eta-\mu)}} & \text{if } \eta\geq 3\mu.\end{cases} \tag{4.13}$$

If we set

$$\Lambda\equiv\frac{1}{2\,\|Q\|\,\|\alpha_\beta\|_{H^2(\delta)}},$$

$$G(a,\varepsilon)\equiv\left\{z:\mathrm{Im}(z)>\mathrm{Im}(a);\ \mathrm{dist}\left(z,\sigma(H_0(a))\right)>\varepsilon\right\},$$

one easily verifies that, for $\varepsilon<|\mathrm{Im}(a)|$, the bound (4.13) implies

$$\sup_{z\in G(a,\varepsilon)}\left\|\lambda H_I(a)\left(H_0(a)-z\right)^{-1}\right\|\leq\frac{|\lambda|}{\Lambda\,\varepsilon}. \tag{4.14}$$

Consequently, if $|\lambda|<\Lambda\,\varepsilon$, the identity (4.10) holds on $G(a,\varepsilon)$. Moreover the following estimate holds for $N\geq 0$

$$\sup_{z\in G(a,\varepsilon)}\left\|\left(z-H_\lambda(a)\right)^{-1}-\sum_{j=0}^{N-1}\left(z-H_0(a)\right)^{-1}\Big(\lambda H_I(a)\big(z-H_0(a)\big)^{-1}\Big)^j\right\|\leq\frac{1}{\varepsilon}\frac{\left(\frac{\lambda}{\Lambda\varepsilon}\right)^N}{1-\left(\frac{\lambda}{\Lambda\varepsilon}\right)}.$$

It follows that any z in the set

$$\mathrm{P}(\mathrm{Im}(a)) \setminus \sigma(H_A) = \bigcup_{\varepsilon>0} G(a, \varepsilon),$$

is in the resolvent set of $H_\lambda(a)$ for small λ. Therefore, the discrete spectrum of $H_0(a)$ is stable, and analytic perturbation theory applies. The first statement of Theorem 4.6 follows, except for the independence of the eigenvalues on the parameter a. Fix (λ_0, a_0) satisfying (4.9). Since $H_{\lambda_0}(a)$ is an analytic family in a, its discrete eigenvalues are (branches of) analytic functions with at most algebraic singularities in a neighborhood of a_0. On the other hand, $H_{\lambda_0}(a_0)$ and $H_{\lambda_0}(a)$ are unitarily equivalent if $a - a_0$ is real. Thus the discrete eigenvalues are independent of a.

To prove the second statement, assume that $2\varepsilon < |\mathrm{Im}(a)|$ and $|\lambda| < \Lambda\varepsilon$. Let $\gamma_\pm$ be the contours defined by $\{z : \mathrm{Im}(z) = \pm\mathrm{Im}(a)/2\}$, and set $\gamma \equiv \gamma_+ - \gamma_-$. We formally define

$$P_\lambda(a) = \oint_\gamma \frac{dz}{2\pi i} \left(z - H_\lambda(a)\right)^{-1}. \tag{4.15}$$

We shall prove below that, as a weak integral, and after extraction of explicit zeroth and first order terms, the above integral becomes absolutely convergent. Therefore, $P_\lambda(a)$ is analytic and, by a standard argument, is the spectral projection of $H_\lambda(a)$ corresponding to the part of its spectrum contained in the strip bounded by γ_+ and γ_-. Iterating the resolvent identity we get

$$P_\lambda(a) = P_0 + \lambda\Pi^{(1)}(a) + \lambda^2\Pi^{(2)}_\lambda(a), \tag{4.16}$$

where

$$\begin{aligned}
P_0 \equiv P_0(a) &= I \otimes \Omega\,(\Omega,\,\cdot\,),\\
\Pi^{(1)}(a) &\equiv \oint_\gamma \frac{dz}{2\pi i} \left(H_0(a) - z\right)^{-1} H_I(a) \left(H_0(a) - z\right)^{-1},\\
\Pi^{(2)}_\lambda(a) &\equiv -\oint_\gamma \frac{dz}{2\pi i} \left(H_0(a) - z\right)^{-1} H_I(a) \left(H_\lambda(a) - z\right)^{-1} H_I(a) \left(H_0(a) - z\right)^{-1}.
\end{aligned}$$

An explicit calculation shows that $\Pi^{(1)}(a)$ can be written as

$$\Pi^{(1)}(a) = \frac{1}{i} \int_{-\infty}^{\infty} \exp(-\mu|t|)\, \Xi_t\, dt, \tag{4.17}$$

where

$$\Xi_t \equiv \begin{cases} P_0 \exp(iH_0 t) H_I(a) \exp(-iH_0 t) & \text{for } t < 0,\\ \exp(iH_0 t) H_I(a) \exp(-iH_0 t) P_0 & \text{for } t > 0. \end{cases}$$

Another simple calculation yields

$$\|\Xi_t\| = 2^{-3/2}\Lambda^{-1}.$$

Thus we conclude from Equation (4.17) that

$$\left\|\lambda\Pi^{(1)}(a)\right\| \leq \frac{|\lambda|}{\Lambda\,\mu}. \tag{4.18}$$

To estimate $\Pi_\lambda^{(2)}(a)$, we proceed as follows: By Cauchy-Schwarz inequality we have, for any $\Phi, \Psi \in \mathcal{H}$,

$$\left|\left(\Phi, \Pi_\lambda^{(2)}(a)\Psi\right)\right| \leq \sup_{z\in\gamma}\left\|H_I(a)\left(H_\lambda(a)-z\right)^{-1}H_I(a)\right\|\,\nu(\Phi)\,\nu(\Psi), \tag{4.19}$$

where

$$\nu(\Phi)^2 \equiv \int_\gamma \frac{d|z|}{2\pi}\left\|\left(H_0(a)-z\right)^{-1}\Phi\right\|^2.$$

By the spectral theorem (recall that $H_0(a)$ is normal), this quantity is easily seen to be bounded by

$$\nu(\Phi) \leq \sqrt{\frac{2}{\mu}}\,\|\Phi\|. \tag{4.20}$$

We now deal with the supremum in Expression (4.19). We start by the simpler case $\lambda = 0$. There we can apply the method which leads to Inequality (4.14). Leaving the details to the reader, we quote the resulting bound

$$\sup_{z\in\gamma}\left\|H_I(a)\left(H_0(a)-z\right)^{-1}H_I(a)\right\| \leq \frac{2}{\Lambda^2\,\mu}. \tag{4.21}$$

Using the resolvent formula (4.10), a simple calculation shows that

$$\begin{gathered} H_I(a)\left(H_\lambda(a)-z\right)^{-1}H_I(a) = \\ \left(1-\lambda H_I(a)\left(H_0(a)-z\right)^{-1}\right)^{-1}H_I(a)\left(H_0(a)-z\right)^{-1}H_I(a). \end{gathered}$$

Therefore, Inequalities (4.14) and (4.21) yield

$$\sup_{z\in\gamma}\left\|H_I(a)\left(H_\lambda(a)-z\right)^{-1}H_I(a)\right\| \leq \frac{2}{\Lambda^2\,\mu}\left(1-\left(\frac{|\lambda|}{\Lambda\varepsilon}\right)\right)^{-1}.$$

Optimizing the last expression over ε, and combining it with the estimate (4.20) gives the desired bound

$$\left\|\lambda^2\Pi_\lambda^{(2)}(a)\right\| \leq \left(\frac{2|\lambda|}{\Lambda\mu}\right)^2\left(1-\left(\frac{2|\lambda|}{\Lambda\mu}\right)\right)^{-1}. \tag{4.22}$$

Putting together (4.18) and (4.22), we finally get

$$\|P_\lambda(a) - P_0(a)\| \le x\left(\frac{1+2x}{1-2x}\right),$$

with $x \equiv |\lambda|/\Lambda\mu < 1/2$, from which the required inequality follows easily. ■

Remark. In the case of unbounded H_A and bounded Q, the above argument shows that the spectrum of $H_\lambda(a)$ decomposes into a first part $\sigma_0 \subset \{z : |\mathrm{Im}(z)| \le |\lambda|/\Lambda\}$, and a second part in $\{z : \mathrm{Im}(z) \le \mathrm{Im}(a) + |\lambda|/\Lambda\}$. Statement (ii) of Theorem 4.6 still holds in this case. However σ_0 need not be purely discrete. In general we can only assert that, given any bounded region $\mathcal{R}$, there exists a $\Lambda(a, \mathcal{R})$ such that the spectrum in $\sigma_0 \cap \mathcal{R}$ is discrete for $|\lambda| < \Lambda(a, \mathcal{R})$. On the other hand, if we assume that the spectrum of H_A is well separated

$$d_0 \equiv \liminf_{j\to\infty} (E_{j+1} - E_j) > 0,$$

and has bounded multiplicity, then one easily shows that σ_0 is discrete provided $|\lambda| < \Lambda \min(|\mathrm{Im}(a)|, d_0)$.

The previous result allows us to apply reduction theory to the discrete spectrum of resonances, and to construct the quasi-energy operator by transforming $\mathrm{Ran}(P_\lambda(a))$ back to $\mathcal{H}_A$ with a linear isomorphism $S_\lambda(a)$. We follow here the developments of [HP]. If

$$|\lambda| < \Lambda\, |\mathrm{Im}(a)|/4, \tag{4.23}$$

Theorem 4.6 implies that

$$\|P_\lambda(a) - P_0\| < 1.$$

It immediately follows that the maps

$$\begin{aligned} P_0 &: \mathrm{Ran}(P_\lambda(a)) \to \mathcal{H}_A, \\ P_\lambda(a) &: \mathcal{H}_A \to \mathrm{Ran}(P_\lambda(a)), \end{aligned}$$

are isomorphisms. Consequently, setting

$$T_\lambda \equiv P_0 P_\lambda(a) P_0, \tag{4.24}$$

one easily checks that the operator

$$S_\lambda(a) \equiv T_\lambda^{-1/2} P_0 P_\lambda(a),$$

from $\mathrm{Ran}(P_\lambda(a))$ to $\mathcal{H}_A$ has inverse

$$S_\lambda(a)^{-1} \equiv P_\lambda(a) P_0 T_\lambda^{-1/2}.$$

We use the isomorphism $S_\lambda(a)$ to transport the reduced operator $P_\lambda(a)H_\lambda(a)P_\lambda(a)$ back in the space $\mathcal{H}_A$. A simple calculation yields

$$\Sigma_\lambda \equiv S_\lambda(a)P_\lambda(a)H_\lambda(a)P_\lambda(a)S_\lambda(a)^{-1} = T_\lambda^{-1/2} M_\lambda T_\lambda^{-1/2}, \tag{4.25}$$

with

$$M_\lambda \equiv P_0 P_\lambda(a)H_\lambda(a)P_\lambda(a)P_0. \tag{4.26}$$

Finally we remark that since $U(a)P_0 = P_0U(a) = P_0$ for any $a \in \mathbf{C}$, the operators T_λ and M_λ are independent of a as long as Condition (4.23) holds.

The following lemma explores some properties of the quasi-energy (4.25).

Proposition 4.7. *The quasi-energy operator depends analytically on λ for $|\lambda| < \Lambda\, |\mathrm{Im}(a)|/4$. It has a Taylor series of the form*

$$\Sigma_\lambda = H_A + \sum_{n=1}^{\infty} \Sigma^{(2n)} \lambda^{2n}.$$

The first non-trivial coefficient in this expansion is

$$\Sigma^{(2)} \equiv -\frac{1}{2} \sum_j \left(Q\, h_\beta(E_j - H_A)\, Q P_j + P_j Q\, h_\beta(E_j - H_A)\, Q \right),$$

where the function $h_\beta(z)$, analytic in $\mathrm{P}(-\delta)$, is given for $\mathrm{Im}(z) > 0$ by the formula

$$h_\beta(z) \equiv \int_{\mathbf{R}\times \mathrm{S}^2} \frac{|\alpha_\beta(s, \hat{k})|^2}{s - z}\, ds\, d\sigma(\hat{k}).$$

Proof. The analyticity of T_λ follows from its definition (4.24) and Theorem 4.6 (ii). By the same result, $\|T_\lambda - I\| < 1$ holds for $|\lambda| < \Lambda\, |\mathrm{Im}(a)|/4$. Therefore $T_\lambda^{-1/2}$ is also analytic. The Taylor series of T_λ is obtained by inserting the Neumann series for the resolvent of $H_\lambda(a)$ in Equation (4.15). In this way we obtain

$$T_\lambda = I + \sum_{n=1}^{\infty} T^{(n)} \lambda^n,$$

with coefficients given by

$$T^{(n)} \equiv \oint_\gamma \frac{dz}{2\pi i} \left(z - H_A\right)^{-1} P_0 H_I(a) \left(\left(z - H_0(a)\right)^{-1} H_I(a) \right)^{n-1} P_0 \left(z - H_A\right)^{-1}.$$

In a completely similar way we can write

$$M_\lambda = H_A + \sum_{n=1}^{\infty} M^{(n)} \lambda^n,$$

with the following coefficients

$$M^{(n)} \equiv \oint_\gamma \frac{dz}{2\pi i} z \left(z - H_A\right)^{-1} P_0 H_I(a) \left(\left(z - H_0(a)\right)^{-1} H_I(a)\right)^{n-1} P_0 \left(z - H_A\right)^{-1}.$$

The fact that odd powers of λ drop out of this expansions is an easy consequence of photon number conservation (recall that P_0 projects on the zero photon subspace). By Definition (4.25), the first non-trivial coefficient in the Taylor series of Σ_λ is

$$\Sigma^{(2)} = M^{(2)} - \frac{1}{2}\left(T^{(2)} H_A + H_A T^{(2)}\right).$$

An explicit calculation gives

$$\Sigma^{(2)} = \frac{1}{2} \oint_\gamma \frac{dz}{2\pi i} \left(K(z) \left(z - H_A\right)^{-1} + \left(z - H_A\right)^{-1} K(z)\right), \tag{4.27}$$

where

$$K(z) \equiv P_0 H_I(a) \left(z - H_0(a)\right)^{-1} H_I(A) P_0.$$

Remark that the resolvent in $K(z)$ is restricted to the one-photon sector, therefore $K(z)$ is analytic in P($Im(a)$). Another explicit calculation shows that, for Im(z) > 0,

$$K(z) = -\frac{1}{2} Q\, h_\beta(z - H_A)\, Q.$$

Therefore, applying the Cauchy integral formula to Equation (4.27) gives

$$\Sigma^{(2)} = -\frac{1}{2} \sum_j Q\, h_\beta(E_j - H_A)\, Q P_j + P_j Q\, h_\beta(E_j - H_A)\, Q,$$

as required. ■

Remark 1. In the case of unbounded H_A and bounded Q, the quasi-energy is an analytic family of type A with domain $D(H_A)$. In fact one can show that the commutator $[H_A, T_\lambda]$ is bounded. It follows easily that $\Sigma_\lambda - H_A$ is bounded and analytic.

Remark 2. The above argument also yields an expression for the Lamb shifts of the energy level E_j.

Now that we have got some understanding of the family $\{H_\lambda(a) \mid a \in \mathfrak{S}^{\pm}(\delta)\}$, we shall relate it to the physical operator H_λ. This is the content of the next result.

Lemma 4.8. *For* $\lambda \in \mathbf{R}$ *and* $\mathrm{Im}(z)$ *sufficiently large, we have*

$$\mathrm{s}-\lim_{\mathrm{Im}(a)\uparrow 0} \left(H_\lambda(a) - z\right)^{-1} = \left(H_\lambda(\mathrm{Re}a) - z\right)^{-1}.$$

Proof. Clearly we may assume $\mathrm{Re}(a) = 0$. Note that, by Proposition 4.4 (ii), the resolvent of $H_\lambda(a)$ is uniformly bounded as $\mathrm{Im}(a) \uparrow 0$ when $\lambda \in \mathbf{R}$. Therefore, it suffices to show strong convergence on a dense subspace. We will prove that

$$\lim_{\mathrm{Im}(a)\uparrow 0} \left\| \left(\left(H_\lambda(a) - z\right)^{-1} - \left(H_\lambda - z\right)^{-1} \right) (N+1)^{-1} \right\| = 0.$$

As usual, we denote by $F \subset \mathfrak{F}(\mathcal{H}_s)$ the subspace of finite particle vectors. We define

$$\mathcal{D}_0 \equiv \left\{ \left(H_\lambda(a) - z\right) \Psi : \Psi \in \mathcal{H}_A \otimes (F \cap D(\mathrm{d}\Gamma(|s|))) \right\},$$

which is a dense subspace by Proposition 4.4 and the remark which follows it. Since $\mathcal{H}_A \otimes (F \cap D(\mathrm{d}\Gamma(|s|)))$ is a core of $\mathrm{d}\Gamma(|s|)$, it is also a core of H_λ by Lemma 3.2. It follows that, for $\Phi \in \mathcal{D}_0$,

$$\begin{aligned} L(a)\Phi &\equiv \left(H_\lambda(a) - z\right)^{-1} \Phi - \left(H_\lambda - z\right)^{-1} \Phi \\ &= \left(H_\lambda - z\right)^{-1} \left(H_\lambda - H_\lambda(a)\right) \left(H_\lambda(a) - z\right)^{-1} \Phi \\ &= \left(H_\lambda - z\right)^{-1} \left(\lambda Q \otimes \frac{1}{\sqrt{2}} \left(a(\alpha_\beta - \alpha_\beta^{\bar{a}}) + a^*(\alpha_\beta - \alpha_\beta^{a}) \right) - aN \right) \left(H_\lambda(a) - z\right)^{-1} \Phi. \end{aligned}$$

Since $\mathcal{D}_0$ is dense, the above formula extends by continuity to arbitrary Φ. By Proposition 3.3 (i) and Lemma 4.1 (iii), we further have

$$\|L(a)(N+1)^{-1}\| \leq \frac{|a|}{\mathrm{Im}(z)} \left(1 + \frac{\sqrt{2}|\lambda| \, \|Q\| \, \|\alpha_\beta\|_{H^2(\delta)}}{\delta - |\mathrm{Im}(a)|} \right) \left\| (N+1)\left(H_\lambda(a) - z\right)^{-1} (N+1)^{-1} \right\|.$$

We end the proof by showing that $(N+1)\left(H_\lambda(a) - z\right)^{-1}(N+1)^{-1}$ is uniformly bounded as $\mathrm{Im}(a) \uparrow 0$. Indeed, a simple calculation shows that

$$(N+1)\, H_\lambda(a)\, (N+1)^{-1} = H_\lambda(a) + \lambda\, Q \otimes \frac{1}{\sqrt{2}} \left(a^*(\alpha_\beta^{a}) - a(\alpha_\beta^{\bar{a}}) \right) (N+1)^{-1},$$

which, by Proposition 3.3, is a uniformly bounded perturbation of $H_\lambda(a)$. ■

Remark. If H_A is unbounded and Q bounded, we only need to replace $\mathcal{H}_A$ by $D(H_A)$ in the definition of $\mathcal{D}_0$, and the above proof still holds.

Let $E \subset \mathfrak{F}(\mathcal{H}_s)$ be the set of entire vectors for the group $U(a)$. We recall that E consists of all $\Psi \in \mathfrak{F}(\mathcal{H}_s)$ such that the vector-valued function $U(a)\Psi$ has an entire analytic extension. Define $\mathcal{E} = \mathcal{H}_A \otimes E$. It follows from the Paley-Wiener theorem that $\mathcal{E}$ is a dense set of vectors in $\mathcal{H}$.

Proof of Theorem 2.2. For $\Phi, \Psi \in \mathcal{E}$, $\lambda \in \mathbf{R}$ and Im(z) sufficiently large, the function

$$a \mapsto f(a) \equiv \left(U(\bar{a})\Phi, \left(H_\lambda(a) - z\right)^{-1} U(a)\Psi\right),$$

is analytic in $\mathfrak{S}^-(\delta)$. Since it is obviously independent of Re(a), f is actually constant on $\mathfrak{S}^-(\delta)$. Let us show that f is continuous on $\mathfrak{S}^-(\delta) \cup \mathbf{R}$. Indeed,

$$\begin{aligned} f(a) - f(0) = &\left(\Phi, \left(\left(H_\lambda(a) - z\right)^{-1} - \left(H_\lambda - z\right)^{-1}\right) \Psi\right) \\ &+ \left(U(\bar{a})\Phi, \left(H_\lambda(a) - z\right)^{-1} (U(a) - I)\, \Psi\right) \\ &+ \left((U(\bar{a}) - I)\, \Phi, \left(H_\lambda(a) - z\right)^{-1} \Psi\right). \end{aligned}$$

The first term vanishes as Im$(a) \uparrow 0$ by Lemma 4.8. The two other terms also tend to zero in this limit since Φ and Ψ are entire vectors for translations, and $(H_\lambda(a) - z)^{-1}$ is uniformly bounded. Therefore, $f(a) = f(0)$ holds on $\mathfrak{S}^-(\delta) \cup \mathbf{R}$. Since by Theorem 4.6 the function

$$z \mapsto \left(U(\bar{a})\Phi, \left(H_\lambda(a) - z\right)^{-1} U(a)\Psi\right),$$

is meromorphic on P(Im$(a) + |\lambda|/\Lambda$), it does provide the required extension. The properties of this extension follow from Theorem 4.6 and Proposition 4.7. In particular the expression for Im$(\Sigma^{(2)})$ is obtained using the well known formula

$$\mathrm{Im}\left(\frac{1}{x - io}\right) = i\pi\delta(x),$$

in the definition of $h_\beta(z)$. ■

Remark. Here again we did not use the fact that H_A is bounded. Of course the constructed extension will only be meromorphic if the spectrum of Σ_λ is discrete.

Proof of Theorem 2.5. Let $\Phi, \Psi \in \mathcal{E}$, and define

$$f(t) \equiv \left(\Phi, \exp(-iH_\lambda t)\Psi\right).$$

Then for $\mathrm{Im}(z) > 0$, the Fourier-Laplace transform

$$\widehat{f}(z) \equiv \int_0^\infty f(t)\exp(izt)dt = \frac{1}{i}\left(\Phi, \left(H_\lambda - z\right)^{-1}\Psi\right),$$

is well defined. For any $\eta > 0$, the inverse relation

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(E + i\eta)\exp(-i(E + i\eta)t)\frac{dE}{2\pi}, \tag{4.28}$$

holds for $t > 0$. Now let us set $a = -i\mu$ with $0 < \mu < \delta$, and assume that $4\varepsilon < |\mathrm{Im}(a)|$, and $|\lambda| < \Lambda\varepsilon$. As in the proof of Theorem 2.2, $\widehat{f}(z)$ has an extension to the lower half-plane given by

$$\widehat{f}(z) = \frac{1}{i}\left(U(\bar{a})\Phi, \left(H_\lambda(a) - z\right)^{-1} U(a)\Psi\right).$$

By the resolvent identity and estimates (4.20) (4.14), $\widehat{f}$ belongs to the Hardy class of the strip $\{z : -\mu + \varepsilon < \mathrm{Im}(z) < -\varepsilon\}$. It follows that we can rewrite the inversion formula (4.28) as

$$f(t) = \oint_\gamma \widehat{f}(z)\exp(-izt)\frac{dz}{2\pi} + \int_{-\infty}^{\infty} \widehat{f}(E - i(\mu - \varepsilon'))\exp(-i(E - i(\mu - \varepsilon'))t)\frac{dE}{2\pi},$$

where the contour γ is as in the proof of Theorem 4.6, and $\varepsilon' > \varepsilon$. The first term in the above expression is easily identified as

$$f_d(t) = \left(U(\bar{a})\Phi, S_\lambda(a)^{-1}\exp(-i\Sigma_\lambda t)S_\lambda(a)U(a)\Psi\right),$$

whereas the second term is of the order $\exp(-(\mu - \varepsilon'')t)$ for $\varepsilon'' > \varepsilon'$. The proof is complete. To prove Corollary 2.6 we only need the additional observation that, if $\Psi_j(\lambda, a)$ denote the eigenvector of $H_\lambda(a)$ associated to the eigenvalue $E_j(\lambda)$, then

$$\left(\Psi_i(\lambda, a), \Psi_j(0, a)\right) = \delta_{i,j} + O(\lambda^2).$$

■

Remark. The above proof does not use the fact that H_A is bounded.

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