AN ASYMPTOTIC EXPANSION FOR THE DENSITY OF STATES OF A RANDOM SCHRÖDINGER OPERATOR WITH BERNOULLI DISORDER

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Abstract. In this paper, we study the density of states of a random Schrödinger operators of the form $H(t) = H + \sum_{\gamma\in\mathbb{Z}^d} t_\gamma V_\gamma$. Here H is a periodic Schrödinger operator, V is an exponentially decreasing function and V_γ, its translate by γ; the random variables $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ are chosen i. i. d. with the following common Bernoulli probability measure: $t_\gamma = 1$ with probability p, and $t_\gamma = 0$ with probability $1-p$. We show that $N_p(d\lambda)$, the density of states of $H(t)$, has an asymptotic expansion in p when $p\to 0$. Then, we use this expansion to deduce the behaviour of the integrated density of states of $H(t)$ in the gaps of H when p goes to 0.

Résumé. Dans ce travail, nous étudions la densité d'états d'un opérateur de Schrödinger aléatoire de la forme $H(t) = H + \sum_{\gamma\in\mathbb{Z}^d} t_\gamma V_\gamma$. H est un opérateur de Schrödinger périodique, V est un potentiel exponentiellement décroissant et V_γ, son translaté par γ; les variables aléatoires $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ sont supposées i.i.d avec pour loi de probabilité commune la loi de Bernoulli suivante: $t_\gamma = 1$ avec probabilité p, et $t_\gamma = 0$ avec probabilité $1-p$. Nous démontrons l'existence d'un développement asymptotique en p pour $N_p(d\lambda)$, ceci quand $p\to 0$. Puis nous utilisons ce développement pour estimer la taille de la densité d'états intégrée de $H(t)$ dans les lacunes du spectre de H quand p tend vers 0.

0) Introduction.

The present paper is devoted to the study of the denstity of states of a simple model of random Schrödinger operators. The model (denoted by $H(t)$) consists in a periodic Schrödinger operator (denoted by H) to which impurities (represented by replicas of an exponentially decreasing potential V) have been added in the following way: at each site (represented by a point in $\mathbb{Z}^d$), there is a probability p of finding an impurity, and probability $1-p$ of finding none. p is a parameter controlling the concentration of impurities in our model. The same model has been studied by R. Hempel and W. Kirsch in [H-Ki] which inspired the present work.

Let us call $N_p(d\lambda)$, the density of states for our model. Then N_p is supported by the almost sure spectrum of $H(t)$, which does not depend on p, but, in the gaps of the spectrum of H, the density of states tends to 0 (in a weak sense) as p tends to 0 (cf [H-Ki]). As our main result, we prove

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that $N_p(d\lambda)$ admits an asymptotic expansion (in the distributional sense) in powers of p when p tends to 0, i.e, there exists a sequence of distributions $(n_k)_{k\geq 0}$ such that

$$N_p(d\lambda) \underset{\substack{p\to 0\\ p>0}}{\sim} \sum_{k\geq 0} p^k n_k, \tag{0.1}$$

the precise meaning of this asymptotic expansion being given in Theorem 1.1.

We compute the distributions $(n_k)_{k\geq 0}$. n_0 is the density of states of the periodic Schrödinger operator H. For $k\geq 1$, let Λ be a set of k sites, and H_Λ be the hamiltonian H perturbed by putting exactly one impurity at each site of Λ. This is a relatively compact perturbation of H; moreover one can define $\zeta(\lambda;\Lambda)$, the spectral shift function for the pair (H_Λ, H) (cf [Ya]). Then we get the following formula for n_k

$$n_k(\lambda) = -\frac{1}{k!} \sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k \text{ and } 0\in\Lambda}} \sum_{A\subset\Lambda} (-1)^{\#\Lambda-\#A}\zeta'(\lambda;A). \tag{0.2}$$

In the gaps of H, H_Λ has only discrete eigenvalues. We say that λ is a k-eigenvalue if it is an eigenvalue for some H_Λ with $\#\Lambda = k$. Then, using (0.1) and (0.2), we show that, for I an open interval strictly contained in a gap of H,

(a) $N_p(I) \sim \dfrac{\mathfrak{n}_k(I)}{k!}p^k$ when $p\to 0$, if I contains k-eigenvalues but no j-eigenvalues for $1\leq j<k$, here $\mathfrak{n}_k(I)$ is the number of k-eigenvalues (counted with multiplicity) contained in I,

(b) $N_p(I) = O(p^{k+1})$ when $p\to 0$, if I does not contain any k-eigenvalue.

In particular, we recover the results of [H-Ki]. We see that there is a concentration of the density of states in the neighborhood of the k-eigenvalues. This notion of concentration is made more precise in Theorem 1.4. Such a concentration hints to the fact that there may be singularities in the density of states at the k-eigenvalue. These singularities were studied for a related model in dimension $d=1$ in [Ni-Lu]. In dimension $d=1$, upper bounds on the regularity of the density of states of the Anderson model with Bernoulli disorder were proved in [Si-Ta]. These upper-bounds imply that the density of states looses its regularity as p tends to 0. In [Ca-Kl-Ma], it was proved that, for the Anderson model with Bernoulli disorder, for sufficiently large perturbations, the density of states has a singular continuous part. This result is for arbitrarily chosen and fixed p.

Formally at least, the proof of the asymptotic expansion is very simple. By [Pa-Fi], we know that the density of states of $H(t)$ is given by $N_p(d\lambda) = \mathbb{E}_p\{\mathrm{Tr}(\chi_0 E_t(d\lambda)\chi_0)\}$. Hence we write a Taylor expansion in p at $p=0$ for the measure $\bigotimes_{\gamma\in\mathbb{Z}^d}(p\delta_1 + (1-p)\delta_0)$ and integrate this against the function $\mathrm{Tr}(\chi_0 E_t(d\lambda)\chi_0)$. Of course this cannot be done in such a straight forward way as the measure one obtains through the Taylor expansion are not signed and the sums involved in the expansion are not absolutely convergent.

The method developed here can be applied to many other models of random Schrödinger operators where the randomness is given by Bernoulli disributions of parameter p, for example, random operators on a lattice like the Anderson model, random magnetic Schrödinger operator in 2-dimension.

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I) The main results.

Let W be a $\mathbb{Z}^d$-periodic, bounded, real-valued function on $\mathbb{R}^d$. Let us define the following periodic Schrödinger operator: $H = -\Delta + W$. H is essentially self-adjoint on $L^2(\mathbb{R}^d)$, lower semi-bounded and has $H^2(\mathbb{R}^d)$ as domain. Let $\sigma(H)$ be the spectrum of H. For the sake of simplicity, we may without restriction assume that $\inf(\sigma(H)) > 0$ (just by shifting H by a constant). We recall that the spectrum of H is made of bands of purely absolutely continuous spectrum (cf [Re-Si] or [Sj]).

Let V be a function that is not identically 0 and that satisfies

(H.1) for some $\eta > 0$, and some $C > 0$, $|V(x)| \leq Ce^{-\eta|x|}$ for $x \in \mathbb{R}^d$.

Let $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ be independently identically distributed Bernoulli random variables taking value 1 with probability p and value 0 with probability $1-p$; in other words, their common probability measure is defined by $P(t_0 = 1) = p$ and $P(t_0 = 0) = 1 - p$.

We now consider the following random Schrödinger operator

$$H(t) = H + \sum_{\gamma\in\mathbb{Z}^d} t_\gamma V_\gamma \qquad \text{where } V_\gamma(x) = V(x-\gamma). \tag{1.1}$$

$H(t)$ is a lower semi-bounded, essentially self-adjoint, ergodic random Schrödinger operator. Its domain is $H^2(\mathbb{R}^d)$. As $H(t)$ is ergodic, there exists a closed set Σ such that, with probability 1, $\Sigma = \sigma(H(t))$ (cf [Pa-Fi], [Ca-La]). As the potentials $\sum_{\gamma\in\mathbb{Z}^d} t_\gamma V_\gamma$ are uniformly bounded for all realizations of $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$, by shifting H by a constant, we may assume that $\inf(\Sigma) > 0$. Moreover, as noted in [H-Ki], Σ is independent of p the parameter defining the probability measure of the $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$.

a) The density of states.

Let Λ_l be a cube in $\mathbb{R}^d$ centered at 0 and of sidelentgh l. Define $H_l^D(t) = H(t)_{|\Lambda_l}$ i.e. $H(t)$ restricted to Λ_l with Dirichlet boundary conditions. Define, for $\lambda \in \mathbb{R}$,

$$\mathcal{N}_l(\lambda) = \frac{1}{\mathrm{Vol}(\Lambda_l)} \#\{\lambda_n;\ \lambda_n \text{ is an eigenvalue of } H_l^D(t) \text{ and } \lambda_n \leq \lambda\}$$

and $N_l(d\lambda) = \partial_\lambda \mathcal{N}_l$, the corresponding discrete measure on $\mathbb{R}$. Then, e.g. by Theorem 5.20 of [Pa-Fi], there exists a non-random measure $N_p(d\lambda)$ such that, with probability 1,

$$\lim_{l\to+\infty} N_l(d\lambda) = N_p(d\lambda),$$

and

$$N_p(d\lambda) = \mathbb{E}_p\{\mathrm{Tr}(\chi_0 E_t(d\lambda)\chi_0)\}$$

where $E_t(d\lambda)$ is the spectral resolution of $H(t)$, χ_0 is the characteristic function of the cube centered in 0 with sidelength 1, $\mathbb{E}_p$ is the average with respect to the probability measure defined by the $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ and $\mathrm{Tr}(A)$ is the trace of A.

$N_p(d\lambda)$ is the ***density of states*** of $H(t)$. It is a positive measure supported in Σ. One has

$$\int_{\mathbb{R}} \lambda^{-q} N_p(d\lambda) < +\infty \quad \text{for} \quad q > d/2$$

as $\chi_0(1-\Delta)^{-q}$ is trace class for $q > d/2$ and $H(t)$ is a uniformly bounded perturbation of $-\Delta$ over all realizations of t. Hence $N_p(d\lambda)$ is rapidly decreasing test function,a tempered distribution, that is an element of the Schwartz space $\mathcal{S}'(\mathbb{R})$.

Before stating our main result let us introduce a last definition. Let $A \subset \mathbb{Z}^d$ such that A is finite, and define $H_A = H + \sum_{\gamma \in A} V_\gamma$. H_A is relatively compact perturbation of H; moreover, for $q > d/2$ and $z \notin \sigma(H_A) \cup \sigma(H)$, $(z - H_A)^{-q} - (z - H)^{-q}$ is trace class. By the arguments of [Ki-Ma], we know that $\inf(\sigma(H_A)) \geq \inf(\Sigma) > 0$. Hence, we define $\zeta(\lambda; A)$ to be the *spectral shift function* for the pair of operators H_A and H; $\zeta(\lambda; A)$ is the distribution in $\mathcal{S}'(\mathbb{R})$ defined by, for $\phi \in \mathcal{S}(\mathbb{R})$,

$$Tr(\phi(H_A) - \phi(H)) = \int_0^{+\infty} \zeta(\lambda; A)\phi'(\lambda)d\lambda$$

(see [Ya] chapter 8 section 9).

Let us now state our main result that is

Theorem 1.1. *$N_p(d\lambda)$ admits an asymptotic expansion in $\mathcal{S}'(\mathbb{R})$ when p tends to 0 i.e, there exists a sequence of distributions $(n_k)_{k\geq 0}$ such that*

(a) for any $k \geq 0$, $n_k \in \mathcal{S}'(\mathbb{R})$,

(b) for any $N > 0$, there existe $|\cdot|_N$, a semi-norm in $\mathcal{S}'(\mathbb{R})$ such that, for any $\varphi \in \mathcal{S}(\mathbb{R})$, rapidly decreasing test function, one has

$$|\langle N_p(d\lambda), \varphi\rangle - \sum_{k=0}^{N} p^k \langle n_k, \varphi\rangle| \leq p^{N+1} \mid \varphi \mid_N \quad \textit{for} \quad p \in [0,1].$$

Moreover the distributions $(n_k)_{k\in\mathbb{N}}$ are given by the following formulae:

(c) if $k = 0$, n_0 is the density of states for the unperturbed operator H,

(d) if $k \geq 1$, n_k is given by the following convergent (in $\mathcal{S}'(\mathbb{R})$) series:

$$n_k(\lambda) = -\frac{1}{k!} \sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{A \subset \Lambda} (-1)^{\#\Lambda - \#A} \zeta'(\lambda; A),$$

where $\zeta'(\lambda; A)$ is the derivative of $\zeta(\lambda; A)$ with respect to λ.

Remark. Notice that, as $\text{supp}N_p(d\lambda) \subset \Sigma$, for any $k \geq 0$, one has $\text{supp}n_k \subset \Sigma$. In the proof of Theorem 1.1, we also obtain an upper bound on the order of n_k which is $\text{ord}(n_k) \leq (k+1)(d+1)+2$. The same way one can give an estimate upon the order of the semi-norm used to control the remainder in the asymptotic expansion; if the expansion is made up to order N, then the remainder is at most of order $(N+2)(d+1)+2$.

b) The behaviour of the integrated density of states in the gaps of the spectrum of H when p tend to 0.

Let $\Lambda \subset \mathbb{Z}^d$, Λ finite, and define $H_\Lambda = H + \sum_{\gamma\in\Lambda} V_\gamma$. Let $\sigma(H_\Lambda)$ be the spectrum of H_Λ. We know that $\inf_{\Lambda \subset \mathbb{Z}^d} \inf(\sigma(H_\Lambda)) > 0$. By our assumptions on V, H_Λ is relatively compact perturbation of H; hence, by Weyl's Theorem, for any Λ finite, the essential spectrum of H_Λ is the essential spectrum of H that is the spectrum of H. By $\Sigma_{\text{disc}}(\Lambda)$, we denote the closure of discrete spectrum of H_Λ (i.e. the eigenvalues of H_Λ contained in the gaps of H as well as, possibly, the edges of the gaps). Then we define

Definition. For $k \geq 1$, we define the k-eigenvalues of (H,V) to be the elements of the set $\mathcal{E}_k = \bigcup_{\#\Lambda=k} \Sigma_{\text{disc}}(H_\Lambda)$.

Remark. For $A \in \mathbb{Z}^d$, A finite, we know that $\zeta(\lambda;A)$ is constant in the gaps of $\sigma(H)$ except at the eigenvalues of H_A where it has a jump discontinuity (see [Ya]). Hence, formula (d) of Theorem 1.1 tells us that, for $k \geq 1$, $\operatorname{supp}(n_k) \subset \sigma(H) \bigcup \bigcup_{1\leq j\leq k} \mathcal{E}_k$.

One shows

Proposition 1.2. *$\overline{\mathcal{E}_1} = \mathcal{E}_1$ and for any $k \geq 2$, $\overline{\mathcal{E}_k} = \mathcal{E}_k \bigcup \overline{\mathcal{E}_{k-1}} = \bigcup_{j=1}^{k} \mathcal{E}_j$. Moreover, for I, a closed interval contained in $\mathbb{R} \setminus \sigma(H)$, the points of $\overline{\mathcal{E}_k \cap I} \setminus \overline{\mathcal{E}_{k-1} \cap I}$ are isolated in $\overline{\mathcal{E}_k \cap I}$. Here $\overline{A}$ denotes the closure of $A \subset \mathbb{R}$.*

For $I \subset \mathbb{R} \setminus \sigma(H)$, we define $\mathfrak{n}_k(I)$ to be the number of k-eigenvalues of (H,V) in I counted with multiplicities, that is, if we define $\Pi_\Lambda(\lambda)$ to be the eigenprojector of H_Λ associated to the eigenvalue λ, then

$$\mathfrak{n}_k(I) = \sum_{\substack{\Lambda\subset\mathbb{Z}^d \\ \#\Lambda=k \text{ and } 0\in\Lambda}} \sum_{\lambda\in I\cap\sigma(H_\Lambda)} \operatorname{rank}(\Pi_\Lambda(\lambda)).$$

Remark. Notice that this number may be infinite. In the definition of $\mathfrak{n}_k(I)$, we restrict ourselves to counting the eigenvalues of H_Λ for $\#\Lambda = k$ and $0 \in \Lambda$. The condition "$0 \in \Lambda$" permits us to get rid of a natural infinite multiplicity coming from the fact that, as H is $\mathbb{Z}^d$-periodic, H_Λ and $H_{\Lambda+\gamma}$ are unitarily equivalent for any $\gamma \in \mathbb{Z}^d$.

The behaviour of the integrated density of states in the gaps of the spectrum of H when p tend to 0 is given by

Theorem 1.3. *Let I be a open interval such that $\overline{I} \subset \mathbb{R} \setminus \sigma(H)$. Then,*

(a) if $I \cap \mathcal{E}_k \neq \emptyset$ and for $1 \leq j \leq k-1$, $\overline{I} \cap \mathcal{E}_j = \emptyset$ then, $0 < \mathfrak{n}_k(I) < +\infty$,

$$N_p(I) = \int_I N_p(d\lambda) = \frac{\mathfrak{n}_k(I)}{k!} p^k (1 + O(p)) \quad \textit{when } p \to 0,\ p > 0,$$

(b) if for $1 \leq j \leq k$, $\overline{I} \cap \mathcal{E}_j = \emptyset$ then,

$$N_p(I) = \int_I N_p(d\lambda) = O(p^{k+1}) \quad \textit{when } p \to 0,\ p > 0.$$

Remark. If the $(n_k)_{k\geq 0}$ were distributions of order 0, we could easily deduce the non-regularity of the density of states from Theorem 1.1. Here this is not possible because of the bad estimates we have on the remainder term; bad here means that we do not control the order.

We recover in a more precise form and extend the results of [H-Ki]. As noted in that paper, Theorem 1.3 proves that the integrated density of states concentrates around the k-eigenvalues of (H,V). This notion of concentration can be made more precise for we have

Theorem 1.4. *Let μ be an isolated k-eigenvalue i.e. $\mu \in \mathcal{E}_k$ and $\mu \notin \mathcal{E}_j$ for $1 \le j \le k-1$. Then, for any* $0 < \epsilon < \dfrac{1}{(k+2)(d+1)+2}$,

$$p^{-k} N_p([\mu - p^\epsilon, \mu + p^\epsilon]) \to \frac{\mathfrak{n}_k(\{\mu\})}{k!} \quad \textit{when } p \to 0,\ p > 0.$$

Remark. So we see that, in the limit $p \to 0$, the k-eigenvalues are "asymptotic" singular points of the density of states. In [Ni-Lu], these points were found to be singularities of the density of states for some related 1-dimensional model.

II) The asymptotic expansion for the density of states.

By Theorem 5.20 of [Pa-Fi], we know that, for $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi(\lambda) N_p(d\lambda) = \mathbb{E}_p\{\mathrm{Tr}(\chi_0 \varphi(H(t))\chi_0)\}. \tag{2.1}$$

a) A most useful formula.

For any realization of t, $H(t)$ is essentially self-adjoint, and for a real function $\varphi \in \mathcal{S}(\mathbb{R})$, $\varphi(H(t))$ is well defined by the Spectral Theorem and can expressed (see [He-Sj]) by the following formula

$$\varphi(H(t)) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z)(z - H(t))^{-1} d\overline{z} \wedge dz, \tag{2.2}$$

where $\tilde{\varphi}: \mathbb{C} \to \mathbb{C}$ is an extension of φ such that

(a) for $z \in \mathbb{R}$, $\tilde{\varphi}(z) = \varphi(z)$,
(b) $\mathrm{supp}(\tilde{\varphi}) \subset \{z \in \mathbb{C};\ |\mathrm{Im}(z)| < 1\}$,
(c) $\tilde{\varphi} \in \mathcal{S}(\{z \in \mathbb{C};\ |\mathrm{Im}(z)| < 1\})$,
(d) The family of functions $x \mapsto \dfrac{\partial \tilde{\varphi}}{\partial \overline{z}}(x+iy) \cdot |y|^{-n}$ (for $0 < |y| < 1$) is bounded in $\mathcal{S}(\mathbb{R})$ for any $n \in \mathbb{N}$ and, one has the following estimates: for $n \ge 0$, $\alpha \ge 0$, $\beta \ge 0$, there exists $C_{n,\alpha,\beta} > 0$ such that

$$\sup_{0<|y|\le 1} \sup_{x \in \mathbb{R}} \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} \left(|y|^{-n} \cdot \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(x+iy) \right) \right| \le C_{n,\alpha,\beta} \sup_{\substack{\beta' \le n+\beta+2 \\ \alpha' \le \alpha}} \sup_{x \in \mathbb{R}} \left| x^{\alpha'} \frac{\partial^{\beta'} \varphi}{\partial x^{\beta'}}(x) \right|.$$

Such extensions always exist for $\varphi \in \mathcal{S}(\mathbb{R})$ (see [Mat]).

Hence, for $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\mathrm{supp}\varphi \subset (0, +\infty)$, for $q > d/2$, using (2.1) and (2.2) for the function $\psi(x) = x^q \varphi(x)$ (as $\varphi(H(t)) = \psi(H(t))H(t)^{-q}$), we get

$$\int_{\mathbb{R}} \varphi(\lambda) N_p(d\lambda) = \frac{i}{2\pi} \mathbb{E}_p \left\{ \mathrm{Tr}(\chi_0 \left(\int_{\mathbb{C}} \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z)(z - H(t))^{-1} H(t)^{-q} d\overline{z} \wedge dz \right) \chi_0) \right\} \tag{2.3}$$

as $\chi_0(z - H(t))^{-1} H(t)^{-q} \chi_0$ is trace class. Here $\tilde{\varphi}$ is an analytic extension of ψ.

By Proposition 4.3, we know that, there exists $C > 0$ such that for any realization of t, and for $z \in \mathbb{C} \setminus \mathbb{R}$ and $|\mathrm{Im}(z)| \le 1$,

$$|\mathrm{Tr}(\chi_0(z - H(t))^{-1} H(t)^{-q} \chi_0)| \le C \left(\frac{1 + |z|}{|\mathrm{Im}(z)|} \right)^2.$$

Hence using (2.3), estimate (d) for analytic extensions and Fubini's Theorem, we get

$$\int_{\mathbb{R}} \varphi(\lambda) N_p(d\lambda) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) \mathbb{E}_p \left\{ \mathrm{Tr}(\chi_0(z - H(t))^{-1} H(t)^{-q} \chi_0) \right\} d\overline{z} \wedge dz. \tag{2.4}$$

Finally, we are just left with finding an asymptotic expansion for $\mathbb{E}_p\{\mathrm{Tr}(\chi_0(z - H(t))^{-1} H(t)^{-q} \chi_0)\}$.

b) The asymptotic expansion for $\mathbb{E}_{\mathbf{p}}\{\mathbf{Tr}(\boldsymbol{\chi}_{\mathbf{0}}(\mathbf{z}-\mathbf{H}(\mathbf{t}))^{-\mathbf{1}}\mathbf{H}(\mathbf{t})^{-\mathbf{q}}\boldsymbol{\chi}_{\mathbf{0}})\}$.

For $l>0$ and a realization of t, we define $H_l(t)=H+\sum_{\gamma\in\Lambda_l}t_\gamma V_\gamma$ (here Λ_l is cube in $\mathbb{Z}^d$ of center 0 and sidelength l). By Proposition 4.3, we know that, for any configuration of t and for $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\|\chi_0(z-H(t))^{-1}H(t)^{-q}\chi_0-\chi_0(z-H_l(t))^{-1}H_l(t)^{-q}\chi_0\|_{\mathcal{T}_1}\to 0 \quad \text{when } l\to+\infty$$

where $\|\cdot\|_{\mathcal{T}_1}$ denotes the trace class norm.

Hence using Lebesgue's Dominated Convergence Theorem,

$$\mathbb{E}_p\{\mathrm{Tr}(\chi_0(z-H(t))^{-1}H(t)^{-q}\chi_0)\}=\lim_{l\to+\infty}\mathbb{E}_p\{\mathrm{Tr}(\chi_0(z-H_l(t))^{-1}H_l(t)^{-q}\chi_0)\}.$$

By definition $H_l(t)$ depends only on finitely many random variables (i.e. the one indexed by $\gamma\in\Lambda_l$). Hence

$$\begin{aligned} f_l(z,p)&:=\mathbb{E}_p\{\mathrm{Tr}(\chi_0(z-H_l(t))^{-1}H_l(t)^{-q}\chi_0)\}\\ &=\int_{[0,1]^{\Lambda_l}}\mathrm{Tr}(\chi_0(z-H_l(t))^{-1}H_l(t)^{-q}\chi_0)\bigotimes_{\Lambda_l}(p\delta_1+(1-p)\delta_0).\end{aligned}$$

$f_l(z,p)$ obviously is a polynomial in p of degree $\#\Lambda_l$. Hence we may expand it using Taylor's formula to an arbitrary order N and get

$$f_l(z,p)=\sum_{k=0}^{N}\frac{f_l^{(k)}(z,0)}{k!}p^k+\frac{p^{N+1}}{N!}\int_0^1 f_l^{(N+1)}(z,pu)(1-u)^N\,du.$$

To compute the coefficients in this expansion, we notice that we are dealing with a measure on a finite set that depend analytically in the parameter p; so we just have to write the Taylor expansion in p for this measure and integrate the trace against it; we compute

$$\frac{d^k}{dp^k}\left(\bigotimes_{\Lambda_l}(p\delta_1+(1-p)\delta_0)\right)=\sum_{\substack{\Lambda\subset\Lambda_l\\ \#\Lambda=k}}\sum_{\substack{A\cup B=\Lambda\\ A\cap B=\emptyset}}(-1)^{\#B}\bigotimes_{A}\delta_1\bigotimes_{B}\delta_0\bigotimes_{\Lambda_l\setminus\Lambda}(p\delta_1+(1-p)\delta_0)\,.$$

Hence, for $k\geq 1$, we get

$$f_l^{(k)}(z,p)=\sum_{\substack{\Lambda\subset\Lambda_l\\ \#\Lambda=k}}\sum_{A\subset\Lambda}(-1)^{\#(\Lambda\setminus A)}\mathbb{E}_p\{\mathrm{Tr}(\chi_0(z-H_{\Lambda,A}(u))^{-1}H_{\Lambda,A}(u)^{-q}\chi_0)\},$$

where:

(a) $H_{\Lambda,A}(u)=H+\sum_{\gamma\in A}V_\gamma+\sum_{\gamma\in(\Lambda_l\setminus\Lambda)}u_\gamma V_\gamma$,

(b) $(u_\gamma)_{\gamma\in\mathbb{Z}^d}$ are i.i.d random variables with common probability distribution $p\delta_1+(1-p)\delta_0$,

(c) $\mathbb{E}_p$ is the expectation taken with respect to these random variables.

One easily checks that this equality may be rewritten in the following form

$$f_l^{(k)}(z,p)=\sum_{\substack{\Lambda\subset\Lambda_l\\ \#\Lambda=k}}\int_{[0,1]^\Lambda}\mathbb{E}_p\left\{\mathrm{Tr}\left(\chi_0\partial_\Lambda\left[(z-H_\Lambda(t,u))^{-1}H_\Lambda(t,u)^{-q}\right]\chi_0\right)\right\}dt_\Lambda,$$

where:

(a) $H_\Lambda(t,u) = H + \sum_{\gamma\in\Lambda} t_\gamma V_\gamma + \sum_{\gamma\in(\Lambda_l\setminus\Lambda)} u_\gamma V_\gamma$,

(b) $(t_\gamma)_{\gamma\in\Lambda}$ are variables taking value in $[0,1]$, $\partial_\Lambda = \bigotimes_{\gamma\in\Lambda} \partial_{t_\gamma}$ et $dt_\Lambda = \bigotimes_{\gamma\in\Lambda} dt_\gamma$,

(c) $(u_\gamma)_{\gamma\in\mathbb{Z}^d}$ are defined as above.

Define

$$a_{\Lambda,l}(z,p) = \int_{[0,1]^\Lambda} \mathbb{E}_p\left\{\mathrm{Tr}\left(\chi_0 \partial_\Lambda\left[(z - H_\Lambda(t,u))^{-1} H_\Lambda(t,u)^{-q}\right]\chi_0\right)\right\} dt_\Lambda. \tag{2.5}$$

We show the

Proposition 2.1. *(a) For any $\Lambda \subset \mathbb{Z}^d$ and K, an arbitrary compact of $\mathbb{C}\setminus\mathbb{R}$, the sequence of functions $((z,p)\mapsto a_{\Lambda,l}(z,p))_{l\geq 0}$ converges uniformly in $K\times[0,1]$ to a function $(z,p)\mapsto a_\Lambda(z,p)$ when l tends to $+\infty$.*

(b) $a_\Lambda(z,0)$ is given by

$$a_\Lambda(z,0) = \int_{[0,1]^\Lambda} Tr\left(\chi_0 \partial_\Lambda\left[(z - H_\Lambda(t))^{-1} H_\Lambda(t)^{-q}\right]\chi_0\right) dt_\Lambda,$$

(c) There exists $C > 0$ such that for $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k}} \sup_{\substack{p\in[0,1]\\ l>0}} |a_{\Lambda,l}(z,p)| \leq C\left(\frac{1+|z|}{|Im(z)|}\right)^{(d+1)(k+1)}.$$

Then, using Lebesgue's dominated convergence theorem, we get that, for any $N > 0$, $p \in [0,1]$ and $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\mathbb{E}_p\{\mathrm{Tr}(\chi_0(z - H(t))^{-1}H(t)^{-q}\chi_0)\} = \sum_{k=0}^{N} \frac{f_k(z)}{k!}p^k + p^{N+1}G_N(z,p), \tag{2.6}$$

where $(f_k(z))_{0\leq k\leq N}$ and $G_N(z,p)$ are analytic in z for $z \in \mathbb{C}\setminus\mathbb{R}$. The $(f_k)_{0\leq k\leq N}$ are defined by the following formulae

$$f_0(z) = \mathrm{Tr}((z-H)^{-1}H^{-q}\chi_0), \tag{2.7.a}$$

and, for $1 \leq k \leq N$,

$$f_k(z) = \sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k}} \int_{[0,1]^\Lambda} \mathrm{Tr}\left(\chi_0 \partial_\Lambda\left[(z - H_\Lambda(t))^{-1} H_\Lambda(t)^{-q}\right]\chi_0\right) dt_\Lambda \tag{2.7.b}$$

Moreover one has the following growth estimates at infinity and close to the real axis, for $0 \leq k \leq N$,

$$|f_k(z)| \leq C\left(\frac{1+|z|}{|\mathrm{Im}(z)|}\right)^{(d+1)(k+1)} \quad\text{and}\quad \sup_{p\in[0,1]} |G_N(z,p)| \leq C\left(\frac{1+|z|}{|\mathrm{Im}(z)|}\right)^{(d+1)(N+2)}. \tag{2.8}$$

Using the translation invariance of H, for $k \geq 1$, we may rewrite (2.7.b) in the following form

$$
\begin{aligned}
f_k(z) &= \sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \#\Lambda = k}} \sum_{A \subset \Lambda} (-1)^{\#(\Lambda \setminus A)} \mathrm{Tr}\big(\big[(z-H_A)^{-1} H_A^{-q} - (z-H)^{-1} H^{-q}\big]\, \chi_0\big) \\
&= \sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{\gamma \in \mathbb{Z}^d} \sum_{A \subset \Lambda + \gamma} (-1)^{\#(\Lambda \setminus A)} \mathrm{Tr}\big(\big[(z-H_A)^{-1} H_A^{-q} - (z-H)^{-1} H^{-q}\big]\, \chi_0\big) \\
&= \sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{A \subset \Lambda} (-1)^{\#(\Lambda \setminus A)} \sum_{\gamma \in \mathbb{Z}^d} \mathrm{Tr}\Big(\Big[(z-H_{A-\gamma})^{-1} H_{A-\gamma}^{-q} - (z-H)^{-1} H^{-q}\Big]\, \chi_0\Big) \\
&= \sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{A \subset \Lambda} (-1)^{\#(\Lambda \setminus A)} \sum_{\gamma \in \mathbb{Z}^d} \mathrm{Tr}\big(\big[(z-H_A)^{-1} H_A^{-q} - (z-H)^{-1} H^{-q}\big]\, \chi_\gamma\big) \\
&= \sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{A \subset \Lambda} (-1)^{\#(\Lambda ! s A)} \mathrm{Tr}((z-H_A)^{-1} H_A^{-q} Let us now complete the proof of Theorem 1.1. By the estimate (d) for the analytic extension of \psi,
\end{aligned}
$$

by (2.7.b) and estimate (c) of Proposition 2.1, we know that, for $k \geq 1$,

1

$$
\sum_{A \subset \Lambda} (-1)^{\#(\Lambda \setminus A)} \mathrm{Tr}((z-H_A)^{-1} H_A^{-q} - (z-H)^{-1} H^{-q}) \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) \in L^1(\mathbb{C}; dz d\overline{z}) \text{ for any } \Lambda \subset \mathbb{Z}^d
$$

$$
\text{and} \quad G_N(z,p) \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) \in L^1(\mathbb{C}; dz d\overline{z}),
$$

and

$$
\left(\sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \left| \sum_{A \subset \Lambda} (-1)^{\#(\Lambda \setminus A)} \mathrm{Tr}((z-H_A)^{-1} H_A^{-q} - (z-H)^{-1} H^{-q}) \right| \right) \cdot \left| \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) \right| \in L^1(\mathbb{C}; dz d\overline{z}).
$$

Hence we may apply Lebesgue's Dominated Convergence Theorem to (2.4) and (2.6) to get

$$
\int_{\mathbb{R}} \varphi(\lambda) N_p(d\lambda) = \sum_{k=0}^{N} \frac{p^k}{k!} \left(\frac{i}{2\pi} \int_{\mathbb{C}} f_k(z) \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) d\overline{z} \wedge dz \right) + p^{N+1} \left(\frac{i}{2\pi} \int_{\mathbb{C}} G_N(z,p) \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) d\overline{z} \wedge dz \right)
$$

and, for $k\geq 1$, using Lebesgue's Dominated Convergence Theorem and (2.2),
(2.10)

$$\begin{aligned}
&\frac{i}{2\pi}\int_{\mathbb{C}} f_k(z)\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz\\
&=\sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k\text{ and }0\in\Lambda}}\sum_{A\subset\Lambda}(-1)^{\#(\Lambda\setminus A)}\left\{\frac{i}{2\pi}\int_{\mathbb{C}}\mathrm{Tr}((z-H_A)^{-1}H_A^{-q}-(z-H)^{-1}H^{-q})\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz\right\}\\
&=\sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k\text{ and }0\in\Lambda}}\sum_{A\subset\Lambda}(-1)^{\#(\Lambda\setminus A)}\mathrm{Tr}\left(\frac{i}{2\pi}\int_{\mathbb{C}}(z-H_A)^{-1}H_A^{-q}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz\right.\\
&\qquad\qquad\left.-\frac{i}{2\pi}\int_{\mathbb{C}}(z-H)^{-1}H^{-q}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz\right)\\
&=\sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k\text{ and }0\in\Lambda}}\sum_{A\subset\Lambda}(-1)^{\#(\Lambda\setminus A)}\mathrm{Tr}(\varphi(H_A)-\varphi(H)).
\end{aligned}$$

Moreover, using estimate (2.8) and estimate (d) for analytic extensions, we see that the distribution n_k defined by $\varphi\mapsto\langle n_k,\varphi\rangle=\frac{i}{2\pi}\int_{\mathbb{C}} f_k(z)\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz$ is of order at most $(k+1)(d+1)+2$.

By (2.8) and estimate (d) for analytic extensions, $\varphi\mapsto\frac{i}{2\pi}\int_{\mathbb{C}} G_N(z,p)\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz$ defines a distribution in $\mathcal{S}'(\mathbb{R})$ (of order at most $(N+2)(d+1)+2$) that satisfies the following estimate: there exists $C_N>0$ such that, for $\varphi\in\mathcal{S}(\mathbb{R})$,

$$\left|\frac{i}{2\pi}\int_{\mathbb{C}} G_N(z,p)\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz\right|\leq C_{N,\alpha,\beta}\sup_{\substack{\alpha'\leq(N+2)(d+1)+q\\ \beta'\leq(N+2)(d+1)+2}}\sup_{x\in\mathbb{R}}\left|x^{\alpha'}\frac{\partial^{\beta'}\varphi}{\partial x^{\beta'}}(x)\right|. \tag{2.11}$$

Hence, we proved points (a) and (b) of Theorem 1.1. By the definition of the spectral shift function, (2.10) gives us the formula for n_k for $k\geq 1$. For $k=0$, by (2.7.a), we know that $f_0(z)=\mathrm{Tr}(\chi_0(z-H)^{-1}H^{-q}\chi_0)$, so, using the definition of $n_0(d\lambda)$, the density of states for H (see e.g. [Re-Si] or [Sj]), we get

$$\frac{i}{2\pi}\int_{\mathbb{C}} f_0(z)\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)d\overline{z}\wedge dz=\int_{\mathbb{R}}\varphi(\lambda)n_0(d\lambda).$$

This ends the proof of Theorem 1.1.

c) Proof of Proposition 1.2, Theorems 1.3 and 1.4.

Proof of Proposition 1.2. Assume $(\lambda_n)_{n\geq 0}$ is a sequence in $\mathcal{E}_k$ converging to λ_∞. We may assume that λ_n stays in $\overline{G}$, the closure of a fixed gap G of H. We also may assume that $\overline{G}$ is compact as the operators we consider are uniformly semi-bounded. Then for any $n\geq 0$, there exists $\Lambda_n\subset\mathbb{Z}^d$ such that $\#\Lambda_n=k$ and λ_n is an eigenvalue of H_{Λ_n} (in G). Define $\Lambda_n:=\{\gamma_n^1,\dots,\gamma_n^k\}$.

Either λ_∞ is the edge of G in which case the result is proved. If not so, we may assume that λ_n stays in a compact interval I contained in the interior of G.

We define the following equivalence relation on the set $\{1,\dots,k\}$: $j\sim l$ if $\gamma_n^j-\gamma_n^l$ stays bounded when $n\to+\infty$. This relation induces a partition $\{1,\dots,k\}=\cup_{1\leq a\leq b}U^a$. If we define

$U_n^a=\{\gamma_n^j; j\in U^a\}$, then, when $n\to+\infty$, $\mathrm{dist}(U_n^a,U_n^{a'})\to+\infty$ and, for any $j\in U^a$, $U_n^a-\gamma_n^j$ stays bounded. Using the fact that H is translation invariant and that the eigenfunctions associated to eigenvalues of $H_{U_n^a}$ in I are localized near U_n^a (as I is a compact set contained in the interior of the resolvent set of H), we get, that

$$\mathrm{dist}\left(\sigma(H_{\Lambda_n})\cap G, \bigcup_{1\le a\le b}\sigma(H_{U_n^a})\cap G\right)\to 0 \text{ when } n\to+\infty.$$

So, extracting a subsequence from $(\lambda_n)_{n\ge0}$, we may assume that, for some a, $\mathrm{dist}(\lambda_n,\sigma(H_{U_n^a}))\to 0$ when $n\to+\infty$. By construction, the sets $(U_n^a)_{n\ge0}$ are contained in a bounded set (modulo translation). So $\cup_{n\ge0}\sigma(H_{U_n^a})\cap I$ contains at most a finite number of points. Hence λ_n must converge to one of these points, which is an eigenvalue for some $H_{U_n^a}$. Hence λ_n converges to some point in $\mathcal{E}_j$ for $j<k$.

Now take λ a k-eigenvalue that is not a j-eigenvalue for any $1\le j\le k-1$ (i.e. $\lambda\in\overline{\mathcal{E}_k}\setminus\overline{\mathcal{E}_{k-1}}$). Let $(\lambda_n)_{n\ge0}$ be a sequence in $\mathcal{E}_k$ that converges to λ; in the decomposition we have done above, the partition $(U^a)_{1\le a\le b}$ must be in fact reduced to a single set, otherwise λ would be a limit of j-eigenvalues for some $1\le j\le k-1$. Hence, $U^1=\{1,\dots,k\}$ and $U_n^1-\gamma_n^1$ stays bounded so there only finitely many possible values for the sequence $(\lambda_n)_{n\ge0}$; hence this sequence is constant equal to λ for n large enough. So λ is isolated in $\overline{\mathcal{E}_k}$.

Proof of Theorem 1.3. Assume that I be a open interval such that $\overline{I}\subset\mathbb{R}\setminus\sigma(H)$ and such that $I\cap\mathcal{E}_k\neq\emptyset$ and for $1\le j\le k-1$, $I\cap\mathcal{E}_j=\emptyset$ then, any point $\lambda\in I\cap\mathcal{E}_k$ is isolated in $\overline{\mathcal{E}_k}$, so by the proof of Proposition 1.2, it is an eigenvalue of H_Λ for a finite number of sets $\Lambda\subset\mathbb{Z}^d$ such that $\#\Lambda=k$; hence, as for any of these H_Λ, it is of finite multiplicity, we get $1\le\mathfrak{n}_k(I)<+\infty$.

$\overline{I\cap\mathcal{E}_k}$ consists in a finite number of point and $\overline{I\cap\mathcal{E}_k}\cap\overline{\mathcal{E}_{k-1}}=\emptyset$. Hence, we can pick $\delta>0$ and I' and I'' two open intervals such that, $I'+(-\delta,\delta)\subset I$, $I+(-\delta,\delta)\subset I''$, $I''\cap\mathcal{E}_k=I'\cap\mathcal{E}_k=I\cap\mathcal{E}_k$ and $\overline{I''}\cap\overline{\mathcal{E}_{k-1}}=\overline{I'}\cap\overline{\mathcal{E}_{k-1}}=\emptyset$. Let χ_I, $\chi_{I'}$ and $\chi_{I''}$ be respectively the characteristic functions of I, I' and I''. Pick $0\le\psi\in\mathcal{C}_0^\infty(\mathbb{R})$ such that $\psi(x)=1$ for $|x|<1/2$, $\psi(x)=0$ for $|x|>1$ and $\int_{\mathbb{R}}\psi(x)dx=1$. For $\epsilon>0$, set $\psi_\epsilon(x)=\frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$ and $\chi_\epsilon=\chi_I*\psi_\epsilon$, $\chi'_\epsilon=\chi_{I'}*\psi_\epsilon$ and $\chi''_\epsilon=\chi_{I''}*\psi_\epsilon$. Then $0\le\chi'_\epsilon\le\chi_I\le\chi''_\epsilon$ for $0<\epsilon<\delta$. Moreover, we choose $\epsilon>0$ small enough such that $\mathrm{supp}(\chi_\epsilon)\cap\overline{\mathcal{E}_{k-1}}=\emptyset$, $\mathrm{supp}(\chi_\epsilon)\cap\mathcal{E}_k=I\cap\mathcal{E}_k$, for $\lambda\in I\cap\mathcal{E}_k$, $\chi_\epsilon(\lambda)=1$, and such that the same properties hold for χ'_ϵ and χ''_ϵ.

As $N_p(d\lambda)$ is a positive measure

$$\langle N_p,\chi'_\epsilon\rangle\le N_p(I)\le\langle N_p,\chi''_\epsilon\rangle.$$

Let us compute the left and right hand side of this inequality. By Theorem 1.1,

$$\langle N_p(d\lambda),\chi'_\epsilon\rangle=\sum_{j=0}^{k}p^j\langle n_j,\chi'_\epsilon\rangle+p^{k+1}\langle R_k(p),\chi'_\epsilon\rangle, \tag{2.12}$$

where, for $|\cdot|_k$, some semi-norm in $\mathcal{S}(\mathbb{R})$, we have for any $\varphi\in\mathcal{S}(\mathbb{R})$,

$$\sup_{p\in[0,1]}|\langle R_k(p),\varphi\rangle|\le|\varphi|_k. \tag{2.13}$$

As χ'_ϵ is supported in a gap of $\sigma(H)$, we know that (cf [Ya]) for $A\in\mathbb{Z}^d$, A finite,

$$\langle\zeta'(\lambda;A),\chi'_\epsilon\rangle=-\sum_{\lambda\in\sigma(H_A)\cap\mathrm{supp}(\chi_\epsilon)}\mathrm{rank}(\Pi_\Lambda(\lambda))\cdot\chi'_\epsilon(\lambda).$$

Hence as $\mathrm{supp}\chi'_\epsilon\cap\overline{\mathcal{E}_{k-1}}=\emptyset$ and $\mathrm{supp}(\chi'_\epsilon)\cap\mathcal{E}_k=I\cap\mathcal{E}_k$, by formula (d) of Theorem 1.1, we get $\langle n_j,\chi'_\epsilon\rangle=0$ for $0\le j\le k-1$, and

$$\langle n_k,\chi'_\epsilon\rangle=\frac{1}{k!}\sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k\text{ and }0\in\Lambda\\ \sigma(H_\Lambda)\cap I\neq\emptyset}}\left(\sum_{\lambda\in\sigma(H_\Lambda)\cap I}\mathrm{rank}(\Pi_\Lambda(\lambda))\right)=\frac{1}{k!}\mathfrak{n}_k(I). \tag{2.14}$$

The same way one proves that $\langle n_j,\chi''_\epsilon\rangle=0$ for $0\le j\le k-1$ and $\langle n_k,\chi''_\epsilon\rangle=\dfrac{\mathfrak{n}_k(I)}{k!}$. Plugging this and (2.14) into (2.12), and using (2.13), we get point point (a) of Theorem 1.3. Point (b) is obtained by noticing that under the new assumptions, all $(\langle n_j,\chi'_\epsilon\rangle)_{0\le j\le k}$ and $(\langle n_j,\chi''_\epsilon\rangle)_{0\le j\le k}$ vanish.

Proof of Theorem 1.4. Without loss of generality we may assume $\mu=0$. Take ψ as in the proof of Theorem 1.3, and set $\psi_\delta(x)=\psi(x/\delta)$. Then, for any $n\ge 0$, $\sup_{x\in\mathbb{R}}|(\partial^n\psi_\delta)(x)|\le C_n\delta^{-n}$. Obviously $0\le\psi_\delta\le\chi_{[-\delta,\delta]}\le\psi_{2\delta}$, hence

$$\langle N_p,\psi_\delta\rangle\le N_p([-\delta,\delta])\le\langle N_p,\psi_{2\delta}\rangle.$$

Then by the same computations (and with the same notations) as in the proof of Theorem 1.3, using (2.11) to estimate the rest, we get

$$|p^{-k}N_p([-\delta,\delta])-\mathfrak{n}(\{0\})|\le p\cdot\delta^{-(k+2)(d+1)+2}.$$

Hence, if we choose $\delta=p^\epsilon$ pour $0<\epsilon<\dfrac{1}{(k+2)(d+1)+2}$, we get Theorem 1.4.

III) Proof of Proposition 2.1.

Let us first prove point (c) of Proposition 2.1. Obviously, one has, by (2.5),

$$|a_{\Lambda,l}(z,p)|\le\sup_{t\in[0,1]^\Lambda}\sup_{u\in\{0,1\}^{\Lambda_l\setminus\Lambda}}|\mathrm{Tr}\left(\chi_0\partial_\Lambda\left[(z-H_\Lambda(t,u))^{-1}H_\Lambda(t,u)^{-q}\right]\chi_0\right)|.$$

We compute

$$\partial_\Lambda\left[(z-H_\Lambda(t,u))^{-1}H_\Lambda(t,u)^{-q}\right]=\bigotimes_{\gamma\in\Lambda}\partial_{t_\gamma}\left[(z-H_\Lambda(t,u))^{-1}H_\Lambda(t,u)^{-q}\right]$$

$$=\sum_{\substack{\bigcup_{j=1}^{q+1}A_j=\Lambda\\ A_j\cap A_l=\emptyset\text{ if }j\neq l}}\left(\sum_{\sigma_1\in\mathfrak{S}(A_1)}(z-H_\Lambda(t,u))^{-1}\left(\prod_{\gamma_1\in A_1}V_{\sigma_1(\gamma_1)}(z-H_\Lambda(t,u))^{-1}\right)\cdot\right. \tag{3.1}$$

$$\left.\cdot\prod_{j=2}^{q+1}\left(\sum_{\sigma_j\in\mathfrak{S}(A_j)}H_\Lambda(t,u)^{-1}\left(\prod_{\gamma_j\in A_j}V_{\sigma_j(\gamma_j)}H_\Lambda(t,u)^{-1}\right)\right)\right),$$

where $\mathfrak{S}(A_j)$ is the group of permutations of A_j.
Hence, using the estimates of Proposition 4.2 and the exponential decrease of V to control the trace-class norm of the right hand side of (3.1), we get, for some $C>0$ and for any $z\in\mathbb{C}\setminus\mathbb{R}$,

$$
\begin{aligned}
(3.2)\qquad &\sup_{\substack{p\in[0,1]\\ l>0}} |a_{\Lambda,l}(z,p)| \leq \\
&\leq C^{q+1} \sum_{\substack{\bigcup_{j=1}^{q+1} A_j=\Lambda\\ A_j\cap A_l=\emptyset \text{ if } j\neq l}} \left(\sum_{\substack{\sigma_1\in\mathfrak{S}(A_1),\dots\\ \dots,\sigma_{q+1}\in\mathfrak{S}(A_{q+1})}} \left(1+\frac{1}{\eta(z)}\right)^{\#A_1} e^{-\mathrm{diam}(z;A_1,\dots,A_{q+1};\sigma_1,\dots,\sigma_{q+1})} \right)
\end{aligned}
$$

where $\eta(z)$ is given in Proposition 4.3 and $\mathrm{diam}(z;A_1,\dots,A_{q+1};\sigma_1,\dots,\sigma_{q+1})$ is defined in the following way: if for $1\leq j\leq q+1$, we write $A_j=\{\gamma_1^j,\dots,\gamma_{m_j}^j\}$ (i.e $\#A_j=m_j$) then

$$
\mathrm{diam}(z;A_1,\dots,A_{q+1};\sigma_1,\dots,\sigma_{q+1}) = \eta(z)\sum_{l=0}^{m_1}|\sigma_1(\gamma_l^1)-\sigma_1(\gamma_{l+1}^1)| + \sum_{j=2}^{q+1}\frac{1}{C}\sum_{l=0}^{m_j}|\sigma_j(\gamma_l^j)-\sigma_j(\gamma_{l+1}^j)|,
$$

with (a) $\sigma_j(\gamma_{m_j+1}^j)=\sigma_{j+1}(\gamma_1^{j+1})$ for $1\leq j\leq q$,
(b) $\sigma_{q+1}(\gamma_{m_{q+1}+1}^{q+1})=0$ and $\sigma_0(\gamma_0^1)=0$.
Hence we get

$$
\begin{aligned}
(3.3)\qquad \mathrm{diam}(z;A_1,\dots,A_{q+1};\sigma_1,\dots,\sigma_{q+1}) &\geq \frac{\inf(\eta(z),1/C)}{k}\cdot\sum_{1\leq j\leq q+1}\sum_{l=1}^{m_j}|\sigma_j(\gamma_l^j)| \\
&= \frac{\inf(\eta(z),1/C)}{k}\sum_{\gamma\in\Lambda}|\gamma|.
\end{aligned}
$$

Summing the estimates (3.2) and (3.3) over all possible $\gamma_1,\dots,\gamma_k$ in $\mathbb{Z}^d$ and replacing $\eta(z)$ by an obvious lower bound, we get, for some $C>0$ (depending on k but not on $z\in\mathbb{C}\setminus\mathbb{R}$)

$$
\sum_{\substack{\Lambda\subset\mathbb{Z}^d\\ \#\Lambda=k}}\sup_{\substack{p\in[0,1]\\ l>0}}|a_{\Lambda,l}(z,p)|\leq C\left(\frac{1+|z|}{|\mathrm{Im}(z)|}\right)^{(k+1)(d+1)}.
$$

This proves point (c) of Proposition 2.1.

Let us now prove the point (a). It suffices to show that $(a_{\Lambda,l}(z,p))_{l\geq 0}$ is a Cauchy sequence of continuous functions for (z,p) in $K\times[0,1]$. Let $l'>l>0$ large such that $\Lambda\subset\Lambda_l$. Then

$$
\begin{aligned}
(3.4)\qquad |a_{\Lambda,l'}(z,p)-a_{\Lambda,l}(z,p)| \leq \int_{[0,1]^\Lambda}\mathbb{E}_p\Big\{\Big|\mathrm{Tr}\Big(\chi_0\partial_\Lambda\Big[&(z-H_{\Lambda,l'}(t,u))^{-1}H_{\Lambda,l'}(t,u)^{-q}- \\
&-(z-H_{\Lambda,l}(t,u))^{-1}H_{\Lambda,l}(t,u)^{-q}\Big]\chi_0\Big)\Big|\Big\}\,dt_\Lambda
\end{aligned}
$$

where $H_{\Lambda,l'}(t,u)=H+\sum_{\gamma\in\Lambda}t_\gamma V_\gamma+\sum_{\gamma\in(\Lambda_{l'}\setminus\Lambda)}u_\gamma V_\gamma$ and the random variables $u=(u_\gamma)_{\gamma\in\mathbb{Z}^d}$ and the variables $t=(t_\gamma)_{\gamma\in\Lambda}$ are defined as in section 2.

By Proposition 4.3, for any realization of u and t, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\|\chi_0 \left[(z - H_{\Lambda,l'}(t,u))^{-1} H_{\Lambda,l'}(t,u)^{-q} - (z - H_{\Lambda,l}(t,u))^{-1} H_{\Lambda,l}(t,)^{-q}\right] \chi_0\|_{\mathcal{T}_1} \leq C\left(1 + \frac{1}{\eta(z)}\right)^2 e^{-\epsilon \inf(\eta,\eta(z))\cdot|l|}.$$

It is clear that the arguments given in section 4 stay valid for z in K, some compact of $\mathbb{C} \setminus \mathbb{R}$ and the perturbations V having a small imaginary part (the size depending on K). Hence as $(z - H_{\Lambda,l'}(t,u))^{-1} H_{\Lambda,l'}(t,u)^{-q} - (z - H_{\Lambda,l}(t,u))^{-1} H_{\Lambda,l}(t,u)^{-q}$ is analytic in t (uniformly for $z \in K$ and $|t|$ small enough), we get, using a Cauchy estimate, for some $C > 0$ (depending on K and η), for $z \in K$ and $p \in [0,1]$,

$$\|\partial_\Lambda \left(\chi_0 \left[(z - H_{\Lambda,l'}(t,u))^{-1} H_{\Lambda,l'}(t,u)^{-q} - (z - H_{\Lambda,l}(t,u))^{-1} H_{\Lambda,l}(t,u)^{-q}\right] \chi_0\right) \|_{\mathcal{T}_1} \leq C e^{-|l|/C}.$$

This gives then by (3.4), for some $C > 0$, for $z \in K$ and $p \in [0,1]$,

$$|a_{\Lambda,l}(z,p) - a_{\Lambda,l'}(z,p)| \leq C e^{-|l|/C}.$$

Hence this ends the proof of point (a) of Proposition 2.1. The formula for $a_\Lambda(z,0)$ is obvious taking $p = 0$.

IV) Some estimates for the resolvent of H(t).

Let χ_0 be the characteristic function of the cube of center 0 and sidelength 1 in $\mathbb{R}^d$. Let χ_α be its translated by the vector α i.e $\chi_\alpha(x) = \chi_0(x - \alpha)$. Then we have

Proposition 4.1. *For $q > d/2$, there exists $\epsilon > 0$ such that, for any $V \in L^\infty(\mathbb{R}^d)$ and for any $z \notin \sigma(-\Delta + V)$, for any $(\alpha,\beta) \in \mathbb{Z}^d \times \mathbb{Z}^d$,*

$$\|\chi_\alpha \cdot (z - (-\Delta + V))^{-1} \cdot \chi_\beta\|_{\mathcal{T}_q} \leq C\left(1 + \frac{1}{\eta(z,V)}\right) e^{-\epsilon\eta(z,V)\cdot|\alpha-\beta|}$$

$$\text{where } \eta(z,V) = \frac{dist(z, \sigma(-\Delta + V))}{|z| + |V|_\infty + 1},$$

here $\|\cdot\|_{\mathcal{T}_q}$ denotes the norm in the q-th Schatten class, $|V|_\infty$, the supremum norm of V and $dist(z,z')$ denotes the distance in $\mathbb{C}$.

Proof. By [Si] section B.9, we know that $\chi_\alpha \cdot (z - (-\Delta + V))^{-1} \cdot \chi_\beta \in \mathcal{T}_q$. To prove the estimate on its norm we will use the idea used in [Co-Th] or [Si]. Let $H = -\Delta + V$, and for $a \in \mathbb{R}^d$, define

$$H_a = e^{a\cdot x} H e^{-a\cdot x} = (i\nabla - ia)^2 + V.$$

Then, for $z \notin \sigma(H) \cup \sigma(H_a)$, one has

$$\chi_\alpha (z - H)^{-1} \chi_\beta = \left(e^{-ax}\chi_\alpha\right) \chi_\alpha (z - H_a)^{-1} \chi_\beta \left(e^{ax}\chi_\beta\right). \tag{4.1}$$

Expanding H_a, we get

$$(z - H_a) = z - H + |a|^2 - 2a\cdot\nabla = (z - H)(1 + (z - H)^{-1}(-2a\cdot\nabla + |a|^2)).$$

We now estimate the last term of this product; first, for some $C>0$, independent of z and V,

$$\|(z-H)^{-1}\nabla\| = \|(\Delta-1)^{-1}\nabla-(z-H)^{-1}(z+1-V)(\Delta-1)^{-1}\nabla\| \le C\left(1+\frac{|z|+|V|_\infty+1}{\mathrm{dist}(z,\sigma(-\Delta+V))}\right)$$

(here $\|\cdot\|$ denotes the norm of bounded operators).

Now, choose a such that $|a| = \epsilon\left(1+\frac{\mathrm{dist}(z,\sigma(-\Delta+V))}{|z|+|V|_\infty+1}\right)$ and $\epsilon>0$ such that $8\epsilon(C+1)<1$ then

$$\|(z-H)^{-1}(-2a\cdot\nabla+|a|^2)\| < \frac{3}{4}$$

hence $(z-H_a)$ is invertible and for some $C>0$,

$$\begin{aligned}\|\chi_\alpha\cdot(z-H_a)^{-1}\cdot\chi_\beta\|_{\mathcal{T}_q} &\le 4\|(z-H)^{-1}\cdot\chi_\beta\|_{\mathcal{T}_q}\\ &\le 4\|(\Delta-1)^{-1}\chi_\beta-(z-H)^{-1}(z+1-V)(\Delta-1)^{-1}\chi_\beta\|_{\mathcal{T}_q}\\ &\le C(1+\|(z-H)^{-1}(z+1-V)\|).\end{aligned}$$

Hence, by (4.1) and [Si] section B.9, we get, for some $C>0$,

$$\|\chi_\alpha\cdot(z-(-\Delta+V))^{-1}\cdot\chi_\beta\|_{\mathcal{T}_q} \le C\left(1+\frac{|z|+|V|_\infty+1}{\mathrm{dist}(z,\sigma(-\Delta+V))}\right)e^{a\cdot(\alpha-\beta)},$$

which becomes the estimate given in Proposition 4.1 if we choose $a=\frac{|a|}{|\alpha-\beta|}(\alpha-\beta)$.

We now can apply Proposition 4.1 to estimate powers of the resolvent and get

Proposition 4.2. *Let p, q be integers such that $p\cdot q>d/2$. Then, there exists $\epsilon>0$ and $C_{p,q}>0$, such that for any $V\in L^\infty(\mathbb{R}^d)$ and for any $z\notin\sigma(-\Delta+V)$, for any $(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d$, $\chi_\alpha\cdot(z-(-\Delta+V))^{-p}\cdot\chi_\beta\in\mathcal{T}_q$ and*

$$\|\chi_\alpha\cdot(z-(-\Delta+V))^{-p}\cdot\chi_\beta\|_{\mathcal{T}_q} \le C_{p,q}\left(1+\frac{1}{\eta(z,V)}\right)^{p(d+1)} e^{-\epsilon\eta(z,V)\cdot|\alpha-\beta|}$$

where $\eta(z,V)$ is given in Proposition 4.1.

Proof. One writes

$$\chi_\alpha\cdot(z-H)^{-p}\cdot\chi_\beta = \sum_{\substack{\alpha_1,\dots,\alpha_{p-1}\\ \alpha_j\in\mathbb{Z}^d}}\left(\chi_\alpha\cdot(z-H)^{-1}\cdot\chi_{\alpha_1}\right)\cdot\left(\chi_{\alpha_1}\cdot(z-H)^{-1}\cdots\right.$$
$$\left.\cdots\chi_{\alpha_{p-1}}\right)\left(\chi_{\alpha_{p-1}}\cdot(z-H)^{-1}\cdot\chi_\beta\right),$$

and uses the norm estimates given in Proposition 4.1 and the product rule for elements in $\mathcal{T}_{p\cdot q}$.

We apply this to the realization of the random Schrödinger operator we are studying to get

Proposition 4.3. *(1) Let $q > d/2$. There exists $C > 0$ such that, for any realization of t, any $z \in \mathbb{C} \setminus \mathbb{R}$ and any $(\alpha, \beta) \in \mathbb{Z}^d \times \mathbb{Z}^d$,*

$$\|\chi_\alpha (z - H(t))^{-1} H(t)^{-q} \chi_\beta\|_{\mathcal{T}_1} \leq C \left(1 + \frac{1}{\eta(z)}\right)^2 e^{-\epsilon \eta(z) \cdot |\alpha - \beta|}.$$

(2) Let $H_l(t) = H + \sum_{\gamma \in \mathbb{Z}^d \cap \Lambda_l} t_\gamma V_\gamma$. Let $q > d/2$. There exists $C > 0$ such that, for any realization of t, any $z \in \mathbb{C} \setminus \mathbb{R}$ and any $l > 0$,

$$\|\chi_0 (z - H(t))^{-1} H(t)^{-q} \chi_0 - \chi_0 (z - H_l(t))^{-1} H_l(t)^{-q} \chi_0\|_{\mathcal{T}_1} \leq C \left(1 + \frac{1}{\eta(z)}\right)^2 e^{-\epsilon \inf(\eta, \eta(z)) \cdot |l|}.$$

Here η is the parameter controlling the exponential decrease of V and $\eta(z)$ is chosen as the infimum of $\eta(z, V(t))$ (given in Proposition 4.1) when t runs over all possible configurations. Notice that $\eta(z) > 0$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Part (1) is clear by Proposition 4.2. To get part (2), we write
(4.2)

$$\begin{aligned}
(z - H(t))^{-1} H(t)^{-q} &- (z - H_l(t))^{-1} H_l(t)^{-q} \\
&= ((z - H(t))^{-1} - (z - H_l(t))^{-1}) H(t)^{-q} + \\
&\qquad + (z - H_l(t))^{-1} (H(t)^{-q} - H_l(t)^{-q}) \\
&= (z - H(t))^{-1} (H(t) - H_l(t)) (z - H_l(t))^{-1} H(t)^{-q} + \\
&\qquad + (z - H_l(t))^{-1} \sum_{k=1}^{q} H_l(t)^{k-q} (H(t) - H_l(t)) H(t)^{-k}.
\end{aligned}$$

Then, using the exponential decrease for V and the boundedness of t, we get that

$$\|H(t) - H_l(t)\| \leq C e^{-\eta (d(\alpha, \mathbb{Z}^d \setminus \Lambda_l) + d(\beta, \mathbb{Z}^d \setminus \Lambda_l))}.$$

Plugging this into (4.2) and using Proposition 4.2, we get the announced estimate.

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