Canonical Quantization of Yang Mills on a Circle

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Abstract

The canonical quantization of YM on a circle is developed in two different formulations. In the first case the basic coordinates are the gauge potentials and we give a complete construction, but find no observables. In the second case the basic coordinates are the holonomy operators (Wilson loop operators) and here we also give a construction. A key ingredient is a proof that the irreducible characters of the holonomy operators are eigenvectors for the Hamiltonian.

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1 Introduction

Let G be a compact connected semi-simple Lie group with Lie algebra $\mathcal{G}$. In this paper we discuss the canonical quantization of the $\mathcal{G}$-valued Yang-Mills field theory on a circle. One begins with a certain classical Hamiltonian system for the gauge fields augmented by a family of constraints. In carrying out the quantization one has a choice of first imposing the constraint and then quantizing, or else of first quantizing and then imposing the constraint. The two approaches are not expected to be equivalent in general. In the first approach the present problem has been discussed by Rajeev [17] who shows that it all reduces to a system with a finite number of degrees of freedom. In this paper we consider the second approach and do obtain exactly the same structure

At first we need not restrict the discussion to one dimension. Consider the spacetime $\mathbf{R}\times\mathbf{M}$ where $\mathbf{M}=\mathbf{T}^d$ is the d-dimensional torus. (One could also take $\mathbf{M}$ to be any compact Riemannian manifold). A gauge potential A_μ on this spacetime is a Lie algebra valued one form, and has a field strength which is the 2-form $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$. The Yang Mills equations are $\partial^\mu F_{\mu\nu}-[A^\nu,F_{\mu\nu}]=0$ (summation convention). We specialize to a temporal gauge with $A_0=0$; this is always possible. Then the the equations become a pair of equations for a gauge potential A_k on $\mathbf{M}$ together with its time derivative $\Pi_k=\partial_0 A_k$. With field strength $F_{ij}=\partial_i A_j-\partial_i A_j+[A_i,A_j]$ the equations are :

$$\partial_i\Pi_i+[A_i,\Pi_i] = 0 \tag{1}$$

$$\partial_0\Pi_k-\partial_j F_{jk}-[A_j,F_{jk}] = 0 \tag{2}$$

We next give the problem a Hamiltonian formulation. Let $\Omega^k=\Omega^k(M,\mathcal{G})$ be the smooth $\mathcal{G}$-valued k-forms on $\mathbf{M}$. The phase space is $\Omega^1\times\Omega^1$ with a symplectic form defined as follows. There is a canonical positive definite inner product $(\cdot,\cdot)$ on the Lie algebra which is the negative of the Killing form. For any pair A,Π in $\Omega^1\times\Omega^1$ define an inner product

$$(A,\Pi)=\int_{\mathbf{M}}(A_i,\Pi_i) \tag{3}$$

and an associated norm $\|A\|$. There are similar inner products for the other Ω^k. The symplectic form on the phase space is the

$$\sigma(A,\Pi;A',\Pi')=(A,\Pi')-(A',\Pi) \tag{4}$$

so A and Π are conjugate variables.

The Hamiltonian for the system is

$$H=\|\Pi\|^2/2+\|F\|^2/4 \tag{5}$$

The dynamics generated by this Hamiltonian are $dA_k/dt = \Pi_k$ and $d\Pi_k/dt = \partial_j F_{jk} + [A_j, F_{jk}]$ just as in equation (2) above. Equation (1) is not dynamical but is a constraint equation preserved by the time evolution. If $J = \partial_i \Pi_i + [A_i, \Pi_i] \in \Omega^0$ the constraint is that $(J, h) = 0$ for all $h \in \Omega^0$. One can also write

$$(J, h) = -(\Pi, dh) + (A, [\Pi, h]) \tag{6}$$

where dh is the exterior derivative.

Here is the formal quantization procedure. Consider the functions (A, u) and (Π, v) for $u, v \in \Omega^1$. We ask for corresponding families of operators denoted $A(u)$ and $\Pi(v)$ on some complex Hilbert space which commute with themselves and statisfy the canonical commutation relations:

$$[A(u), \Pi(v)] = i(u, v) \tag{7}$$

Then (J, h) is also an operator denoted $J(h)$, as is the Hamiltonian H. The constraint is imposed as a condition on state vectors: the physical states Ψ are those satisfying $J(h)\Psi = 0$. Observables are self-adjoint operators that commute with $J(h)$ and they act on the space of physical states. The Hamiltonian H should be a positive self-adjoint operator satisfiying $[H, J(h)] = 0$. The Hamiltonian generates time evolution through the unitary group $\exp(-iHt)$. Our goal is to see how much of this can actually be realized.

Let us ignore the difficult question of time evolution for the moment, and see what structure we have. One computes formally

$$\begin{aligned} [J(h), A(u)] &= iA([h, u]) + i(u, dh) && (8) \\ [J(h), \Pi(v)] &= i\Pi([h, v]) && (9) \\ [J(h), J(h')] &= iJ([h, h']) && (10) \end{aligned}$$

These equations say that we have a representation of a certain infinite dimensional Lie algebra. The first two say that the $-iJ(h)$ generates gauge tranformations and the last says $-iJ(h)$ is a representation of the gauge algebra.

(Because one has to Wick order to define the $J(h)$ there is the possibility that there will be extra terms on the right side of (10). This turns out not to be the case. The extra terms, if present, would be called anomalies or central extensions, and for $d = 1$ we would be talking about Kac-Moody algebras.)

It will be more useful to consider unitary representations of the Lie group obtained by exponentiating the Lie algebra. This avoids problems with specifying domains of unbounded operators. The representation is a collection of unitary operators $W(u, v)$ defined for $u, v \in \Omega^1$ and $U(g)$ defined for g in the gauge group, the smooth functions from $\mathbf{M}$ to G. These are required to satisfy

$$W(u, v)W(u', v') = W(u + u', v + v') \exp(-i\sigma(u, v; u', v')/2) \tag{11}$$

$$U(g)W(u,v)U(g^{-1}) = W(gug^{-1}, gvg^{-1})\exp(i(v, g^{-1}dg)) \tag{12}$$
$$U(g)U(g') = U(gg') \tag{13}$$

Our group contains the Heisenberg group (Weyl form of the canonical commutation relations) and gauge group as subgroups. One recovers operators A, Π, J satisying (7),(8),(9),(10) by

$$i(A(v) - \Pi(u)) = d/dt\ W(tu, tv)|_{t=0} \tag{14}$$
$$-iJ(h) = d/dtU(e^{th})|_{t=0} \tag{15}$$

The physical states are those which are invariant under all the $U(g)$ and the observables are operators which commute with all the $U(g)$

Next we show that examples of (11),(12),(13) exist. We give two examples. The first is based on a Π representation of the Weyl relations, that is Π is represented by a multiplication operator in a standard manner. Let $\mathcal{H}_r(\mathbf{M}, \mathcal{G})$ be the Sobolev space obtained by completing $\Omega^1(\mathbf{M}, \mathcal{G})$ in the norm $\|u\|_r = \|(-\Delta + 1)^{r/2}u\|$ Then $\mathcal{H}_r$ is decreasing in r and one can identity the dual space of $\mathcal{H}_r$ with $\mathcal{H}_{-r}$. Let $\hat{\mathcal{G}} = \mathcal{H}_{-r}$ with r sufficiently large so that the injection from $\mathcal{H}_0 \to \mathcal{H}_r$ is trace class. Then for every $s > 0$ there is a Gaussian measure μ_s on $\hat{\mathcal{G}}$ such that for all $u \in \Omega^1 \subset \mathcal{H}_r$

$$\int_{\hat{\mathcal{G}}} exp(i(\Pi, u))d\mu_s(\Pi) = exp(-s\|u\|^2/2) \tag{16}$$

The Hilbert space for the representation is then $L^2(\hat{\mathcal{G}}, d\mu_s)$. The unitary representation is based on the facts that μ_s is is invariant under $\Pi \to g^{-1}\Pi g$ (since $\|u\|$ is invariant), and quasi-invariant under $\Pi \to \Pi + v$. The latter means that the induced measure is absolutely continuous with respect to the original measure. One can explicitly compute the derivative

$$\frac{d\mu_s(\Pi + v)}{d\mu_s(\Pi)} = \exp(-\|v\|^2/2s\ - (\Pi, v)/s) \tag{17}$$

The representation is defined by

$$(W(u,v)\Psi)(\Pi) = \exp(-i(\Pi, u) + i(u,v)/2)\sqrt{\frac{d\mu_s(\Pi - v)}{d\mu_s(\Pi)}}\Psi(\Pi - v) \tag{18}$$
$$(U(g)\Psi)(\Pi) = \exp(i(\Pi, g^{-1}dg))\Psi(g^{-1}\Pi g) \tag{19}$$

In checking that this is a representation one needs to use the fact that $\beta(g) = g^{-1}dg$ is a cocycle, that is $\beta(gg') = \beta(g) + g\beta(g')g^{-1}$.

This representation of the gauge group has been studied by many authors and is known to be irreducible in dimension $d \geq 3$ [1] [11]. This means that in this case there are no physical states, and the only observables are multiples of the identity (by Schur's lemma).

There is also an A-representation. This is also defined on $L^2(\hat{\mathcal{G}}, d\mu)$ but now A is given as a multiplication operator. We have

$$
\begin{aligned}
(W(u,v)\Psi)(A) &= \exp(i(A,v) - i(u,v)/2)\sqrt{\frac{d\mu_s(A-u)}{d\mu_s(A)}}\Psi(A-u) && (20)\\
(U(g)\Psi)(A) &= \sqrt{\frac{d\mu_s(A^g)}{d\mu_s(A)}}\Psi(A^g) && (21)
\end{aligned}
$$

where $A^g = g^{-1}Ag + g^{-1}dg$ The A-representation of the gauge group is equivalent to the Π representation [11] and suffers from the same difficulties.

The representation we actually want to consider is the $s \to \infty$ limit of the A-representation (for d=1). This corresponds to taking Lesbesgue measure on the function space. In order to accomodate this singular Hilbert space we need to formulate the problem in terms of C^*-algebras. We digress to review this approach as it applies to our problem

An algebraic approach. A general reference is [6]. Let $W(u,v)$ be a representation of the canonical commution relations (11). We define the Weyl algebra $\mathcal{F}$ to be the C^* algebra generated by the $W(u,v)$. This algebra is independent of the particular representation. Every element is a norm limit of finite sums $\sum_i c_i W(u_i, v_i)$ for some complex numbers c_i. States are now positive linear functionals on the C^* algebra, for example the expectation with respect to a unit vector in the representation space.

By the GNS reconstruction theorem any state determines a representation of the algebra. The theorem states that given a state ω on $\mathcal{F}$ there is a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathcal{F}$ by operators on $\mathcal{H}_\omega$, and a cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$ such that $\omega(F) = (\Omega_\omega, \pi_\omega(F)\Omega_\omega)$. for any $F \in \mathcal{F}$. Any two such representations are unitarily equivalent.

The operators $U(g)$ are not necessarily in $\mathcal{F}$, but they can still be accomodated as automorphisms β_g of the algebra. In any representation of (11),(12),(13) define β_g on $\mathcal{F}$ by

$$\beta_g(F) = U(g)FU(g)^{-1} \tag{22}$$

We have $\beta_{g'}\beta_g = \beta_{g'g}$ and since

$$\beta_g(W(u,v)) = W(gug^{-1}, gvg^{-1})\exp(i(v, g^{-1}dg)) \tag{23}$$

it follows that each β_g defines an automorphism of the algebra. We say that the automorphism is unitarily implementable in a particular representation of the algebra if there is a unitary representation $U(g)$ of the gauge group such that (22) is satisfied.

A physical state (gauge invariant state) is now a state ω on $\mathcal{F}$ such that $\omega \circ \beta_g = \omega$. After reconstruction in such a state the gauge automorphisms will be unitarily implementable by operators $U_\omega(g)$ satisfying $U_\omega(g)\Omega_\omega = \Omega_\omega$. Observables are now elements of the algebra left invariant by β_g. These form a subalgebra of $\mathcal{F}$ and one can take the attitude that one only needs a state on the observables. However in this case one has to do something further to show that the constraint is satisfied by the state.

One would like to treat time evolution in a similar way by asking for a one paramenter group of automorphisms α_t so that in some sense $\alpha_t(F) = e^{iHt}Fe^{-iHt}$. There is not much hope that this is true in general: the quantum Hamiltonian is just too singular. However in the case $d = 1$ it does work out this way as we now show.

2 YM on a circle: fields

Now suppose that the spacetime is $\mathbf{R} \times \mathbf{S}^1$ where $\mathbf{S}^1$ is the interval $[0, L]$ with the endpoints identified. In this case we can carry out the program. The main simplification is that on $\mathbf{S}$ the one forms are just functions and there are no two forms. Thus the classical Hamiltionian is just $H = \|\Pi\|^2/2$ and the dynamical equations are $dA/dt = \Pi$ and $d\Pi/dt = 0$. These have solutions $(A_t, \Pi_t) = T(t)(A, \Pi) = (A + t\Pi, \Pi)$. Since the equations are linear this is also the solution for the quantum problem. Taking into account that $W(u, v)$ is formally $\exp(i\sigma(A, \Pi; u, v))$ and that $T(t)$ is sympectic (i.e. leaves σ invariant), a time evolution automorphism α_t should satisfy

$$\alpha_t W(u, v) = W(T(-t)(u, v)) = W(u - tv, v) \tag{24}$$

We define the automorphism by this equation: since $T(t)$ is symplectic, $W(T(-t)(u, v))$ is a new representation of (11) generating $\mathcal{F}$, and hence there is a a unique automorphism α_t of $\mathcal{F}$ satisfying (24). Furthermore since $T(t)$ is a one parameter group of operators we get a one parameter group of automorphisms α_t. Note that the α_t commute with the gauge automorphisms β_g. We seek a state invariant under both.

We could go directly to the answer, but at the same time we want to develop a more regular approximation to these dynamics. Instead of adding a gauge fixing term as is usual for gauge theories, we add a mass term to the Hamiltonian. Thus we consider

$$H_s = 1/2(\|\Pi\|^2 + \|A\|^2/4s^2) \tag{25}$$

which has a mass $m = 1/2s$. This generates a classical time evolution given by

$$T_s(t) = \begin{pmatrix} \cos(t/2s) & 2s\sin(t/2s) \\ -1/2s\sin(t/2s) & \cos(t/2s) \end{pmatrix}$$

This operator is also symplectic and so for the quantum problem we may define a one-parameter group of time evolution automorphisms by

$$\alpha_t^s W(u,v) = W(T_s(-t)(u,v)) \tag{26}$$

Note that this reduces to the true time evolution as $s \to \infty$, and that it does not commute with gauge transformations.

Now suppose $\mathcal{F}$ is generated by the representation (20) of $W(u,v)$ on $L^2(\hat{\mathcal{G}}, d\mu_s)$. Let $\Omega_s = 1$ and define a state ω_s on $\mathcal{F}$ by $\omega_s(F) = (\Omega_s, F\Omega_s)_s$. Our main result states that ω_s is the physical vacuum for approximate dynamics, that these states have a limit ω as $s \to \infty$, and that ω is the physical vacuum for the true dynamics.

Theorem 2.1

1. *The state ω_s satisfies*

$$\omega_s(W(u,v)) = \exp(-\|u\|^2/8s - \|v\|^2 s/2) \tag{27}$$

and $\omega_s \circ \alpha_t^s = \omega_s$. After reconstruction (or on $L^2(\hat{\mathcal{G}}, d\mu_s)$ time evolution is unitarily implemented with positive energy; i.e. there is a positive self-adjoint operator H_s so

$$\alpha_t^s(F) = \exp(iH_s t)F\exp(-iH_s t) \tag{28}$$

2. *$\omega(F) = lim_{s\to\infty}\omega_s(F)$ exists for every $F \in \mathcal{F}$ and satisfies*

$$\omega(W(u,v)) = \begin{cases} 1 & v = 0 \\ 0 & v \neq 0 \end{cases} \tag{29}$$

The state ω satisfies $\omega \circ \beta_g = \omega$ and $\omega \circ \alpha_t = \omega$. After reconstruction gauge tranformations are unitarily implemented and time evolution is unitarily implemented with positive energy.

Remark. The state ω is not as trivial as it might look. The reconstruction theorem still generates a representation of the Weyl algebra. Furthermore it is known that the Weyl algebra is simple and so every representation is faithful. What is lost is that the representation is not regular in that the map $(u,v) \to W(u,v)$ is not continuous. This means one cannot define field operators A, Π in this representation, but only their exponentials.

Proof.
(1.) The first part is standard. The evaluation of ω_s on $W(u,v)$ is a computation and these values determine the state completely. It then suffices to check the invariance on the $W(u,v)$ where it is a simple computation. The unitary implementability of time translations follows. To see the positive energy one can check that $\omega_s(W(u',v')\alpha^s_t W(u,v))$ has a (distribution) Fourier transform in t with non-negative support. Since this function is equal to $(\Omega_s, W(u',v')\exp(iH_s t)W(u,v)\Omega_s)$ and since linear combinations of the $W(u,v)\Omega_s$ are dense the result follows.

(An alternative is to make a Fock space construction of the state ω_s. Then H_s is a Wick ordered version of the classical Hamiltonian and the positivity of H_s is manifest. This is just the standard construction of a free scalar field except that there is no term $\|\partial A/\partial x\|^2$ in the Hamiltonian. This means H_s is just a number operator that counts each particle $m = 1/2s$.)
(2) The fact that $\omega_s(W(u,v))$ has a limit equal to (29) is immediate. The limit exists on general F by making a uniform approximation by linear combinations of the $W(u,v)$'s. The property of being a positive linear functional carries over to the limit.

The invariance of the state ω under time translation and gauge transformations is easy to check on $W(u,v)$ using (29) and follows generally.

For the positivity of the energy we check that $\omega(W(u',v')\alpha_t W(u,v))$ has a (distribution) Fourier transform with non-negative support. Indeed if $v' \neq -v$ the function is zero while if $v' = -v$ it is

$$\begin{aligned}
& \omega(W(u',-v)W(u-tv,v) \\
= \;& \omega(W(u'+u-tv,0))\exp(-i\sigma(u',-v;u-tv,v)/2) \\
= \;& \text{const}\exp(it\|v\|^2/2)
\end{aligned}$$

This completes the proof.

More on the Hamiltonian H_s: H_s regarded as an operator on $L^2(\hat{\mathcal{G}}, d\mu_s)$ is an infinite dimensional differential operator, a generalization of the finite dimensional operator

$$L = 1/2(-\Delta + s^{-1}x\cdot\nabla) \tag{30}$$

This is well-known, but we develop a version suited to our purposes.

Start with the finite dimensional case. Consider the Hilbert space $L^2(\mathbf{R}^k, d\mu_s)$ with $d\mu_s(x) = \exp(-|x|^2/2s)dx$, at first for $k=1$. The operator L is symmetric and has a complete set of eigenfunctions. These are the Hermite polynomials defined by

$$He_n(x) = (-s)^n e^{x^2/2s} d^n/dx^n e^{-x^2/2s} \tag{31}$$

with eigenvalues $n/2s$. Normalized eigenfunctions are $(2\pi s)^{-1/4}(n!)^{-1/2}He_n$. The eigenfunction expansion defines L as a self-adjoint operator in a standard way, and if a function in the domain of L is sufficiently differentiable this definition agrees with the definition as a differential operator. Similarly we have for any k that L has a complete set of eigenfunctions $He_{n_1} \otimes ... \otimes He_{n_k}$ with eigenvalues $\sum_j n_j/2s$, and this defines L as a self-adjoint operator in this case.

We need the technical result:

Let $\mathcal{D}$ be finite linear combinations of exponentials $\exp(i\lambda x)$ in $L^2(\mathbf{R}^k, d\mu_s)$. Then $\mathcal{D}$ is a core for L, i.e. L is essentially self-adjoint on $\mathcal{D}$.

Proof. Take $k = 1$ for notational simplicity. $\mathcal{D}$ is a dense set in the domain of L. To prove it is a core it suffices to show that it is invariant under the action of e^{-Lt}, this is a corollary of Stone's theorem. The Hermite polynomials are related to the exponentials by the generating functional

$$\exp(i\lambda x - \lambda^2 s/2) = \sum_n \lambda^n He_n(x)/n! \tag{32}$$

with convergence also in L^2. ¿From this we have

$$e^{-Lt}\exp(i\lambda x - \lambda^2) = \exp(i(e^{-t}\lambda)x - (e^{-t}\lambda)^2)$$

and hence the invariance.

$e^{i(A,v)}$ is in $D(H_s)$ and

$$H_s e^{i(A,v)} = (\|v\|^2/2 + i(A,v)/2s)e^{i(A,v)}$$

Proof. Since $\Omega_s = 1$ and $H_s\Omega_s = 0$ we have

$$\begin{aligned}\exp(-iH_s t)e^{i(A,v)} &= \alpha^s_{-t}(W(0,v))\Omega_s \\ &= W(u_t, v_t)\Omega_s \\ &= \exp(i(A, v_t) - i(u_t, v_t)/2 - \|u_t\|^2 - (A, u_t)/2s)\end{aligned}$$

where $u_t = 2s\sin(t/2s)v$ and $v_t = \cos(t/2s)v$. The last expression is strongly differentiable in t hence $e^{i(A,v)}$ is in $D(H_s)$. Computing the derivative at $t = 0$ gives the result.

Let $f \in D(L)$ in $L^2(\mathbf{R}^k, d\mu_s)$. and let

$$F(A) = f((A, \phi_1), ..., (A, \phi_k))$$

where $\phi_1, ..., \phi_k$ is an orthonormal set in $L^2(\mathbf{S}, \mathcal{G})$. Then $F \in D(H_s)$ and

$$(H_s F)(A) = (Lf)((A, \phi_1), ..., (A, \phi_k))$$

Proof. By lemma 2 the result holds for $f \in \mathcal{D}$. By lemma 2 we can find for any $f \in D(L)$ a sequence $f_n \in \mathcal{D}$ so $f_n \to f$ and $Lf_n \to Lf$. If we define define $F_n(A) = f_n((A, \phi_1), ..., (A, \phi_k))$ it follows that $(H_s F_n)(A) = (Lf_n)((A, \phi_1), ..., (A, \phi_k))$. The convergence of f_n and Lf_n implies that $F_n \to F$ and $H_s F_n \to (Lf)((A, \phi_1), ..., (A, \phi_k))$. Since H_s is closed $F \in D(H_s)$ and $H_s F$ is as claimed.

3 YM on a circle: observables

The defect with the theory so far is that we have no observables. The natural candidate for an observable would be the Wilson loop operator or holonomy operator for parallel transport around the circle.[1] We cannot construct this operator in the C^* algebra $\mathcal{F}$. The second choice would be identify the operator in the vonNeumann algebra $\pi_\omega(\mathcal{F})''$ generated by $\mathcal{F}$ in the physical representation. This also seems difficult. Instead we abandon the fields and let the holonomy operators themselves play the fundamental role; in some sense they are new coordinates for the problem. Having made this decision one might proceed by attempting to construct a C^* algebra based on the holonomy operators, see [5]. However we take a somewhat different route.

The strategy is as follows. We know that the physical theory can be approximated by the more regular theory on $L^2(\hat{\mathcal{G}}, d\mu_s)$ with Hamiltonian H_s. We first construct the holonomy operators on this space and give them the time evolution generated by H_s. Then we show that their vacuum correlation functions have a limit when $s \to \infty$. Finally we use a Wightman type reconstruction theorem to obtain the physical theory. The last step is similar to a construction for Euclidean field theories due to to Frohlich, Seiler, and collaborators [10] [18].

We remark that there are a number of covariant/path-space/imaginary time treatments of YM in two-dimensions in which the holonomy operators also play a key role [2], [3], [13], [7], [8], [19], [20]. Since the loops live in two dimensions these theories are somewhat richer. They do have some of the same flavor as our canonical/operator/real time treatment, and it would be interesting to understand the connection better.

Let us assume that the group G is a closed subgroup of $U(n)$ so that the Lie algebra $\mathcal{G}$ is a subalgebra of the skew-adjoint $n \times n$ matrices. Consider the parallel transport operator for a connection A on $\mathbf{S}^1$, at first as given by a smooth $\mathcal{G}$- valued function on [0,L] with $A(0) = A(L)$. The parallel transport $U(x, A) \in G$ from 0 to x is the unique solution of the equation

$$dU/dx = AU \qquad U(0) = 1 \tag{33}$$

One way to construct the solution is as follows. For any interval I define $A(I) = \int_I A(x)dx$. Let $I_k = [(k-1)L/2^n, kL/2^n]$ and let $I_k(x) = [(k-1)L/2^n, x]$. Then define an approximate solution $U_N(x, A)$ by defining for $x \in I_k$

$$U_N(x, A) = \exp(A(I_k(x))) \exp(A(I_{k-1}))... \exp(A(I_0)) \tag{34}$$

[1]One might also consider multiple loops around the circle. For this paper we focus on the single loop

The actual solution is then given by

$$U(x, A) = \lim_{N\to\infty} U_N(x, A) \tag{35}$$

Of course $U(L, A) \neq U(0, A)$ in general.

Now let A be a $\mathcal{G}$-valued white noise random variable on $\mathbf{S}^1$ with variance s. That is suppose for each $u \in L^2(\mathbf{S}, \mathcal{G}) = L^2([0, L], \mathcal{G})$ there is a measurable function $A(u)$ on some measure space so $\int A(u)A(v) = s(u, v)$. For us the measure space is $(\hat{\mathcal{G}}, \mu_s)$ and $A(u) = (A, u)$ if u is smooth, and is defined by L^2 limits in general. Parallel transport with connection A is now defined by the stochastic differential equation

$$dU/dx = (A - sC/2)U \qquad U(0) = 1 \tag{36}$$

Here C is the Casimir operator defined as the matrix

$$C = -\sum_\alpha e_\alpha e_\alpha \tag{37}$$

where $\{e_\alpha\}$ is any orthonormal basis for $\mathcal{G}$. This extra term is necessary to keep the solution in G. An approximate solution $U_N(x, A)$ can still be defined by (34) provided we interpret

$$A(I) = \sum_\alpha e_\alpha A(e_\alpha \chi_I) \tag{38}$$

where χ_I is the characteristic function of I. As first shown by McKean [15] (see also [14], [9]) the limit $U(x, A) = \lim_{N\to\infty} U_N(x, A)$ still exists almost everywhere and is the unique solution to (36). Furthermore the G-valued random variables $U(x, A)$ are a realization of Brownian motion with variance s on the group G.

We expand on this last statement. Brownian motion on G (starting at the identity) can be characterized as the Markov process with infinitesimal generator $s\Delta$, where Δ is the Laplacian on the group, i.e. it is minus the Casimir operator (37) but with the basis vectors $\{e_\alpha\}$ interpreted as left-invariant vector fields on G. Our statement that $U(x, A)$ is a Brownian motion means that for functions $f_1, ..., f_n$ on G and for $0 \leq x_1 \leq x_2 \leq ... \leq L$:

$$\begin{aligned} & \int f_1(U(x_1, A))...f_n(U(x_n, A))d\mu_s(A) \\ = \; & (\exp(x_1 s\Delta/2) f_1 \exp((x_2 - x_1)s\Delta/2) f_2 ... f_n)(1) \end{aligned} \tag{39}$$

Here $\exp(xs\Delta)f$ is the solution of the heat equation $\partial u/\partial x = s\Delta u$ with initial condition $u(0) = f$.

The parallel transport operator has the covariance property

$$U(x, A^g) = g(x)^{-1} U(x, A) g(0)$$

This follows from the uniqueness of solutions. The holonomy operator itself is defined as parallel transport once around the full circle: it is the G-valued random variable $U(L, A)$. If g is a smooth gauge function on $\mathbf{S^1}$ then $g(L) = g(0)$ and $U(L, A^g) = g(0)^{-1}U(L, A)g(0)$. Thus if χ is a function on G invariant under conjugation, i.e. a class function, we have that

$$Y(\chi, A) = \chi(U(L, A)) \tag{40}$$

is gauge invariant. If χ is also real then we have an observable, that is a bounded self-adjoint operator on $L^2(\hat{\mathcal{G}}, d\mu_s)$ commuting with the gauge transformation $U(g)$ defined in (21). We study the expectation values of these objects in the vacuum $\Omega_s = 1$.

Proposition 3.1 *Let χ_i be smooth class functions. Then*

$$\lim_{s\to\infty}(\Omega_s, Y(\chi_1)...Y(\chi_n)\Omega_s) = \int_G \chi_1(g)...\chi_n(g)dg \tag{41}$$

where dg is Haar measure on G.

Proof. by (39) we have

$$(\Omega_s, Y(\chi_1)...Y(\chi_n)\Omega_s) = (e^{sL\Delta/2}\chi_1...\chi_n)(1)$$

The proposition now follows since for any $h \in G$ and any smooth function f on G we have

$$\lim_{s\to\infty}(e^{s\Delta}f)(h) = \int f(g)dg \tag{42}$$

This result holds on any compact Riemannian manifold. The limit (42) is equivalent to the statement that $\lim_{s\to\infty} e^{s\Delta}(f - (f, 1)) = 0$ at the at the point. The convergence holds in $L^2(G, dg)$ since 0 is a simple eigenvalue of the Laplacian with eigenvector 1, and all other eigenvalues are negative. The convergence also holds in any Sobolev space since these spaces may be defined by the Laplacian. The result now follows by a Sobolev inequality. (For Sobolev spaces on a manifold see for example [12]).

Time evolution. The time evolved holonomy operators should also be observables. For any class function χ we define

$$Y^s(t, \chi) = e^{iH_st}Y(\chi)e^{-iH_st} \tag{43}$$

Since H_s does not commute with gauge transformations this is not yet an observable, however we expect to recover the invariance in the $s \to \infty$ limit, at least formally.

It will be convenient to restrict the choice of class functions to irreducible characters. Let the irreducible representations of the group G be indexed by λ and let χ_λ be the character in the representation λ. Then χ_λ is an eigenvector of $-\Delta$ on G with some eigenvalue c_λ. The next result says that $Y(\lambda) \equiv Y(\chi_\lambda)$ is an approximate eigenvector of H_s with eigenvalue

$$E_\lambda = Lc_\lambda/2 \tag{44}$$

$$\|(e^{-itH_s} - e^{-itE_\lambda})Y(\lambda)\| \leq \mathcal{O}(s^{-1/2})$$

The proof is given in the next section. Assuming this result we can now prove our second main result.

Theorem 3.2

1. *The following limit exists:*

$$W(t_1,\lambda_1,...,t_n,\lambda_n) = lim_{s\to\infty}(\Omega_s, Y^s(t_1,\lambda_1)...Y^s(t_n,\lambda_n)\Omega_s)$$

2. *There exists a Hilbert space $\mathcal{H}$ with operators $Y(t,\lambda)$ and a cyclic vacuum Ω so that*

$$W(t_1,\lambda_1,...,t_n,\lambda_n) = (\Omega, Y(t_1,\lambda_1)...Y(t_n,\lambda_n)\Omega)$$

   There is a self adjoint operator H on $\mathcal{H}$ so that $H \geq 0$, $H\Omega = 0$, and

$$Y(\lambda,t) = e^{iHt}Y(\lambda)e^{-iHt}$$

   where $Y(\lambda) = Y(0,\lambda)$ are the time zero fields.

3. *$Y(\lambda)\Omega$ is an eigenvector of H with eigenvalue E_λ.*

4. *The vacuum is cyclic for the time zero fields.*

Remark. The fact that the holonomy operator is an eigenvector of the Hamiltonian was anticipated by Witten [19].

Proof. The proof of (1) is by induction on n. Since $H_s\Omega_s = 0$ there is no time dependence for $n = 1$ and the result follows from Proposition 3.1. In the general case we write

$$\begin{aligned}
& (\Omega_s, Y^s(t_1,\lambda_1)...Y^s(t_n,\lambda_n)\Omega_s) \\
= \ & (\Omega_s, Y(\lambda_1)e^{-iH_s(t_1-t_2)}Y(\lambda_2)e^{-iH_s(t_2-t_3)}...Y(\lambda_n)\Omega_s) \\
= \ & (\Omega_s, Y(\lambda_1)(e^{-iH_s(t_1-t_2)} - e^{-iE_\lambda(t_2-t_1)})Y(\lambda_2)e^{-iH_s(t_2-t_3)}...Y(\lambda_n)\Omega_s) \\
+ \ & e^{-iE_\lambda(t_2-t_1)}(\Omega_s, Y(\lambda_1)Y(\lambda_2)e^{-iH_s(t_2-t_3)}...Y(\lambda_n)\Omega_s)
\end{aligned} \tag{45}$$

By lemma 3 $\|(e^{-iH_st} - e^{-iE_\lambda t})Y(\lambda)\Omega_s\|$ converges to zero as $s \to \infty$, and since the other factors in the first term are bounded uniformly in s this implies that the first term converges to zero. For the second term we use the fact that the product of two characters is a finite sum of characters. This means that

$$Y(\lambda_1)Y(\lambda_2) = \sum_\lambda C(\lambda_1,\lambda_2;\lambda)Y(\lambda). \tag{46}$$

for some constants $C(\lambda_1,\lambda_2;\lambda)$. Then the second term can be reduced to a correlation function with $n-1$ fields, and hence converges by the inductive hypothesis.

The proof of (2) is a version of the standard Wightman reconstruction theorem. We omit the details.

For (3) note that

$$((e^{-iHt}-e^{-iE_\lambda t})Y(\lambda)\Omega, Y(t_1,\lambda_1)...Y(t_n,\lambda_n)\Omega)=O$$

since it is the limit of the same expression with s and as already noted such a limit is zero. Thus $(e^{-iHt}-e^{-iE_\lambda t})Y(\lambda)\Omega$ is orthogonal to a dense set and hence is zero.

For (4) we observe that states $Y(t_1,\lambda_1)...Y(t_n,\lambda_n)\Omega$ can be written as finite combinations of time zero fields . This follows from part (3) of the theorem and the fact that (46) holds also at $s=\infty$. The latter is established by limits from $s<\infty$. This completes the proof.

Remarks One can actually compute all these correlation functions by reducing them to time zero fields as indicated above and then using Proposition 3.1.

Now we can prove the following result which shows the equivalence with Rajeev [17] who constrains before quantizing. He finds that the Hilbert space is $L^2_{inv}(G,dg)$, the square integrable class functions on G, and the Hamiltonian is $L/2(-\Delta)$

Corollary 3.3 *There is a unitary operator $U : L^2_{inv}(G,dg)\to\mathcal{H}$ such that $U\cdot 1=\Omega$ and $U^{-1}Y(\lambda)U=\chi_\lambda$ and $U^{-1}e^{-iHt}U=e^{iL\Delta t/2}$*

Proof. The χ_λ form an orthonormal basis for $L^2_{inv}(G,dg)$ by the Peter-Weyl theorem. We define U on finite sums by

$$U(\sum_\lambda a_\lambda\chi_\lambda)=\sum_\lambda a_\lambda Y(\lambda)\Omega$$

This is norm preserving by Propostion 3.1 and has dense range by the theorem. Thus it extends to a unitary operator. The identities are easily checked on vectors $\sum_\lambda a_\lambda\chi_\lambda$ and since the operators are bounded they hold generally.

4 Proof of lemma 3.2

Fix an unitary irreducible representation of G which associates to each $g\in G$ a unitary matrix $g'\in U(r)$. This induces a representation of the Lie algebra $\mathcal{G}$ by skew-symmetric $r\times r$ matrices. If $X\in\mathcal{G}$ is represented by X' then $(e^X)'=e^{X'}$. Let e_α be an orthonormal basis for $\mathcal{G}$. Any $X\in\mathcal{G}$ is written $X=\sum_\alpha x_\alpha e_\alpha$ and $X'=\sum_\alpha x_\alpha e'_\alpha$. We have $|X|^2=|x|^2=\sum_\alpha|x_\alpha|^2$ and the matrix X' has an operator norm satisfying $|X'|\le\mathcal{O}(1)|x|$. The Casimir operator in this representation is $-\sum_\alpha e'_\alpha e'_\alpha$ and must be a multiple of the identity $c'I$.

For a constants $\mathcal{O}(1)$ independent of x:

$$\sum_\alpha x_\alpha \frac{\partial(e^{X'})}{\partial x_\alpha} = e^{X'} X' \tag{47}$$

$$|\frac{\partial(e^{X'})}{\partial x_\alpha} - e^{X'} e'_\alpha| \leq \mathcal{O}(1)|x|e^{\mathcal{O}(1)|x|} \tag{48}$$

$$|(\sum_\alpha \frac{\partial^2}{\partial x_\alpha^2} + c')e^{X'}| \leq \mathcal{O}(1)|x|e^{\mathcal{O}(1)|x|} \tag{49}$$

Proof. We compute (see for example [16] p.160)

$$\begin{aligned} \frac{\partial(e^{X'})}{\partial x_\alpha} &= e^{X'} \sum_{j=0}^\infty \frac{(-1)^j}{(j+1)!}(AdX')^j \frac{\partial X'}{\partial x_\alpha} \\ &= e^{X'}(e'_\alpha + \sum_{j=1}^\infty \frac{(-1)^j}{(j+1)!}(AdX')^j e'_\alpha) \end{aligned}$$

If we multiply by x_α and sum over α all the terms in the sum are zero and this gives (47).

Next use $|AdX'(Y')| \leq \mathcal{O}(1)|x||y|$ to estimate

$$\begin{aligned} |\frac{\partial(e^{X'})}{\partial x_\alpha} - e^{X'} e_\alpha| &\leq \mathcal{O}(1) \sum_{j=1}^\infty \frac{1}{(j+1)!}(\mathcal{O}(1)|x|)^j \\ &\leq \mathcal{O}(1)|x|e^{\mathcal{O}(1)|x|} \end{aligned}$$

This proves (48).

Finally we compute

$$\begin{aligned} \sum_\alpha \frac{\partial^2(e^{X'})}{\partial x_\alpha^2} &= e^{X'} \sum_\alpha (e'_\alpha + \sum_{j=1}^\infty \frac{(-1)^j}{(j+1)!}(AdX')^j e'_\alpha)^2 \\ &+ e^{X'} \sum_\alpha \sum_{j=2}^\infty \sum_{k=0}^{j-1} \frac{(-1)^j}{(j+1)!}(AdX')^{j-k-1}(Ade'_\alpha)(AdX')^k e'_\alpha \end{aligned}$$

Identify the leading term $-e^{X'}c'$ and bound the remainder as before. This gives (49) and completes the proof.

The next result concerns $U'_N = U_N(L, A)'$. We regard it as an element of $L^2(\hat{\mathcal{G}}, M_r, d\mu_s)$ where M_r is all $r \times r$ matrices, and the space is all square integrable M_r-valued functions ψ on $\hat{\mathcal{G}}$ with norm

$$\|\psi\|^2 = \int Tr(\psi(A)^*\psi(A))d\mu_s(A) \tag{50}$$

$$= \int \sum_{\beta,\beta'=1}^r |\psi_{\beta,\beta'}(A)|^2 d\mu_s(A) \tag{51}$$

Since U_N' is unitary $\|U_N'\| = \sqrt{r}$. The operator H_s operates on this space by acting on each entry.

U_N' is in the domain of H_s. With $E' = Lc'/2$ and $2^N \geq s$

$$\|(H_s - E')U_N'\| \leq \mathcal{O}(2^{-N/2}s^{1/2}) + \mathcal{O}(s^{-1/2})$$

Proof. We write

$$U_N' = \exp(A_{2^N})... \exp(A_1)$$

where

$$A_j = \sum_\alpha L^{1/2}2^{-N/2}e_\alpha' A(\phi_{\alpha,j})$$

and where

$$\phi_{\alpha,j} = L^{-1/2}2^{N/2}e_\alpha \chi_{I_j}$$

is an orthonormal set in $L^2(\mathbf{S}, \mathcal{G})$.

The matrix elements of U_N' are smooth functions of $A(\phi_{\alpha,j})$, bounded by one, with derivatives which are integrable for a Gaussian measure. Hence by lemma 2, U_N' is in the domain of H_s and

$$\begin{aligned}(H_s - E')U_N' &= 1/2 \sum_{j=1}^{2^N} [\prod_{k>j} e^{A_k}] \, (-\sum_\alpha \frac{\partial^2}{\partial x_{\alpha,j}^2} - 2^{-N}Lc')e^{A_j}[\prod_{k<j} e^{A_k}] \\ &+ 1/2s \sum_{j=1}^{2^N} [\prod_{k>j} e^{A_k}] \, \sum_\alpha x_{\alpha,j} \frac{\partial(e^{A_j})}{\partial x_{\alpha,j}} [\prod_{k<j} e^{A_k}] \\ &= I + II \end{aligned} \tag{52}$$

where now we understand that $A_j = \sum_\alpha L^{1/2}2^{-N/2}e_\alpha' x_{\alpha,j}$ and that the expression is to be evaluated at $x_{\alpha,j} = A(\phi_{\alpha,j})$.

We first give a bound pointwise in x. Taking into account the extra factor $L^{1/2}2^{-N/2}$ in A_j the bound (49) says

$$|(\sum_\alpha \frac{\partial^2}{\partial x_{j\alpha}^2} + 2^{-N}Lc')e^{A_j}| \leq \mathcal{O}(1)2^{-3N/2}|x_j| \exp(\mathcal{O}(1)2^{-N/2}|x_j|)$$

It follows that

$$|I| \leq \mathcal{O}(1)2^{-N/2}|x_j| \exp(\mathcal{O}(1)2^{-N/2}|x_j|)$$

The bound is is on the operator norm but this implies a bound of the same form on $Tr(I^*I)^{1/2}$.

Now compute the full norm. The integral can be evaluated directly in the variables $x_{\alpha,j}$ and we find

$$\begin{aligned}
\|I\|^2 &\leq \mathcal{O}(1)2^{-N}\int |x_j|^2 \exp(\mathcal{O}(1)2^{-N/2}|x_j|)d\mu_s(x) \\
&\leq \mathcal{O}(1)2^{-N}(\int |x_j|^4 d\mu_s(x))^{1/2}(\int \exp(\mathcal{O}(1)2^{-N/2}|x_j|)d\mu_s(x))^{1/2} \\
&\leq \mathcal{O}(1)2^{-N}s\exp(\mathcal{O}(1)2^{-N}s) \\
&\leq \mathcal{O}(1)2^{-N}s
\end{aligned}$$

Thus $\|I\| \leq \mathcal{O}(1)2^{-N/2}s^{1/2}$.

Now consider the second term in (52). By (47) this can be written

$$II = (2s)^{-1}\sum_j[\prod_{k>j} e^{A_k}]e^{A_j}A_j[\prod_{k<j} e^{A_k}] = (2s)^{-1}U_N\sum_j V_j^* A_j V_j$$

where V_j is the unitary matrix

$$V_j = \prod_{k<j} e^{A_k}$$

The norm is

$$\|II\|^2 = (2s)^{-2}\sum_{j,j'}\int Tr(V_j^* A_j^* V_j V_{j'}^* A_{j'} V_{j'})d\mu_s(A)$$

But if $j \neq j'$, say $j > j'$, then the only dependence on $A(\phi_{\alpha,j}) = x_{\alpha,j}$ is as a factor $x_{\alpha,j}$. Since $\int x_{\alpha,j}d\mu_s(x) = 0$ the term vanishes. Thus we have

$$\|II\|^2 = (2s)^{-2}\sum_j\int Tr(A_j^* A_j)d\mu_s(A)$$

Since $|A_j| \leq \mathcal{O}(1)2^{-N/2}|x_j|$ this yields

$$\|II\|^2 \leq \mathcal{O}(1)2^{-N}s^{-2}\sum_{j=1}^{2^N}\int |x_j|^2 d\mu_s(x) \leq \mathcal{O}(1)s^{-1}$$

and hence $\|II\| \leq \mathcal{O}(1)s^{-1/2}$. This completes the proof.

Proof of lemma 3.2: Start with the representation

$$(e^{-iH_s t} - e^{-iE't})U'_N = -i\int_0^t e^{-iH_s u}(H_s - E')e^{-iE'(t-u)}U'_N du$$

By lemma 4 for $2^N \geq s$ and fixed t:

$$\|(e^{-iH_s t} - e^{-iE't})U'_N\| \leq \mathcal{O}(2^{-N/2}s^{1/2}) + \mathcal{O}(s^{-1/2})$$

Now let $N \to \infty$. We have $U_N \to U$ a.e., hence $U'_N \to U'$ a.e. by the continuity of the representation, and hence $\|U'_N - U'\| \to 0$. It follows that

$$\|(e^{-iH_s} - e^{-iE't})U'\| \leq \mathcal{O}(s^{-1/2})$$

Then $\chi'(U) = TrU'$ satisfies the same bound in $L^2(\hat{\mathcal{G}}, d\mu_s)$ which is our result.

5 Discussion

By working exclusively with "observables" in section 3 we have given up gauge transformations. This makes it difficult to establish that the "observables" and the vacuum really are gauge invariant, although this is formally true. To really establish the invariance it would be nice to combine the observable construction of section 3 with the field construction of section 2.

The most straightforward way to combine the two approaches would be to show that vacuum expectation values of arbitrary products of Weyl operators W and loop operators Y have a limit as $s \to \infty$. Then a reconstruction theorem would give a structure in which our two theories could be embedded.

It is not particularly easy to obtain this limit. The separate proofs we have given for the W's and the Y's are not very compatible. Since we have plenty of boundedness one might hope that a convergent subsequence could be found. But even this requires a little uniformity in s which is hard to find.

Perhaps one should not be too optimistic. For a discussion of related difficulties in the abelian case see [10], [4].

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