Notices Amer. Math. Soc. 42 (1995), 26-31.

SYMMETRY AND TILINGS *

Charles Radin
University of Texas
Austin, TX 78712
radin@math.utexas.edu

* Research supported in part by NSF Grant No. DMS-9304269 and Texas ARP Grant 003658-113

A "pinwheel" tiling of the plane (as in Figure 1) does not have any translational or rotational symmetry in the usual sense. But in a tantalizing, unconventional way it is *highly* symmetric, and this unusual form of symmetry has significant applications, for instance in classifying the internal structures of solids.

Tilings, which are decompositions of space into finite-size subsets called tiles, are a traditional medium for exhibiting symmetries. Usually, a pattern in space exhibits a symmetry by being invariant under the corresponding rigid motion of space. As an example consider an infinite length "necklace", a mixture of equidistant cubic and spherical beads embedded along a line. Such an object has a translational symmetry if the pattern is periodic, for instance if the bead-shapes alternate. A very different example is a paraboloid of revolution, which has as symmetries the rotations about its axis. By decomposing space into tiles, a tiling can be thought of as a "multidimensional necklace", with the "beads" fitting smoothly together like a jigsaw puzzle. As a link between continuous surfaces such as paraboloids and discrete sequences such as necklaces, tilings can exhibit an unusual richness of symmetries.

The pinwheel and Penrose tilings of the plane (see Figure 2 for a portion of a Penrose tiling) belong to a special class of tilings in which symmetries are not exhibited in the above traditional sense, but in a more subtle manner, only recently understood. We will sketch the main ideas involved in the analysis of such tilings and their symmetries. This will prove to be an interdisciplinary endeavor, employing intuition from logic and condensed matter physics as well as mathematics.

There are two features which together define the class of tilings, called hierarchical tilings, that we will consider. First, each tiling is made of polyhedral tiles congruent to those in some fixed finite set S. For the Penrose tilings S contains just two such tiles, called the kite and the dart, shown in Figure 3. (The graphics of Figures 1 and 2 are convenient oversimplifications: they do not show the small bumps and dents that are more properly part of the tiles, as shown in Figure 3.) Second we require that *every* tiling that can be made with tiles from S display a certain type of hierarchical structure. The hierarchical structure of the pinwheel and Penrose tilings is suggested by Figures 4 and 5. The structural fact which is suggested is that each such tiling "displays", in a *unique* way, another such tiling of space by the original tiles but enlarged by some factor: $(\sqrt{5}+1)/2$ for the Penrose tilings and $\sqrt{5}$ for the pinwheels. Furthermore, this larger scale tiling displays another unique tiling at the next larger scale, *etc.,* so each tiling displays an infinite hierarchy of tilings at larger and larger scales. (We note that the bumps and dents of Figure 3 are needed to ensure that the tiles can *only* fit together to form tilings with the desired hierarchical structure. With quadrilateral tiles the tiling of Figure 2 still displays a hierarchical structure, but only if the bumps and dents are added does it become a hierarchical tiling.)

Think of tilings as fixed in space, and let X be the set of all (hierarchical) tilings made from some finite set S of polyhedral tiles. We now use X to carry a representation of the special Euclidean group G of rigid motions, "special" meaning without reflections. For g in G, and thinking of a tiling x as a set of tiles P fixed in space, $x = \{P\}$, we define $T^g(x) = \{g(P)\}$; that is, we let g move each of the tiles in x. We now note something much less obvious, using the hierarchical structure of the tilings. Given a tiling x, consider the collection of "large tiles" associated with the next higher level of the hierarchical structure of x (such as that drawn with thick lines in Figure 4), and let $y(x)$ be the tiling made by shrinking these large tiles about the origin by the appropriate factor λ. This defines a map T^s on X, by $T^s(x) = y(x)$. From the assumed uniqueness of the hierarchy, T^s is invertible. It is important to note that T^s can and should be thought of as a representation of that *similarity*, s, of space which shrinks space linearly about the origin by the factor λ; in particular, T^s has the correct group relations with all T^g, $g \in G$, such as $T^sT^tT^{s^{-1}} = T^{\lambda t}$ for any translation t. One use of this group representation is the following simple argument showing that no hierarchical tiling can be invariant under any nonzero translation. For assume $T^t(x) = x$ for some translation t. It is easy to show that $T^{s^n}(x)$ is then invariant under translation by $T^{\lambda^n t}$. But taking n large enough this would imply invariance under a translation by less than the size of any tile, which is only possible if $t = 0$.

The above group representations on X are also useful in understanding the rotational symmetries of these tilings. There are two ways of understanding these symmetries: as geometric features of the tilings and in terms of the mathematical structures of analysis. We will note both points of view, as they give different insight.

As you might guess from Figure 2, kites and darts each appear in precisely ten different orientations in a Penrose tiling, and in fact they appear "equally often" in each of these ten orientations in the following sense. Take any large circle in any Penrose tiling, and divide the number of times a kite appears, in a certain orientation, within that circle by the area of the circle. This fraction has a limit, as the circle grows to the full plane, which is independent of the choice of tiling; furthermore this limit is *the same* for each of the ten orientations. Intuitively, we are saying that the Penrose tilings exhibit ten-fold *statistical* rotational symmetry; not just single kites and darts, but any finite collection of tiles will appear in every tiling in ten, and only ten, orientations (if it appears at all), and with the *same frequency* in each orientation.

The symmetry of the pinwheel tilings is more surprising. Such tilings exhibit complete statistical rotational symmetry: given any two equal-size intervals I_1 and I_2 of orientations, and any finite collection of tiles, then in any tiling the collection will appear having an orientation in I_1 with the same frequency as with an orientation in I_2.

This statistical type of symmetry can be put in the usual form of the invariance of something with respect to a group of rotations, but this requires some analysis. First we put a metric ρ on X such that two tilings are close if, within some large ball centered at the origin, wherever one tiling has a tile the other also has one, which almost coincides. Then we lift the action of the rotations and translations from the space X of tilings to the space $B(X)$ of Borel probability measures on X, and define the subspace $B_T(X)$ of those measures which are *translation invariant*.

With this notation, the above statistical symmetry is precisely the invariance of each element of $B_T(X)$ under the appropriate group of rotations – the group of rotations by multiples of $2\pi/10$ for the Penrose tilings, and the group of all rotations for the pinwheel tilings. So "statistical symmetry" is a weakening of the usual notion of symmetry of a pattern, replacing the invariance of the pattern by the invariance of all invariant measures associated with the pattern.

We now discuss the origins of hierarchical tilings and their statistical symmetries. The subject began 30 years ago when the philosopher Hao Wang was working on a decidability problem in propositional calculus. He reformulated his problem into the following one. Assume we are interested in arbitrary (not necessarily hierarchical) tilings of the plane by tiles which are all basically unit squares, but have different patterns of bumps and dents on their edges. Furthermore, assume as above that all the tiles are congruent to those from some finite set S. The question Wang asked is: Can there be an *algorithm* by which, given as input any such finite set S, we can determine whether or not we can actually tile the plane with such tiles? Now imagine there exists a set S from which we can tile the plane but *only* in complicated ways, in particular only without any translation symmetry; Wang showed that an algorithm for tiling exists if and only if there does not exist such a set S. His student Robert Berger then constructed an example of such a set S (producing hierarchical tilings which, as we saw above, cannot be invariant under a nonzero translation), thereby creating the subject. Logicians extended the result by constructing sets S from which one could tile the plane but only by tilings which were each themselves not the output of any algorithm. (The connection between such "square-ish" tilings and algorithms is obtained by thinking of consecutive rows of a tiling as the tape of a Turing machine at consecutive time steps.)

So the subject began in an attempt to make "complex" tilings, in the technical sense referring to complexity theory and algorithms. There was a different motivation in condensed matter physics. There, tilings like those of Penrose have been used to model the structure of unusual solids. For practical reasons, the relevant feature of the material which is of interest is the X-ray (or electron) scattering patterns produced by the materials. A chunk of a solid produces scattering patterns when illuminated by beams of waves (photons or electrons) of appropriate wavelength; in general one is interested in the family of patterns obtained for a range of wavelengths, and when viewed from a range of angles relative to that of the incoming beam. The pattern is produced by the constructive and destructive interference of the individual scatterings of the beam from each of the large number (order 10^{24}) of constituent atoms. For an ordinary crystal, the atoms are roughly in a periodic configuration, and it can be proven (the so-called "classification of crystallographic groups") that the scattering patterns from any such configuration can only have certain rotational symmetries. Alan Mackay showed if we associate "scatterers" with each tile of a Penrose tiling, then such a configuration would produce a scattering pattern with a ten-fold rotational symmetry *unobtainable by any crystal.* And this became of much greater interest when real materials ("quasicrystals") were found which actually produce such patterns. It is still not known if there is a good reason to model such real materials by means of (3-dimensional versions of) Penrose-like tilings, but much is being written on the subject. The point to be emphasized is that the symmetries of the scattering patterns associated with a tiling are precisely the statistical symmetries of the tilings, as discussed above; scattering patterns are not sensitive to the positions of every scatterer, but only sensitive to statistical properties of the positions. In summary, condensed matter physicists are interested in hierarchical tilings precisely for the statistical symmetries associated with them.

The information flow from tilings to condensed matter physics is not just one-way. Once the above connection was made, it became interesting to ask for instance whether there was an analog in tilings to the phenomena associated with the temperature dependence of crystals, such as phase transitions. For solids it is believed (not proven) that solids melt, as temperature is raised, "because" with increasing temperature the periodic structure of the crystal contains more and more defects, which destroy the structure at some fixed temperature. It is then natural to investigate "imperfect tilings", in which defects are allowed. There is not yet much progress in this direction. (There is precious little proven in physics models either; phase transitions is a difficult subject. There is an interesting discussion of the value of this subject to mathematics in the 1988 Gibbs Lecture of David Ruelle.)

Mathematically, we are studying the class of tilings generated by some family S of polyhedra. In a way this is like solving a differential equation, where a (global) function is determined by the local constraints represented by the differential equation; for tilings by some S, the global tiling is determined by the local restrictions dictating how the tiles can fit together. And just as initial or boundary conditions play the role of reducing to a singleton the set of solutions of a differential equation, for tilings the hierarchical assumption plays such a role. From the hierarchical assumption it follows that all the tilings associated with an S must be locally indistinguishable, and this is the appropriate notion of uniqueness for tilings. We will illustrate the general argument with the pinwheel tilings.

We begin by encoding the information of the hierarchical structure into matrices, in such a way that the different levels of the hierarchy correspond to powers of the matrices. We will say that a composite object making up a large scale tile, congruent to some tile P but larger by a power k of the scale factor, is the tile P "at level k". For the pinwheel we label the two tiles in S as "type 1" and "type 2", as in Figure 6a; note they are just reflections of one another. Then we define a one parameter family of matrices, $A[m]$, with parameter $m \in \mathbf{Z}$, by

$$A[m]_{jk} = \sum_n e^{ima_n(j,k)} \qquad 1)$$

where $a_n(j,k)$ is the angle of rotation, of the n^{th} copy of the tile of type j which appears in a tile at level 1 of type k, the angle being with respect to some standard orientation. (See Figure 6b; note for instance that in a tile at level 1 of type 2 there are three tiles of type 1, and two tiles of type 2.) For the pinwheel tilings the matrices are:

$$\begin{pmatrix} e^{-ims}+e^{-im(s+\pi)} & 2e^{-im(s+\pi)}+e^{-im(s+3\pi/2)} \\ 2e^{im(s+\pi)}+e^{im(s+3\pi/2)} & e^{ims}+e^{im(s+\pi)} \end{pmatrix}$$

Setting $m=0$, it is easy to see that ${A[0]^N}_{jk}$ is the number of copies of tiles of type j in a large tile of type k corresponding to level N of the hierarchy. Setting $m=1$, we see that ${A[1]^N}_{jk}$ keeps track of all the angles of all the tiles in such a large tile. Since every tiling is hierarchical, to show that all tilings associated with some S are statistically the same (ignoring angles for the moment) all one needs to show is that all large scale tiles are statistically the same. But that follows from the above matrix analysis using a theorem of Perron and Frobenius, which shows for instance that

$$\frac{{A[0]^N}_{jk}}{{A[0]^N}_{j'k}} \longrightarrow_{N\to\infty} f(j,j') \neq 0$$

And finally we can apply the "Weyl criterion", used for proving that a sequence of numbers is uniformly distributed on an interval, to the sequence (labelled by N) of angles in the matrix element ${A[1]^N}_{jk}$, obtaining a simple geometric criterion, satisfied for the pinwheel, for uniform distribution.

In the above analysis of the pinwheel we treated the system as if, like the Penrose system, there were only two tiles in S, a right triangle and its reflection, with some pattern of bumps and dents on them. This was mildly misleading, as in fact S is a very large (but finite) set, with many different patterns of bumps and dents on each of those two triangles. And this is a story in itself.

The way one creates a set S as in Figure 3 is to start with an appropriate given tiling with simple shapes, like that shown in Figure 2 for the Penrose system. Of course the quadrilaterals in Figure 2 are not in themselves satisfactory for S since not *all* tilings with such tiles would have the hierarchical structure. So one needs to alter them, by adding bumps and dents on the edges, so that the *only* way the tiles can tile space is like the given tiling with the original simpler shapes. For the Penrose system it is easy to see that the bumps and dents of Figure 3 do the job. For the pinwheel system it is much harder to determine *if and how* we can make such alterations to the basic triangle so the tiles only tile as in Figure 1. However, the case of "square-ish" tiles, discussed above with regard to Wang's thesis, has been well studied and it has been shown rather generally, by Shahar Mozes, how to make appropriate alterations for a given "square-ish" hierarchical structure.

To be more specific assume we have a set $\mathcal{A}$ of four unit square "colored" tiles, {A, B, C, D}, and the hierarchical rule of Figure 7 (analogous to Figure 6b for the pinwheel), which associates an array of four unit square colored tiles to each element of $\mathcal{A}$. This rule determines a family of tilings of the plane with the four colored tiles, as follows. Using the convention adopted earlier we refer to each of the four composite objects in Figure 7 as tiles of level 1, and extend the notion to tiles at all levels by repeating the rule. We can then consider the special class of tilings, by the tiles of $\mathcal{A}$, which displays associated tilings at all levels; Figure 8 shows parts of levels 0 and 1 of such a tiling. What we want to do next is alter the edges of the four squares in $\mathcal{A}$ so that the altered tiles can *only* tile the plane in this hierarchical way. This is harder than for the Penrose system, and rests on the pioneering work of Berger, Raphael Robinson and others. Unfortunately we do have to enlarge the set $\mathcal{A}$ greatly in this process, having many different "versions" of each of the original four elements, each with various bumps and dents on each of their four sides.

The above method does not seem to generalize directly to non-square hierarchical structures; the first test case was the pinwheel, and the method used for it was much more complicated. But it is hoped that now that the pinwheel is "solved" a truely broad method can be produced. Effectively this would give us a general way to realize hierarchical structures as produced from local rules, which could be useful, for example, in applied areas where at present hierarchical structures are mostly put into models in an *ad hoc* manner, without any attempt to show how they could arise from (local) physical laws.

In conclusion, we have used tilings of space to illustrate a nontraditional, statistical form of symmetry, motivated in part by ideas from logic and physics. This mathematics has been slow to develop over the past thirty years, but seems at present to fit comfortably as a geometric extension (geometric in its use of rotations and similarities) of parts of ergodic theory, a rich extension which should prove fruitful to all the overlapping subjects.

FURTHER READING

Notes. The Penrose and pinwheel tilings are presented in [2] and [6] respectively. Square-ish tilings are introduced *beautifully* in [8], together with their connection with certain decidability problems. Chapters 10 and 11 in [3] are a general introduction to the literature on the sort of tilings used in this article, comprehensive until the mid-1980's.

The above works are mainly combinatorial geometry, and require little background. After Penrose found his tilings, conceptual progress required the use of tools from analysis, chiefly ergodic theory. In particular [4] gives a general method for creating hierarchical tilings (that is, finding the bumps and dents) for appropriate given square-ish tilings; and [7] gives a general method to determine the statistical symmetry of hierarchical tilings.

A connection with statistical mechanics appears to be intrinsic to the subject. There is a fine introduction to the use of statistical mechanics in mathematics in [9], and a more specialized study of the connection between statistical mechanics and tilings in [5]. They both require a range of tools from analysis. [1] gives the connection between the way a beam of waves scatters off a configuration of scatterers, and the dynamical spectrum associated with the configuration. It is written in the language of physics.

[1] S. Dworkin, 'Spectral theory and x-ray diffraction', *J. Math. Phys.* **34** (1993), 2965-2967.

[2] M. Gardner, 'Extraordinary nonperiodic tiling that enriches the theory of tiles', *Sci. Am. (USA)* **236** (1977), 110-119.

[3] B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, Freeman, New York, 1986.

[4] S. Mozes, 'Tilings, substitution systems and dynamical systems generated by them', *J. d'Analyse Math.* **53** (1989), 139-186.

[5] C. Radin, 'Global Order from Local Sources', *Bull. Amer. Math. Soc.* **25** (1991), 335-364.

[6] C. Radin, 'The pinwheel tilings of the plane', *Annals of Math.* **139** (1994), 661-702.

[7] C. Radin, 'Space tilings and substitutions', *Geometriae Dedicata*. To appear.

[8] R.M. Robinson, 'Undecidability and nonperiodicity for tilings of the plane', *Invent. Math.* **12** (1971), 177-209.

[9] D. Ruelle, 'Is our mathematics natural? The case of equilibrium statistical mechanics', *Bull. Amer. Math. Soc.* **19** (1988), 259-268.

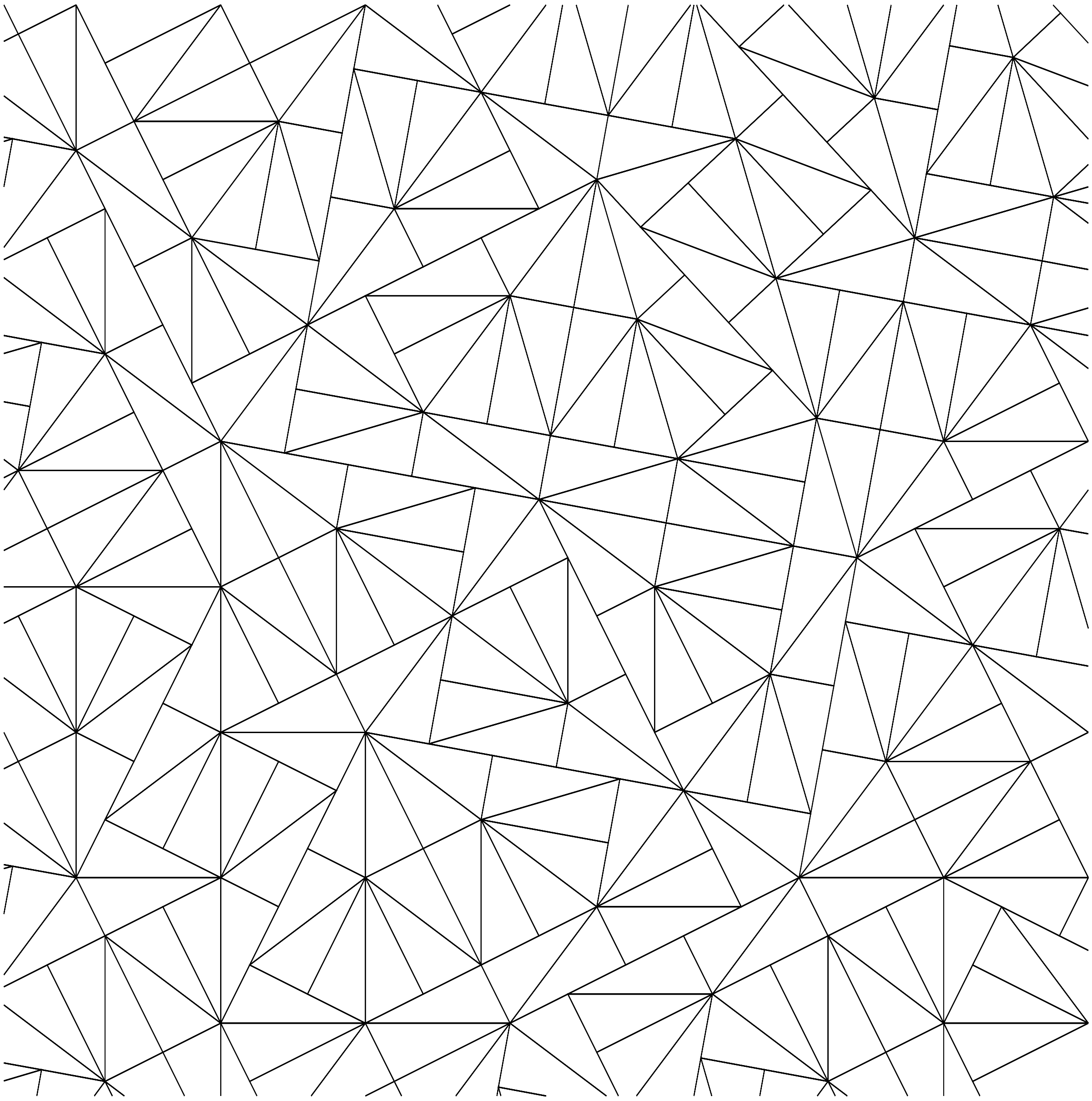


Figure 1

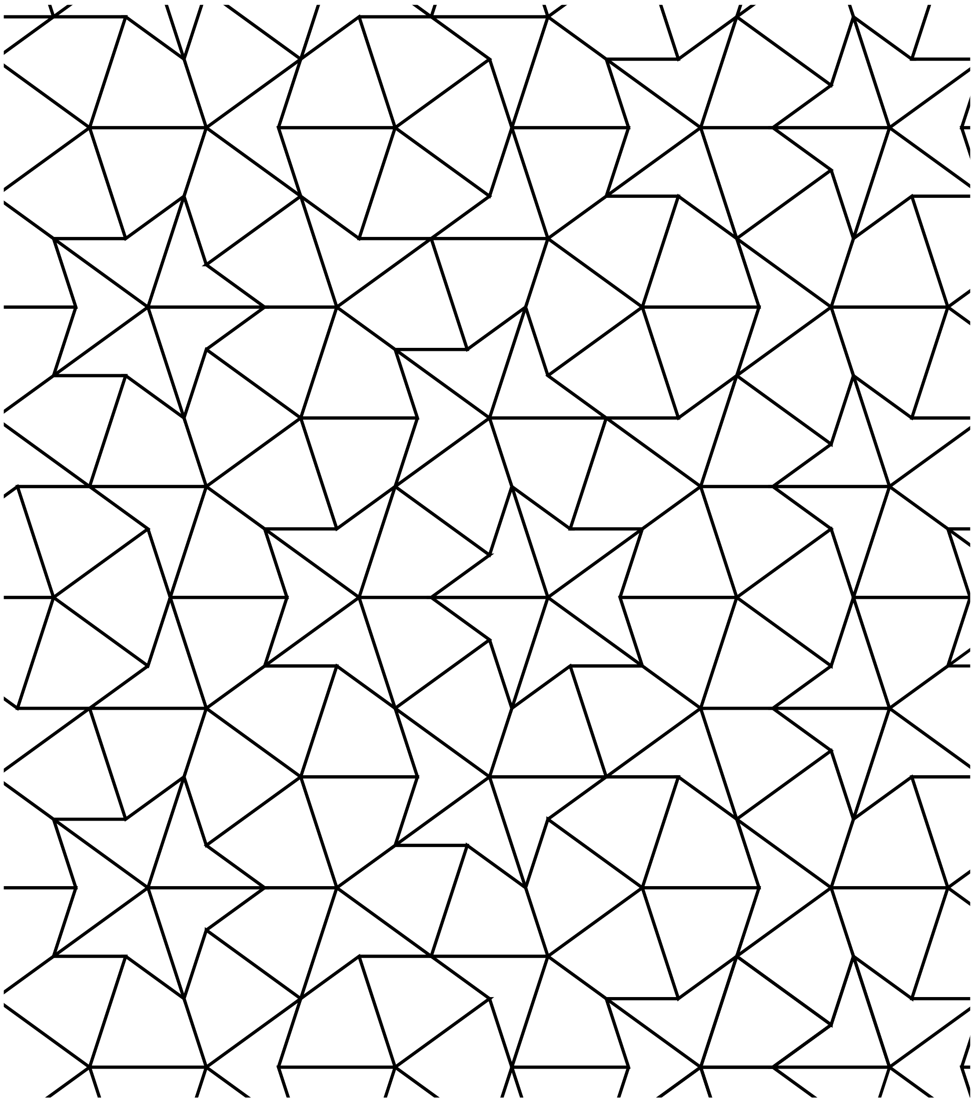
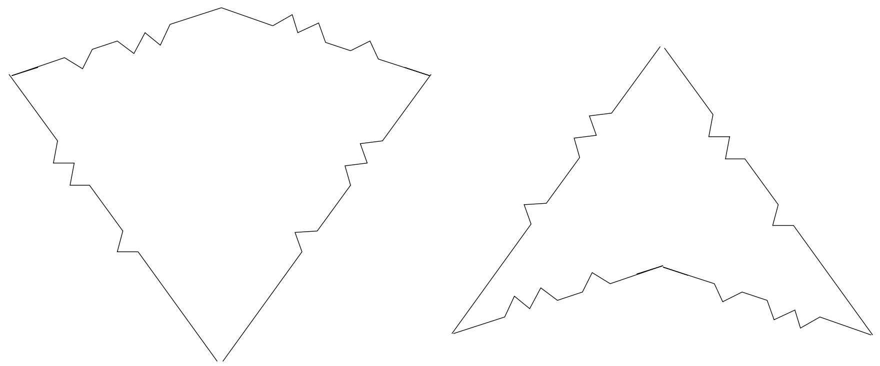


Figure 2



kite dart

Figure 3

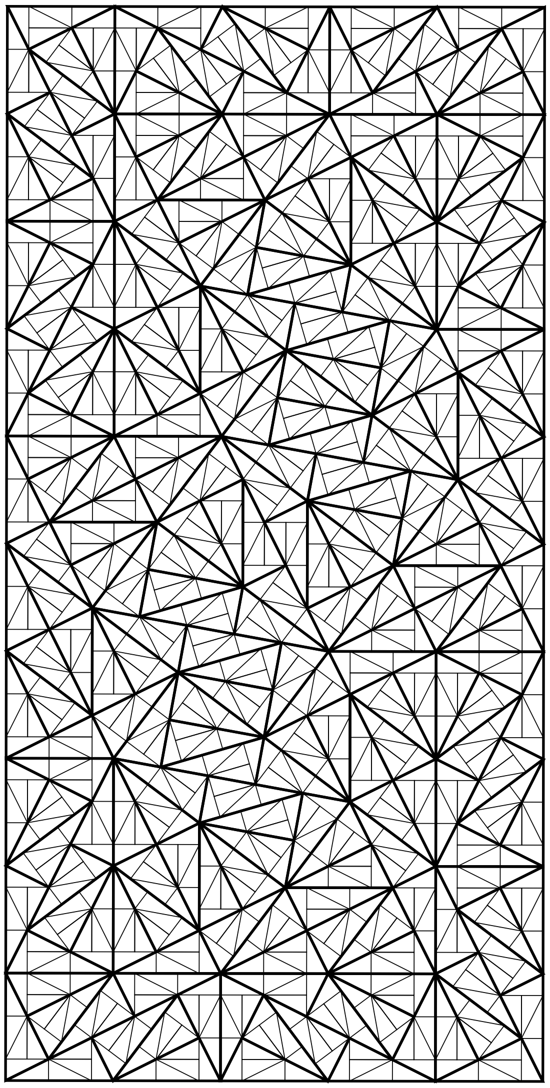


Figure 4

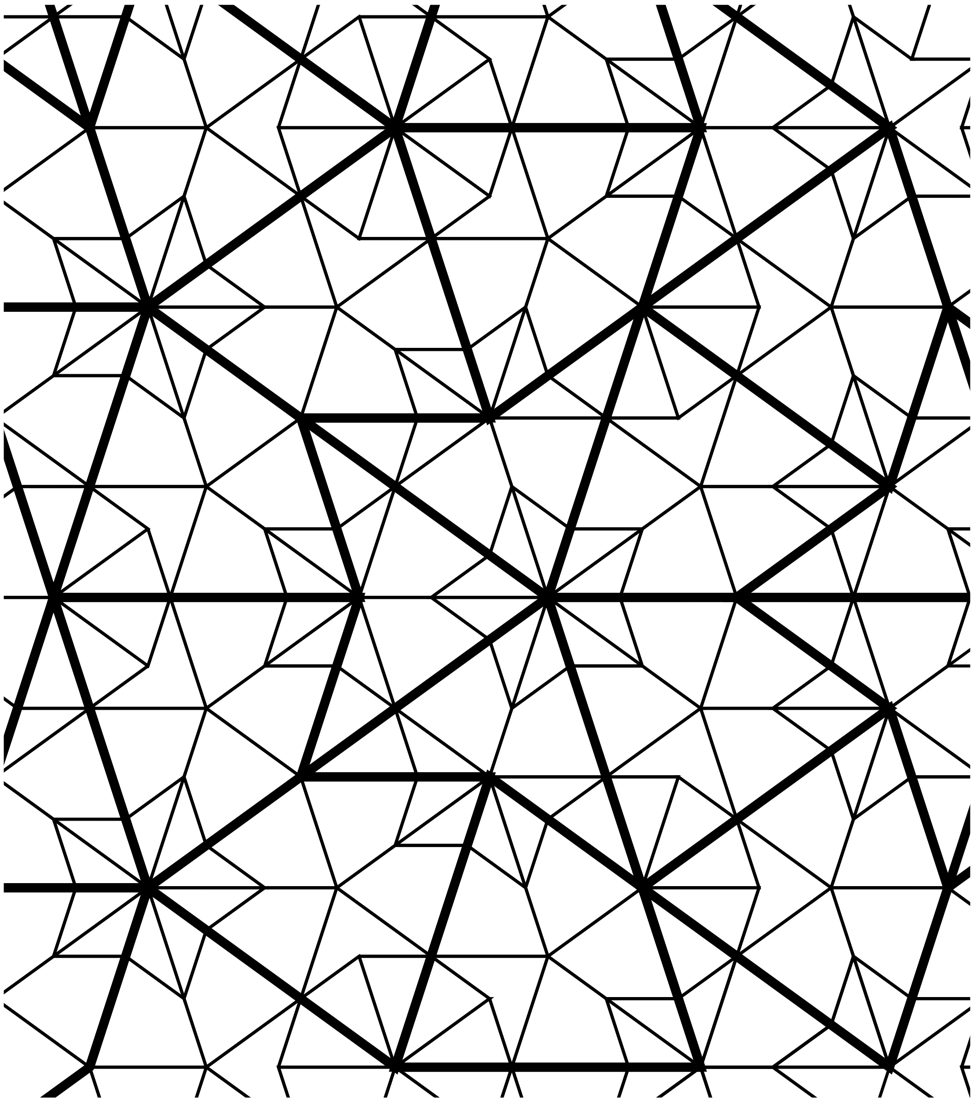


Figure 5

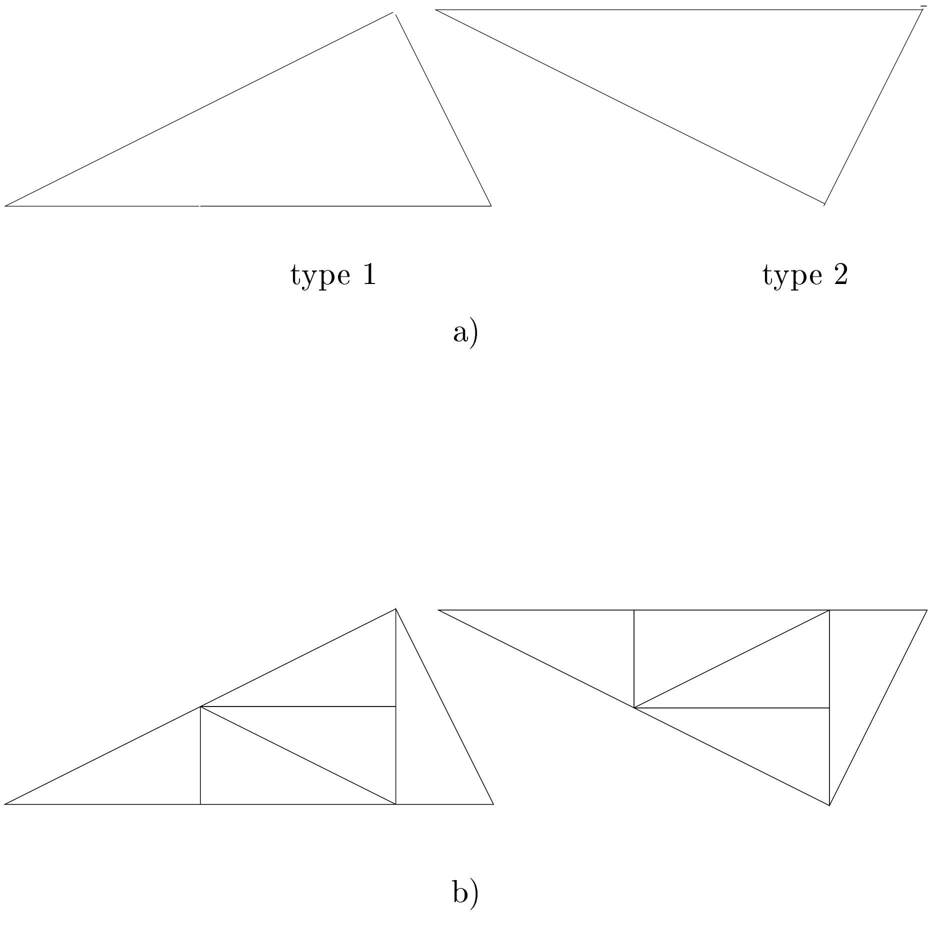


Figure 6

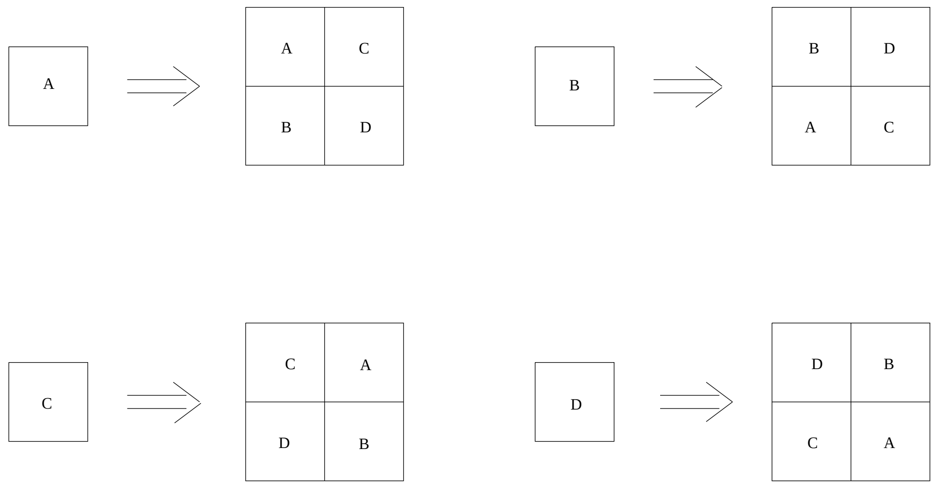


Figure 7

D	B	B	D	B	D	D	B	B	D	D	B	D	B	B	D
C	A	A	C	A	C	C	A	A	C	C	A	C	A	A	C
C	A	A	C	A	C	C	A	A	C	C	A	C	A	A	C
D	B	B	D	B	D	D	B	B	D	D	B	D	B	B	D
C	A	A	C	A	C	C	A	A	C	C	A	C	A	A	C
D	B	B	D	B	D	D	B	B	D	D	B	D	B	B	D
D	B	B	D	B	D	D	B	B	D	D	B	D	B	B	D
C	A	A	C	A	C	C	A	A	C	C	A	C	A	A	C
C	A	A	C	A	C	C	A	A	C	C	A	C	A	A	C
D	B	B	D	B	D	D	B	B	D	D	B	D	B	B	D
D	B	B	D	B	D	D	B	B	D	D	B	D	B	B	D
C	A	A	C	A	C	C	A	A	C	C	A	C	A	A	C
D	B	B	D	B	D	D	B	B	D	D	B	D	B	B	D
C	A	A	C	A	C	C	A	A	C	C	A	C	A	A	C

Figure 8