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Spectral stochastic processes arising in quantum mechanical models with a non–L^2 ground state

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Abstract. A functional integral representation is given for a large class of quantum mechanical models with a non–L^2 ground state. As a prototype the particle in a periodic potential is discussed: a unique ground state is shown to exist as a state on the Weyl algebra, and a functional measure (*spectral stochastic process*) is constructed on trajectories taking values in the spectrum of the maximal abelian subalgebra of the Weyl algebra isomorphic to the algebra of almost periodic functions. The thermodynamical limit of the finite volume functional integrals for such models is discussed, and the superselection sectors associated to an "observable" subalgebra of the Weyl algebra are described in terms of boundary conditions and/or topological terms in the finite volume measures.

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1. Introduction

The deep connection between (euclidean) stochastic processes and Quantum Mechanics (QM) relies on two basic properties: i) the representation of the kernel of $\exp -\tau H$ as the integral of a conditional Wiener measure over trajectories (Feynman–Kac formula), ii) the existence of the "thermodynamical limit" ($T \nearrow \infty$) of the "euclidean" correlation functions $< q(\tau_1) \, \dots \, q(\tau_n) >_{-T,T}$ [1], [2].

The latter property, which is in general not so much emphasized as the first, is essential in order to construct a *stationary measure* over trajectories, and plays an essential rôle in the standard reconstruction of a QM Hilbert space, with time translations described by unitary operators. ¿From a QM point of view, the existence of a stationary measure is equivalent to the existence of a ground state for the Hamiltonian H, an essential ingredient of the euclidean formulation by analytic continuation of the real time correlation functions, and in fact a structural feature of Nelson's and Osterwalder–Schrader's formulation of QM and Quantum Field Theory (QFT) [1], [2], [3].

Such property is satisfied by a large class of QM models (including QFT with polynomial interactions), since the growth of the potential at infinity implies the existence of a ground state in $L^2(\mathbb{R}^s, d^s q)$, and therefore of a stationary measure over real trajectories $q(\tau) \in \mathbb{R}^s$, $\tau \in \mathbb{R}$ the euclidean time.

The above property however fails for an interesting class of QM models, which include the free particle, particles in periodic potentials and also QFT models (gauge QFT in the temporal gauge are an example, due to the continuous spectrum of the Gauss law operator). Such models can be more generally defined in terms of representations of CCR Weyl algebras, and then a ground state exists (and it is unique) as a non–regular state over the Weyl algebra, $\mathcal{A}$, even if the Hamiltonian has no point spectrum in the regular (Schrödinger) representation (see Sect.3).

For a functional integral representation of the correlation functions of such ground states, one is then led to choose a maximal abelian subalgebra $\mathcal{A}_c$ of $\mathcal{A}$ and to construct a measure on the trajectories $\sigma(\tau)$ taking values in the Gelfand spectrum of $\mathcal{A}_c$; the result is a measure on the product over (euclidean) times of the spectrum of $\mathcal{A}_c$ (*spectral functional measure*).

If $\mathcal{A}_c$ is the C^* algebra $\mathcal{A}_q$ generated by $\exp(i\alpha \cdot q)$, $\alpha \in \mathbb{R}^s$, then the functional measure is defined on trajectories in the spectrum of the algebra of almost periodic functions, to which $\mathcal{A}_q$ is isomorphic; the variables $q_i(\tau)$ are well defined on the support of the measure, and describe the spectrum of QM operators, *only* when the representation of $\mathcal{A}_q$ is *regular*, i.e. $\exp(i\alpha \cdot q)$ is strongly continuous in α.

¿From a general functional–measure theoretical point of view, the construction of measures is usually done on the basis of Minlos' theorem [2], [4], starting from (positive) functionals, i.e. expectations of $\exp(i\alpha\Phi(f))$, $f \in \mathcal{S}$, continuous in α; when such continuity property fails, the above functionals cannot be represented by measures on the variable Φ in a space of distributions. Examples, which occurs in the models discussed below, are those of functionals defined by an ergodic mean ("flat distribution") in some real variable. Quite generally, however, the Gelfand and Riesz–Markov theorems imply that such "non–regular" functionals can be represented by measures defined *on the spectrum of the algebra generated by the above exponentials*, and $\Phi(f)$ can then be interpreted as a *"non–regular random variable"*.

The above problems have strong implications on the infinite "volume" limit ($T \nearrow \infty$) of stochastic processes constructed with some fixed "end point" $x(T) = \bar{x}$; already the brownian motion (free particle in the quantum mechanical interpretation) is an example, but problems become acute when the control on the model is not explicit. In fact, for QM models, the functional measures corresponding to different Hamiltonians have disjoint support in an infinite time interval, and the standard strategy for their construction is to work in finite "volume", $-T \leq \tau \leq T$, give boundary conditions in time, and then take the limit $T \nearrow \infty$. In the "non–regular" case, the finite volume measures do not converge as measures on real trajectories $x(\tau)$; they do however converge (see Sect.4) as measures on $Q \equiv \prod_\tau \Sigma$, the product over all times of the spectrum Σ of the algebra of almost periodic functions.

In the theory of stochastic processes similar considerations and results apply, if one looks for the existence of stationary measures: they must in general be defined on the spectrum of a suitable algebra of bounded functions of the trajectories.

The aim of this letter is:

i) to provide a general mathematical framework for the construction of functional measures defined on trajectories $\sigma(\tau)$ taking values in the spectrum of the algebra of almost periodic functions;

ii) to provide a functional integral representation of the non–regular states over the Weyl algebra arising as ground states of Hamiltonians with periodic potentials;

iii) to discuss the thermodinamical limit ($T \nearrow \infty$) of the finite volume functional measures for such models;

iv) to describe the superselection sectors which arise associated to an "observable" subalgebra (of the Weyl algebra), and their construction in the thermodynamical limit as a result of boundary conditions and/or "boundary terms" (corresponding to *topological terms* in the action) in the finite volume functional measure.

2. Spectral stochastic processes.

The basic structure for the algebraic formulation of QM is the CCR algebra generated by the Weyl operators $U(\alpha) = \exp iq\cdot\alpha$, $V(\beta) = \exp ip\cdot\beta$, $\alpha,\beta \in \mathbb{R}^s$, and defined by the Weyl relations

$$U(\alpha)V(\beta) = V(\beta)U(\alpha)\exp i\alpha\cdot\beta \quad . \tag{2.1}$$

There is a unique C^* norm on the CCR algebra [5], the operator norm in the Schrödinger representation; by $\mathcal{A}$ we will denote the C^* algebra obtained by norm completion.

States are defined as positive linear functionals on $\mathcal{A}$. Each state ω defines, by the usual Gelfand–Naimark–Segal (GNS) construction, a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}_\omega$. A representation is called regular if the unitary groups $U(\alpha)$, $V(\beta)$ are (strongly) continuous in their parameters. We are interested in representations in which the group $U(\alpha)$ is not continuous. For simplicity we put $s = 1$ in the following.

The (maximal commutative) C^* subalgebra $\mathcal{A}_q$ generated by $U(\alpha) = \exp i\alpha q$, $\alpha \in \mathbb{R}$, is the algebra of almost periodic functions $f(x)$, x a real variable [6]; by the Gelfand isomorphism, it coincides with the algebra of continuous functions on a compact space, Σ, which we will characterize below.

To construct functional integral representations of a state ω, without regularity assumptions, given a time evolution defined by a Hamiltonian (bounded below) in $\mathcal{H}_\omega$, we start from the expectation of products of elements of the abelian subalgebra $\mathcal{A}_q$, at ordered imaginary times; we write such expectations as

$$\omega(A_1(t_1))\ldots A_n(t_n)) \quad .$$

The algebra $\mathcal{A}_q$ plays here the rôle usually taken by the algebra of bounded (continuous) functions of a real variable q, which is represented in $\mathcal{H}_\omega$ only under regularity assumptions (strong continuity of $U(\alpha)$).

If such expectations define a *positive* linear functional on the ("euclidean") abelian algebra $\mathcal{A}_E(T_1,T_2)$ generated, through linear combinations and closure in the Sup norm, by the functions

$$\exp i\sum_i \alpha_j q(t_j) \quad , \quad q(t_j) \in \mathbb{R} \quad , \quad a_j \in \mathbb{R} \quad , \quad t_j \in [T_1,T_2] \quad , \tag{2.2}$$

then the Riesz–Markov theorem represents such a functional in terms of a Baire measure, with a unique regular Borel extension, on the spectrum Q of $\mathcal{A}_E$ (here $[T_1,T_2]$ may be a finite or infinite interval); see, e.g., [7]. It is easy to see that Q is the product over $t \in \mathbb{R}$ of copies of the spectrum Σ of $\mathcal{A}_q$. It can therefore be identified with a space of *spectral trajectories* $\sigma(t)$, i.e. functions of the real variable t taking values in the spectrum Σ of $\mathcal{A}_q$.

The above arguments show that in general, without regularity assumptions, *Nelson positivity* of the euclidean correlation functions of $\mathcal{A}_q$ implies their representation as integrals over *spectral* trajectories: if $f_i(\sigma)$ is the Gelfand image of the element A_i of $\mathcal{A}_q$, then

$$\omega(A_1(t_1))\ldots A_n(t_n)) = \int d\mu(\sigma(t)) f_1(\sigma(t_1))\ldots f_n(\sigma(t_n)) \tag{2.3}$$

The above considerations on the construction of measures on trajectories in the spectrum of the algebra of quasi–periodic functions are largely independent from their motivations in Quantum Mechanics and apply to a (a priori) much larger class of problems like, e.g., the construction of a stationary measure for the brownian motion.

To characterize measures on spectral trajectories it is convenient to have an explicit construction of the spectrum Σ of $\mathcal{A}_q$. We recall that the Gelfand spectrum of a commutative C^* algebra $\mathcal{B}$ (with identity) is the space $\mathcal{M}$ of multiplicative linear functionals $M : \mathcal{B} \to \mathbb{C}$, with the weak topology defined by $\mathcal{B}$ on $\mathcal{M}$. By Gelfand's theorem, $\mathcal{B}$ is isomorphic to the algebra $C(\mathcal{M})$ of the continuous functions over its spectrum, with the Sup norm [8]. The proof of the following results is given in ref. [9].

Proposition 1. *The Gelfand spectrum Σ of $\mathcal{A}_q$ has the following representation: given a Hamel basis $\{\alpha_j\}$, for the reals over the rationals, a point $\sigma \in \Sigma$ is a generalized sequence of real numbers*

$$\varphi_{j,m} \in [0, 2\pi m!/\alpha_j) \ , \quad \varphi_{j,m} = \varphi_{j,n} \mod 2\pi\,(\min(m,n))!/\alpha_j \ , \quad n, m \in \mathbb{N}$$

The corresponding functional M_σ is given by

$$M_\sigma(U(\alpha_j/m!)) = \exp(i\alpha_j\varphi_{j,m}/m!) \equiv \exp[\frac{i}{m!}(\alpha_j\varphi_j + 2\pi \sum_{l=1}^{m} N_{j,l}(l-1)!)] \tag{2.4}$$

with $\varphi_j \in [0, 2\pi/\alpha_j)$, and the integers $N_{j,l} \in [0, l-1]$.
Σ is the topological product of the spaces Σ_j, consisting of the above sequences with j fixed. The topology is defined on Σ_j by the basis of neighbourhoods

$$I_{n,\epsilon}(\{\varphi^0_{j,m}\}) = \{\{\varphi_{j,m}\} : |\varphi_{j,n} - \varphi^0_{j,n}| < \epsilon \ \} \quad . \tag{2.5}$$

Σ and all the spaces Σ_j are compact.
The points σ of Σ such that $M_\sigma(U(\alpha))$ is continuous in α, and therefore of the form $M_\sigma(U(\alpha)) = \exp i\alpha x$, $x \in \mathbb{R}$, define a dense set of "real points" of Σ, of the form

$$\varphi_{j,m} = x \mod 2\pi m!/\alpha_j \quad , \quad x \in \mathbb{R} \quad . \tag{2.6}$$

We will write $x(\sigma)$ for the (unique) real number x corresponding, via eq.(2.6), to a "real point" $\sigma \in \Sigma$.
The translation automorphisms $T_z : \exp i\alpha q \mapsto \exp i\alpha(q+z)$ define on Σ, by duality, a continuous one–parameter group T^*_z, $z \in \mathbb{R}$:

$$T^*_z\{\varphi_{j,m}\} = \{\varphi_{j,m} + z \mod 2\pi m!/\alpha_j\} \quad . \tag{2.7}$$

$T^*_z\sigma$ will also be denoted by $\sigma + z$.

The use of the Hamel basis [10] is necessary for the spectrum of the algebra of all periodic functions. One may however consider separable subalgebras, e.g., the algebras generated by exponentials with periods which are *rational* combinations of a finite numbers of periods, L_i; Proposition 1 then applies with the Hamel basis replaced by $\{\alpha_i = 1/L_i\}$. Most of the phenomena and results can also be discussed on the basis of such algebras; their spectra are finite products of spaces Σ_j, which have a much richer structure then the spectra of the subalgebras defined by *integer* combinations of α_i (which are tori).

By Proposition 1, the Borel sets of Σ are generated (by countable union and complements) by the open sets in a finite number of variables $\varphi_{j,m}$. Regular Borel measures on Σ are therefore identified by their restriction to tori of the form

$$\{\varphi_{j,m} \in [0, 2\pi m!/\alpha_j) \quad , \quad (j,m) = (j_1, m_1)\ldots(j_k, m_k)\} \quad . \tag{2.8}$$

It is not difficult to show [9] that the set of "real points" of Σ is a Borel set in Σ, and that the Borel structure defined on the "real points" by the spectral topology coincides with the standard Borel structure of the real line. Therefore, every Borel measure on the real line defines a Borel measure on Σ with "real" support. There are, however, many (regular) Borel measures on Σ with support disjoint from the real points. The following ones are of special interest: given any positive almost periodic function $M(x)$, the ergodic mean

$$\lim_{L\to\infty} 1/2L \int_{-L}^{L} M(x)f(x)dx \tag{2.9}$$

defines a positive functional on $f \in \mathcal{A}_q$, and therefore a measure on Σ; for $M(x) = 1$ this measure, denoted by $d\nu_0$, is given by the usual (normalized) Lebesgue measure on the tori (2.8). The support of all such measures is disjoint from the set of the "real points" of Σ [9].

A class of measures on $Q = \prod_{t \in \mathbb{R}} \Sigma$, corresponding to "locally real stochastic processes" on Σ can be constructed as follows: given any stochastic process defined on the real line and starting at $z(0) = 0$, identified by the measures $d\rho(z(t_1), \ldots, z(t_n))$, and any Borel measure $d\nu(\sigma)$ on Σ, a measure on Q is defined by the measures

$$d\mu(\sigma(0), \sigma(0) + z(t_1), \ldots, \sigma(0) + z(t_n)) \equiv d\nu(\sigma(0))\, d\rho(z(t_1), \ldots, z(t_n)) \quad . \tag{2.10}$$

Such a measure is supported on trajectories of the form $\sigma(t) = \sigma(0) + z(t)$, with spectral translations $z(t)$ in the support of $d\rho$.

For example, if $d\rho$ defines the brownian motion starting at $z(0) = 0$, and $d\nu = d\nu_0$, then $d\mu(\{\sigma(t)\})$ defines a stationary Markov process on Σ (an explicit control follows from the above characterization of $d\nu_0$ on all tori (2.8)). This construction formalizes the fact that "the ergodic mean on the real line is invariant under the brownian motion".

In the following section we show that measures of the form (2.10), with $d\nu = d\nu_0$, represent the ground state of any quantum mechanical Hamiltonian of a particle in a periodic potential.

3. Ground states for periodic potential and their functional integral representation.

Existence of the Hamiltonian and of the ground state for particles in periodic potentials is controlled by the following results [9]:

Proposition 2. *i) Given any bounded, measurable and periodic potential $W(x)$ (of period 1), there exists one and only one irreducible representation of the Weyl algebra in which the Hamiltonian*

$$H = p^2/2 + W$$

is well defined as a strong limit of elements of $\mathcal{A}$ on a dense domain, and has a ground state.

ii) Such a representation is independent of W and is the unique (non–regular) representation π_0 in which the subgroup $V(\beta)$, $\beta \in \mathbb{R}$ is regularly represented and its generator p has a discrete spectrum.

iii) The Hilbert space $\mathcal{H}$ of π_0 is given by the Gelfang–Naimark–Segal (GNS) construction over the state Ω_0 on $\mathcal{A}_q$ defined by the ergodic mean in q; its vectors are formal sums

$$\psi(x) = \sum_{n\in\mathbb{Z}} a_n \exp i\alpha_n x \quad , \qquad x \in \mathbb{R} \ , \quad \{a_n\} \in l^2 \quad , \tag{3.1}$$

and the Weyl operators are represented as

$$U(\alpha)\psi(x) = \exp(i\alpha x)\ \psi(x) \quad , \quad V(\beta)\psi(x) = \psi(x+\beta) \quad . \tag{3.2}$$

The scalar product in $\mathcal{H}$ is given by

$$(\psi,\psi) = \sum_n |a_n|^2 = \text{ergodic mean } (|\psi|^2) \quad . \tag{3.3}$$

It follows that $\mathcal{H}$ contains a dense subspace described by wave *functions* $\psi(x)$. Moreover, $\mathcal{H}$ is the direct *sum*, over $\alpha \in [0, 2\pi]$, of spaces $\mathcal{H}_\alpha$ of wave functions of the form $\exp(i\alpha x)f(x)$, with $f(x)$ locally L^2 and $f(x+1) = f(x)$. In $\mathcal{H}_\alpha$ the operator p is represented by $-id/dx$, with boundary conditions $\psi(1) = \exp(i\alpha)\,\psi(0)$.

The above *direct sum* over α corresponds to the reduction of π_0 into irreducible representations of the algebra $\mathcal{A}_{per} \subset \mathcal{A}$, generated by $\{\exp i(\beta p + nq),\ \beta \in \mathbb{R},\, n \in \mathbb{Z}\}$. The Hamiltonian H is affiliated to the strong closure of $\mathcal{A}_{per}$ in π_0 and leaves the spaces $\mathcal{H}_\alpha$ invariant. We denote by $\mathcal{D}_0$ the subspace of the vectors of the form $A \cdot 1,\ A \in \mathcal{A}_q$, and by $\mathcal{D}$ the linear span of the spaces $\mathcal{H}_\alpha$.

The following Proposition extends the ordinary path integral representation of the kernel of $\exp -tH$ to the non regular representation π_0; uniqueness of the ground state in $\mathcal{H}$ then follows by a Perron–Frobenius argument.

Proposition 3.[9] *For all $\psi \in \mathcal{D}$,*

$$(e^{-tH}\ \psi)(x) = \int_{-\infty}^{+\infty} dy\, K_t(x,y)\psi(y) \tag{3.4}$$

where $K_t(x,y)$ is the kernel of $\exp -tH$ in the Schrödinger representation, which is strictly positive by the usual Feynman–Kac representation.
The ground state ψ_0 is unique (up to a phase); it belongs to $\mathcal{D}_0 \cap \mathcal{H}_0$ and its wave function can be chosen to be positive; it is cyclic for the algebra $\mathcal{A}_q$. In each $\mathcal{H}_\alpha$, H has a unique lowest energy state ψ_α ($H\psi_\alpha = E(\alpha)\psi_\alpha$).

Positivity of $\psi_0(x)$ and of the kernel of $\exp -tH$ in π_0 imply Nelson positivity for the ordered imaginary time correlation functions of $\mathcal{A}_q$ on ψ_0. In fact, normalizing to zero the ground state energy $E(0)$, they are given by

$$\omega(e^{i\alpha_1 q(t_1)} \ldots e^{i\alpha_n q(t_n)}) \equiv$$

$$\equiv (\psi_0\,,\, e^{i\alpha_1 q} e^{-H(t_2-t_1)} e^{i\alpha_2 q} \ldots e^{-H(t_n-t_{n-1})} e^{i\alpha_n q}\psi_0) \quad . \tag{3.5}$$

By the group property of $\exp -tH$ and the normalization of the ground state energy, eq.(3.5) defines a functional on $\mathcal{A}_E$. The representation of the scalar product in $\mathcal{H}$, eq.(3.3), gives the r.h.s. of eq.(3.5) the form

$$\text{ergodic mean } (x_1) \int_{-\infty}^{+\infty} dx_2 \ldots dx_n\; \psi_0(x_1)\, e^{i\alpha_1 x_1}$$

$$K_{t_2-t_1}(x_1, x_2)\, e^{i\alpha_2 x_2} \ldots K_{t_n-t_{n-1}}(x_{n-1}, x_n)\, e^{i\alpha_n x_n} \psi_0(x_n)) \quad , \tag{3.6}$$

The functional ω is therefore positive, and by the Riesz–Markov theorem it defines a measure $d\mu$ on Q, i.e. on *spectral trajectories* $t \to \sigma(t) \in \Sigma$.

The measure $d\mu$ can be written explicitely, making use of the measure $d\nu_0$ on Σ corresponding to the ergodic mean, eq.(2.9) with $M = 1$. In fact, from Proposition 2 it follows that $\mathcal{H}$ can be represented as $L^2(\Sigma, d\nu_0)$, and Proposition 3 implies that ψ_0 is represented by a continuous function $F_0(\sigma)$, which is determined by its values $\psi_0(x)$ on the "real points" $x \in \Sigma$.

Moreover, from Proposition 3 it follows that for all continuous functions F on Σ and all real points σ,

$$(e^{-tH} F)(\sigma) = \int_{-\infty}^{+\infty} dz\; K_t(x(\sigma), x(\sigma + z))\; F(\sigma + z) \tag{3.7}$$

where we have used the translation group on Σ, eq.(2.7).

Eq.(3.7) extends to all points $\sigma \in \Sigma$, since the l.h.s. is continuous ($\exp -tH$ maps $\mathcal{D}_0$ into $\mathcal{D}_0$) and the resulting kernel $K_t(\sigma, \sigma + z)$ is continuous in σ and L^1–bounded in z, uniformly in σ. Furthermore, the Feynman–Kac representation of

$K_t(x,y)$, implies, for real points σ in Σ,

$$K_\tau(\sigma,\sigma+z)=\int d\rho_{[0,\tau],0,z}(z(t))\ \exp(-\int_0^\tau W(\sigma+z(s))\,ds)\quad , \tag{3.8}$$

where $d\rho_{[0,\tau].0,z}(z(t))$ is the conditional Wiener measure on paths $z(t)$, $0\le t\le\tau$, with $z(0)=0$, $z(\tau)=z$. The extension of eq.(3.8) to all points of Σ is immediate for continuous potentials, to which we will now restrict for simplicity.

By inserting eq.(3.8) in eq.(3.4), we obtain a (generalized) Feynman–Kac representation of the propagator in $\mathcal{H}$, in terms of a functional measure on the spectrum of $\mathcal{A}_E$, i.e. on trajectories $\sigma(t)$ with values in Σ. Eq.(3.6) now gives the following

Theorem 4. *The unique ground state ω of a particle in a continuous periodic potential has the representation*

$$\omega(e^{i\alpha_1 q(t_1)}\,\dots\,e^{i\alpha_n q(t_n)})=$$

$$=\int d\nu_0(\sigma)\,F_0(\sigma)\int d\rho_{[t_1,t_n],0}(z(t))\;e^{-\int_{t_1}^{t_n}W(\sigma+z(s))\,ds}$$

$$e^{i\alpha_1 q}(\sigma)\,e^{i\alpha_2 q}(\sigma+z(t_2))\,\dots\,e^{i\alpha_n q}(\sigma+z(t_n))\,F_0(\sigma+z(t_n))\quad , \tag{3.9}$$

where $d\rho_{[t_1,t_n],0}(z(t))$ is the Wiener measure on paths $z(t)$, $t_1\le t\le t_n$, starting at 0 at t_1, F_0 the representation of the ground state in $L^2(d\nu_0)$, and $e^{i\alpha q}(\sigma)$ the Gelfand image of $e^{i\alpha q}$.

The r.h.s. of eq.(3.9) defines a (Borel) measure $d\mu$ on Q with support on trajectories of the form $\sigma(t)=\sigma(0)+z(t,\sigma(0))$, with $z(t,\sigma(0))$ Brownian trajectories in $\mathbb{R}$. The restrictions of $d\mu$ to bounded time intervals $[-T,T]$ are given by

$$d\mu_{[-T,T]}(\sigma+z(t))=$$

$$=d\nu_0(\sigma)\;F_0(\sigma)\;d\rho_{[-T,T],0}(z(t))\;e^{-\int_{-T}^{T}W(\sigma+z(s))\,ds}\;F_0(\sigma+z(T))\quad . \tag{3.10}$$

By using the boundedness of $F_0(\sigma)^{-1}$, $d\mu$ can be put in the form (2.10).

$d\mu$ is the weak limit, for $T\nearrow\infty$ of spectral functional measures $d\mu^f_{[-T,T]}(\sigma+z(t))$

in finite time intervals, defined by

$$Z(f,T)^{-1}\, d\nu_0(\sigma)\, f(\sigma)\, d\rho_{[-T,T],0}(z(t))\, e^{-\int_{-T}^{T} W(\sigma+z(s))\,ds}\, f(\sigma+z(T)) \tag{3.11}$$

for any non–negative $f(\sigma)\in L^2(\Sigma, d\nu_0)$, *in particular* $f=1$. $Z(f,T)$ *is given by* $(f, \exp -2TH\, f)$ *and converges to* $|\int d\nu_0\, F_0(\sigma) f(\sigma)|^2$ *as* $T \nearrow \infty$.

Proof. The last point follows from the fact that the integral of the generators of $\mathcal{A}_E$ with the measure (3.11) has the quantum mechanical interpretation

$$(\psi_f\,,\, e^{-H(t_1+T)}\, e^{i\alpha_1 q},\, e^{-H(t_2-t_1)}\, \dots\, e^{i\alpha_n q}\, e^{-H(T-t_n)}\, \psi_f) \quad ,$$

which converge, for $T\nearrow\infty$, to $|(\psi_f,\psi_0)|^2\, \omega(e^{i\alpha_1 q(t_1)}\, \dots\, e^{i\alpha_n q(t_n)})$, as a consequence of the spectral properties of the Hamiltonian in $\mathcal{H}$ [9]. Since $\psi_0(x)$ is positive, one can take any $f(\sigma)\geq 0$.

4. Infinite volume limit of measures over real trajectories

¿From a constructive point of view it is important to recover the above spectral functional measure in the limit $T\nearrow\infty$ starting from standard functional measures over real trajectories in bounded time intervals $[-T,T]$. As we shall see, it is crucial to realize that such measures do not converge as measures on real trajectories, but only as measures on Q (measures on real paths uniquely define measures on Q with support on "real trajectories" in Q, see Sect.2).

Given any Hamiltonian of a particle in a bounded measurable periodic potential, if the times are restricted to a bounded interval $[-T,T]$, the expectation of the euclidean variables $f(q(t))$, on any state $\psi\in L^2$ have the usual representation

$$(e^{-H(T+t_1)}\,\psi\,,\, f_1\, e^{-H(t_2-t_1)}\, f_2 \dots e^{-H(t_n-t_{n-1})}\, f_n\, e^{-H(T-t_n)}\,\psi)/||e^{-HT}\psi||^2 =$$

$$= Z(\psi,T)^{-1}\int dx_-\, dx_+\, \overline{\psi}(x_-)\,\psi(x_+)\, \int d\mu_T(x(t))\, f_1(x(t_1))\dots f_n(x(t_n)) \tag{4.1}$$

with

$$d\mu_T(x(t)) \equiv d\rho_{[-T,T],x_-,x_+}(x(t))\; e^{-\int_{-T}^{T} W(x(s))\,ds}$$

and $Z(\psi,T)$ the usual normalization factor.

In the limit $T \nearrow \infty$ the correlation functions (4.1) of the variable $f(x) = x$ are in general divergent; in fact, as it also follows from the results given below, the expectations of all bounded functions of x vanishing at infinity converge to zero in the infinite time limit, so that, e.g., the expectation of $x(0)^2$ has a divergent lower bound as T goes to infinity. The point is that brownian trajectories in a bounded potential wander over larger and larger regions for large times. A more appropriate choice of variables is therefore essential, and in fact the expectations (4.1) have a limit in the formulation based on the Weyl algebra, i.e., for $f \in \mathcal{A}_q$.

Theorem 5. *The l.h.s. of eq.(4.1), with f_i in $\mathcal{A}_q$ and $\psi(x)$ in $L^1 \cap L^2$ such that $\int \psi_0(x)\psi(x)\,dx \neq 0$ converges for $T \nearrow \infty$ to the correlation functions, eq.(3.5), of $\mathcal{A}_q$ on the unique ground state ψ_0. The measures defined by eq.(4.1) converge therefore weakly on $\mathcal{A}_E$ to the measure $d\mu(\sigma(t))$ constructed in Sect.3, eqs.(3.9),(3.10).*

Proof. The proof follows [9] from the integral decomposition of L^2 in spaces $\mathcal{H}_\alpha$ (see Sect.3), the spectral analysis of the Hamiltonian, the representation of $\mathcal{H}_{\pi_0}$ given in Sect.3, the uniqueness of the ground state and some estimates in T of scalar products and normalization factors.

The measures $d\mu_T$ do not therefore converge as measures over trajectories $x(t)$, but only as ("locally brownian") measures on spectral trajectories $\sigma(t)$. Also when restricted to the σ–algebra generated by the variables at a fixed time, e.g. $t = 0$, the limiting measure has support disjoint from the support of the finite time measures, since the latter, eq.(4.1), are supported, as measures on Σ, on its "real points". ¿From the characterization, eq.(3.10), of the support of $d\mu$, it also follows that the problems of convergence of the measures (4.1) can be "reduced to one time"; i.e., they arise in the construction of a measure on Σ, the spectrum of the algebra $\mathcal{A}_q$ at one time, invariant under an evolution defined by (perturbed) brownian translations of Σ. In quantum mechanical terms, the problem is the ground state (a non–regular state on $\mathcal{A}_q$), *not* the Feynman–Kac representation of $\exp -tH$.

The disjointness of the measure defined by the vacuum on the spectrum of the algebras generated by the variables at fixed time, more generally in *bounded* time intervals, which is typically related to ultraviolet problems, occurs here for *infrared* reasons, and one may speak of an *infrared renormalization.* It corresponds to a change of representation for the *local* algebras, and has nothing to do with the usual fact that in infinite volume (or time) the functional measures with different interactions are disjoint; the latter property is based on ergodicity in time and is a functional measure version of Haag's theorem.

5. Superselection sectors, boundary terms, winding numbers.

The construction of a functional integral representation of non regular states over the Weyl algebra allows for the discussion of models where the Weyl algebra plays the rôle of a *field* algebra, and a subalgebra is identified with the *observable* algebra $\mathcal{A}_{obs}$.

We consider the case where $\mathcal{A}_{obs} = \mathcal{A}_{per}$. As mentioned in Sect.2, $\mathcal{H}_{\pi_0}$ decomposes into irreducible representations of $\mathcal{A}_{per}$, labelled by an angle $\theta \in [0, 2\pi)$ and defined by the "ground states" ω_θ, $\omega_\theta(A) = (\psi_\theta, A\psi_\theta)$, $A \in \mathcal{A}_{per}$.

¿From eqs.(3.5)–(3.10) a functional representation of ω_θ on trajectories in the circle (the spectrum of the maximal abelian subalgebra $\mathcal{A}_{q,per}$ of $\mathcal{A}_{per}$ generated by $\{\exp i2\pi nq \;\; n \in \mathbb{Z}\}$) follows by taking as boundary condition the wave function ψ_θ of the unique ground state in $\mathcal{H}_\theta$. In fact, $\psi_\theta(\sigma)$ is of the form $(\exp i\theta q)(\sigma)\, R_\theta(\sigma)$ with R_θ a function depending only on one "spectral angle" $\varphi \in [0,1) = S^1/2\pi$; the factors $\exp i\theta q$ can be inserted in the kernels K_t, where they become functions of z; since the Hamiltonian commutes with periodic translations, the kernel K_t is a function of φ and z; $d\nu_0$ becomes $d\varphi$ and $\omega_\theta(e^{i2\pi n_1 q(t_1)} \ldots e^{i2\pi n_k q(t_k)})$ has the integralrepresentation

$$\int \prod_{i=1}^{k} d\varphi_i \; R_\theta(\varphi_1)\, \mathcal{L}^\theta_{t_2-t_1}(\varphi_1,\varphi_2) \; \ldots \; \mathcal{L}^\theta_{t_k-t_{k-1}}(\varphi_{k-1},\varphi_k) \; \prod_{j=1}^{k} e^{i2\pi n_j \varphi_j} \; R_\theta(\varphi_k) \qquad (5.1)$$

where

$$\mathcal{L}^\theta_t(\varphi_1,\varphi_2) \equiv \exp E_\theta t \sum_{n=-\infty}^{\infty} K_t(\varphi_1, \varphi_2+n)\, \exp(i\theta(\varphi_2-\varphi_1+n)) \quad . \qquad (5.2)$$

Proposition 6. *Eq.(5.1) defines a bounded complex measure $d\gamma^\theta_T$ on trajectories $t \in [-T,T] \to \varphi(t) \in S^1/2\pi$, with $\int d\gamma^\theta_T = 1$. In the case $\theta \neq 0$ its total variation, which is bounded by $\exp 2TE(\theta)$, diverges exponentially for $T \nearrow \infty$ [9]. Moreover, Nelson positivity does not hold for the correlation functions defined by ω_θ, $\theta \neq 0$, even for variables in $\mathcal{A}_{per}$.*

The representation (5.1) is given e.g. in [11], where θ corresponds to the apparently "irrelevant" addition of a time derivative $i\theta\, d\varphi(t)/dt$ to the Lagrangean. The point is that the variable $\exp i\theta\varphi$ belongs to $\mathcal{A}_q$, but not to $\mathcal{A}_{per}$, and therefore the "topological" term $\exp i\theta \int_{-T}^{T} d\varphi(t)/dt$ is equivalent to a change of boundary conditions only in the formulation based on $\mathcal{A}_q$, i.e. for functional integrals based on the spectrum of $\mathcal{A}_q$. On the contrary, in the formulation based on trajectories in the circle, (the spectrum of $\mathcal{A}_{q,per}$) the rôle of θ is that of an additional term in the Hamiltonian (see the definition of $\mathcal{L}_t^\theta$, eq.(5.2)), and there is no relation with boundary conditions.

Such relation can be made explicit also in terms of infinite time limit of functional measures over real trajectories $\varphi(t)$. For boundary conditions $\psi \in L^1 \cap L^2$, one obtains [9] the correlation functions on the lowest energy state(s) of $\mathcal{H}_{\pi_0}$ which appear in the integral decomposition of ψ into components $\psi_\theta \in \mathcal{H}_\theta$ In particular, the result is the ground state if ψ is positive or of compact support.

The construction of θ states thus naturally requires, see eqs.(5.1) and (3.6), "non–local boundary conditions", i.e. an ergodic mean over boundary variables, integrated with almost periodic functions $\psi(q) = \exp i\theta q \, R(q)$, $R(q)$ periodic.

For finite time intervals, the functional measure $d\gamma_T^\theta$ on the circle, eq. (5.1), has an interpretation in terms of winding numbers of the trajectories $\varphi(t) \in S^1/2\pi$, $-T \leq t \leq T$. ¿From eq.(5.1) one obtains in fact

$$d\gamma_T^\theta = \sum_{n=-\infty}^{\infty} e^{in\theta} \, d\gamma_{n,T} \, e^{2E_\theta T} \tag{5.3}$$

where $d\gamma_{n,T}$ is the restriction to the trajectories with winding number n of the positive measure defined by the kernel $\sum_{m \in \mathbb{Z}} K_t(\varphi_1, \varphi_2 + 2\pi m)$ (the factor $\exp 2E_\theta T$ is necessary for the normalization $\int d\gamma_T^\theta = 1$). The restriction to finite volume is crucial: the mean square winding number $< n^2 > \equiv \sum_n n^2 \int d\gamma_{n,T} \, / \sum_n \int d\gamma_{n,T}$ goes to infinity as $T \nearrow \infty$, and a winding number representation never exists in infinite volume. Moreover, since $\sum_n \int d\gamma_{n,T} \; e^{2E_\theta T} \;\; \sim \;\; e^{2E_\theta T}$, the sum over the winding numbers in eq.(5.3) is never done with a probability measure, for $\theta \neq 0$.

For $\theta = 0$, eq.(5.3) do define a probability measure over $n \in \mathbb{Z}$; the positive functional defined by such a measure on the algebra $\mathcal{A}_n$ generated by $\exp i\alpha n$, $\alpha \in [0, 2\pi)$, converges, for $T \nearrow \infty$, to the functional given by the ergodic mean over n. A (generalized) probabilistic interpretation, (as a measure on the Gelfand spectrum of $\mathcal{A}_n$) is therefore allowed, *only for* $\theta = 0$, for the winding number in infinite volume, as a variable "uniformly distributed over the integers".

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