TOPOLOGICAL ENTROPY FOR APPROPRIATELY APPROXIMATED C*-ALGEBRAS

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To the memory of Alfred Wehrl: teacher, colleague, friend

ABSTRACT. The "classical" topological entropy is one of the main numerical invariants in topological dynamics on compact spaces. Here, the author's recent development of a non–commutative generalization of topological entropy, in the natural setting of general C*–algebras as the non–commutative counterpart of continuous function algebras on compact spaces, is presented in a slightly modified and improved form. This includes both a survey of earlier results with some important corrections, and also new general results in response to (and inspired by) a more recent counter–proposal for a non–commutative topological entropy by K. Thomsen. Finally, some partially new examples for the calculation of the defined topological entropy are shown. The rather self–evident physical interpretation in the framework of (operator–algebraic) quantum statistical mechanics and of "chaotic" quantum dynamical systems is briefly touched upon.

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I. Introduction

The notion of topological entropy had been introduced in topological dynamics by Adler, Konheim and McAndrew [1] in 1965, first by purely formal analogy with the Kolmogorov–Sinai (KS) entropy of measure–theoretic ergodic theory that had been created by the two named mathematicians about ten years earlier. Since then, however, on the one hand the topological entropy has become one of the main numerical invariants in topological dynamics, and until quite recently still, it has been more and more successfully applied to deterministically chaotic *classical* physical systems, see in particular [2].

On the other hand, the KS entropy (among other measure–theoretic entropy–like quantities) has been more and more successfully generalized to "quantum ergodic theory" in the framework of the operator–algebraic approach to quantum statistical mechanics (cf. for example [3, 4]). We can already now refer the reader to at least two recent books on these quantum generalizations of measure–theoretic (dynamical) entropy: From the more mathematical point of view, the book by Petz and Ohya [5] is a complete introduction to the subject of quantum entropy theory, whereas the book by Benatti [6] concentrates on the quantum ergodic theory aspects, more from the point of view of mathematical physics. In particular, the work by Connes [7] and Connes, Narnhofer and Thirring [8] (CNT for short, and also subsequent work) reviewed in both books has been a breakthrough in the non–commutative generalization of the classical measure–theoretic KS entropy; see also [9] for a short overview of these recent mathematical developments.

This latter CNT theory of dynamical (state–dependent, as "measure–theoretic") entropy for non–commutative C*–algebras (or von Neumann algebras) was the starting point for the work on "quantum topological entropy" presented here. As is well known (cf. for example [3, 10]), *commutative* C*–algebras with unit correspond exactly to the compact Hausdorff spaces, and in this sense parts of the theory of non–commutative C*–algebras (of bounded operators on Hilbert space) can be considered as "quantized topology" (cf. [11]).

As to the physical interpretation of the theory in the following sections II and III, within the framework of operator–algebraic quantum statistical mechanics [3, 4], we have to refer the reader to the still very condensed section (1.2) of the (unpublished, and German) Thesis [12], which section had to be omitted in the published form of the preliminary short version [13] because of lack of space there (as it is actually also the case here, for the lengthy discussion of the physical interpretation in all its details). A very short account of this interpretation for this theory first published in [13] can be found in [6, p. 207] (after a whole section on the main parts of this theory in Benatti's book); and the forthcoming publication [14] by the present author will contain this interpretation in full detail. But for the experienced reader, the physical interpretation will presumably be almost self–evident; even more so if we now cite already here the two closely related notions of "operations" ([15], used in a *certain* sense) respectively of "instruments" (see [16] and the references there).

The mathematical prerequisites required from the reader are on the one hand rather elementary set–theoretic topology as used in the classical theory of topological entropy (see (1.1) and (1.2) below); and on the other hand a good knowledge of the (mainly) linear algebra aspects of basic C*–algebra theory, for which all notions not explained or cited in the text here can be easily found in [3, 4, 10] for example, or also in [5, 8, 6] for special aspects of the C*–entropy theory.

As basic references for the mathematical entropy theory in the classical case we recommend for example [17] particularly for the measure–theoretic (KS) entropy theory (which is not necessary for the understanding of what follows, but as secondary reading), and [18] for the *topological* entropy theory, our starting point for the non–commutative generalization in the following two sections. For the convenience of the reader and to fix our notation for further use, we briefly recall the definitions and general properties of the "classical" topological entropy as in the original paper by Adler, Konheim and McAndrew [1], which contains already most of the essential parts of the classical theory (in this approach).

Definition (1.1) . Let (X, T) be a topological dynamical system, given by a compact Hausdorff space X and a continuous selfmap $T : X \to X$. By definition, any open cover of X possesses a *finite* subcover, and we can restrict ourselves to the latter from the very beginning, denoting by $\mathcal{O}(\mathcal{X})$ the set (or class) of finite open covers of X.

For $\mathcal{U}, \mathcal{V} \in \mathcal{O}(\mathcal{X})$ we define their join $\mathcal{U} \vee \mathcal{V} = \{\mathcal{U} \cap \mathcal{V} | \mathcal{U} \in \mathcal{U},\ \mathcal{V} \in \mathcal{V}\} \in \mathcal{O}(\mathcal{X})$, and the action of T on $\mathcal{U} \in \mathcal{O}(\mathcal{X})$ by $T^{-1}(\mathcal{U}) = \{\mathcal{T}^{-\infty}(\mathcal{U}) | \mathcal{U} \in \mathcal{U}\} \in \mathcal{O}(\mathcal{X})$ (as T is onto).

(i) $N(\mathcal{U}) = \min\{\mathrm{card}\,\mathcal{U}' | \mathcal{U}' \subset \mathcal{U} : \mathcal{U}' \in \mathcal{O}(\mathcal{X})\}$ denotes the cardinality of a minimal subcover $\mathcal{U}'$ of $\mathcal{U} \in \mathcal{O}(\mathcal{X})$. The "topological" *entropy* of $\mathcal{U} \in \mathcal{O}(\mathcal{X})$ is defined by $H(\mathcal{U}) = \log \mathcal{N}(\mathcal{U})$.

(ii) The entropy of T with respect to $\mathcal{U} \in \mathcal{O}(\mathcal{X})$ is defined as $h(T, \mathcal{U}) = \lim_{\backslash \to \infty} \frac{\infty}{\backslash} \mathcal{H}(\mathcal{U} \vee \mathcal{T}^{-\infty}\mathcal{U} \vee \cdots \vee \mathcal{T}^{-\backslash+\infty}\mathcal{U})$.

(iii) The *topological entropy* of T is $h(T) = \sup_{\mathcal{U} \in \mathcal{O}(\mathcal{X})} h(T, \mathcal{U})$.

A cover $\mathcal{V} \in \mathcal{O}(\mathcal{X})$ is said to be a *refinement* of a cover $\mathcal{U} \in \mathcal{O}(\mathcal{X})$ if $\forall V \in \mathcal{V}\ \exists \mathcal{U} \in \mathcal{U}$ such that $V \subset U$, which gives a partial pre–order on $\mathcal{O}(\mathcal{X})$.

Proposition (1.2) . *The entropy functionals (1.1,i–iii) have the following general properties:* $\forall \mathcal{U}, \mathcal{V} \in \mathcal{O}(\mathcal{X})$,

(i) $N(\mathcal{U} \vee \mathcal{V}) \leqslant \mathcal{N}(\mathcal{U}) \cdot \mathcal{N}(\mathcal{V})$, *or* $H(\mathcal{U} \vee \mathcal{V}) \leqslant \mathcal{H}(\mathcal{U}) + \mathcal{H}(\mathcal{V})$.

(ii) $H(T^{-1}\mathcal{U}) \leqslant \mathcal{H}(\mathcal{U})$, *and so equality holds if T is a homeomorphism.*

(iii) $h(T, \mathcal{U}) \leqslant \mathcal{H}(\mathcal{U})$.

(iv) *If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $H(\mathcal{U}) \leqslant \mathcal{H}(\mathcal{V})$ (the same for N), and $h(T, \mathcal{U}) \leqslant \langle(\mathcal{T}, \mathcal{V})$. From the latter monotonicity it follows that if $(\mathcal{U}_\backslash)_{\backslash \in \mathbb{N}}$ is a* cofinal *and* refining *sequence of open covers $\mathcal{U}_\backslash \in \mathcal{O}(\mathcal{X})$ with respect to the natural partial order from (1.1) above, then $h(T) = \lim_{n\to\infty} h(T, \mathcal{U}_\backslash)$. This is always achievable in a* metric *space (X, d), where each sequence $(\mathcal{V}_{||})_{|| \in \mathbb{N}}$ with diameters*

$d(\mathcal{V}_{\|}) \equiv \max_{\mathcal{V}\in\mathcal{V}_{\|}} \lceil(\mathcal{V})$ shrinking to zero, $d(\mathcal{V}_{\|}) \to \imath$, gives a cofinal *and then even* refining *sequence $(\mathcal{U}_{\backslash})_{\backslash\in\mathbb{N}}$ with $\mathcal{U}_{\backslash} = \bigvee_{\|=\infty}^{\backslash} \mathcal{V}_{\|}$ by Lebesgue's covering lemma.*

(v) *$h(T^k) = k \cdot h(T)$ $\forall k \in \mathbb{N}$, and if T is a homeomorphism, then $h(T^{-1}) = h(T)$.*

(vi) *For a* factor *system (Y, S) of (X, T), i.e. a continuous surjection $\Phi : X \to Y$ such that $\Phi \circ T = S \circ \Phi$, we have $h_Y(S) \leqslant h_X(T)$ (where we add the subscripts for the spaces).*

(vii) *For a closed, T–invariant subset $Y \subseteq X$, i.e. such that $T(Y) \subseteq Y$, we have $h_Y(T\restriction_Y) \leqslant h_X(T)$.*

(viii) *Let $\Phi : X \to X'$ be a homeomorphism onto a compact (Hausdorff) space X', then $h(T) = h(\Phi \circ T \circ \Phi^{-1})$, as follows from (vi).*

(ix) *Denote by $M(X, T)$ the (non–empty) convex compact set of T–invariant Borel probability measures μ on X, i.e. such that $\mu \circ T^{-1} = \mu$. Then $h(T) = \sup_{\mu\in M(X,T)} h_\mu(T)$, where the right hand side is the measure–theoretic KS–entropy of T with respect to μ.*

See now [13, 12] for a very detailed motivation of the approach chosen for the non–commutative generalization of this classical theory. But here we plunge directly into the realm of non–Abelian C*–algebras and will return to the classical case not earlier than in (2.10) below. The paper is organized in three main sections II-IV: In section II, the definitions and basic properties are given, together with the corrected proof of the main Theorem (2.15). All other proofs are essentially left as simple exercises for the reader, who can find some more details in the Thesis [12]. We also review the earlier versions [13, 12] as compared to the modified and improved theory presented here. In section III, we prove additional properties of the defined topological entropy for C*–algebras, all of which are the direct analogues of the additional properties proved by K. Thomsen [19] for his counter–proposal to the theory presented here. In order to prove the analogous modest continuity properties as in [19], we have to impose a restriction on the original theory of section II, which is of independent interest, though (particularly concerning the physical interpretation). After that, we compare our (two) definitions with Thomsen's [19] and add some partly critical remarks concerning his theory. In section IV, we first reproduce a slightly different example from [19] with our definitions, and then we proceed by reviewing the examples for shift automorphisms on AF algebras from [13, 12, 19], now also for endomorphisms. Finally, we conclude by some remarks concerning the non–commutative analogue of the classical so–called "variational principle" (1.2,ix) above (and some side–remarks).

II. Definitions and basic properties

Throughout this paper, $\mathcal{A}$ will denote a unital C*–algebra with unit $\mathbf{1} \in \mathcal{A}$.

Definition (2.1) . We define the following elementary structures and notations:

(i) An *operator cover* for $\mathcal{A}$ is a finite subset $\alpha \subset \mathcal{A}$ such that for $\alpha = \{A_i \in \mathcal{A} | \rangle = \infty, \ldots, \backslash\}$, both $\sum_{i=1}^{n} A_i^* A_i > 0$ (*strictly* positive, i.e. invertible) *and* $\sum_{i=1}^{n} A_i^* A_i \leqslant \mathbf{1}$, the latter condition being a suitable normalization. The set (or class) of all operator covers for $\mathcal{A}$ is denoted by $\mathcal{O}(\mathcal{A})$.

(ii) A *positive* operator cover for $\mathcal{A}$ is a finite set $\alpha \subset \mathcal{A}^+$ such that with above notations $\sum_{i=1}^{n} A_i > 0$ and $\sum_{i=1}^{n} A_i \leqslant \mathbf{1}$; and $\mathcal{O}^+(\mathcal{A})$ denotes the collection of all these.

(iii) For two finite subsets $\alpha, \beta \subset \mathcal{A}$, we denote by $\alpha \vec{\vee} \beta$ their *ordered* operator product, where we choose the order $\alpha \vec{\vee} \beta = \{B \cdot A | \forall B \in \beta, A \in \alpha\}$.

(iv) For $\alpha \in \mathcal{O}(\mathcal{A})$, set $\bar{N}(\alpha) = \min\{\operatorname{card} \alpha' | \alpha' \subseteq \alpha, \ \ \alpha' \in \mathcal{O}(\mathcal{A})\}$.

Remarks and Observations (2.2) . For the corresponding parts of (2.1), the following is important to note:

ad (i) For $\alpha = \{A_i \in \mathcal{A} | \rangle = \infty, \ldots, \backslash\}$ with $A \equiv \sum_{i=1}^{n} A_i^* A_i > 0$ but $A \nleqslant \mathbf{1}$, define $\hat{\alpha} = \{\hat{A}_i \equiv A_i \cdot A^{-1/2} | i = 1, \ldots, n\}$, then obviously $\hat{\alpha} \in \mathcal{O}(\mathcal{A})$ and even $\sum_{i=1}^{n} \hat{A}_i^* \hat{A}_i = \mathbf{1}$, illustrating the mentioned normalization for $\mathcal{O}(\mathcal{A})$. Let us denote by $\mathcal{O}_\infty(\mathcal{A})$ the set of all such $\hat{\alpha}$ (with the corresponding sum equal to $\mathbf{1}$).

On the other hand, for a general $\alpha \in \mathcal{O}(\mathcal{A})$ with elements as above, the normalization condition in (i) implies that $\mathbf{1} - A \equiv B \geqslant 0$, meaning that for $\alpha \in \mathcal{O}(\mathcal{A})$ there exists $\check{\alpha} \in \mathcal{O}_\infty(\mathcal{A})$ such that $\alpha \subseteq \check{\alpha}$, and we define $\check{\alpha}$ by the obvious choice $\check{\alpha} = \alpha \cup \{B^{1/2}\}$.

ad (ii) Rather obviously, for $\alpha \in \mathcal{O}^+(\mathcal{A})$ also its element–wise square $\alpha^2 \equiv \{A_i^2 | i = 1, \dots, n\}$ stays $\alpha^2 \in \mathcal{O}^+(\mathcal{A})$ as well, i.e. we have $\mathcal{O}^+(\mathcal{A}) \subset \mathcal{O}(\mathcal{A})$.

On the other hand, there exists a positive number $c(\alpha) < 1$ such that also $c(\alpha) \cdot \alpha^{1/2} \equiv \{c(\alpha) \cdot A_i^{1/2} | i = 1, \dots, n\}$ is still in $\mathcal{O}^+(\mathcal{A})$, where the $c(\alpha)$ is necessary only for the normalization condition in $\mathcal{O}^+(\mathcal{A})$.

ad (iii) Again rather obviously, for $\alpha, \beta \in \mathcal{O}(\mathcal{A})$ (respectively in $\mathcal{O}_\infty(\mathcal{A})$), also $\alpha \vec{\nabla} \beta \in \mathcal{O}(\mathcal{A})$ (resp. in $\mathcal{O}_\infty(\mathcal{A})$); but note that $\vec{\nabla} : \mathcal{O}^+(\mathcal{A}) \times \mathcal{O}^+(\mathcal{A}) \to \mathcal{O}(\mathcal{A})$ with image *more* than $\mathcal{O}^+(\mathcal{A})$ for non–Abelian $\mathcal{A}$.

ad (iv) Note that in the first remark above, it is easily seen that $\bar{N}(\hat{\alpha}) = \bar{N}(\alpha)\ \forall \alpha \in \mathcal{O}(\mathcal{A})$; but on the other hand, only $\bar{N}(\check{\alpha}) \leqslant \bar{N}(\alpha)$ holds, where strict inequality is well possible. And the second remark above amounts to the observation that $\bar{N}(\alpha^2) = \bar{N}(\alpha)\ \forall \alpha \in \mathcal{O}^+(\mathcal{A})$, where we should add the further observation that for $\alpha \in \mathcal{O}^+(\mathcal{A})$, equivalently $\bar{N}(\alpha) = \min\{\operatorname{card} \alpha' | \alpha' \subseteq \alpha,\ \alpha' \in \mathcal{O}^+(\mathcal{A})\}$ (instead of $\alpha' \in \mathcal{O}(\mathcal{A})$ as second condition).

Proposition (2.3) . *The following properties are easy to deduce:*

(i) $\bar{N}(\alpha) \leqslant \bar{N}(\alpha \vec{\nabla} \beta) \leqslant \bar{N}(\alpha) \cdot \bar{N}(\beta)\ \forall \alpha, \beta \in \mathcal{O}(\mathcal{A})$, *but* not *necessarily $\bar{N}(\beta) \leqslant \bar{N}(\alpha \vec{\nabla} \beta)$ in general.*

(ii) $\bar{N}(\alpha \vec{\nabla} \alpha \vec{\nabla} \dots \vec{\nabla} \alpha) = \bar{N}(\alpha)\ \forall \alpha \in \mathcal{O}^+(\mathcal{A})$, *but* not *necessarily for $\alpha \in \mathcal{O}(\mathcal{A}) \setminus \mathcal{O}^+(\mathcal{A})$. Note that this follows already from the observation $\bar{N}(\alpha^2) = \bar{N}(\alpha)$ in (2.2,iv) above, together with (i).*

(iii) *For a positive unital map $\gamma : \mathcal{B} \to \mathcal{A}$ from a unital C^*–algebra $\mathcal{B}$ into $\mathcal{A}$, and $\beta \in \mathcal{O}^+(\mathcal{B})$, obviously $\gamma(\beta) \in \mathcal{O}^+(\mathcal{A})$ and $\bar{N}(\gamma(\beta)) \leqslant \bar{N}(\beta)$.*

Note, furthermore, that from Kadison's inequality $\gamma(B^2) \geqslant \gamma(B)^2\ \forall B = B^* \in \mathcal{B}$ for positive maps γ [20], *it follows that*

$\bar{N}(\gamma(\beta^2)) \leqslant \bar{N}(\gamma(\beta))$ *(again using (2.2,iv) above); and the reverse inequality is again rather obvious because of the positivity of* γ*, i.e. even* $\bar{N}(\gamma(\beta^2)) = \bar{N}(\gamma(\beta))$ *holds for* $\beta \in \mathcal{O}^+(\mathcal{A})$*, also if* γ *is not a homomorphism.*

(iv) *For a unital* $*$*–endomorphism* $\theta : \mathcal{A} \to \mathcal{A}$*, we have that* $\bar{N}(\theta(\alpha)) \leqslant \bar{N}(\alpha),\ \forall \alpha \in \mathcal{O}(\mathcal{A})$ *(not only for* $\alpha \in \mathcal{O}^+(\mathcal{A})$*!).*

Lemma (2.4) . *For a finite–dimensional* C^**–algebra* $\mathcal{B}$*, define* $D(\mathcal{B})$ *to be the dimension of a maximal Abelian (*C^**–)subalgebra of* $\mathcal{B}$*. Then* $\bar{N}(\alpha) \leqslant D(\mathcal{B})\ \forall \alpha \in \mathcal{O}(\mathcal{B})$*; in other words, if* $\mathcal{B} \cong \bigoplus_{k=1}^n M_{d_k}(\mathbb{C})$*, then* $\bar{N}(\alpha) \leqslant \sum_{k=1}^n d_k$*. We leave the proof to the reader as a simple exercise (cf. also* [12]*).*

Obviously, there exists $\beta \in \mathcal{O}^+(\mathcal{B})$ *such that* $\bar{N}(\beta) = D(\mathcal{B})$*; and together we have that* $\max_{\alpha \in \mathcal{O}(\mathcal{B})} \bar{N}(\alpha) = D(\mathcal{B}) = \max_{\beta \in \mathcal{O}^+(\mathcal{B})} \bar{\mathcal{N}}(\beta)$ *(note that one can obviously choose a* $\beta \in \mathcal{O}^+(\mathcal{B})$ *such that even* $\operatorname{card} \beta = \bar{N}(\beta) = D(\mathcal{B})$ *holds).*

Definition (2.5) . We now proceed to the central definition of the entropy of several positive unital maps, generalizing the entropy of (the join of) finite open covers as in (1.1,i).

(i) To simplify the notation in the following, we call two positive unital maps $\gamma_k : \mathcal{B}_\| \to \mathcal{A}$ $(\| = \infty, \in)$ from (not yet necessarily finite-dimensional) C*–algebras $\mathcal{B}_\|$ into $\mathcal{A}$ *equivalently covering*, denoted by $\gamma_1 \approxeq \gamma_2$, if $\gamma_1 = \gamma_2 \circ \theta_{12}$ with a $*$–isomorphism $\theta_{12} : \mathcal{B}_\infty \to \mathcal{B}_\in$.

(ii) Let $(\gamma_1, \ldots, \gamma_n)$ be a tuple of positive unital maps $\gamma_k : \mathcal{B}_\| \to \mathcal{A}$ from finite–dimensional C*–algebras $\mathcal{B}_\|$ into $\mathcal{A}$, then their *entropy* is defined by

$$\bar{H}(\gamma_1, \ldots, \gamma_n) = \max_{\{(\alpha_1, \ldots, \alpha_n)\}} \log \bar{N}(\gamma_1(\alpha_1) \vec{\nabla} \ldots \vec{\nabla} \gamma_n(\alpha_n)),$$

where the maximum is taken over the set of n–tuples with entries $\alpha_k \in \mathcal{O}^+(\mathcal{B}_\|)$, but with the additional restriction: $\gamma_k \approxeq \gamma_\ell,\ k < \ell \Longrightarrow \alpha_\ell \equiv \{\mathbf{1}_\ell \in \mathcal{B}_\ell\}$.

Remarks and Observations (2.6) . The following remarks and simple results will eludicate this main definition:

1. The additional restriction in the variational expression (2.5,ii) may be even simpler expressed verbally: Repeated arguments of $\bar{H}$ (even repeated up to covering equivalence only) do not contribute any more, i.e. $\bar{H}$ is a functional on the *ordered set* of its arguments (even of their covering–equivalence classes only), with the order inherited from the original tuple of arguments. This additional restriction is necessary for reasons to become clear in (2.9,iv) and (2.13,iii) below, whereas at first sight, admittedly, it seems to be unnatural.

   Also without this restriction, however, the functional $\bar{H}$ would generally not be continuous in norm (for the linear maps $\gamma_k : \mathcal{B}_\| \to \mathcal{A}$, $\mathcal{B}_\|$ fixed), as we will see in (2.12,2) below. This seems to be a good excuse for the additional "discontinuity" of $\bar{H}$ due to this necessary restriction; and clearly, $\bar{H}$ remains unchanged on the "generic" set of n–tuples $(\gamma_1, \ldots, \gamma_n)$ with $\gamma_i \not\approx \gamma_j$ for all pairs $i \neq j \in \{1, \ldots, n\}$.
2. For a *single* positive unital map $\gamma : \mathcal{B} \to \mathcal{A}$ from a finite–dimensional C*–algebra $\mathcal{B}$, we get as an immediate corollary of (2.3,iii) in (2.5,ii), resp. of Lemma (2.4): $\bar{H}(\gamma) \leqslant \log D(\mathcal{B}) = \bar{\mathcal{H}}(\mathcal{B})$, where we use on the right hand side the general notation $\bar{H}(\mathcal{B}_\infty, \ldots, \mathcal{B}_\backslash) \equiv \bar{\mathcal{H}}(\imath_{\mathcal{B}_\infty}, \ldots, \imath_{\mathcal{B}_\backslash})$ for the inclusion $*$–homomorphisms $\imath_{\mathcal{B}_\|} : \mathcal{B}_\| \hookrightarrow \mathcal{A}$ of finite–dimensional C*–subalgebras $\mathcal{B}_\|$ of $\mathcal{A}$.
3. Note that in (2.5,ii) above, $\bar{H}(\gamma_1, \ldots, \gamma_n) = \log \bar{N}(\beta)$, with $\beta \in \mathcal{O}(\mathcal{A})$ given by $\beta = \gamma_1(\bar{\alpha}_1)\vec{\vee} \ldots \vec{\vee}\gamma_n(\bar{\alpha}_n)$, where $(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$ maximizes to $\bar{H}(\gamma_1, \ldots, \gamma_n)$ (note that this maximizing n–tuple of $\bar{\alpha}_k \in \mathcal{O}^+(\mathcal{B}_\|)$ always exists, see (2.7,ii) below). By (2.2,iv), we know that for the corresponding $\hat{\beta} \in \mathcal{O}_\infty(\mathcal{A})$ as defined in (2.2,i), $\bar{H}(\gamma_1, \ldots, \gamma_n) = \log \bar{N}(\hat{\beta})$. –

   On the other hand, we know that for $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_k \in \mathcal{O}^+(\mathcal{B}_\|) \subset \mathcal{O}(\mathcal{B}_\|)$, there always exist $\check{\alpha}_k \in \mathcal{O}_\infty(\mathcal{B}_\|)$ defined as in (2.2,i) but even with $\check{\alpha}_k \subset \mathcal{B}_\|^+$, such that $\alpha_k \subseteq \check{\alpha}_k$ $(k = 1, \ldots, n)$.

   Denoting generally by $\mathcal{O}^+_\in(\mathcal{A})$ the set of all $\check{\beta} \in \mathcal{O}_\infty(\mathcal{A})$ such that $\check{\beta} \subset \mathcal{A}^+$ (i.e. explicitly of the form $\check{\beta} = \{B_i \in \mathcal{A}^+ | \sum_\rangle \mathcal{B}_\rangle^\in =$

$\mathbf{1}\}$), we can thus equivalently replace in definition (2.5,ii) above the $\max_{\{(\alpha_1,\ldots,\alpha_n)\}}$ by the twofold optimization

$$\max_{\{(\breve{\alpha}_1,\ldots,\breve{\alpha}_n)\}} \quad \max_{\{\alpha_k\subseteq\breve{\alpha}_k,\ k=1,\ldots,n|\alpha_k\in\mathcal{O}^+(\mathcal{B}_\|)\}},$$

where now $\breve{\alpha}_k \in \mathcal{O}^+_\in(\mathcal{B}_\|)$, but again with the additional restriction $\gamma_k \approxeq \gamma_\ell,\ \ k<\ell \Longrightarrow \breve{\alpha}_\ell \equiv \{\mathbf{1}_\ell \in \mathcal{B}_\ell\}$.

4. At this point, we should compare the definition (2.5,ii) with its earlier versions in [13] respectively [12]: In both cases, we had taken the maximum as in (2.5,ii) only over the smaller set $\{(\alpha_1,\ldots,\alpha_n)|\alpha_k \in \mathcal{O}^+_\in(\mathcal{B}_\|)\}$ for rather obvious "physical" reasons (cf. [16, 15] and the references in the Introduction); where we had still chosen a different restriction for repeated arguments of $\bar{H}(\gamma_1,\ldots,\gamma_n)$ in [13] than the improved condition (2.5,i & ii) in [12] and here. In addition, we had still tried in [13] to avoid the also "physically" less appealing fact that even then, $\alpha = \gamma_1(\alpha_1)\vec{\vee}\ldots\vec{\vee}\gamma_n(\alpha_n) \in \mathcal{O}(\mathcal{A})$ will *not* be a generalized partition of unity in general: $\alpha \notin \mathcal{O}_\infty(\mathcal{A})$ (what we had denoted by $\mathcal{O}_\in(\mathcal{A})$ in [13, 12]), using the trick that we had replaced α in (2.5,ii), as defined just before, by $\gamma_1[\alpha_1]_2\vec{\vee}\ldots\vec{\vee}\gamma_n[\alpha_n]_2 \in \mathcal{O}_\infty(\mathcal{A})$ with the deformed, non–linear "modulus"–application $\gamma_k[\centerdot]_2 : \mathcal{B}_\| \to \mathcal{A}^+$ as defined in [13], where $\gamma_k[\mathcal{O}^+_\in(\mathcal{B}_\|)]_\in \subset \mathcal{O}^+_\in(\mathcal{A})$ holds true.

   But the first part of (3) above shows that this unnatural deformation is not necessary, as we can always "renormalize" the final α as above to $\hat{\alpha} \in \mathcal{O}_\infty(\mathcal{A})$ with the same entropy.

   On the other hand, the second part of (3) shows that $\bar{H}(\gamma_1,\ldots,\gamma_n)$ as defined in (2.5,ii) here is generally greater (or equal) than $\bar{H}(\gamma_1,\ldots,\gamma_n)$ of [12], although we are again starting with the "physically appealing" $\breve{\alpha}_k \in \mathcal{O}^+_\in(\mathcal{B}_\|)$ in (3) above. Actually, for a *general* $\alpha \in \mathcal{O}(\mathcal{A})$ as above (but with $\alpha_k \in \mathcal{O}^+(\mathcal{B}_\|)$) in (2.5,ii), obviously $\alpha \subseteq \tilde{\alpha} \equiv \gamma_1(\breve{\alpha}_1)\vec{\vee}\ldots\vec{\vee}\gamma_n(\breve{\alpha}_n) \in \mathcal{O}(\mathcal{A})$ and hence $\bar{N}(\alpha) \geqslant \bar{N}(\tilde{\alpha})$ by (2.2,iv); meaning that not only the maximum over the *larger set* in (2.5,ii) increases $\bar{H}(\gamma_1,\ldots,\gamma_n)$ when compared to [12], but even for fixed $(\breve{\alpha}_1,\ldots,\breve{\alpha}_n)$ with $\breve{\alpha}_k \in \mathcal{O}^+_\in(\mathcal{B}_\|)$, we can always improve the resulting "entropy" by choosing *any* "incomplete but still sufficient" operations $\alpha_k \subset \check{\alpha}_k,\ \ \alpha_k \in \mathcal{O}^+(\mathcal{B}_\|)$ (for $k = 1, \dots, n$), in the twofold optimization as in (3) above (cf. [15] for the notion of "operation").

Proposition (2.7) . *The following properties of the entropy functional (2.5,ii) are easy to deduce:*

(i) *For $\theta_k : \mathcal{B}_\| \to \mathcal{A}_\|$ and $\gamma_k : \mathcal{A}_\| \to \mathcal{A}$ both positive unital maps with finite–dimensional C^*–algebras $\mathcal{A}_\|, \mathcal{B}_\|$ $(\| = \infty, \dots, \backslash)$, such that for $\gamma_i \approxeq \gamma_j$ (as defined in (1.5,i)) also $\gamma_i \circ \theta_i \approxeq \gamma_j \circ \theta_j$ $(\forall i, j = 1, \dots, n)$, we have:*

$$\bar{H}(\gamma_1 \circ \theta_1, \dots, \gamma_n \circ \theta_n) \leqslant \bar{H}(\gamma_1, \dots, \gamma_n).$$

This follows from $\theta_k(\mathcal{O}^+(\mathcal{B}_\|)) \subset \mathcal{O}^+(\mathcal{A}_\|)$, cf. (2.3,iii).

Note that this is the first main advantage of the definition (2.5,ii) when compared to that in [12], *as discussed in (2.6,4) above.*

(ii) *For $\gamma_k : \mathcal{A}_\| \to \mathcal{A}$ positive unital maps with finite–dimensional C^*–algebras $\mathcal{A}_\|$, we have*

$$\bar{H}(\gamma_1, \dots, \gamma_m) \leqslant \bar{H}(\gamma_1, \dots, \gamma_n) \leqslant \bar{H}(\gamma_1, \dots, \gamma_m) + \bar{H}(\gamma_{m+1}, \dots, \gamma_n),$$

$\forall m < n$ (where literally the order of $(1, \dots, n)$ has to be respected), which follows from (2.3,i).

If in addition $\gamma_i \not\approxeq \gamma_j$ $\forall i \neq j \in \{1, \dots, n\}$, then even for $1 \leqslant i_1 < i_2 < \dots < i_\ell \leqslant n$ and $\ell < n$:

$$\bar{H}(\gamma_{i_1}, \dots, \gamma_{i_n}) \leqslant \bar{H}(\gamma_1, \dots, \gamma_n),$$

as follows directly from Def. (2.5,ii). Note that the right hand side "subadditivity" above implies that actually $\bar{H}(\gamma_1, \dots, \gamma_n)$ is always finite, what we used implicitely in (2.5,ii) with the max.

(iii) *For a unital $*$–endomorphism θ of $\mathcal{A}$ and γ_k $(k = 1, \dots, n)$ as in (ii), we have the inequality $\bar{H}(\theta \circ \gamma_1, \dots, \theta \circ \gamma_n) \leqslant \bar{H}(\gamma_1, \dots, \gamma_n)$, which follows from (2.3,iv).*

(iv) *If there exists a finite–dimensional C^*–subalgebra $\mathcal{B} \subset \mathcal{A}$ such that the image $Im(\gamma_k) \subseteq \mathcal{B}$ $\forall \| = \infty, \dots, \backslash$, then $\bar{H}(\gamma_1, \dots, \gamma_n) \leqslant$*

$\log D(\mathcal{B}) \equiv \bar{\mathcal{H}}(\mathcal{B})$, generalizing (2.6,ii). This follows again from Lemma (2.4).

(v) *For $\gamma_k \approxeq \gamma'_k$ $(k = 1, \ldots, n)$, we have $\bar{H}(\gamma_1, \ldots, \gamma_n) = \bar{H}(\gamma'_1, \ldots, \gamma'_n)$, which follows directly from (2.5).*

Definition (2.8) . For a positive unital map $\gamma : \mathcal{B} \to \mathcal{A}$ from a finite–dimensional C^*–algebra $\mathcal{B}$, and for a unital $*$–endomorphism θ of $\mathcal{A}$, the entropy of θ with respect to γ is defined by $\hbar(\theta, \gamma) = \lim_{n\to\infty} \frac{1}{n} \bar{H}(\gamma, \theta \circ \gamma, \ldots, \theta^{n-1} \circ \gamma)$, where the limit exists by (2.7,ii & iii), and actually equals the *infimum* over $n \in \mathbb{N}$ (by standard arguments).

Proposition (2.9) . *The entropy (2.8) has the following properties (as corollaries of (2.7)): For $\gamma : \mathcal{B} \to \mathcal{A}$ and θ as in (2.8),*

(i) *$\hbar(\theta, \gamma) \leqslant \bar{H}(\gamma)$, as follows from (2.7,ii & iii).*

(ii) *For positive unital maps $\gamma_2 : \mathcal{B}_\in \to \mathcal{A}$ and $\gamma_1 : \mathcal{B}_\infty \to \mathcal{B}_\in$ with finite–dimensional $\mathcal{B}_\|$ $(\| = \infty, \in)$, we have $\hbar(\theta, \gamma_2 \circ \gamma_1) \leqslant \hbar(\theta, \gamma_2)$, as follows from (2.7,i).*

(iii) *$\hbar(\sigma \circ \theta \circ \sigma^{-1}, \sigma \circ \gamma) = \hbar(\theta, \gamma)$ for a $*$–isomorphism $\sigma : \mathcal{A} \to \sigma(\mathcal{A})$, as follows from (2.7,iii).*

(iv) *If there exists a $k \in \mathbb{N}$ such that $\gamma \approxeq \theta^k \circ \gamma$, i.e. θ acts periodically on γ (up to pre-image isomorphisms, to be precise) then $\hbar(\theta, \gamma) = 0$. This follows directly from definitions (2.5) and (2.8).*

(v) *$\hbar(\theta^n, \gamma) \leqslant n \cdot \hbar(\theta, \gamma)$ $\forall n \in \mathbb{N}$. Note that if the right hand side is equal to zero because of the situation (iv) above, then so is the left hand side; and otherwise, the inequality follows from the second one in (2.7,ii).*

Lemma (2.10) . *Let (X, d) be a* metric *compact space and $\mathcal{A} = \mathcal{C}(\mathcal{X})$. Let $(\mathcal{U}_\backslash)_{\backslash\in\mathbb{N}}$ be a sequence of finite open covers $\mathcal{U}_\backslash = \{\mathcal{U}^{(\backslash)}_{\rangle_\backslash} \subset \mathcal{X} | \rangle_\backslash = \infty, \ldots, \mathcal{N}_\backslash : \bigcup_{\rangle_\backslash} \mathcal{U}^{(\backslash)}_{\rangle_\backslash} = \mathcal{X}\}$ of X, such that $d(\mathcal{U}_\backslash) \to \prime$, where $d(\mathcal{U}_\backslash) = \max_{\rangle_\backslash} \lceil(\mathcal{U}^{(\backslash)}_{\rangle_\backslash})$ is the maximal d–diameter of the sets in $\mathcal{U}_\backslash$. For each $n \in \mathbb{N}$, choose a partition of unity β_n in $\mathcal{A}^+$ such that $\beta_n = \{f^{(n)}_{i_n} \in \mathcal{A}^+ | \rangle_\backslash = \infty, \ldots, \mathcal{N}_\backslash : \sum_{\rangle_\backslash} \{^{(\backslash)}_{\rangle_\backslash} = \mathbf{1}\}$ is "supported" exactly on the elements of $\mathcal{U}_\backslash$, i.e. $f^{(n)}_{i_n}(x) > 0 \iff x \in U^{(n)}_{i_n}$ $\forall x \in X$, $\forall i_n = 1, \ldots, N_n$. Note that we have $\beta_n \in \mathcal{O}^+(\mathcal{A})$ as defined in (2.1,ii), and define new partitions* *of unity* $\alpha_n = \beta_1 \vec{\nabla} \beta_2 \vec{\nabla} \dots \vec{\nabla} \beta_n$ *with* $\vec{\nabla}$ *as in (2.1,iii); but note that now for* $\alpha_n = \{g^{(n)}_{I_n} \equiv \prod_{k=1}^n f^{(k)}_{i_k} | \forall I_n = (i_1, \dots, i_n)\}$, *we have again* $\alpha_n \in \mathcal{O}^+(\mathcal{A})$, *as* $\mathcal{A}$ *is Abelian.*

Define positive unital maps $\tau_n : \mathcal{B}_\backslash = \bigoplus_{\mathcal{I}_\backslash} (\mathbb{C})_{\mathbb{I}_\ltimes} \to \mathcal{A}$, *on the Abelian direct sum* C^**–algebra* $\mathcal{B}_\backslash$ *of as many copies of* $\mathbb{C}$ *as there are occurring multi–indices* I_n, *by* $\tau_n(e^{(n)}_{I_n}) = g^{(n)}_{I_n}$ *with the minimal projectors* $e^{(n)}_{I_n} \in \mathcal{B}_\backslash$, $\forall \mathcal{I}_\backslash = (\rangle_\infty, \dots, \rangle_\backslash)$, $\forall \backslash \in \mathbb{N}$. *Then the following is true:*

(i) $(\tau_n)_{n\in\mathbb{N}}$ *is a "cover–increasing" sequence in the sense that* $\tau_n(\mathcal{O}^+(\mathcal{B}_\backslash)) \subseteq \tau_{\backslash+\infty}(\mathcal{O}^+(\mathcal{B}_{\backslash+\infty}))$ $(\subset \mathcal{O}^+(\mathcal{A}))$. *This is obvious from the fact that, by construction,*

$$g^{(n)}_{I_n} = \sum_{i_{(n+1)}} g^{(n+1)}_{I_{(n+1)}} \qquad \forall I_{(n+1)} = (i_1, \dots, i_{(n+1)}), \ \forall n \in \mathbb{N}.$$

(ii) $(\tau_n)_{n\in\mathbb{N}}$ *is a sequence "approximating for nuclearity" (of* $\mathcal{A}$*) in the following sense: There exists a sequence* $(\sigma_n)_{n\in\mathbb{N}}$ *of positive unital maps* $\sigma_n : \mathcal{A} \to \mathcal{B}_\backslash$ *with the Choi–Effros–Lance "approximation property"* $\|\tau_n \circ \sigma_n(A) - A\| \to 0, \ \forall A \in \mathcal{A}$ *(see* [21] *and the reference there to Lance's work, cf. also for example* [8]*). See* [13, 12] *for the simple proof, left to the reader after the essential hint that the maps* σ_n *may be chosen to be*

$$\sigma_n(A) = \sum_{I_n} A(x^{(n)}_{I_n}) \cdot e^{(n)}_{I_n} \qquad \forall A \in \mathcal{A} = \mathcal{C}(\mathcal{X}),$$

for any chosen sets of points $x^{(n)}_{I_n} \in X$ *such that* $g^{(n)}_{I_n}(x^{(n)}_{I_n}) > 0 \ \forall I_n, n \in \mathbb{N}$.

(iii) *For any continuous map* $T : X \to X$ *from* X *onto itself, we denote by* θ_T *the unital* $*$*–endomorphism of* $\mathcal{A}$ *induced by* T *(via* $\theta_T(g) = g \circ T \ \forall g \in \mathcal{A} = \mathcal{C}(\mathcal{X})$*, which gives an automorphism of* $\mathcal{A}$ *iff* T *is a homeomorphism). Then the entropy (2.8) gives exactly the classical entropy (1.1,ii) of the refining covers* $\mathcal{V}_\backslash \equiv \bigvee_{\|=\infty}^{\backslash} \mathcal{U}_\|$ *of* X*:* $\hbar(\theta_T, \tau_n) = h(T, \mathcal{V}_\backslash)$*, and thus the topological entropy* [1] *of* T *can be computed as* $h(T) = \lim_{n\to\infty} h(\theta_T, \tau_n)$*, where it follows here from (i) that the sequence* $(\hbar(\theta_T, \tau_n))_{n\in\mathbb{N}}$ *is monotonically non–decreasing in* $\mathbb{R}^+$ *(but possibly divergent with* $h(T) = \infty$*). See* *again* [13, 12] *for the simple but tedious proof of the equivalence above.*

Definition (2.11) . Again for a general C*–algebra $\mathcal{A}$, we say that $\mathcal{A}$ is "appropriately approximated" (AA for short) by a sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$ of positive unital maps $\tau_n : \mathcal{B}_\backslash \to \mathcal{A}$ from (general) finite–dimensional C*–algebras $\mathcal{B}_\backslash$, if *one* of the following two (independent) conditions is fulfilled:

(i) τ is a "cover–increasing" sequence, i.e. $\tau_n(\mathcal{O}^+(\mathcal{B}_\backslash)) \subseteq \tau_{\backslash+\infty}(\mathcal{O}^+(\mathcal{B}_{\backslash+\infty}))$, $\forall\backslash \in \mathbb{N}$. Note that this is the case whenever $\tau_n = \tau_{n+1} \circ \sigma_{n,(n+1)}$ with positive unital maps $\sigma_{n,(n+1)} : \mathcal{B}_\backslash \to \mathcal{B}_{\backslash+\infty}$, $\forall\backslash \in \mathbb{N}$ (for example if $\mathcal{B} \subset \mathcal{A}$ is an AF algebra, i.e. $\mathcal{B} = \overline{\bigcup_{\backslash\in\mathbb{N}} \mathcal{B}_\backslash}$ is the norm–closure of the increasing inductive limit of finite–dimensional C^*–algebras $\mathcal{B}_\backslash \subset \mathcal{B}_{\backslash+\infty}$ $\forall\backslash \in \mathbb{N}$, and $\tau_n = \imath_{\mathcal{B}_\backslash} : \mathcal{B}_\backslash \hookrightarrow \mathcal{A}$ are the inclusion $*$–homomorphisms, where we can choose $\sigma_{n,(n+1)} : \mathcal{B}_\backslash \hookrightarrow \mathcal{B}_{\backslash+\infty}$ to be the inclusion maps, too).

(ii) $\mathcal{A}$ is separable and $\tau = (\tau_n)_{n\in\mathbb{N}}$ is "range approximating" $\mathcal{A}$, i.e. $\overline{\bigcup_{n\in\mathbb{N}} \tau_n(\mathcal{B}_\backslash)} = \mathcal{A}$. Note that this is the case in particular whenever $\mathcal{A}$ is nuclear and τ is a sequence "approximating for nuclearity" of $\mathcal{A}$ as in (2.10,ii), there for Abelian $\mathcal{A}$, above (for example if $\mathcal{B} = \mathcal{A}$ in (i)).

If $\mathcal{A}$ is AA, by the sequence τ (i.e. either (i) or (ii) is true), we define the "τ–topological" entropy of a unital $*$–endomorphism θ of $\mathcal{A}$ by $\hbar_\tau(\theta) = \limsup_{n\to\infty} \hbar(\theta, \tau_n)$ with (2.8) on the right hand side. Note that if τ is "cover–increasing", that is (i) is realized, it follows that $\hbar_\tau(\theta) = \lim_{n\to\infty} \hbar(\theta, \tau_n)$ as an *increasing* limit.

Remarks and Observations (2.12) . The following remarks should again eludicate the above definition (1.11):

1. Again (cf. the remark in (2.6,1)), the definition (2.11) seems to be not very "canonical" at first sight. The canonical approach to the "topological" entropy $\hbar(\theta)$ of a $*$–endomorphism θ of $\mathcal{A}$, directly following our "guiding" theory [8] of entropy for automorphisms with respect to an invariant state, would be to define

$\hbar(\theta) = \sup_\gamma \hbar(\theta, \gamma)$, where the supremum is taken over *all* completely positive unital maps γ into $\mathcal{A}$ with finite rank (i.e. with finite–dimensional pre–image algebra). Let us henceforth denote by $\mathcal{CP}_\infty(\mathcal{A})$ the set of all such maps γ; and by $\mathcal{P}_\infty(\mathcal{A})$ we denote the subset of maps with (finite–dimensional) *Abelian* pre–image algebra.

Then, however, we would have to *prove* an "approximation theorem" of the form $\hbar(\theta) = \lim_{n\to\infty} \hbar(\theta, \tau_n)$, for example in the case that $(\tau_n)_{n\in\mathbb{N}}$ is a sequence "approximating for nuclearity" of $\mathcal{A}$. But, as pointed out in [13, 12], the entropy $\hbar(\theta, \gamma)$ of (2.8) is *not* continuous in norm with respect to the varying linear maps $\gamma : \mathcal{B} \to \mathcal{A}$, $\mathcal{B}$ fixed (for example with $\gamma \in \mathcal{CP}_\infty(\mathcal{A})$); and this makes it impossible to proof such a theorem along the lines of [8].

2. Let us briefly recall recall from [13, 12] that for a general C*–algebra $\mathcal{A}$ (also if it is *non*–Abelian, by a similar argument as for Abelian $\mathcal{A}$), the set $\{\gamma \in \mathcal{P}_\infty(\mathcal{A}) | \gamma : \mathcal{B} \to \mathcal{A}\}$ with $\mathcal{B}$ fixed has even an *open, dense* subset (in the norm of the linear maps γ) of $\gamma' : \mathcal{B} \to \mathcal{A}$, on which the entropy (2.8) vanishes for *any* $*$–endomorphism θ, i.e. $\hbar(\theta, \gamma') = 0$.

   In particular, for $\mathcal{A} = \mathcal{C}(\mathcal{X})$ as in (2.10) above, one can easily construct sequences $\tau = (\tau_n)_{n\in\mathbb{N}}$ "approximating for nuclearity" as in (2.10,ii), with $\tau_n : \mathcal{B}_\backslash \to \mathcal{A}$ and $\mathcal{B}_\backslash$ *Abelian* (i.e. $\tau_n \in \mathcal{P}_\infty(\mathcal{A})$), but still in such a way that $\hbar(\theta_T, \tau_n) \equiv 0 \;\; \forall n \in \mathbb{N}$ (no matter how T respectively θ_T is chosen).

   In other words, the "approximation" condition (2.10,ii) is by construction necessary but not at all sufficient for the result $\hbar(\theta_T, \tau_n) \to h(T)$ in (2.10,iii), which is rather due to the natural partial order on the set of finite open covers of X, for which the sequence $\mathcal{V}_\backslash = \bigvee_{\|=\infty}^{\backslash} \mathcal{U}_\|$ in (2.10) is cofinal, and even monotonically "refining".

3. But even the additional "cover–increase" condition (2.10,i), which in a way represents the latter order monotonicity of the sequence $\mathcal{V}_\backslash$, is not sufficient together with (2.10,ii) to imply the

result $\hbar(\theta_T, \tau_n) \to h(T)$, as still the same counter–examples with $\hbar(\theta, \tau_n) \equiv 0$ as in (2) apply (again an easy exercise).

¿From that algebraic point of view, this convergence in (2.10,iii) is rather implied by the fact that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in \mathcal{O}^+(\mathcal{A})$ as in (2.10) is again cofinal (and even monotonic) in $\mathcal{O}^+(\mathcal{A})$ with respect to the natural partial pre–order on $\mathcal{O}^+(\mathcal{A})$ "inherited" from that on the set of finite open covers of X, see [13, 12].

But for a *non*–Abelian C*–algebra $\mathcal{A}$, this argument fails for (at least) two different reasons: First, this mentioned partial pre–order on $\mathcal{O}^+(\mathcal{A})$ relies on the usual partial order of $\mathcal{A}^+$ by positivity, and there is no way that this latter order is preserved in (2.5,ii) when taking the operator products of the $\gamma_k(\alpha_k) \in \mathcal{O}^+(\mathcal{A})$ which would be order–related to other $\beta_k \in \mathcal{O}^+(\mathcal{A})$, say $(k = 1, \ldots, n)$.

And secondly, for non–commutative (even separable) $\mathcal{A}$ there seems to be generally no *cofinality* possible for this "natural" partial pre–order on $\mathcal{O}^+(\mathcal{A})$, when considering a fixed sequence of $\tau_n \in \mathcal{CP}_\infty(\mathcal{A})$ as used in (2.11) and in the Abelian case (2.10) above (cf. [12, Appendix]. In other words, we lack a "non–commutative Lebesgue's covering theorem", whereas in (2.10) it follows from the basic assumption $d(\mathcal{U}_\backslash) \to \prime$ that the sequence $(\alpha_n)_{n\in\mathbb{N}}$, and already $(\beta_n)_{n\in\mathbb{N}}$, is cofinal in $\mathcal{O}^+(\mathcal{A})$ as mentioned; or there even equivalently, that $(\tau_n)_{n\in\mathbb{N}}$ is increasing also "cofinally" in $\mathcal{P}_\infty(\mathcal{A})$ in a certain sense).

4. It should now be clear that our not very "canonical" definition (2.11) is an (admittedly, provisional) attempt to retain as much of the essential structure used for the classical computations of the topological entropy, here in (2.10, iii), as is possible by now for non–Abelian C*–algebras, and to do so in the least restrictive way by imposing only one of the two conditions (2.11,i & ii) alternatively. Actually, with only the first (2.11,i) being fulfilled, it seems to be an "abuse of language" to call $\mathcal{A}$ A. *approximated* by τ, and it would perhaps be more appropriate to call $\mathcal{A}$ A. "approached" by τ in that case.

However, note that at least all AF algebras $\mathcal{A}$ are AA in the "strong" sense that *both* (2.11,i & ii) are fulfilled, if τ and the AF–structure are chosen as noted in (2.11). But still then, and even "more so" generally, we have first only the trivial inequality for an AA-sequence $\tau \subset \mathcal{CP}_\infty(\mathcal{A})$, i.e. with *completely* positive maps τ_n: $\hbar_\tau(\theta) \leq \hbar(\theta)$, defining the right hand side as in (1) above by $\hbar(\theta) = \sup_{\gamma \in \mathcal{CP}_\infty(\mathcal{A})}$. Note that for $\mathcal{A}$ an AF algebra as before, in fact $\tau \subset \mathcal{CP}_\infty(\mathcal{A})$.

5. It should be emphasized at this point, however, that this latter upper bound is *not* necessarily a direct generalization of the classical topological entropy $h(T)$ as in (2.10,iii) above:

   Whereas it is easily seen that for $\mathcal{A} = \mathcal{C}(\mathcal{X})$ generally

$$h(T) = \sup_{\gamma \in \mathcal{P}_\infty(\mathcal{A})} \hbar(\theta_T, \gamma) \leqslant \hbar(\theta_T)$$

   with the notations from (2.10,iii) and (1) above (cf. [13, 12] for the left hand side equality), we have not been able yet to either prove or disprove the equality $h(T) = \hbar(\theta_T)$ in this $\mathcal{A}$–Abelian case, where the problem comes from the additional non–commutative freedom for $\alpha_k \in \mathcal{O}^+(\mathcal{B}_\|)$ in definition (2.5,ii) for different $k \in \mathbb{N}$ (used for $\theta^k \circ \gamma : \mathcal{B} \to \mathcal{A}$ with $\mathcal{B}$ *non*–Abelian in definition (2.8) of $\hbar(\theta, \gamma)$ with $\gamma \in \mathcal{CP}_\infty(\mathcal{A})$).

   Although the analogous equality in the case of Abelian $\mathcal{A}$ holds for the state–dependent entropy of [8], identifying it with the classical measure–theoretic entropy even when taking the analogous supremum over $\mathcal{CP}_\infty(\mathcal{A})$ for Abelian $\mathcal{A}$ (cf. [22]), we do not regard this property as really essential; but on the other hand, we consider that non–commutative freedom in Def. (2.5,ii) as indispensable for non–Abelian $\mathcal{A}$ (cf. also section III below, after the proof of (3.1)).

   While this "hybrid" $\mathcal{A}$–Abelian and $\mathcal{B}$–noncommutative case seems to be an interesting problem *per se*, we have now circumvented also this obstacle with our pragmatic definition (2.11), as we can always choose the AA–sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$ for *Abelian* $\mathcal{A}$ such that only $\tau_n \in \mathcal{P}_\infty(\mathcal{A})$ enters, $\forall n \in \mathbb{N}$ (implying that we have in turn the general inequality $\hbar_\tau(\theta_T) \leqslant h(T)$, where equality holds in the situation of (2.10) for separable $\mathcal{A}$). We do not force this choice for Abelian $\mathcal{A}$ to be part of Definition (2.11), however, because there *are* some rather obvious examples of AA–sequences τ with truly "non–commutatively mapping" $\tau_n \in \mathcal{CP}_\infty(\mathcal{A})$ for Abelian $\mathcal{A}$, but still $\hbar_\tau(\theta_T) = h(T)$; see section IV below (in particular Example (4.13), where it should be an easy exercise for the reader to find such a sequence τ).

Unfortunately, the possible *counter*–examples are not obvious enough, if they exist at all. We leave this problem touched upon here for further study.

Proposition (2.13) . *If $\mathcal{A}$ is AA with a sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$ as in (2.11), and $\sigma : \mathcal{A} \to \mathcal{B}$ is a $*$–isomorphism, then rather obviously also $\mathcal{B}$ is AA by the sequence denoted as $\sigma(\tau) \equiv (\sigma\circ\tau_n)_{n\in\mathbb{N}}$ (i.e. both (2.11,i & ii) are preserved by σ). The entropy (2.11) of a $*$–endomorphism θ of $\mathcal{A}$ has the following rather obvious properties (as corollaries of (2.9)):*

(i) $\hbar_{\sigma(\tau)}(\sigma\circ\theta\circ\sigma^{-1}) = \hbar_\tau(\theta)$ $\forall\sigma : \mathcal{A}\to\mathcal{B}$, *$*$–isomorphisms.*
(ii) $\hbar_\tau(\theta^n) \leqslant n\cdot\hbar_\tau(\theta)$ $\forall n\in\mathbb{N}$.
(iii) *For a* periodic *$*$–automorphism $\theta = \theta^k$ $(k\in\mathbb{N})$, we have $\hbar_\tau(\theta) = 0$.*

Note that it is only at this point (iii) (and at the intermediate step (2.9,iv)) that the additional restriction in Def. (2.5) is really necessary, as discussed already in (2.6,i). In other words, this restriction in (2.5) is a necessary correction to exclude the additional "quantum stochasticity" which could otherwise lead to a positive $\hbar_\tau(\theta)$ also for trivial or periodic θ, due to the repeated "operations" $\gamma(\alpha_1), \theta\circ\gamma(\alpha_2), \ldots, \theta^{n-1}\circ\gamma(\alpha_n)$ with $\alpha_k \in \mathcal{O}^+(\mathcal{B})$ and $\mathcal{B}$ *non*–Abelian finite–dimensional in (2.5,ii).

Definition (2.14) . We define the following two notions for the later Theorem (2.15):

(i) We call a positive unital map $\gamma : \mathcal{B}\to\mathcal{A}$ from a finite–dimensional C*–algebra $\mathcal{B}$ *faithfully covering*, if there exists $\alpha' \in \mathcal{O}^+(\mathcal{B})$ such that α' at the same time maximizes to $\bar{H}(\gamma)$ as in (2.5), and is already "γ–minimal", i.e. both equations

$$\exp \bar{H}(\gamma) = \bar{N}(\gamma(\alpha')) = \operatorname{card} \alpha'$$

are satisfied.

(ii) We call n positive unital maps $\gamma_k : \mathcal{B}_\| \to \mathcal{A}$ $(\| = \infty, \ldots, \backslash)$ ($\mathcal{B}_\|$ here not necessarily finite–dimensional) *independently covering*, if for $B_k \in \mathcal{B}_\|^+$ but $\gamma_k(B_k)$ not invertible $(k = 1, \ldots, n)$ there exists a state $\omega \in S_{\mathcal{A}}$ which simultaneously annihilates all $\gamma_k(B_k)$: $\omega \circ \gamma_k(B_k) = 0 \;\; \forall k = 1, \ldots, n$.

We say that two maps γ_1 and γ_2 are "commuting" (denoted symbolically $[\gamma_1, \gamma_2] = 0$) if $[\gamma_1(B_1), \gamma_2(B_2)] = 0 \;\; \forall B_k \in \mathcal{B}_\|$ $(\| = \infty, \in)$.

Theorem (2.15) . *If $\gamma_k : \mathcal{B}_\| \to \mathcal{A}$ $(\| = \infty, \ldots, \backslash)$ are positive unital maps from finite–dimensional C^*–algebras $\mathcal{B}_\|$ into $\mathcal{A}$, which are* independently *and (each individually)* faithfully *covering, and pairwise commuting, i.e. $[\gamma_i, \gamma_j] = 0 \;\; \forall i \neq j \in \{1, \ldots, n\}$, then the optimal upper bound of (2.7,ii) is attained: $\bar{H}(\gamma_1, \ldots, \gamma_n) = \sum_{k=1}^n \bar{H}(\gamma_k)$.*

Proof. Using the notation α' of (2.14,i), we choose $\alpha'_k \in \mathcal{O}^+(\mathcal{B}_\|)$ such that by assumption also $\bar{N}(\gamma_k(\alpha'_k)) = \operatorname{card} \alpha'_k \equiv N_k$ holds $(k = 1, \ldots, n)$. Clearly, $\beta = \gamma_1(\alpha'_1) \vec{\vee} \ldots \vec{\vee} \gamma_n(\alpha'_n) \in \mathcal{O}(\mathcal{A})$ is a possible choice inside the maximum in definition (2.5,ii) of $\bar{H}(\gamma_1, \ldots, \gamma_n)$.

We now claim that $\bar{N}(\beta) = \prod_{k=1}^n N_k$, which implies by (2.5,ii) that $\bar{H}(\gamma_1, \ldots, \gamma_n) \geqslant \log \bar{N}(\beta) = \sum_{k=1}^n \bar{H}(\gamma_k)$ again by the faithfulness assumption on the γ_k, and then (2.7,ii) gives the converse.

To prove that $\bar{N}(\beta) = \prod_{k=1}^n N_k$, put $\alpha'_k = \{A^{(k)}_{i_k} \in \mathcal{B}_\|^+ | \rangle_\| = \infty, \ldots, \mathcal{N}_\|\}$ $(\| = \infty, \ldots, \backslash)$ and use the following abbreviation:

$$B_{(i_1, \ldots, i_n)} = \gamma_1(A^{(1)}_{i_1}) \gamma_2(A^{(2)}_{i_2}) \cdot \cdots \cdot [\gamma_n(A^{(n)}_{i_n})]^2 \cdot \cdots \cdot \gamma_1(A^{(1)}_{i_1}) \in \mathcal{A}^+$$

$\forall I_n \equiv (i_1, \ldots, i_n)$, where $i_k \in \{1, \ldots, N_k\} \;\forall k$. The statement $\beta \in \mathcal{O}(\mathcal{A})$ means that the following sum is strictly positive:

$$B \equiv \sum_{I_n} B_{I_n} = B_< + \sum_{k=1}^n \sum_{j_k \neq k} B_{\leqslant}(k, j_k) + B_{(N_1, \ldots, N_n)} > 0,$$

where we decomposed the sum using the notations

$$B_< = \sum_{\substack{\{(i_1,\ldots,i_n)| \\ i_k \lneqq N_k \ \ \forall k\}}} B_{(i_1,\ldots,i_n)}, \qquad B_\leqslant(k, j_k) = \sum_{\substack{\{(i_1,\ldots,i_n)| \\ i_k \lneqq N_k, \\ i_{j_k}=N_{j_k}\}}} B_{(i_1,\ldots,i_n)}$$

(note that in the summation defining $B_\leqslant(k, j_k)$ for $k \neq j_k$ the other indices are free: $i_j \leqslant N_j \ \ \forall j \neq k, j_k$).

That we have $\bar{N}(\gamma_k(\alpha'_k)) = \operatorname{card}\alpha'_k$ implies in particular (but without loss of generality, by renumbering) that the following elements $C_k \in \mathcal{A}$ are *not* strictly positive: $C_k = \sum_{i_k=1}^{(N_k-1)} \gamma_k(A^{(k)}_{i_k}) \not> 0$ $(\forall k = 1, \ldots, n)$: Remember from (2.2, ad (iv)) that for general $\alpha \in \mathcal{O}^+(\mathcal{A})$, equivalently $\bar{N}(\alpha) = \min\{\operatorname{card}(\alpha' \subset \alpha) | \alpha' \in \mathcal{O}^+(\mathcal{A})\}$.

By the assumption that the γ_k are independently covering, there exists a state $\omega \in S_\mathcal{A}$ such that $\omega(C_k) = 0 \ \ \forall k = 1, \ldots, n$. This is *a priori* not clear from definition (2.14,ii) for $D_k \equiv \sum_{i_k=1}^{(N_k-1)} [\gamma_k(A^{(k)}_{i_k})]^2 \not> 0$ $(\forall k = 1, \ldots, n)$; but because of $A^2 \leqslant A \cdot \|A\|$ (for $A \in \mathcal{A}^+$) it follows obviously that also $\omega(D_k) = 0 \ \ \forall k$ and hence $\omega(D_k^2) = 0 \ \ \forall k$.

Using the pairwise commutativity assumption ($[\gamma_i, \gamma_j] = 0 \ \ \forall i \neq j$), it is also obvious that the individual terms in the decomposed sum above can be written as:

$$B_< = \prod_{k=1}^{n} D_k, \qquad B_\leqslant(k, j_k) = D_k \cdot (E_{j_k} \cdot \prod_{\ell \neq k, j_k} F_\ell) \quad \forall k \neq j_k,$$

where we used the notations $E_k = [\gamma_k(A^{(k)}_{N_k})]^2$ and $F_k = \sum_{i_k=1}^{N_k} [\gamma_k(A^{(k)}_{i_k})]^2 > 0$ (the *full* sum), $\forall k = 1, \ldots, n$.

¿From the Cauchy-Schwarz inequality for the state ω it follows in particular that $|\omega(D_k \cdot A)|^2 \leqslant \omega(D_k^2) \cdot \omega(A^*A) = 0 \ \ \forall A \in \mathcal{A}, \ \ \forall \|$, and hence by summation that $\omega(B - B_{(N_1,\ldots,N_n)}) = 0$, but on the other hand we know $\omega(B) > 0$ because of $B > 0$. This implies both $\bar{N}(\beta) = \operatorname{card}\beta$ *and* $B_{(N_1,\ldots,N_n)} \neq 0$, hence $B_{I_n} \neq 0 \ \ \forall I_n$ (by renumbering), so $\operatorname{card}\beta = \prod_{k=1}^{n} N_k$ and the claim is proved. □

Remarks and Observations (2.16) . The following remarks and simple results refer to (2.14) and (2.15):

1. The first condition (2.14,i) of "faithfully covering" γ (which was still missing in [13, 12] for reasons to become clear in (4) below) is always fulfilled for $\mathcal{B} \subset \mathcal{A}$ a finite–dimensional unital (C*–)subalgebra and $\gamma = \imath_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{A}$ the unital inclusion $*$–homomorphism: This is obvious from (2.4).

   Furthermore, in the Abelian case $\mathcal{A} = \mathcal{C}(\mathcal{X})$ as in (2.10), this condition means the following: Starting with an *arbitrary* open cover $\mathcal{U}_\infty = \{\mathcal{U}_\rangle \subset \mathcal{X} | \rangle = \infty, \ldots, \backslash : \bigcup_\rangle \mathcal{U}_\rangle = \mathcal{X}\}$ and choosing *any* corresponding positive unital map $\tau_1 : \mathcal{B}_\infty = \bigoplus_{\rangle=\infty}^{\mathcal{N}} (\mathbb{C})_\beth \to \mathcal{A}$ as in (2.10) (where we take $n = 1$ to avoid the additional refinements and multi–indices), the condition that $\tau_1 \in \mathcal{P}_\infty(\mathcal{A})$ (with the notation (2.12,i)) be *faithfully* covering means that there exists a "cover" $\{\mathfrak{X}_1, \ldots, \mathfrak{X}_{\mathfrak{M}}\} \subset 2^{\mathfrak{X}} \equiv \mathcal{P}(\mathfrak{X})$ of the index set $\mathfrak{X} = \{\imath, \ldots, \mathfrak{N}\}$ of $\mathcal{U}_\infty$ (i.e. $\bigcup_{j=1}^{M} \mathfrak{X}_{\mathfrak{j}} = \mathfrak{X}$), such that for the resulting *coarser* cover (than $\mathcal{U} \equiv \mathcal{U}_\infty$) $\mathcal{W}$ of X given by the resulting unions:

$$\mathcal{W} = \{\mathcal{W}_| = \bigcup_{\rangle_| \in \mathfrak{X}_{\mathfrak{j}}} \mathcal{U}_{\rangle_|} | \forall | = \infty, \ldots, \mathcal{M}\},$$

   we have the corresponding *classical* relation $N(\mathcal{U}) = \mathcal{N}(\mathcal{W}) = \operatorname{card} \mathcal{W}$ (note that generally we know only the inequalities $N(\mathcal{U}) \geqslant \mathcal{N}(\mathcal{W}) \leqslant \operatorname{card} \mathcal{W}$, see (1.2,iv)). This is obviously a very natural condition on $\mathcal{U}$ for requiring that $\mathcal{U}$ has not *too* much "redundancy" in covering X, although it need *not* be itself "minimal" at all (i.e. $N(\mathcal{U}) \leqslant \operatorname{card} \mathcal{U}$ is well possible). Also in this classical context, we call $\mathcal{U}$ *faithfully* covering in this case.
2. But also more generally (now again for non–Abelian $\mathcal{A}$), there are very natural *non*–subalgebra examples of faithfully covering maps $\gamma \in \mathcal{CP}_\infty(\mathcal{A})$, i.e. with non–commutative (pre–)image: Choose any concrete C*–algebra (i.e. in a Hilbert space representation) $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ with two unitaries $U, V \in \mathcal{A}$ such that $[U, V] \neq 0$ but U and V have two common eigenvectors $\xi_\pm \in \mathcal{H}$ with eigenvalues ± 1 (thus belonging to the pure point spectrum of U, V), respectively: $U\xi_\pm = V\xi_\pm = \pm\xi_\pm$. Such examples almost obviously exist, and

then the “formal almost–Mathieu Hamiltonian” (just to give it a name, cf. [23]) $h = \frac{1}{4}(U + V + U^* + V^*)$ has of course again ± 1 in its (pure point) spectrum: $h\xi_\pm = \pm\xi_\pm$.

On the other hand, we can always define a *completely* positive unital map $\gamma : \mathcal{B} = \mathcal{M}_{\in}(\mathbb{C}) \to \mathcal{A}$ by $\gamma(e_{ij}) = A_{ij}$, where e_{ij} are the canonical matrix units in $M_2 = \mathcal{B}$ and $A_{ij} \in \mathcal{A}$ are the elements of a *positive* matrix $A = (A_{ij})$ in the C*–algebra $M_2(\mathcal{A})$ of (2×2)–matrices over $\mathcal{A}$, given by:

$$A = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U + V^* \\ V + U^* & \mathbf{1} \end{pmatrix} \in M_2(\mathcal{A})^+.$$

Note that with $B = \left(\begin{smallmatrix} \mathbf{1} & U \\ V & \mathbf{1} \end{smallmatrix}\right)$, and unitary U, V, we have $A = B^*B \in M_2(\mathcal{A})$; and thus γ is a *completely* positive map by [21] (see the further reference in the introduction there).

Now define $\beta' \in \mathcal{O}^+(\mathcal{B})$ by $\beta' = \{p_+, p_-\}$ where $p_\pm = \frac{1}{2}\left(\begin{smallmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{smallmatrix}\right)$; note that β' is even a (square) partition of unity: $p_+^{(2)} + p_-^{(2)} = \mathbf{1}_2$ (i.e. $\beta \in \mathcal{O}^+_{\in}(\mathcal{B})$ as defined in (2.6,3)).

Then $\gamma(p_\pm) = \frac{1}{2}(\mathbf{1} \pm h)$, and hence by the above choice $\gamma(p_\pm) \not> 0$, which implies that $\bar{N}(\gamma(\beta')) = \log 2$. Thus obviously $\bar{H}(\gamma) = \log 2$ by (2.6,2), and so, as said before, this $\gamma \in \mathcal{CP}_\infty(\mathcal{A})$ is a *faithfully* covering map (with $\beta' \in \mathcal{O}^+(\mathcal{B})$ as the defining entity of (2.14,i), which is even uniquely determined in this simple example).

The important point here is of course that $\gamma(\mathcal{B}) = \mathrm{span}\{\mathbf{1}, (\mathcal{U} + \mathcal{V}^*), (\mathcal{V} + \mathcal{U}^*)\} \subset \mathcal{A}$ is generally *not* contained in any finite–dimensional C*–subalgebra of $\mathcal{A}$, nor will $C^*(U + V^*)$ be Abelian (more or less by assumption) regardless that in this simple example with $\beta' \in \mathcal{O}^+_{\in}(\mathcal{M}_{\in})$ necessarily $[\gamma(p_+), \gamma(p_-)] = 0$.

3. The second condition in (2.14,ii) of “independently covering” maps $\gamma_1, \ldots, \gamma_n$ has at least a natural *terminology*, as it is obviously equivalent to the following reformulations: For $B_k \in \mathcal{B}^+_{\|}$ but $\gamma_k(B_k) \not> 0$ $(k = 1, \ldots, n)$, there exists an $\omega \in S_{\mathcal{A}}$ such that $\omega(\sum_{k=1}^n \gamma_k(B_k)) = 0$; which is in turn equivalent to its negation:

For $B_k \in \mathcal{B}_{\|}^{+}$ $(\| = \infty, \dots, \backslash)$ such that $\{\gamma_k(B_k) | k = 1, \dots, n\} \in \mathcal{O}^{+}(\mathcal{A})$, there exists already a *single* $\gamma_k(B_k) > 0$.

But the condition (2.14,ii) is also equivalent to the "natural" notion of C*–independence [24] for *two* C*–subalgebras $\mathcal{A}_\infty, \mathcal{A}_\in \subset \mathcal{A}$ which are commuting $([\mathcal{A}_\infty, \mathcal{A}_\in] = \prime)$, but still not necessarily finite–dimensional: As extensively reviewed in [24], generally $\mathcal{A}_\infty$ and $\mathcal{A}_\in$ are *C^*–independent* if *one* of the following two equivalent conditions is fulfilled (among other possible characterizations): For all pairs of states $\phi_i \in S_{\mathcal{A}_\rangle}$ $(i = 1, 2)$ there exists an extending state $\varphi \in S_{\mathcal{A}}$ with $\varphi \restriction_{\mathcal{A}_\rangle} = \phi_i$ $\forall i = 1, 2$; or also equivalently: $0 \neq A_i \in \mathcal{A}_\rangle$ $(\rangle = \infty, \in)$ implies $0 \neq A_1 A_2 (\in \mathcal{A})$.

It is easy to see that, given $[\mathcal{A}_\infty, \mathcal{A}_\in] = \prime$, the inclusion $*$–homorphisms $\imath_{\mathcal{A}_\infty}$ and $\imath_{\mathcal{A}_\in}$ are independently covering *iff* $\mathcal{A}_\infty$ and $\mathcal{A}_\in$ are C*–independent; see [12] for the simple proof of the one non–trivial implication, running along the line of the proof of [24]: (3.2,1) $\Longrightarrow$ (3.2,2) there.

If $\mathcal{A}_\infty$ and $\mathcal{A}_\in$ are in addition *finite*–dimensional, it is clear (cf. [24]) that $\mathrm{C}^*(\mathcal{A}_\infty, \mathcal{A}_\in) \cong \mathcal{A}_\infty \otimes \mathcal{A}_\in$; and consequently, if $\mathcal{B}_\infty, \dots, \mathcal{B}_\backslash \subset \mathcal{A}$ are *finite*–dimensional and pairwise commuting, it is again equivalent that $\imath_{\mathcal{B}_\infty}, \dots, \imath_{\mathcal{B}_\backslash}$ are independently covering, to that the natural $*$–homomorphism mapping $\bigotimes_{k=1}^{n} \mathcal{B}_\| \to \mathrm{C}^*(\mathcal{B}_\infty, \dots, \mathcal{B}_\backslash)$ is an isomorphism. More generally, all conditions (2.14) of the additivity result (2.15) are met if the $\gamma_k : \mathcal{B}_\| \to \mathcal{A}$ ($\mathcal{B}_\|$ finite–dimensional, $k = 1, \dots, n$) are faithfully covering maps to $\mathcal{A} = \mathcal{A}_\infty \otimes \cdots \otimes \mathcal{A}_\backslash$, for example with nuclear $\mathcal{A}_\| \ni \mathbf{1}_\|$, and $\gamma_k : \mathcal{B}_\| \to \mathbf{1}_\infty \otimes \cdots \otimes \mathcal{A}_\| \otimes \cdots \otimes \mathbf{1}_\backslash$ $(\forall \| = \infty, \dots, \backslash)$.

4. The reason why the first condition (2.14,i) was still missing in the earlier preliminary versions [13, 12] of the theory presented here is the following: It had escaped the attention of the author that there had been a gap in the very first paper on ("classical") topological entropy by Adler, Konheim and McAndrew [1], namely the claim that for a product cover $\mathcal{U}_\infty \times \mathcal{U}_\in = \{\mathcal{U} \times \mathcal{V} | \mathcal{U} \in \mathcal{U}_\infty,\ \mathcal{V} \in \mathcal{U}_\in\}$ of a product space $X = X_1 \times X_2$ (with product topology), the covering

counting functional $N(\centerdot)$ would be multiplicative: $N_X(\mathcal{U}_\infty \times \mathcal{U}_\in) = \mathcal{N}_{\mathcal{X}_\infty}(\mathcal{U}_\infty) \cdot \mathcal{N}_{\mathcal{X}_\in}(\mathcal{U}_\in)$, not only (rather trivially) *sub*–multiplicative.

This seemingly innocent statement would have been again analogous to the additivity of the measure–theoretic partition entropy for independent partitions (in particular, product partitions of product probability spaces, cf. [17]); but at this point the simple analogy fails: It was then apparently first published by Goodwyn [25], that "without much difficulty one can construct counter–examples where this equality does not hold"; and with the right intuition, this is indeed true.

We leave it to the reader as a simple exercise to construct a counter–example for two three–point spaces $X_1 = X_2 = \{p_1, p_2, p_3\}$, both with discrete topology (or, rather obviously "equivalently", for $X_1 = X_2 = \mathbf{S^1}$ and $X = \mathbb{T}^{\not\models}$ with usual topology and a product cover by $9 = 3^2$ *connected* open sets); and actually, this flaw in the argument of [1] had been immediately pointed out by R. Bowen [26], leading to his alternative definition of topological entropy (see [18, 17] and the references there), but unfortunately the latter seems to be not (yet) apt for non–commutative generalization.

Anyway, the proof of the earlier version of (2.15) in [12] (implicit in [13]) had still contained the same old mistake, which was kindly brought to the author's attention by Klaus Thomsen (cf. section III below).

Note that the fact that now this gap in the proof can be easily avoided, by the additional condition (2.14,i) of *faithfully* covering maps, is the second main advantage of the definition (2.5,ii) when compared to that in [12] (i.e. taking the maximum with $\alpha_k \in \mathcal{O}^+(\mathcal{B}_\|)$ instead of $\mathcal{O}^+_\in(\mathcal{B}_\|)$; cf. (2.7,i)). Of course, the result (2.15) holds in particular for Abelian $\mathcal{A} = \mathcal{C}(\mathcal{X})$, where it can be retranslated in the classical situation $X = X_1 \times X_2$ of (1) above as follows: If both $\mathcal{U}_\infty$ and $\mathcal{U}_\in$ are even *faithfully* covering (in the sense of (1)), then $N_X(\mathcal{U}_\infty \times \mathcal{U}_\in) = \mathcal{N}_{\mathcal{X}_\infty}(\mathcal{U}_\infty) \cdot \mathcal{N}_{\mathcal{X}_\in}(\mathcal{U}_\in)$.

Let us repeat in simpler terms the proof of (2.15) in this classical case:

Proof. By definition, $N(\mathcal{U}_\rangle) = \operatorname{card}\mathcal{W}_\rangle = \mathcal{N}(\mathcal{W}_\rangle)$ where by construction $\mathcal{U}_\rangle$ *refines* $\mathcal{W}_\rangle$ in the natural partial (pre–)order of open covers ($i = 1, 2$), hence obviously $\mathcal{U}_\infty \times \mathcal{U}_\in$ refines $\mathcal{W}_\infty \times \mathcal{W}_\in$, and by (1.2,iv) we know $N(\mathcal{U}_\infty \times \mathcal{U}_\in) \geq \mathcal{N}(\mathcal{W}_\infty \times \mathcal{W}_\in) = \mathcal{N}(\mathcal{W}_\infty) \cdot \mathcal{N}(\mathcal{W}_\in)$, where the factorization for the $\mathcal{W}_\rangle$ is almost obvious here. The reverse inequality is rather trivial (see (1.2,i)), as already mentioned before. □

5. Unfortunately, however, the "superadditivity" of the entropy (2.8) repectively (2.11) for a tensor product $\mathcal{A} = \mathcal{A}_\infty \otimes \mathcal{A}_\in$ with factorizing $*$–endomorphism $\theta = \theta_1 \otimes \theta_2$ ($\theta_i : \mathcal{A}_\rangle \to \mathcal{A}_\rangle$), with respect to a product map $\gamma = \gamma_1 \otimes \gamma_2$ (respectively a product sequence $\tau = \tau_1 \otimes \tau_2$), cannot be proven any more, as the gap in that proof of [12] still remains also here.

III. Additional properties and Thomsen's modified approach

We first prove additional properties of the entropy (2.11) for endomorphisms of AA C*–algebras (with respect to a corresponding sequence τ). Again, generally $\mathcal{A} \in \mathbf{1}$ denotes a unital C*–algebra.

Proposition (3.1) . *The following are non–commutative generalizations of additional properties of the classical topological entropy (cf. in the Introduction I):*

(i) *Let $\mathbf{1} \in \mathcal{B} \subset \mathcal{A}$ be an AA C*–subalgebra with corresponding sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$, and θ be a unital $*$–endomorphism of $\mathcal{A}$ such that $\theta(\mathcal{B}) \subset \mathcal{B}$. If there exists a sequence $\pi = (\pi_k)_{k\in\mathbb{N}}$ of positive unital maps $\pi_k : \mathcal{C}_\| \to \mathcal{A}$ ($\mathcal{C}_\|$ finite–dimensional C*) which renders also $\mathcal{A}$ AA, in such a way that*

$$\exists N \in \mathbb{N} : \forall \ltimes \geqslant \mathbb{N}, \quad \tau_\ltimes(\mathcal{O}^+(\mathcal{B}_\setminus)) \subseteq \pi_\setminus(\mathcal{O}^+(\mathcal{C}_\setminus)),$$

then $\hbar_\tau(\theta \restriction_\mathcal{B}) \leqslant \hbar_\pi(\theta)$.

Note that this is the case in particular if (2.11,i) is valid for τ*, where for* $\pi \equiv \tau$ *even equality holds.*

(ii) *Let* $\mathcal{A}$ *be AA with corresponding sequence* $\tau = (\tau_n)$ *as in (2.11),* $q : \mathcal{A} \to \mathcal{C}$ *a surjective* $*$*–homomorphism onto a unital* C^**–algebra* $\mathcal{C}$ *and* $\sigma : \mathcal{C} \to \mathcal{C}$ *a unital* $*$*–endomorphism such that* $\sigma \circ q = q \circ \theta$ *for* θ *as in (i).*

Then also $\mathcal{C}$ *is AA with the sequence* $q(\tau) \equiv (q \circ \tau_n)_{n\in\mathbb{N}}$*, and* $\hbar_{q(\tau)}(\sigma) \leqslant \hbar_\tau(\theta)$*.*

(iii) *Let* $\mathcal{A}_\infty \ni \mathbf{1}_\infty$ *and* $\mathcal{A}_\in \ni \mathbf{1}_\in$ *be unital AA* C^**–algebras, with corresponding sequences* $\tau^1 = (\tau^1_n)_{n\in\mathbb{N}}$ *and* $\tau^2 = (\tau^2_n)_{n\in\mathbb{N}}$*, respectively, where we assume that* one *of the two conditions (2.11,i & ii) is true for* both $(\mathcal{A}_\infty, \tau^\infty)$ *and* $(\mathcal{A}_\in, \tau^\in)$*.*

Let $\theta_i : \mathcal{A}_\rangle \to \mathcal{A}_\rangle$ *be unital* $*$*–endomorphisms* $(i = 1, 2)$ *and* $\theta_1 \oplus \theta_2$ *the direct sum endomorphism of* $\mathcal{A}_\infty \oplus \mathcal{A}_\in$ *(considered as unital* C^**–algebra with unit* $\mathbf{1}_1 \oplus \mathbf{1}_2$*); then the latter is again AA with respect to the sequence* $\tau = \tau^1 \oplus \tau^2 \equiv (\tau^1_n \oplus \tau^2_n)_{n\in\mathbb{N}}$*, and we have that* $\hbar_\tau(\theta_1 \oplus \theta_2) = \max\{\hbar_{\tau^1}(\theta_1), \hbar_{\tau^2}(\theta_2)\}$*.*

Proofs and Remarks for (3.1) . The respective proofs of (i) and (ii) are followed by some remarks:

(i) is obvious; and correspondingly, this result is not very powerful, although not meaningless. Note, however, that the conditions (2.11,i & ii) on a sequence π (as here, where in the case (2.11,ii) we have to restrict to *increasing* range spaces: $\pi_n(\mathcal{C}_\backslash) \subseteq \pi_{\backslash+\infty}(\mathcal{C}_{\backslash+\infty})$, at this point) are stable with respect to taking a *sub*sequence, hence we can always relate τ_n to π_n for the same $n \geqslant N$ (if at all) as already done in (i).

In the Abelian case $\mathcal{A} = \mathcal{C}(\mathcal{X})$ and $\tau_n, \pi_n \in \mathcal{P}_\infty(\mathcal{A})$ $\forall\backslash \in \mathbb{N}$ (cf. (2.12,1)), (i) means the following: Given a factor system (Y, S) of $(X, T_\theta \equiv T)$ as in (2.10), i.e. a continuous surjection $\Phi : X \to Y$ such that $\Phi \circ T = S \circ \Phi$, and calculating the entropy of (Y, S) with respect to a sequence of open covers $(\mathcal{U}_\backslash)_{\backslash\in\mathbb{N}}$ of Y which should be *cofinal* or at least monotonically "refining" (cf. (2.12,2 & 3)), then if we choose a sequence $(\mathcal{V}_\backslash)_{\backslash\in\mathbb{N}}$ of open covers of X which should again fulfill one of these two requirements, by putting $\mathcal{V}_\backslash = \Phi^{-\infty}(\mathcal{U}_\backslash) \vee \mathcal{W}_\backslash \ (\forall \backslash \geqslant \mathcal{N})$ for some sequence $(\mathcal{W}_\backslash)_{\backslash \in \mathbb{N}}$ of X–covers, it follows that $\limsup_{n\to\infty} h_Y(S, \mathcal{U}_\backslash) \leqslant \limsup_{\backslash\to\infty} \langle_{\mathcal{X}}(\mathcal{T}, \mathcal{V}_\backslash)$.

Clearly, this implies in particular that $h_Y(S) \leqslant h_X(T)$ (for the full topological entropies, cf. [18]), by choosing *cofinal* (and refining) sequences for *both* $(\mathcal{U}_\backslash)_{\backslash\in\mathbb{N}}$ in Y and $(\mathcal{W}_\backslash)_{\backslash\in\mathbb{N}}$ in X; but for non–Abelian $\mathcal{A}$, we do not (yet?) have these tools at hand, remember (2.12,3).

(ii) First, it is easy to see that both (2.11,i) or (2.11,ii) are preserved by q: As q is a $*$–homomorphism, clearly again $q \circ \tau_n(\mathcal{O}^+(\mathcal{B}_\backslash)) \subset \mathcal{O}^+(\mathcal{C})$, and trivially $q \circ \tau_n(\mathcal{O}^+(\mathcal{B}_\backslash)) \subset \text{II} \circ \tau_{\backslash+\infty}(\mathcal{O}^+(\mathcal{B}_{\backslash+\infty})) \ \forall \backslash \in \mathbb{N}$, by assumption (2.11,i).

If on the other hand $\overline{\bigcup_{n\in\mathbb{N}} \tau_n(\mathcal{B}_\backslash)} = \mathcal{A}$ by (2.11,ii), it follows obviously from the automatic continuity and the surjectivity of q that also $\overline{\bigcup_{n\in\mathbb{N}} q \circ \tau_n(\mathcal{B}_\backslash)} = \mathcal{C}$. So, the result follows from the definitions involved in $\hbar_\tau(\theta)$ as in (2.12) and the fact that $\bar{N}(q(\alpha)) \leqslant \bar{N}(\alpha) \ \forall \alpha \in \mathcal{O}(\mathcal{A})$, cf. (2.3,iv). $\square$

Again back in the Abelian case $\mathcal{A} = \mathcal{C}(\mathcal{X})$, the meaning of (ii) is that for a closed, T–invariant subset $Y \subseteq X$ such that $T(Y) \subseteq Y$, and for a cofinal(–refining) sequence of open covers $(\mathcal{U}_\backslash)_{\backslash\in\mathbb{N}}$ for X with resulting sequence $(\mathcal{V}_\backslash)_{\backslash\in\mathbb{N}}$ of open covers $\mathcal{V}_\backslash = \mathcal{U}_\backslash \cap \mathcal{Y}$ (symbolically, but clear) for Y, we have $\lim(\sup)_{n\to\infty} h_Y(T \restriction_Y, \mathcal{V}_\backslash) \leqslant \lim(\sup)_{\backslash\to\infty} \langle_{\mathcal{X}}(\mathcal{T}, \mathcal{U}_\backslash)$.

(iii) It is clear that each one of (2.11,i) and (2.11,ii) is preserved by the direct sum; and it follows from (ii) that $\max\{\hbar_{\tau^1}(\theta_1), \hbar_{\tau^2}(\theta_2)\} \leqslant \hbar_\tau(\theta_1 \oplus \theta_2)$, where the q_i as used in (ii) is given by the respective canonical projection onto the respective direct summand $\mathcal{A}_\rangle$. For completeness it should be noted for the purists that here actually $q_i(\tau) = (q_i \circ (\tau_n^1 \oplus \tau_n^2) : \mathcal{B}_\backslash^\infty \oplus \mathcal{B}_\backslash^\in \to \mathcal{A}_\rangle)$, but of course $q_i \circ (\tau_n^1 \oplus \tau_n^2)(\mathcal{B}_\backslash^\infty \oplus \mathcal{B}_\backslash^\in) = \tau_\backslash^\rangle(\mathcal{B}_\backslash^\rangle)$ anyway $(i = 1, 2)$.

To prove the reverse inequality, let us assume for the sake of notational simplicity that (2.11,i) is fulfilled for both $(\mathcal{A}_\rangle, \tau^\rangle)$ $(\rangle = \infty, \in)$, such that the general $\limsup$ in definition (2.11) of $\hbar_{\tau^i}(\theta_i)$ is actually an increasing limit. It will then be clear at the end of the proof that in the other case (2.11,ii) it is sufficient to choose suitable subsequences in the subsequent.

Now fix $n \in \mathbb{N}$ and also $k \in \mathbb{N}$ and use Def. (2.5,ii) for $\theta = \theta_1 \oplus \theta_2,\ \ \tau = (\tau_n)_{n\in\mathbb{N}}$ to get

$$\frac{1}{k}\bar{H}(\tau_n, \theta\circ\tau_n, \ldots, \theta^{k-1}\circ\tau_n) = \frac{1}{k}\log \bar{N}(\tau_n(\alpha_1)\vec{\nabla}\theta\circ\tau_n(\alpha_2)\vec{\nabla}\ldots\vec{\nabla}\theta^{k-1}\circ\tau_n(\alpha_k)),$$

where $\alpha_j = \alpha_j(k,n) \in \mathcal{O}^+(\mathcal{B}_\backslash^\infty \oplus \mathcal{B}_\backslash^\in)$ with corresponding direct summands $\alpha_j^i \in \mathcal{O}^+(\mathcal{B}_\backslash^\rangle) : \alpha_| = \alpha_|^\infty \oplus \alpha_|^\in$ (symbolically, but clear). It is easily seen that from the right hand side we get:

$$\begin{aligned}
&\bar{N}(\tau_n(\alpha_1)\vec{\nabla}\theta\circ\tau_n(\alpha_2)\vec{\nabla}\ldots\vec{\nabla}\theta^{k-1}\circ\tau_n(\alpha_k)) \leqslant \\
&\qquad\leqslant \bar{N}(\tau_n^1(\alpha_1^1)\vec{\nabla}\theta_1\circ\tau_n^1(\alpha_2^1)\vec{\nabla}\ldots\vec{\nabla}\theta_1^{k-1}\circ\tau_n^1(\alpha_k^1)) + \\
&\qquad\qquad + \bar{N}(\tau_n^2(\alpha_1^2)\vec{\nabla}\theta_2\circ\tau_n^2(\alpha_2^2)\vec{\nabla}\ldots\vec{\nabla}\theta_2^{k-1}\circ\tau_n^2(\alpha_k^2)) \equiv
\end{aligned}$$

$\equiv N_1(k,n) + N_2(k,n)$. On the other hand, we know, again by (2.5,ii), that each

$$\log N_i(k,n) \leqslant \bar{H}(\tau_n^i, \theta_i\circ\tau_n^i, \ldots, \theta_i^{k-1}\circ\tau_n^i) \equiv H_i(k,n),$$

for $i = 1, 2$. Together it follows by Definition (2.8) as $k \to \infty$ on the very first left hand side above (making a choice, without loss of generality):

$$\hbar(\theta, \tau_n) \leqslant \frac{1}{k}H_1(k,n) + \frac{1}{k}\log(1 + e^{[H_2(k,n) - H_1(k,n)]}),$$

where we used that the limit $k \to \infty$ in (2.8) is actually the infimum. To repeat it, by (2.8) and (2.12) we know $\hbar_{\tau^i}(\theta_i) = \lim_{n\to\infty}\lim_{k\to\infty}\frac{1}{k}H_i(k,n)$; and first assuming that $\hbar_{\tau^1}(\theta_1) \neq \hbar_{\tau^2}(\theta_2)$, we choose (again without loss of generality) that $\hbar_{\tau^1}(\theta_1) \gneqq \hbar_{\tau^2}(\theta_2)$. Then it is easily seen that there exists $N \in \mathbb{N}$ such that $\forall n \geqslant N,\ \ \forall k \geqslant K(n)$ we have $H_1(k,n) \geqslant H_2(k,n)$. Thus $\forall n \geqslant N$ the right hand side above tends to $\hbar(\theta_1, \tau_n^1)$ as $k \to \infty$. In the other case $\hbar_{\tau^1}(\theta_1) = \hbar_{\tau^2}(\theta_2)$ it follows from $\log(1+x) \leqslant \log x + 1\ \ \forall x \geqslant 1$ that we can still bound $\hbar(\theta, \tau_n)$ by $\max\{\hbar(\theta_1, \tau_n^1), \hbar(\theta_2, \tau_n^2)\}$ (using both choices for the stronger inequality before, alternatively). In both cases, performing the limit $n \to \infty$ first on the right hand, then on the left hand side yields the result $\hbar_\tau(\theta) \leqslant \hbar_{\tau^1}(\theta_1)$. $\square$

As already noted in the Introduction I, the three additional properties (3.1) are the direct analogues of the (remaining) basic properties proved by Thomsen [19] for his slightly modified topological entropy for local C*–algebras; in particular (i), (ii) respectively (iii) of (3.1) corresponds to (ii), (iii) of Theorem (1.2) in [19] respectively to Proposition (1.3) in [19].

At this point the reader will ask also for the analogues here of the "modest" coninuity properties proved by Thomsen for his modified entropy (section 2 in [19]), and in the following we will answer this question to the affirmative; however, we are able to do so only with a severe restriction of the theory discussed so far, which is of great independent interest, though (particularly concerning the physical interpretation):

It turns out (see (3.5,iv) and the subsequent remarks below) that the non–commutative freedom in Def. (2.5,ii), admitting *different* and *general* $\alpha_k \in \mathcal{O}^+(\mathcal{B}_\|)$ for $k = 1, \ldots, n$ (cf. the discussion in (2.12,5)) is too much when trying to control $\hbar(\theta, \gamma)$ as in (2.8) via any norm–continuity along the lines of [19]. But on the other hand, we have to consider it as inadequate to completely abandon that freedom in the $\mathcal{B}_\|$'s, what is done in [19] by taking the joint entropy of the images (under the iterated $*$–endomorphism θ) of one and the same "partition" α, i.e. something of the form $\alpha \vee \theta(\alpha) \vee \cdots \vee \theta^n(\alpha)$.

Both for (mathematically) conceptual and for physical reasons, we want to maintain the "non–commutative open covers" represented by the positive unital maps $\gamma : \mathcal{B} \to \mathcal{A}$, with the finite–dimensional C*–algebra $\mathcal{B}$ representing some "operational" quantum–mechanical (e.g. spin) degrees of freedom being transformed by θ (in the "macroscopic realization" represented by γ).

Surprisingly, now, it is *exactly* the "traditional" (or more conventional) notion of a finite "measurement" performed on this degree of freedom $\mathcal{B}$, i.e. a partition of the unit $\mathbf{1}_\mathcal{B}$ by mutually orthogonal projections, which has to be substituted for the more general $\alpha \in \mathcal{O}^+(\mathcal{B})$ used so far (but again, for *each* k in (2.5,ii) separately), in order to make the continuity arguments work. As remarked in more detail below (3.3), not too much of the other desirable properties of the entropy functionals gets lost due to this restriction; and for the hitherto *computable* examples, that modification makes not any difference (see section IV).

In this sense, the following can be viewed as an alternative "theory within the theory"; and at this stage we can still leave it to the personal taste of the reader as to which definitions or properties should be preferred.

Definition (3.2) . We define the following alternatives (i) for (2.1,ii), (ii) for (2.5) and (iii) for (2.8):

(i) For a finite–dimensional unital C*–algebra $\mathcal{B} \ni \mathbf{1}_\mathcal{B}$, we denote by $\mathcal{OP}_\infty(\mathcal{B}) \subset \mathcal{O}^+_\in(\mathcal{B})$ the set of all partitions of the unit $\mathbf{1}_\mathcal{B}$ by mutually orthogonal projections:

$$\mathcal{OP}_\infty(\mathcal{B}) \ni \beta = \{ \surd_\rangle = \surd_\rangle^* = \surd_\rangle^\in | \surd_\rangle \surd_| = \prime \ \forall\rangle \neq |, \ \sum_{\rangle=\infty}^{\operatorname{card}\beta} \surd_\rangle = \mathbf{1}_\mathcal{B}\},$$

where it follows trivially that $\operatorname{card}\beta \leqslant D(\mathcal{B})$ as defined in (1.4).

(ii) In the situation of (2.5,ii), the "partition" (or "projection") entropy of $(\gamma_1, \ldots, \gamma_n)$ is defined as

$$\bar{H}_P(\gamma_1, \ldots, \gamma_n) = \max_{\{(\beta_1,\ldots,\beta_n)\}} \log \bar{N}(\gamma_1(\beta_1)\vec{\nabla} \ldots \vec{\nabla}\gamma_n(\beta_n)),$$

where $\beta_k \in \mathcal{OP}_\infty(\mathcal{B}_\|) \ \forall \| = \infty, \ldots, \backslash$; and with the *same* additional restriction as in (1.5,ii): For $\gamma_k \approxeq \gamma_\ell$ $(k < \ell)$, fix $\beta_\ell \equiv \{\mathbf{1}_\ell\}$.

(iii) In the situation of (2.8), the P–entropy of θ with respect to γ is $\hbar_P(\theta, \gamma) = \lim_{n\to\infty} \frac{1}{n} \bar{H}_P(\gamma, \theta \circ \gamma, \ldots, \theta^{n-1} \circ \gamma)$, which is again the infimum of the sequence.

Remarks (3.3) . Note first that obviously $\bar{H}_P(\gamma_1, \ldots, \gamma_n) \leqslant \bar{H}(\gamma_1, \ldots, \gamma_n)$ and consequently $\hbar_P(\theta, \gamma) \leqslant \hbar(\theta, \gamma)$. Nerverthe-less, the following properties of the right hand side functionals remain valid for the P–entropies: (2.6,2), (2.7;ii, iii, v) and also (2.7,iv), where for the latter we still have to use (2.4) (with $\alpha \in \mathcal{O}(\mathcal{A})$!) to get the inequality; hence also (2.9;i, iii, iv, v) and all of (2.13) still hold. Unfortunately, (2.7,i) respectively (2.9,ii) hold only for

$*$–*homomorphisms* θ_k respectively γ_1, instead of general positive unital maps (without the P–restriction).

In (2.11,i) respectively (2.14,i), the $\mathcal{O}^+(\mathcal{B}_\backslash)$ respectively $\mathcal{O}^+(\mathcal{B})$ have to be replaced by $\mathcal{OP}_\infty(\mathcal{B}_\backslash)$ respectively $\mathcal{OP}_\infty(\mathcal{B})$, rendering the respective condition more restrictive. Note, however, that the non–trivial example (2.16,2) for (2.14,i) is of the form with $\beta' \in \mathcal{OP}_\infty(\mathcal{B})$. Apart from that, (2.15) and the following discussion remain fully valid. Finally, (3.1) remains valid with suitable changes of $\mathcal{O}^+(\mathcal{B})$'s into $\mathcal{OP}_\infty(\mathcal{B})$'s.

Definition (3.4) . The following definitions are adaptions of the definitons in [19] for our needs:

(i) For $\varepsilon \in [0,1)$, $\varepsilon \neq 1$, we define the set of "ε–operator covers" $\mathcal{O}(\mathcal{A},\varepsilon) = \{\alpha \in \mathcal{O}(\mathcal{A}) | \forall \omega \in \mathcal{S}_\mathcal{A} \ \exists \mathcal{A} \in \alpha : \omega(\mathcal{A}^*\mathcal{A}) > \varepsilon\}$.
Note that $\mathcal{O}(\mathcal{A}) = \mathcal{O}(\mathcal{A}, \prime) \supseteq \mathcal{O}(\mathcal{A},\varepsilon) \supseteq \mathcal{O}(\mathcal{A},\delta)$ for $\varepsilon \leqslant \delta < 1$.

(ii) For $\alpha \in \mathcal{O}(\mathcal{A},\varepsilon)$, define $\bar{N}(\alpha,\varepsilon) = \min\{\operatorname{card}\alpha' | \alpha' \subseteq \alpha, \ \alpha' \in \mathcal{O}(\mathcal{A},\varepsilon)\}$.
Note that $\bar{N}(\alpha) = \bar{N}(\alpha,0) \leqslant \bar{N}(\alpha,\delta) \leqslant \bar{N}(\alpha,\varepsilon)$ for $0 < \delta < \varepsilon < 1$ and $\alpha \in \mathcal{O}(\mathcal{A},\varepsilon)$.

(iii) Again (cf. definition (2.1)), we can define also $\mathcal{O}^+(\mathcal{A},\varepsilon) = \{\alpha \in \mathcal{O}^+(\mathcal{A}) | \forall \omega \in \mathcal{S}_\mathcal{A} \ \exists \mathcal{A} \in \alpha : \omega(\mathcal{A}) > \varepsilon\}$.
Note that it follows easily from the Schwartz inequality for the state ω (in particular, from $\omega(A^2) \geqslant \omega(A)^2 \ \forall A \in \mathcal{A}^+$) that $\mathcal{O}^+(\mathcal{A},\varepsilon) \subset \mathcal{O}(\mathcal{A},\varepsilon^\in) \ \forall \varepsilon < \infty$ (note that there is no $^+$ on the right hand side $\mathcal{O}$!).

Proposition (3.5) . *The following properties are, in the case of (i) and (ii), again adaptions of the corresponding properties proved in* [19], *and in the case of (iii) and (iv) additional properties important in our context here.*

(i) $\bar{N}(\alpha\vec{\nabla}\beta, \varepsilon\delta) \leqslant \bar{N}(\alpha,\varepsilon) \cdot \bar{N}(\beta,\delta)$ *for* $\alpha \in \mathcal{O}(\mathcal{A},\varepsilon)$, $\beta \in \mathcal{O}(\mathcal{A},\delta)$.

(ii) $\bar{N}(\theta(\alpha),\varepsilon) \leqslant \bar{N}(\alpha,\varepsilon)$ *for a unital* $*$–*endomorphism* θ *of* $\mathcal{A}$ *and* $\alpha \in \mathcal{O}(\mathcal{A},\varepsilon)$, *whence also* $\theta(\alpha) \in \mathcal{O}(\mathcal{A},\varepsilon)$.

(iii) *For a positive unital map* $\gamma : \mathcal{B} \to \mathcal{A}$, *and* $\beta \in \mathcal{O}^+(\mathcal{B},\varepsilon)$, *obviously also* $\gamma(\beta) \in \mathcal{O}^+(\mathcal{A},\varepsilon)$, *and* $\bar{N}(\gamma(\beta),\varepsilon^2) \leqslant \bar{N}(\beta,\varepsilon^2)$.

(iv) *Let $\mathcal{B}$ be a finite–dimensional unital C^*–algebra. For $\varepsilon < D(\mathcal{B})^{-\infty}$ with $D(\mathcal{B})$ as in (1.4), the set of all "projection–partitions" $\mathcal{OP}_\infty(\mathcal{B}) \subset \mathcal{O}^+(\mathcal{B}, \varepsilon)$ is included as shown (where for these* pro-jections, *actually also $\mathcal{OP}_\infty(\mathcal{B}) \subset \mathcal{O}(\mathcal{B}, \varepsilon)$; and in other words, more explicitly: for $\beta \in \mathcal{OP}_\infty(\mathcal{B})$, $\beta = \{{\surd\rangle}\}$, $\forall \omega \in \mathcal{S}_\mathcal{B}\ \exists\ {\surd\rangle}$: $\omega({\surd\rangle}) > \varepsilon$).*

Proof. The proof of (ii) and (iii) is rather trivial, and also (iv) is left as an easy exercise for the reader. For the sake of completeness, we reproduce the simple proof of (i) from [19]:

For $\alpha' \subset \alpha$ respectively $\beta' \subset \beta$ such that $\bar{N}(\alpha, \varepsilon) = \operatorname{card} \alpha$ respectively $\bar{N}(\beta, \delta) = \operatorname{card} \beta$, obviously $\alpha' \vec{\nabla} \beta'$ is a subset of $\alpha \vec{\nabla} \beta$; but it is even such that $\alpha' \vec{\nabla} \beta' \in \mathcal{O}(\mathcal{A}, \varepsilon\delta)$: For $\omega \in S_\mathcal{A}$, $A \in \alpha'$ such that $\omega(A^*A) > \varepsilon$, we get the state $\varphi_A = \omega(A^* \centerdot A)\omega(A^*A)^{-1} \in S_\mathcal{A}$ and thus there exists a $B \in \beta' : \varphi_A(B^*B) > \delta$. Then $(BA) \in \alpha' \vec{\nabla} \beta'$ and $\omega((BA)^*BA) > \varepsilon\delta$. □

Note, however, that for $\mathcal{O}^+(\mathcal{B})$ instead of $\mathcal{OP}_\infty(\mathcal{B})$ (as we had used it in section II and (3.1)), we could not prove anything like (iv) again: $\forall \varepsilon > 0$, it is *not* true that $\mathcal{O}^+(\mathcal{B}) \subset \mathcal{O}^+(\mathcal{B}, \varepsilon)$ (only the trivial converse inclusion holds), but even also *not* that $\mathcal{O}^+_\infty(\mathcal{B}) \subset \mathcal{O}^+(\mathcal{B}, \varepsilon)$: The *trivial* example is $\beta \in \mathcal{O}^+_\infty(\mathcal{B})$ given by $\beta = \{\delta_i \cdot \mathbf{1}_\mathcal{B} | \delta_i \neq \delta_j,\ \ \delta_i < \varepsilon\ \ \forall i,\ \ \sum_i \delta_i = \mathbf{1}_\mathcal{B}\} \notin \mathcal{O}^+(\mathcal{B}, \varepsilon)$; but of course, this would give only zero entropies ((2.5,ii), for example).

It is not hard to realize, however, that for any $\alpha \in \mathcal{O}^+_{(\infty)}(\mathcal{B})$ maximizing "some" entropy (as in (2.5,ii)), and particularly for one argument $\gamma : \mathcal{B} \to \mathcal{A}$ where we have to have an analogue of Lemma (2.4) (as we will get it in Remark (3.7) below, here by the automatic bound on card$[\beta \in \mathcal{OP}_\infty(\mathcal{A})]$ by definition), there would be no guaranty that $\alpha \in \mathcal{O}^+(\mathcal{B}, \varepsilon)$ for any fixed $\varepsilon > 0$.

This is the "obstruction" to using the norm–continuity arguments as in [19] with the definition from section II, as we had announced it before (3.2) above.

Definition (3.6) . The following are again the analogous definitions to (2.5) and (2.8), now using the "ε–covering" definitions from above:

(i) Again in the situation of (2.5,ii), let $\varepsilon < \min\{D(\mathcal{B}_{\|})^{-\infty} \| \| = \infty, \dots, \backslash\}$ be fixed, then we can define (by (3.5))

$$\bar{H}_P^{\varepsilon}(\gamma_1, \dots, \gamma_n) = \max_{\{(\beta_1, \dots, \beta_n)\}} \log \bar{N}(\gamma_1(\beta_1) \vec{\vee} \dots \vec{\vee} \gamma_n(\beta_n), \varepsilon^{(2n)}),$$

where $\beta_k \in \mathcal{OP}_{\infty}(\mathcal{B}_{\|})\ \forall \|$, with the same restriction as in (2.5,ii) or (3.2,ii).

Note that we have $\bar{H}_P(\gamma_1, \dots, \gamma_n) = \bar{H}_P^0(\gamma_1, \dots, \gamma_n) \leqslant \bar{H}_P^{\delta}(\gamma_1, \dots, \gamma_n) \leqslant \bar{H}_P^{\varepsilon}(\gamma_1, \dots, \gamma_n)$ for $0 < \delta < \varepsilon$ (using the max in above definition, this is really obvious).

(ii) $\hbar_P^{\varepsilon}(\theta, \gamma) = \lim_{n \to \infty} \frac{1}{n} \bar{H}_P^{\varepsilon}(\gamma, \theta \circ \gamma, \dots, \theta^{n-1} \circ \gamma)$ for a $*$–endomorphism θ of $\mathcal{A}$ with respect to γ as in (2.8) has again the analogous properties. In particular, the limit is the *infimum.*

Remark (3.7) . For a single map $\gamma : \mathcal{B} \to \mathcal{A}$ as before, it follows from (3.5,iii) that $\bar{H}_P^{\varepsilon}(\gamma) \leqslant \bar{H}_P^{\varepsilon}(\mathcal{B}) = \log \mathcal{D}(\mathcal{B})$, where the right hand side equality is rather trivial for $\varepsilon < D(\mathcal{B})^{-\infty}$ as assumed (and where we used again the general notation as in (2.6,2)).

In the following results (3.8)–(3.10), we can now in turn "imitate" Thomsen's results on continuity properties (section 2 in [19]).

Lemma (3.8) . *For $\gamma : \mathcal{B} \to \mathcal{A}$ with $\mathcal{B}$ finite–dimensional, the map $[0, D(\mathcal{B})^{-\infty}) \ni \varepsilon \mapsto \hbar_{\mathcal{P}}^{\varepsilon}(\theta, \gamma)$ is non–decreasing and continuous from the* right*; in particular,* $\lim_{\varepsilon \to 0+} \hbar_P^{\varepsilon}(\theta, \gamma) = \hbar_P(\theta, \gamma)$.

Proof. It follows from the remark in (3.6,i) above that $\hbar_P^{\varepsilon}(\theta, \gamma)$ is non–decreasing in ε. To prove that $\lim_{\varepsilon \to \varepsilon_0 +} \hbar_P^{\varepsilon}(\theta, \gamma) = \hbar_P^{\varepsilon_0}(\theta, \gamma)$ for $0 \leqslant \varepsilon_0 < \varepsilon < D(\mathcal{B})^{-\infty}$, let $\delta > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{n} \bar{H}_P^{\varepsilon_0}(\gamma, \theta \circ \gamma, \dots, \theta^{n-1} \circ \gamma) \leqslant \hbar_P^{\varepsilon_0}(\theta, \gamma) + \delta$. For all $\varepsilon \geqslant \varepsilon_0$ sufficiently close to ε_0,

$$\bar{H}_P^{\varepsilon}(\gamma, \theta \circ \gamma, \dots, \theta^{n-1} \circ \gamma) = \log \bar{N}(\gamma(\beta_1) \vec{\vee} \dots \vec{\vee} \theta^{n-1} \circ \gamma(\beta_n), \varepsilon^{(2n)}) =$$

$$= \log \bar{N}(\gamma(\beta_1) \vec{\vee} \dots \vec{\vee} \theta^{n-1} \circ \gamma(\beta_n), \varepsilon_0^{(2n)}) \leqslant \bar{H}_P^{\varepsilon_0}(\gamma, \theta \circ \gamma, \dots, \theta^{n-1} \circ \gamma)$$

(and hence also equality on the right; with some $\beta_k \in \mathcal{OP}_{\infty}(\mathcal{B}_{\|}),\ \| = \infty, \dots, \backslash$). Thus $\hbar_P^{\varepsilon}(\theta, \gamma) \leqslant \hbar_P^{\varepsilon_0}(\theta, \gamma) + \delta$. $\square$

Theorem (3.9) . *Let $\gamma : \mathcal{B} \to \mathcal{A}$ be a positive unital map, and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of such maps $\gamma_n : \mathcal{B} \to \mathcal{A}$ ($\mathcal{B}$ finite–dimensional) such that* $\lim_{n\to\infty} \frac{1}{n} \log \|\gamma - \gamma_n\| = -\infty$ *(in other words, $\|\gamma-\gamma_n\|^{\frac{1}{n}} \to 0$); and such that θ does* not *act periodically on any of the γ, γ_n $(n \in \mathbb{N})$. Then $(\forall \varepsilon, \delta_n < D(\mathcal{B})^{-\infty})$:*

(i) $\sup_{\delta<\varepsilon} \hbar^{\delta}_{P}(\theta, \gamma) \leqslant \liminf_{n\to\infty} \frac{1}{n} \bar{H}^{\varepsilon}_{P}(\gamma_n, \theta \circ \gamma_n, \ldots, \theta^{n-1} \circ \gamma_n)$, and $\limsup_{n\to\infty} \frac{1}{n} \bar{H}^{\varepsilon}_{P}(\gamma_n, \theta \circ \gamma_n, \ldots, \theta^{n-1} \circ \gamma_n) \leqslant \hbar^{\varepsilon}_{P}(\theta, \gamma)$.
In particular, $\hbar_P(\theta, \gamma) = \lim_{\varepsilon\to 0}(\limsup_{n\in\mathbb{N}} \frac{1}{n} \bar{H}^{\varepsilon}_{P}(\gamma_n, \theta \circ \gamma_n, \ldots, \theta^{n-1} \circ \gamma_n))$.

(ii) $\hbar_P(\theta, \gamma) = \lim_{n\to\infty} \frac{1}{n} \bar{H}^{\delta_n}_{P}(\gamma_n, \theta \circ \gamma_n, \ldots, \theta^{n-1} \circ \gamma_n)$ for $\delta_n = c \cdot \|\gamma - \gamma_n\|^{\frac{1}{(2n)}}$, for any $c > 1$, $c \neq 1$.

Proof. Define the generalized Hausdorff distance (metric) on $\mathcal{O}(\mathcal{A}) \ni \alpha, \beta$ by

$$D(\alpha, \beta) = \max\{\max_{A\in\alpha} \min_{B\in\beta} \|A - B\|, \ \max_{B\in\beta} \min_{A\in\alpha} \|A - B\|\}.$$

(i) The following general inequality in the situation of (i), with *arbitrary* $\alpha_k \in \mathcal{O}^+(\mathcal{B})$ $(\forall \| = \infty, \ldots, \backslash)$ is easy to deduce:

$$D(\gamma(\alpha_1)\vec{\vee}\theta\circ\gamma(\alpha_2)\vec{\vee} \ldots \vec{\vee}\theta^{n-1}\circ\gamma(\alpha_n), \gamma_n(\alpha_1)\vec{\vee}\theta\circ\gamma_n(\alpha_2)\vec{\vee} \ldots \vec{\vee}\theta^{n-1}\circ\gamma_n(\alpha_n)) \leqslant n \cdot \|\gamma - \gamma_n\|.$$

Now let $0 < \delta_1 < \varepsilon < \delta_2 < D(\mathcal{B})^{-\infty}$, all inequalities strict. By assumption, $\|\gamma - \gamma_n\| \leqslant \frac{1}{2n}\delta_1^{(2n)}$ for all sufficiently large n. It follows that for $\omega \in S_{\mathcal{A}}$ a state, and a suitably chosen (see below) n–tuple $(\alpha_1, \ldots, \alpha_n) \in \mathcal{OP}_\infty(\mathcal{B})^\backslash$, in particular, for any $A \in \gamma_n(\alpha_1)\vec{\vee} \ldots \vec{\vee}\theta^{n-1} \circ \gamma_n(\alpha_n)$ such that $\omega(A^*A) > \varepsilon^{(2n)}$, there exists $B \in \gamma(\alpha_1)\vec{\vee} \ldots \vec{\vee}\theta^{n-1} \circ \gamma(\alpha_n)$ such that $\omega(B^*B) > \varepsilon^{(2n)} - \delta_1^{(2n)} > \delta_1^{(2n)}$ for sufficiently large n.

Hence, by the above definitions:

$$\frac{1}{n}\bar{H}_P^{\delta_1}(\gamma,\theta\circ\gamma,\dots,\theta^{n-1}\circ\gamma) =$$
$$= \frac{1}{n}\log\bar{N}(\gamma(\alpha_1)\vec{\nabla}\theta\circ\gamma(\alpha_2)\vec{\nabla}\dots\vec{\nabla}\theta^{n-1}\circ\gamma(\alpha_n),\delta_1^{(2n)}) \leqslant$$
$$\leqslant \frac{1}{n}\log\bar{N}(\gamma_n(\alpha_1)\vec{\nabla}\theta\circ\gamma_n(\alpha_2)\vec{\nabla}\dots\vec{\nabla}\theta^{n-1}\circ\gamma_n(\alpha_n),\varepsilon^{(2n)}) \leqslant$$
$$\leqslant \frac{1}{n}\bar{H}_P^{\varepsilon}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)$$

for all sufficiently large n, where the last inequality uses the non–periodicity assumption on θ with respect to γ_n.

Similarly, we can choose a tuple $(\beta_1,\dots,\beta_n)\in\mathcal{OP}_\infty(\mathcal{B})^\setminus$ such that

$$\frac{1}{n}\bar{H}_P^{\varepsilon}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n) =$$
$$= \frac{1}{n}\log\bar{N}(\gamma_n(\beta_1)\vec{\nabla}\theta\circ\gamma_n(\beta_2)\vec{\nabla}\theta^{n-1}\circ\gamma_n(\beta_n),\varepsilon^{(2n)}) \leqslant$$
$$\leqslant \frac{1}{n}\bar{H}_P^{\delta_2}(\gamma,\theta\circ\gamma,\dots,\theta^{n-1}\circ\gamma),$$

for all sufficiently large n, now by the non–periodicity assumption on θ with respect to γ.

Hence we have $\hbar_P^{\delta_1}(\theta,\gamma)\leqslant\liminf_{n\to\infty}\frac{1}{n}\bar{H}_P^{\varepsilon}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)$, and on the other hand, $\limsup_{n\to\infty}\frac{1}{n}\bar{H}_P^{\varepsilon}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)\leqslant\hbar_P^{\delta_2}(\theta,\gamma)$. Since $\delta_1<\varepsilon$ was arbitrary, and on the other hand $\lim_{\delta_2\to\varepsilon+}\hbar_P^{\delta_2}(\theta,\gamma)=\hbar_P^{\varepsilon}(\theta,\gamma)$ by Lemma (3.8), we get the desired inequalities.

(ii) Let $\varepsilon>0$ be fixed. By Lemma (3.8), there exists $\delta>0$ such that $\hbar_P^{\delta}(\theta,\gamma)\leqslant\hbar_P(\theta,\gamma)+\varepsilon$. For a chosen n–tuple $(\alpha_1,\dots,\alpha_n)\in\mathcal{OP}_\infty(\mathcal{B})^\setminus$, a state $\omega\in S_{\mathcal{A}}$ and $A\in\gamma_n(\alpha_1)\vec{\nabla}\theta\circ\gamma_n(\alpha_2)\vec{\nabla}\dots\vec{\nabla}\theta^{n-1}\circ\gamma_n(\alpha_n)$ such that $\omega(A^*A)>\delta_n^{(2n)}$, there exists $B\in\gamma(\alpha_1)\vec{\nabla}\theta\circ\gamma(\alpha_2)\vec{\nabla}\dots\vec{\nabla}\theta^{n-1}\circ\gamma(\alpha_n)$ such that $\omega(B^*B)>\delta_n^{(2n)}-2n\|\gamma-\gamma_n\|>0$ for all sufficiently large n, by definition of δ_n (note at this point that (ii) still holds for $\delta_n=c_n\|\gamma-\gamma_n\|^{\frac{1}{(2n)}}$ with a *sequence* $(c_n)_{n\in\mathbb{N}}$, $c_n\to1$ but $c_n>(2n)^{\frac{1}{(2n)}}$ for all sufficiently large n; for example $c_n=(2n)^{\frac{1}{(2n)}}+\left(\frac{1}{2n}\right)^{2n}!$).

On the other hand, for a chosen tuple $(\beta_1,\dots,\beta_n)\in\mathcal{OP}_\infty(\mathcal{B})\backslash$ and a state $\omega\in S_{\mathcal{A}}$, take $A\in\gamma(\beta_1)\vec{\nabla}\theta\circ\gamma(\beta_2)\vec{\nabla}\dots\vec{\nabla}\theta^{n-1}\circ\gamma(\beta_n)$ such that $\omega(A^*A)>(2\delta_n)^{2n}$; then there exists $B\in\gamma_n(\beta_1)\vec{\nabla}\theta\circ\gamma_n(\beta_2)\vec{\nabla}\dots\vec{\nabla}\theta^{n-1}\circ\gamma_n(\beta_n)$ such that $\omega(B^*B)>(2\delta_n)^{2n}-2n\|\gamma-\gamma_n\|>\delta_n^{(2n)}+(\delta_n^{(2n)}-2n\|\gamma-\gamma_n\|)>\delta_n^{(2n)}$ for all sufficiently large n. Thus

$$\begin{aligned}&\bar{H}_P(\gamma,\theta\circ\gamma,\dots,\theta^{n-1}\circ\gamma)\equiv\\&\quad\equiv\bar{H}_P^0(\gamma,\theta\circ\gamma,\dots,\theta^{n-1}\circ\gamma)\leqslant\bar{H}_P^{\delta_n}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)\leqslant\\&\qquad\leqslant\bar{H}_P^{(2\delta_n)}(\gamma,\theta\circ\gamma,\dots,\theta^{n-1}\circ\gamma)\leqslant\bar{H}_P^{\delta}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)\end{aligned}$$

for all sufficiently large n, such that also $2\delta_n<\delta$ from above (chosen to be $\delta<D(\mathcal{B})^{-\infty}$).

Consequently, $\hbar(\theta,\gamma)\leqslant\liminf_{n\to\infty}\frac{1}{n}\bar{H}_P^{\delta_n}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)$ *and*

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\bar{H}_P^{\delta_n}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)\leqslant\\&\quad\leqslant\lim_{n\to\infty}\frac{1}{n}\bar{H}_P^{\delta}(\gamma,\theta\circ\gamma,\dots,\theta^{n-1}\circ\gamma)=\hbar_P^{\delta}(\theta,\gamma)\leqslant\hbar_P(\theta,\gamma)+\varepsilon.\qquad\square\end{aligned}$$

Corollary (3.10) . *The following are important consequences of Theorem (3.9):*

(a) *Under the assumptions of the Theorem (3.9) we get that* $\hbar_P(\theta,\gamma)\geqslant\limsup_{n\to\infty}\bar{H}_P(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)$.

(b) *If, furthermore, the following limit exists as an* infimum *for* all *ε small enough (i.e. also for $\varepsilon=0$):* $\lim_{n\to\infty}\frac{1}{n}\bar{H}_P^{\varepsilon}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)$, *then we have that* $\hbar_P(\theta,\gamma)=\lim_{n\to\infty}\frac{1}{n}\bar{H}_P(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)$.

Proof. We give the very simple proof, for the sake of completeness:

(a) By (i) of (3.9), $\hbar_P^{\varepsilon}(\theta,\gamma)\geqslant\limsup_{n\to\infty}\frac{1}{n}\bar{H}_P(\gamma_n,\dots,\theta^{n-1}\circ\gamma_n)$, and for $\varepsilon\to0$ the left hand side tends to $\hbar_P(\theta,\gamma)$ by (3.8).

(b) Assuming this, we can prove that (cf. the proof of (3.8)):

$$\begin{aligned}&\lim_{\varepsilon\to0+}(\lim_{n\to\infty}\frac{1}{n}\bar{H}_P^{\varepsilon}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n))=\\&\qquad\qquad=\lim_{n\to\infty}\bar{H}(\gamma_n,\theta\circ\gamma_n,\dots,\theta^{n-1}\circ\gamma_n).\end{aligned}$$

The equality therefore follows by the $\varepsilon \to 0$ part of (3.9,i). $\square$

Remarks and Observations (3.11) . The following remarks and simple results should clarify the relation between this "P–entropy" approach in section III, our original approach as in section II, and Thomsen's approach [19] as cited above.

1. Theorem (3.9) and Corollary (3.10) are particularly "natural" for a separable *nuclear* C*–algebra $\mathcal{A}$ with a Choi–Effros–Lance sequence $(\sigma_n, \tau_n)_{n\in\mathbb{N}}$ of completely positive unital maps $\sigma_n : \mathcal{A} \to \mathcal{B}_\backslash$, $\tau_\backslash : \mathcal{B}_\backslash \to \mathcal{A}$ with finite–dimensional (multi–)matrix algebras $\mathcal{B}_\backslash$, such that $\|\tau_n \circ \sigma_n(A) - A\| \to 0 \ \forall A \in \mathcal{A}$ [21]:

   Defining, for given $\gamma \in \mathcal{CP}_\infty(\mathcal{A})$ say, $\gamma_n = \tau_n \circ \sigma_n \circ \gamma$ $(\forall n \in \mathbb{N})$, it is obvious that for any such $\gamma \in \mathcal{CP}_\infty(\mathcal{A})$ there exists a suitable subsequence $(n_k)_{k\in\mathbb{N}}$ for which the condition of (3.9) is met: $\|\gamma - \gamma_{n_k}\|^{\frac{1}{k}} \to 0$. Thus it follows from (3.9,ii) with $\delta_k = c\|\gamma - \gamma_{n_k}\|^{\frac{1}{k}}$ that $\hbar_P(\theta, \gamma) = \lim_{k\to\infty} \frac{1}{k} \bar{H}_P^{\delta_k}(\tau_{n_k} \circ \sigma_{n_k} \circ \gamma, \theta \circ \tau_{n_k} \circ \sigma_{n_k} \circ \gamma, \ldots, \theta^{k-1} \circ \tau_{n_k} \circ \sigma_{n_k} \circ \gamma)$.

   Note, however, that here we can *not* conlude that the right hand side is dominated by $\frac{1}{k}\bar{H}_P^{\delta_k}(\tau_{n_k}, \theta \circ \tau_{n_k}, \ldots, \theta^{k-1} \circ \tau_{n_k})$ as $k \to \infty$, because $\bar{H}_P^{\varepsilon}$ (and $\bar{H}_P$) is known to be monotonic only with respect to compositions with $*$–homomorphisms (which $\sigma_{n_k} \circ \gamma$ are definitely *not*). We *could* use such an argument if by some "miracle" part (b) of (3.10) could be verified for these γ_{n_k}, with a "minimally spaced" subsequence $(n_k)_{k\in\mathbb{N}}$, implying that:

$$\hbar_P(\theta, \gamma) = \lim_{k\to\infty} \frac{1}{k} \bar{H}_P(\tau_{n_k} \circ \sigma_{n_k} \circ \gamma, \theta \circ \tau_{n_k} \circ \sigma_{n_k} \circ \gamma, \ldots, \theta^{k-1} \circ \tau_{n_k} \circ \sigma_{n_k} \circ \gamma) \leqslant$$

$$\leqslant \limsup_{k\to\infty} \frac{1}{k} \bar{H}(\tau_{n_k} \circ \sigma_{n_k} \circ \gamma, \theta \circ \tau_{n_k} \circ \sigma_{n_k} \circ \gamma, \ldots, \theta^{k-1} \circ \tau_{n_k} \circ \sigma_{n_k} \circ \gamma) \leqslant$$

$$\leqslant \limsup_{k\to\infty} \frac{1}{k} \bar{H}(\tau_{n_k}, \theta \circ \tau_{n_k}, \ldots, \theta^{k-1} \circ \tau_{n_k}) \leqslant \infty,$$

   now by (2.7,i) and the first trivial inequality.

   This non–trivial upper bound (if finite) seems to be most interesting for an AF algebra $\mathcal{A} = \overline{\bigcup_{\backslash\in\mathbb{N}} \mathcal{A}_\backslash} \equiv \overline{\mathcal{A}_\infty}$, where $\tau_n = \imath_{\mathcal{A}_\backslash} : \mathcal{A}_\backslash \hookrightarrow \mathcal{A}$ is the inclusion, with an "$\mathcal{A}_\infty$–shifty" automorphism θ as in (4.5) below.

In that case we could use that by assumption on θ, we have $\bar{H}(\mathcal{A}_{\backslash\|}, \theta(\mathcal{A}_{\backslash\|}), \ldots, \theta^{\|-\infty}(\mathcal{A}_{\backslash\|})) \leqslant \bar{\mathcal{H}}(\mathcal{A}_{\mathcal{N}_\|})$ for $N_k = n_k + k$ $(\forall k \in \mathbb{N})$; and at least for sufficiently "$\mathcal{A}_\infty$–smooth" $\gamma : \mathcal{B} \to \mathcal{A}$ (where this is a not defined notion, in contrast to "shifty" above) but with $\gamma(\mathcal{B}) \nsubseteq \mathcal{A}_\infty$, the resulting even larger upper bound $\limsup_{k\to\infty} \frac{1}{k}\bar{H}(\mathcal{A}_{\mathcal{N}_\|})$ could turn out to be *finite*, although on the other hand this might be incompatible with the assumption that (3.10,b) holds for $(\gamma_{n_k})_{k\in\mathbb{N}}$.

2. In any case, this "mere" speculation (1) shows that, also if we (or the reader) would now choose Definition (3.2) instead of the corresponding definitions in section II – either for reasons of the even more concrete physical interpretation of (3.2), or because it enables the modest continuity results (3.9 & 10) above – also then, our still "more non–commutative" definitions (2.5,ii) etc. of section II would not be obsolete at all: Not only in the situation of the "diagonalization trick" above (of the double sequence $(\frac{1}{k}\bar{H}(\tau_n, \theta\circ\tau_n, \ldots, \theta^{k-1}\circ\tau_n))_{k,n}$ in the last line of (1), for example), but *generally*, the definitions with $\mathcal{O}^+$ instead of $\mathcal{OP}_\infty$ give natural upper bounds for the latter "P–entropies" (cf. Remark (3.3)), and the former might be easier to calculate.
3. Although we know by (2.2,i) that for any $\alpha \in \mathcal{O}^+(\mathcal{A})$ as defined in (2.1,ii), there exists a map $\gamma \in \mathcal{P}_\infty(\mathcal{A}) \subset \mathcal{CP}_\infty(\mathcal{A})$ as in (2.12,i) (i.e. $\gamma : \mathcal{B} \to \mathcal{A}$ with $\mathcal{B}$ *Abelian* finite–dimensional) together with a $\beta \in \mathcal{OP}_\infty(\mathcal{B})$ as above such that $\alpha \subseteq \gamma(\beta)$ (this is obvious), we can *not* relate our modified version (3.2) here to Thomsen's modified definition in [19] (which clearly then would have to be a *lower* bound, as [19] uses the same α for all θ^k; cf. before (3.2)), for *two* (related) reasons:

   First, Thomsen uses right away instead of *positive* sets $\alpha \in \mathcal{O}^+(\mathcal{A})$ more general "operator covers" $\alpha \in \mathcal{O}(\mathcal{A})$ for the individual "non–commutative open covers", but in turn with the restriction that $\alpha \in \mathcal{O}(\mathcal{A})$ be *self–adjoint*: $\alpha = \alpha^*$. This additional condition has then the main advantage that the analogue (see below) of (2.3,ii) holds for *arbitrary* such α, not only for the positive $\alpha \in \mathcal{O}^+(\mathcal{A})$.

Secondly, however, to get again an inner operation on these "partitions" $\mathcal{P}(\mathcal{A}) = \{\alpha \in \mathcal{O}(\mathcal{A}) | \alpha = \alpha^*\}$ (in the same way that we have here $\vec{\vee} : \mathcal{O}(\mathcal{A})^{\in} \to \mathcal{O}(\mathcal{A})$, cf. also (2.2,iii)), Thomsen [19] consequently has to replace $\vec{\vee}$ by the operation leading to the *totally symmetrized* set of operator products of the individual "factors", denoted by $\vee : \mathcal{P}(\mathcal{A})^{\in} \to \mathcal{P}(\mathcal{A})$.

On the *one* hand, this latter modification has the formal disadvantage that this operation $\vee$ is not associative any more; but still the analogue of the subadditivity (2.3,i) (there actually submultiplicativity) holds. And because Thomsen takes the supremum in the final definition (for the entropy of a $*$–endomorphism θ) really over this set $\mathcal{P}(\mathcal{A})$, he can use the *same* simple proof for the generalization of the classical inequality $h(T^n) \geqslant n \cdot h(T)\ \ \forall n \in \mathbb{N}$ as used already in the very original paper by Adler et al. [1] (such as if we would define here an $\hbar(\theta)$ by a supremum over $\mathcal{O}(\mathcal{A})$, with our "join" $\vec{\vee}$); whereas this inequality is (still) missing in our approach.

4. On the other hand, however, this totally symmetrized join $\vee : \mathcal{P}(\mathcal{A})^{\in} \to \mathcal{P}(\mathcal{A})$ has the really *serious* drawback that Thomsen [19] can *not* prove the monotonicity $\bar{N}(\alpha) \leqslant \bar{N}(\alpha \vee \beta)$ (or $\bar{N}(\beta) \leqslant \bar{N}(\alpha \vee \beta)$) for all $\alpha, \beta \in \mathcal{P}(\mathcal{A})$ any more, in contrast to our (2.3,i) and even more importantly (2.7,ii) here. This forces [19] to define the "unscaled" entropy of an endomorphism θ by $\hbar\hbar(\theta) = \sup_{k \in \mathbb{N}} \frac{1}{k} \hbar(\theta^k)$ (where we denote here as $\hbar(\theta) = \sup_{\alpha \in \mathcal{P}(\mathcal{A})} \hbar(\theta, \alpha)$; in the notation of [19]: $\hbar(\theta, \alpha) = \hbar_1(\theta, \alpha)$), in order to maintain the essential converse inequality $\hbar\hbar(\theta^n) \leqslant n \cdot \hbar\hbar(\theta)\ \ (\forall n \in \mathbb{N})$.

This second supremum (taken over the scaled entropies of the endomorphism's powers) is obviously a rather "cheap trick"; whereas in our approach here, the corresponding inequality (2.9,v) respectively (2.13,ii) follows from (2.7,ii) by the also simple but *natural* argument analogous to the non–commutative *state*–dependent entropy theory, cf. [8].

¿From the physical point of view, it is clear that the totally *symmetrized* operation $\vee$ as in [19] completely destroys the natural interpretation of the ordered operator products in our $\vec{\vee} : \mathcal{O}_{(\infty)}(\mathcal{A})^{\in} \to \mathcal{O}_{(\infty)}(\mathcal{A})$, from (2.1,iii), as successive quantum–mechanical "measurements" (or so-called "operations", cf. [15, 16]), see the references in the Introduction I and particularly also the forthcoming publication [14].

Then it is not surprising that the remaining partial monotonicity (2.3,i) of the entropy with respect to additional such arguments gets also lost, which is actually against any "physical" intuition if the entropy $\bar{H}$ respectively $\bar{N}$ should be a sensible measure of "information" in some sense (side–remark: the *non*–monotonicity of the quantum–mechanical von Neumann entropy $S(\omega)$, cf. [27, 5], of a state $\omega \in S_{\mathcal{A}}$ with respect to restriction to subalgebras $\mathcal{B} \subset \mathcal{A}$ is not a good excuse at this point, as it refers to a rather different aspect of "quantum information"; see again also [14] in preparation).

5. Finally, we should add the following remarks: Although Thomsen's entropy [19] or actually the corresponding functional $\bar{N}(\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n)$ is clearly *symmetric* in its arguments $\alpha_k \in \mathcal{P}(\mathcal{A})$, it is not (yet?) much better suited for more general (semi–)group actions (than $\mathbb{Z}$– respectively $\mathbb{N}$–actions here) by $*$–auto–(respectively endo–)morphisms of $\mathcal{A}$, than our above approach with the *ordered* operation $\vec{\vee}$, which seems to be not suited at all for that purpose. A bit more explicitly: To proceed again along the lines of [28] (for example), Thomsen's approach [19] would still need (at least) the invariance of $\bar{N}(\alpha_1 \vee \cdots \vee \alpha_n)$ with respect to repetitions of $\alpha_k = \alpha_\ell$ $(k < \ell \leqslant n)$, which is not provable in this general form (contrast with our *built–in* invariance in (2.5,ii) etc.).

But of course, one could use this symmetrized functional to define *independently* some formal "entropy densities" of $\mathbb{Z}^{\ltimes}$–actions by automorphisms, which seems to be not possible with our *ordered* operation $\vec{\nabla}$ any more. In other words, the theory presented here is really dealing with *dynamical* entropy (in a physical sense) of *single* $*$–endomorphisms θ of $\mathcal{A}$ only.

IV. AF algebras and related examples

We first reproduce the following example from [19] with the entropy definitions of section II, noting that throughout this section *all* examples would give the same final results if we would use the restricted "P–entropy" definitions of section III instead. This first example due to Thomsen "pretends" to be more general than AF algebras, although the *positive* and *finite* entropy values result from the AF algebra case only.

Example (4.1) . Let $\mathcal{B}$ be a (unital) AA C*–algebra as defined in (2.11) with corresponding sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$, where we assume that either (2.11,ii) is fulfilled with *increasing* range spaces $\tau_n(\mathcal{B}_\backslash) \subseteq \tau_{\backslash+\infty}(\mathcal{B}_{\backslash+\infty})$ $(n \in \mathbb{N})$, or that $\tau_n : \mathcal{B}_\backslash \hookrightarrow \mathcal{B}$ are even the inclusions of unital *sub*algebras $\mathcal{B}_\backslash \subset \mathcal{B}$ which are *increasing* $(\mathcal{B}_\backslash \subseteq \mathcal{B}_{\backslash+\infty})$ such that *a fortiori* (2.11,i) is fulfilled. The general case of (2.11) is apparently less tractable in this example.

Let then $\mathcal{A} = \bigotimes_{\|\in\mathbb{N}}(\mathcal{B})_\|$ be the infinite tensor product C*–algebra of a countable number of copies of $\mathcal{B}$. If $\mathcal{B}$ is nuclear, this tensor product is uniquely determined; and otherwise we choose for example the injective or projective tensor product (cf. [10]). For each $k \in \mathbb{N}$, the simple tensors of the form $b_1 \otimes b_2 \otimes \cdots \otimes b_k \otimes \mathbf{1}_{k+1} \otimes \dots$ $(b_k \in (\mathcal{B})_\|)$ generate a unital C*–subalgebra $\mathcal{A}_\|$ of $\mathcal{A}$ which is $*$–isomorphic to the tensor product $\mathcal{B} \otimes \mathcal{B} \otimes \cdots \otimes \mathcal{B}$ of k copies of $\mathcal{B}$.

We assume that the AA maps $\tau_n : \mathcal{B}_\backslash \to \mathcal{B}$ are even *completely* positive (which is automatically fulfilled in the special case of (2.11,i) as above), i.e. $\tau_n \in \mathcal{CP}_\infty(\mathcal{B})$ with our notation from (2.12,1), $\forall n \in \mathbb{N}$. Then we can define maps $\pi_n : \bigotimes_{k=1}^{n}(\mathcal{B}_\backslash)_\| \to \mathcal{A}_\backslash$ by the n–fold product map $\bigotimes_{k=1}^{n}(\tau_n)_k$ on the tensor product of n copies of $\mathcal{B}_\backslash$ (which is again completely positive, for example by Stinespring's original representation of completely positive maps [29]) composed still with the $*$–isomorphism $\bigotimes_{k=1}^{n}(\mathcal{B})_{\|} \cong \mathcal{A}_{\backslash}\ \ (\forall\backslash \in \mathbb{N})$.

We first claim that $\mathcal{A}$ is again AA with respect to the sequence $\pi = (\pi_n)_{n\in\mathbb{N}}$.

Now, let $\sigma : \mathbb{N} \to \mathbb{N}$ be an injective map. For each $b \in \mathcal{B}$, let $b(i)$ denote the element $\mathbf{1}_{\mathcal{B}} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}} \otimes b \otimes \mathbf{1}_{\mathcal{B}}$, where b occurs as the i–th tensor factor, $i \in \mathbb{N}$. There is a unique unital $*$–endomorphism θ_σ of $\mathcal{A}$ given by $\theta_\sigma(b(i)) = b(\sigma(i)),\ \ b \in \mathcal{B}\ \ (\forall\rangle \in \mathbb{N})$. Then the entropy $\hbar_\pi(\theta_\sigma)$ depends on $\sigma, \tau, \mathcal{B}$ as follows:

(i) $\hbar_\pi(\theta_\sigma) = 0$ if σ has no infinite orbit in $\mathbb{N}$.

(ii) $\hbar_\pi(\theta_\sigma) \geqslant \limsup_{n\to\infty} \bar{H}(\tau_n)$ if σ has an infinite orbit in $\mathbb{N}$, and where we assume in addition that in case (2.11,ii) τ_n is *faithfully* covering as defined in (2.14,i), $\forall n \in \mathbb{N}$. In particular, if $\mathcal{B}$ is infinite–dimensional and $\bar{H}(\tau_n) \to \infty$, we have $\hbar_\pi(\theta_\sigma) = \infty$.

(iii) If $\mathcal{B}$ is finite–dimensional, we have in the (rather useless) general case the inequality $\hbar_\pi(\theta_\sigma) \leqslant r \cdot \bar{H}(\mathcal{B})$, where r is the number of infinite orbits of σ in $\mathbb{N}$. For the obvious choice $\tau \equiv (\mathrm{Id}_{\mathcal{B}})_{n\in\mathbb{N}},\ \ \mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ the identity map, equality follows.

Proof. We first show that $\mathcal{A}$ is again AA with respect to π. This is obvious if the special case (2.11,i) is fulfilled as assumed above, because then $\mathcal{C}_{\backslash} \equiv \bigotimes_{\|=\infty}^{\backslash}(\mathcal{B}_{\backslash})_{\|}$ is $*$–isomorphic to a *subalgebra* $\pi_n(\mathcal{C}_{\backslash})$ of $\mathcal{A}_{\backslash}$ and these are again increasing, by assumption: $\pi_n(\mathcal{C}_{\backslash}) \subseteq \pi_{\backslash+\infty}(\mathcal{C}_{\backslash+\infty})$, such that we have again *a fortiori* also $\pi_n(\mathcal{O}^+(\mathcal{C}_{\backslash})) \subseteq \pi_{\backslash+\infty}(\mathcal{O}^+(\mathcal{C}_{\backslash+\infty}))$.

And in the other case (2.11,ii), it clearly follows from $\mathcal{A} = \overline{\bigcup_{\|\in\mathbb{N}} \mathcal{A}_{\|}}$ and the assumption $\overline{\bigcup_{n\in\mathbb{N}} \tau_n(\mathcal{B}_{\backslash})} = \mathcal{B}$ that also $\overline{\bigcup_{n\in\mathbb{N}} \pi_n(\mathcal{C}_{\backslash})} = \mathcal{A}$.

Now we compute the entropy of θ_σ with respect to π:

(i) If σ has no infinite orbit in $\mathbb{N}$, it is clear that for each $n \in \mathbb{N}$, there is an integer $N(n)$ such that $\theta_\sigma^{N(n)}$ is the identity on $\mathcal{A}_{\backslash}$, hence $\theta_\sigma^{N(n)} \circ \pi_n \approxeq \pi_n$ by (2.5,i), and by (2.9,iv), $\hbar(\theta_\sigma, \pi_n) = 0$.

(ii) For a finite subset $F \subset \mathbb{N}$, we denote by $\mathcal{A}(\mathcal{F})$ the $*$–algebra generated by elements of the form $b(i),\ \ i \in F$. Let $m \in \mathbb{N}$ be an integer such that $\{\sigma^n(m)|n \in \mathbb{N}\}$ is infinite. Then for all $n \geqslant m$,

$$\hbar(\theta_\sigma, \pi_n) = \lim_{k\to\infty} \frac{1}{k} \bar{H}(\pi_n, \theta_\sigma \circ \pi_n, \ldots, \theta_\sigma^{k-1} \circ \pi_n) \geqslant$$
$$\geqslant \lim_{k\to\infty} \frac{1}{k} \bar{H}(\tau_{n,m}, \theta_\sigma \circ \tau_{n,m}, \ldots, \theta_\sigma^{k-1}(\tau_{n,m})),$$

where we denote by $\tau_{n,m} : \mathcal{B}_\backslash \to \mathcal{A}(\{\Updownarrow\})$ the map given by the composition of π_n *after* the inclusion homomorphism of the m–th tensor factor $(\mathcal{B})_\Updownarrow \hookrightarrow \mathcal{C}_\backslash \equiv \bigotimes_{\|=\infty}^{\backslash}(\mathcal{B}_\backslash)_\|$, such that the inequality follows by (2.7,i), using the fact that $\theta_\sigma^k \circ \pi_n \not\cong \pi_n \;\; \forall k \in \mathbb{N}$.

By the assumption that τ_n (and hence $\tau_{n,m}$) is faithfully covering, it follows from (2.15) that

$$\bar{H}(\tau_{n,m}, \theta_\sigma \circ (\tau_{n,m}), \ldots, \theta_\sigma^{k-1} \circ (\tau_{n,m})) = k \cdot \bar{H}(\tau_{n,m}),$$

because by construction the arguments on the left hand side are independently covering and commuting (cf. (2.16,3)). Thus it follows that $\hbar(\theta_\sigma, \pi_n) \geqslant \bar{H}(\tau_{n,m}) \equiv \bar{H}(\tau_n) \;\; \forall n \geqslant m$.

(iii) We use the following expression for the number r of infinite periodic orbits of σ, derived in [19]:

$$r = \lim_{n\to\infty} \lim_{k\to\infty} \frac{1}{k} \operatorname{card}(F_n \cup \sigma(F_n) \cup \sigma^2(F_n) \cup \cdots \cup \sigma^k(F_n)),$$

where $F_n = \{1, 2, \ldots, n\}$, and it is shown in [19] if not obvious that the first limit is a supremum, the second one an infimum (from left to right).

Note that for a general finite subset $F \subset \mathbb{N}$, with $\mathcal{A}(\mathcal{F})$ as defined in (4.1,ii) above, we have that $\bar{H}(\mathcal{A}(\mathcal{F})) = \operatorname{card} \mathcal{F} \cdot \bar{\mathcal{H}}(\mathcal{B})$, and that the notations in (i) and (ii) are related by $\mathcal{A}_\backslash = \mathcal{A}(\mathcal{F}_\backslash)$. Now use that

$$\hbar(\theta_\sigma, \pi_n) = \lim_{k\to\infty} \bar{H}(\pi_n, \theta_\sigma \circ \pi_n, \ldots, \theta_\sigma^{k-1} \circ \pi_n) \leqslant \lim_{k\to\infty} \frac{1}{k} \bar{H}(\mathcal{A}(\mathcal{G}_{\|,\backslash})),$$

where $G_{k,n} = F_n \cup \sigma(F_n) \cup \sigma^2(F_n) \cup \cdots \cup \sigma^{k-1}(F_n)$ as used for r above ($k, n \in \mathbb{N}$), and where the right hand side inequality follows from (2.7,iv). To repeat it, we know that $\bar{H}(\mathcal{A}(\mathcal{G}_{\|,\backslash})) = \operatorname{card} \mathcal{G}_{\|,\backslash} \cdot \bar{\mathcal{H}}(\mathcal{B})$, and together it follows that $\hbar(\theta_\sigma, \pi_n) \leqslant r \cdot \bar{H}(\mathcal{B}), \;\; \forall \backslash \in \mathbb{N}$, as the first limit for r is the supremum.

To prove the reverse inequality for $\tau_n \equiv \mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}\ \forall\backslash$, let $\beta = \{e_i | i = 1, \ldots, D(\mathcal{B})\} \subset \mathcal{B}$ be a partition of unity $\mathbf{1}_{\mathcal{B}}$ by mutually orthogonal minimal projections e_i (i.e. $\beta \in \mathcal{OP}_\infty(\mathcal{B})$ as defined in (3.2,i) above). For each $m \in \mathbb{N}$, let $\alpha_m = \bigotimes_{k=1}^m (\beta)_k$ (symbolically, but clear) be the m–th tensor power of β, explicitly $\alpha_m = \{e_{i_1} \otimes_{i_2} \otimes \cdots \otimes e_{i_m} | i_k = 1, \ldots, D(\mathcal{B})\ \forall \| = \infty, \ldots, \Updownarrow\}$, which is a $\mathcal{OP}_\infty$–partition of unity in $\bigotimes_{k=1}^m (\mathcal{B})_\| \cong \mathcal{A}_\Updownarrow$; and $\pi_m(\alpha_m) \in \mathcal{O}^+(\mathcal{A})$ is the image under that latter isomorphism. Then

$$\hbar(\theta_\sigma, \pi_m) \geqslant \lim_{k\to\infty} \frac{1}{k} \log \bar{N}(\pi_m(\alpha_m) \vec{\vee} \theta_\sigma \circ \pi_m(\alpha_m) \vec{\vee} \ldots \vec{\vee} \theta_\sigma^{k-1} \circ \pi_m(\alpha_m)),$$

and it is easy to see that the right hand side gives

$$\bar{N}(\pi_m(\alpha_m) \vec{\vee} \theta_\sigma \circ \pi_m(\alpha_m) \vec{\vee} \ldots \vec{\vee} \theta_\sigma^{k-1} \circ \pi_m(\alpha_m)) = D(\mathcal{B})^{\mathrm{card}\, \mathcal{G}_{\|,\Updownarrow}}.$$

Together we get that $\hbar(\theta_\sigma, \pi_m) \geqslant \log D(\mathcal{B}) (\lim_{\| \to \infty} \frac{\infty}{\|} \mathrm{card}\, \mathcal{G}_{\|,\Updownarrow})$, and by letting $m \to \infty$ it follows that $\hbar_\pi(\theta_\sigma) \geqslant r \cdot \bar{H}(\mathcal{B})$. □

In the rest of this final section, we briefly review the theory and examples on the entropy calculation for AF algebras, following the earlier versions [13, 12] but now including the rather trivial extension of these results to $*$–*endo*morphisms of AF algebras (not only automorphisms), again inspired by Choda's results [30] for the Connes–Stormer entropy. This is just meant to provide the reader with something at least formally new, as it is also the case in [19]. For this reason, we do not repeat the detailed proofs from [13, 12], where in the published version [13] the proofs for the general calculation methods are already included (for $*$–automorphisms, not yet $*$–endomorphisms), whereas the calculations for the examples reviewed below are given in full detail only in the unpublished thesis [12]. We can leave it to the reader to repeat these calculations as exercises, and to extend the general results for $*$–endomorphisms as stated below, following [30].

Throughout the remainder of this section, $\mathcal{A} = \overline{\bigcup_{\backslash \in \mathbb{N}} \mathcal{A}_\backslash}$ is a (unital) AF algebra with finite–dimensional $\mathcal{A}_\backslash \subseteq \mathcal{A}_{\backslash + \infty}$ (all with the same unit) and their algebraic inductive limit $\mathcal{A}_\infty = \bigcup_{\backslash \in \mathbb{N}} \mathcal{A}_\backslash$.

Corollary (4.2) . *For the canonical "AA structure" on $\mathcal{A}$ as in (2.11), i.e. $\tau = (\tau_n)_{n \in \mathbb{N}}$ with $\tau_n : \mathcal{A}_\backslash \hookrightarrow \mathcal{A}$ the inclusions, we have* *that* $\hbar_\tau(\theta) \equiv \lim_{n\to\infty} \hbar(\theta, \mathcal{A}_\backslash) = \sup_{\mathcal{B}\subset\mathcal{A}_\infty} \hbar(\theta, \mathcal{B})$*, where the supremum is taken over all finite–dimensional* C^**–subalgebras* $\mathcal{B} \subset \mathcal{A}_\infty$*. Thus we can use the notation* $\hbar_{\mathcal{A}_\infty}(\theta)$ *for the above AF–entropy.*

Proof. This is a direct consequence of (2.9,ii), as any $\mathcal{B} \subset \mathcal{A}_\infty$ is contained in some $\mathcal{A}_\backslash \supset \mathcal{B}$. $\square$

Definition (4.3) . We say with Choda [30] that the sequence $(\mathcal{A}_\backslash)_{\backslash\in\mathbb{N}}$ as above is *periodic* with period p, if $\exists n_0 \in \mathbb{N}$ such that $\forall j \geq n_0$:

(i) The inclusion matrices (cf. [30, 31]) are periodic: $[\mathcal{A}_| \hookrightarrow \mathcal{A}_{|+\infty}] = [\mathcal{A}_{|+\surd} \hookrightarrow \mathcal{A}_{|+\surd+\infty}]$.

(ii) The (hence necessarily square–) matrix $T_j = [\mathcal{A}_| \hookrightarrow \mathcal{A}_{|+\surd}]$ is *primitive* ($\iff \exists \ell \in \mathbb{N} : (T_j^\ell)_{ik} > 0\ \forall i,k \iff$ by the inclusion $\mathcal{A}_| \subset \mathcal{A}_{|+\ell\surd}$ each simple direct summand of $\mathcal{A}_|$ is "contained" in every simple direct summand of $\mathcal{A}_{|+\ell\surd} \implies$ the correpsonding Bratteli diagram [32, 31] of $\mathcal{A}$ is *connected*), which implies that T_j has a unique Perron–Frobenius eigenvalue $\beta_j > 0$ (which, together with (i), is actually independent of $j \geqslant n_0$!).

Lemma (4.4) . *For a periodic sequence* $(\mathcal{A}_\backslash)_{\backslash\in\mathbb{N}}$ *as above with period* p *and Perron–Frobenius eigenvalue* β *of* $T_j = [\mathcal{A}_| \hookrightarrow \mathcal{A}_{|+\surd}]$ *for* $j \geqslant n_0$*, we have* $\lim_{n\to\infty} \frac{1}{n}\bar{H}(\mathcal{A}_\backslash) = \frac{\infty}{\surd}\log\beta$ *(see* [13] *for the correct proof, and cf.* [30] *for the Connes–Størmer entropy).*

Definition (4.5) . A $*$–endomorphism θ of $\mathcal{A}$ is said to be "$\mathcal{A}_\infty$–shifty", if the following conditions are fulfilled (cf. [30, 19]):

(i) For all $j, m \in \mathbb{N}$, the C*–algebra generated by $\mathcal{A}_|, \theta(\mathcal{A}_|), \ldots, \theta^{\Updownarrow-\infty}(\mathcal{A}_|)$ is finite–dimensional (although these need not be pairwise commuting), and $\bar{H}(\mathcal{A}_|, \theta(\mathcal{A}_|), \ldots, \theta^{\Updownarrow-\infty}(\mathcal{A}_|)) \leqslant \bar{\mathcal{H}}(\mathcal{A}_{|+\Updownarrow})$.

(ii) There is a sequence $(n_j \in \mathbb{N})_{\jmath\in\mathbb{N}}$ such that $\forall k \in \mathbb{N}$: $\mathcal{A}_|, \theta^{\backslash_|}(\mathcal{A}_|), \ldots, \theta^{\|\backslash_|}(\mathcal{A}_|)$ are pairwise commuting *and* independently covering as defined in (2.14,ii) and in the sense of (2.16,3) (i.e., here: of tensor product form); and such that $\lim_{j\to\infty} \frac{n_j - j}{j} = 0$.

Theorem (4.6) . *For an "$\mathcal{A}_\infty$–shifty" $*$–endomorphism θ of $\mathcal{A}$, the AF–entropy (4.2) is $\hbar_{\mathcal{A}_\infty}(\theta) = \lim_{n\to\infty}\frac{1}{n}\bar{H}(\mathcal{A}_\backslash)$ (see* [13] *for the correct proof; cf. also* [30, 19]*, and be prepared to use the fact that the limit in our basic definition (2.5,ii) is actually the infimum).*

If θ is even a $*$–automorphism of $\mathcal{A}$, then $\hbar_{\mathcal{A}_\infty}(\theta^{-1}) = \hbar_{\mathcal{A}_\infty}(\theta)$ (because obviously by (4.5) above, also θ^{-1} is "$\mathcal{A}_\infty$–shifty").

Corollary (4.7) . *If $(\mathcal{A}_\backslash)_{\backslash\in\mathbb{N}}$ is a* periodic *sequence with period p and Perron–Frobenius eigenvalue β of the inclusion matrix $[\mathcal{A}_| \hookrightarrow \mathcal{A}_{|+\surd}]$ $\forall| \geqslant \backslash,$ then for any "$\mathcal{A}_\infty$–shifty" $*$–endomorphism θ of $\mathcal{A}$, the AF–entropy (4.2) is $\hbar_{\mathcal{A}_\infty}(\theta) = \frac{1}{p}\log\beta$.*

Example (4.8) . We consider the n^∞–UHF algebra (cf. [33]) $\mathcal{A} = \bigotimes_{\|\in\mathbb{Z}}(\mathcal{M}_\backslash(\mathbb{C}))_\neg$ as the *bilaterally* infinite C*–tensor product of copies of the $(n\times n)$–matrix algebra $M_n(\mathbb{C})$, and we choose the AF structure for $\mathcal{A}$ with the following notation (in contrast to [13], where the notation had been "too short" here): $\forall j \geqslant 0$,

$$\mathcal{A}_{\in|+\infty} = \mathbf{1}_\backslash^{\otimes\infty}\otimes\bigotimes_{\|=-|}^{|}(\mathcal{M}_\backslash(\mathbb{C}))_\neg\otimes\mathbf{1}_\ltimes^{\otimes\infty}, \qquad \mathcal{A}_{\in|} = \mathbf{1}_\backslash^{\otimes\infty}\otimes\bigotimes_{\|=-|+\infty}^{|}(\mathcal{M}_\backslash(\mathbb{C}))_\neg\otimes\mathbf{1}_\ltimes^{\otimes\infty},$$

where we still use a short notation, denoting by the symbolic infinite tensor power of $\mathbf{1}_n \in M_n(\mathbb{C})$ the corresponding unital inclusions into $\mathcal{A}$. The unit shift on $\mathbb{Z}$ determines a $*$–automorphism θ_n of $\mathcal{A}$ by

$$\theta_n(\mathbf{1}_n^{\otimes\infty}\otimes(M_n(\mathbb{C}))_\neg\otimes\mathbf{1}_\ltimes^{\otimes\infty}) = \mathbf{1}_\ltimes^{\otimes(\infty+\nVdash)}\otimes(\mathbb{M}_\ltimes(\mathbb{C}))_{\neg+\nVdash}\otimes\mathbf{1}_\ltimes^{\otimes(\infty-\nVdash)} \qquad \forall\urcorner\in\mathbb{N},$$

and from (4.7) with $p = 1$ and $[\mathcal{A}_\Updownarrow \hookrightarrow \mathcal{A}_{\Updownarrow+\infty}] = \backslash \in \mathbb{N}$ $(\forall> \in \mathbb{N})$ it follows that $\hbar_{\mathcal{A}_\infty}(\theta_n) = \log n$.

Note that this is the same result as in example (4.1) with $\mathcal{B} = \mathcal{M}_\backslash(\mathbb{C})$ and $\sigma(k) = k+1$ $(\forall k\in\mathbb{N})$, choosing τ as the constant identity map on $\mathcal{B}$ in (4.1,iii). This can be cast in words and formulae more generally, in the following complement of (3.1,i&ii):

Corollary (4.9) . *Let θ be an "$\mathcal{A}_\infty$–shifty" $*$–automorphism of $\mathcal{A}$, and assume that there exists an AF–subalgebra $\mathcal{B} \subset \mathcal{A}$ (included with the same unit $\mathbf{1} \in \mathcal{B}$) such that $\theta(\mathcal{B}) \subset \mathcal{B}$, and furthermore that it can* *be represented as* $\mathcal{B} = \overline{\bigcup_{\|\in\mathbb{N}}\mathcal{B}_\|} \equiv \overline{\mathcal{B}_\infty}$ *such that* $\theta\restriction_{\mathcal{B}}$ *is still "*$\mathcal{B}_\infty$*–shifty" as a* $*$*–endomorphism of* $\mathcal{B}$*, and at the same time* $\lim_{n\to\infty}\frac{1}{n}\bar{H}(\mathcal{A}_\backslash) = \lim_{\|\to\infty}\frac{\infty}{\|}\bar{\mathcal{H}}(\mathcal{B}_\|)$*. Then it follows obviously from (4.6) above that we have* $\hbar_{\mathcal{A}_\infty}(\theta) = \hbar_{\mathcal{B}_\infty}(\theta\restriction_{\mathcal{B}})$*.*

Example (4.10) . Let $\mathcal{A}$ be the unital AF algebra generated by the sequence of Jones projections [31] $(e_i)_{i\geqslant 0}$ (in the hyperfinite II_1 factor with trace tr), which satisfy the relations $e_i = e_i^* = e_i^2$, $e_ie_{i\pm1}e_i = \lambda e_i$ for some $\lambda \leqslant 1$, $e_ie_j = e_je_i$ for $|i-j| \geqslant 2$, and the additional relation $\mathrm{tr}(w\cdot e_i) = \lambda\cdot\mathrm{tr}(w)$ when w is a word in $\mathbf{1}, e_1, \dots, e_{i-1}$, $\forall i \geqslant 0$. As shown in [31], such a sequence exists exactly iff $\lambda \in (0, \frac{1}{4}] \cup \{(4\cos^2\frac{\pi}{m})^{-1} | m \in \mathbb{N},\ > \geqslant \nVdash\}$.

Using the same notation as in [31], we denote by $\mathcal{A}_\backslash$ the finite–dimensional C*–algebra generated by $\mathbf{1}, e_0, e_1, \dots, e_{n-1}$ and by $\mathcal{A}_\infty$ the corresponding inductive limit as used before, then the AF–algebra $\mathcal{A}$ is again defined by $\mathcal{A} = \overline{\mathcal{A}_\infty}$ (norm closure within the hyperfinite II_1–factor).

The translation on the index set $\mathbb{N}$ of the Jones sequence determines a (tr–preserving) unital $*$–endomorphism θ_λ of $\mathcal{A}$ by $\theta_\lambda(e_i) = e_{i+1}\ \ \forall i \geqslant 0$. Then it follows again from (4.7) that for $\lambda > \frac{1}{4}$ the entropy (4.2) is given by $\hbar_{\mathcal{A}_\infty}(\theta_\lambda) = -\frac{1}{2}\log\lambda$, whereas for $\lambda \leqslant \frac{1}{4}$ it follows from (4.6) itself, using [31], that $\hbar_{\mathcal{A}_\infty}(\theta_\lambda) \equiv \log 2\ \ \forall\lambda \leqslant \frac{1}{4}$, as explicitly shown in [19].

In [13, 12] we had actually used the automorphic version of this Jones shift, following the results of [34, 30] for the Connes–Størmer entropy, where the sequence $(e_i)_{i\geqslant 0}$ is extended to a bilateral sequence $(e_i)_{i\in\mathbb{Z}}$ (being the reason that above the sequence is indexed by $\mathbb{N}\cup\{\nvdash\}$), again generating the hyperfinite II_1–factor and satisfying the analogous relations. Replacing all the above $\mathcal{A}$'s in (4.10) here by $\mathcal{B}$'s and defining instead $\forall j \geqslant 0$:

$$\mathcal{A}_{\in|+\infty} = \mathrm{C}^*(\{\rceil_\rangle | \rangle = -|, \dots, |\}), \qquad \mathcal{A}_{\in|} = \mathrm{C}^*(\{\rceil_\rangle | \rangle = -| + \infty, \dots, |\}),$$

with corresponding inductive limit $\mathcal{A}_\infty$ and $\mathcal{A} = \overline{\mathcal{A}_\infty}$, it follows from [30] and results cited there that the assumptions of (4.9) above are met for the extension $\theta_\lambda^\pm(e_i) = e_{i+1}\ \ (\forall i \in \mathbb{Z})$, which implies that $\hbar_{\mathcal{A}_\infty}(\theta_\lambda^\pm) =$

$\hbar_{\mathcal{B}_\infty}(\theta_\lambda)$ for all possible values of λ (as shown directly in [13, 12] for $\lambda > \frac{1}{4}$).

Example (4.11) . Let $S \subset \mathbb{N}$ be any finite subset, and choose $n \in \mathbb{N}$. There exists then a sequence $(u_i)_{i\in\mathbb{N}}$ of unitaries, generating the hyperfinite II$_1$–factor, with relations $u_i^n = \mathbf{1}\ \forall i \in \mathbb{N}$, $u_i u_j = \exp(\frac{2\pi i}{n}) u_j u_i$ when $|i-j| \in S$, and $[u_i, u_j] = 0$ when $|i-j| \notin S$ (besides the relation $u_i^* = u_i^{-1}\ \forall i \in \mathbb{N}$). The generated C*–algebras $\mathcal{A}_\backslash = \mathrm{C}^*(\{\sqcap_\rangle | \rangle = \infty, \ldots, \backslash\})$ are all finite–dimensional, and hence again $\mathcal{A} = \overline{\mathcal{A}_\infty}$ is an AF algebra (unitally included in the hyperfinite II$_1$–factor).

Again, the index translation on $\mathbb{N}$ induces a unital $*$–endomorphism θ_n of $\mathcal{A}$, explicitly $\theta_n(u_i) = u_{i+1}\ \forall i \in \mathbb{N}$. Then it follows from (4.7) that $\hbar_{\mathcal{A}_\infty}(\theta_n) = \frac{1}{2}\log n$, see again [30] with the Connes–Størmer entropy and references there, cf. [19]. Again, θ_n can be extended to an automorphism $\theta_n^\pm$ of the AF algebra generated by the bilaterally extended sequence of u_i's, and (4.9) applies to give the same AF–entropy for $\theta_n^\pm$, see again [30].

Example (4.12) . The following example and the ideas for the method of calculation are due to Narnhofer and Thirring [35], see there for further references to its origins in the work of Powers and Price.

Let $\mathcal{A}_\backslash = \mathrm{C}^*(\{\rceil_\rangle | \rangle = -\backslash, \ldots, \backslash\})$ be the (universal, and existing) C*–algebras generated by the "generalized Pauli–matrices" e_i $(i \in \mathbb{Z})$ with the relations $e_i = e_i^*$, $e_i^2 = \mathbf{1}\ \forall i \in \mathbb{Z}$ and $e_i e_j = e_j e_i (-1)^{g(|i-j|)}\ \forall i \neq j$ with $g : \mathbb{N} \to \{\nvdash, \nVdash\}$. Again $\mathcal{A} = \overline{\mathcal{A}_\infty}$ is a unital AF algebra and $\theta_g(e_i) = e_{i+1}$ $(\forall i \in \mathbb{Z})$ induces a $*$–automorphism θ_g of $\mathcal{A}$. Then if *either* $g(n) \equiv 1\ \forall n \in \mathbb{N}$, *or* if $g(n) = 1 \iff n \in S \subset \mathbb{N}$ on a *finite* subset $S \neq \emptyset$ of $\mathbb{N}$ (where in the second case we still have to add the – possibly redundant – assumption that for the *centers* $\mathfrak{C}(\mathcal{A}_\backslash)$, it follows $\frac{1}{n}\log\dim\mathfrak{C}(\mathcal{A}_\backslash) \to \prime$), we get $\hbar_{\mathcal{A}_\infty}(\theta_g) \equiv \frac{1}{2}\log 2$.

See [12] for the correct proof, and cf. [6] for a published version of it in the first case $g \equiv 1$. We leave it to the reader as an interesting problem to relate this example to (4.3–5) above, so as to deduce the result directly from (4.7), again in special cases (and as a second problem, to find out if the additional assumption in the second case for g above is redundant or not).

Example (4.13) . Let θ_A be the shift on an AF algebra $\mathcal{A}$ associated with a topological Markov chain (also called subshift of finite type) as treated by Evans [36] with his "AF–imitation" of the topological entropy (via the Connes–Størmer entropy), and following the constructions of Cuntz and Krieger (see the references in [36]).

Without repeating this lengthy example here (see [12]), we just compare with Evans' notation: Our $(\mathcal{A}, \theta_{\mathcal{A}}) = (\mathcal{C}^{\mathcal{A}}, \sigma_\prime)$ of Evans, where A is a primitive $(n \times n)$–matrix with entries in $\{0, 1\}$ defining the subshift, and our $\mathcal{A}_\backslash = \mathcal{N}_\backslash \;\; (\backslash \in \mathbb{N})$ respectively $\mathcal{A}_\infty = \bigcup_{\backslash \in \mathbb{N}} \mathcal{N}_\backslash$ of Evans.

Then, repeating the proof of the main theorem in [36] with our $\bar{H}$ instead of Evans' H and using the properties (2.7,i & ii) and (2.6,2) of $\bar{H}$ (and also its definition (2.5,ii)), we immediately get the same result: $\hbar_{\mathcal{A}_\infty}(\theta_A) = \log \lambda$, where λ is the spectral radius (i.e. the Perron–Frobenius eigenvalue) of A. See [12] for the details of the proof, left as an exercise here: Note that we do not need at all even the *ingredients* of Evans' Proposition 2 [36] for the second part of the rewritten proof, as the latter amounts in our case to the *same* estimate by "$\log |\mathcal{M}_{\backslash,\backslash+\|}|$" in Evans' notation as the first part, only from below instead from above.

Actually, the same result holds also for the larger AF algebra $\mathcal{B}$ containing the above $\mathcal{A} \subset \mathcal{B}$ as defined in [36] (see [12] for the correspondence of notations) with θ_A naturally extended to $\mathcal{B}$, and with a corresponding sequence $\mathcal{B}_\backslash \subset \mathcal{B}_{\backslash+\infty}$ for $\mathcal{B} = \overline{\mathcal{B}_\infty}$ such that $\mathcal{A}_\backslash \subset \mathcal{B}_\backslash \;\; \forall \backslash \in \mathbb{N}$ holds true: $\hbar_{\mathcal{B}_\infty}(\theta_A) = \log \lambda$. The proof is exactly the same as for the "gauge–invariant observable" algebra $\mathcal{A}$ before, see [12, 36].

On the other hand, however, the *classical* subshift is even contained in $\mathcal{A}$ as the restriction of θ_A to a canonical Abelian AF subalgebra $\mathcal{D} \subset \mathcal{A}$ which is *diagonal* in $\mathcal{A}$ (and also in $\mathcal{B}$) in the sense that $\mathcal{D} = \overline{\bigcup_{\backslash \in \mathbb{N}} \mathcal{D}_\backslash}$ with finite–dimensional (Abelian) $\mathcal{D}_\backslash \subset \mathcal{A}_\backslash, \;\; \forall \backslash \in \mathbb{N}$. The classical topological entropy of the original Markov chain is then again given by $\hbar_{\mathcal{D}_\infty}(\theta_A \restriction_{\mathcal{D}}) = \log \lambda$; and as pointed out also in [19], it follows here from (3.1,i) that for every AF subalgebra $\mathcal{C} \subset \mathcal{B}$ such that $\mathcal{D}_\backslash \subseteq \mathcal{C}_\backslash \subseteq \mathcal{B}_\backslash \ \forall \backslash \in \mathbb{N}$ (for example, take $\mathcal{A} = \mathcal{C}$), we get still $\hbar_{\mathcal{C}_\infty}(\theta_A \restriction_{\mathcal{C}}) = \log \lambda$.

Concluding Remark (4.14) . In all the AF examples considered above, *except* in the "new" case $\lambda < \frac{1}{4}$ of (4.10), we have at least the partial *analogue* of the variational principle (1.2,ix) in the following sense: $\hbar_{\mathcal{A}_\infty}(\theta) = h_{\mathrm{tr}}(\theta)$, where $\mathrm{tr} \in S_{\mathcal{A}}$ is a canonically given, θ-invariant trace state on $\mathcal{A}$, respectively; and where $h_\omega(\theta)$, here with $\omega = \mathrm{tr}$, is the state–dependent C*–dynamical entropy of Connes, Narnhofer and Thirring [8], actually also naturally extended for $*$–endomorphisms θ (along the lines of [30] for the Connes–Størmer entropy). See the respective references in the respective examples for hints how to show this claim.

The desirable but generally "utopic" non–commutative *generalization* of (1.2,ix) in the form $\hbar_{\mathcal{A}_\infty}(\theta) = \sup_{\omega \in S^\theta_{\mathcal{A}}} h_\omega(\theta)$, with $S^\theta_{\mathcal{A}} = \{\omega \in S_{\mathcal{A}} | \omega \circ \theta = \omega\}$, is beyond the scope of this paper, but it will be treated together with the detailed proofs of the above claim for the mentioned examples in the computational part of the forthcoming publication [14].

We should remark still here, however, that for the case $\lambda < \frac{1}{4}$ in (4.10), the problem remains open: Choosing a fixed $\lambda \in (0, \frac{1}{4})$, the algebra $\mathcal{A} = \mathcal{A}_\lambda$ of (4.10), first in the final automorphic version of θ_λ there, has the canonical trace state $\mathrm{tr}_\lambda \in S^{\theta_\lambda}_{\mathcal{A}_\lambda}$ equal to the restriction of the unique trace state tr on the generated hyperfinite II_1–factor containing $\mathcal{A}_\lambda$; and by [8, (VII.2)] it follows that $h_{\mathrm{tr}_\lambda}(\theta_\lambda) \restriction_{\mathcal{A}_\lambda}$ as introduced above is equal to the Connes–Størmer entropy of the extended II_1–factor automorphism $\bar{\theta}_\lambda$ with respect to tr. The latter entropy was shown in [34, 30] to be $h_{\mathrm{tr}}(\bar{\theta}_\lambda) = -t \log t - (1-t)\log(1-t)$, where $\lambda = t(1-t)$; and as pointed out in [19], for all $\lambda \in (0, \frac{1}{4})$ this gives the strict inequality $h_{\mathrm{tr}_\lambda}(\theta_\lambda) < \hbar_{\mathcal{A}_\infty}(\theta_\lambda) \equiv \log 2$ from (4.10), which violates the above claim for this canonical trace state $\mathrm{tr}_\lambda \in S^{\theta_\lambda}_{\mathcal{A}_\lambda}$.

But this still leaves open the possibility that the *strong* form above of the non–commutative generalization of (1.2,ix) could be true, i.e. that the supremum over $h_\omega(\theta_\lambda)$ for $\omega \in S^{\theta_\lambda}_{\mathcal{A}_\lambda}$, invariant states on the *norm*–completed inductive limit $\overline{\bigcup_{n\in\mathbb{N}}\mathcal{A}_\backslash} = \mathcal{A}_\lambda$, would in fact be equal to $\log 2 \equiv \hbar_{\mathcal{A}_\infty}(\theta_\lambda)$ from (4.10), for all $\lambda \in (0, \frac{1}{4})$.

It follows easily from the analysis in [34] and from [8] respectively [7], that $h_\omega(\theta_\lambda) \leqslant \log 2 \ \forall \omega \in S^{\theta_\lambda}_{\mathcal{A}_\lambda}$, for any fixed $\lambda < \frac{1}{4}$: By the explicit construction in [34, (5.5)], $\mathcal{A}_\lambda$ is isomorphically represented as C*–subalgebra of the 2^∞–UHF algebra (cf. [33] and see (4.8) above), such that θ_λ is given by the restriction of the unit shift θ_2 of (4.8) to $\mathcal{A}_\lambda$. For the latter θ_2 it follows from [8, 7] that $\sup_\phi h_\phi(\theta_2) = \log 2$, with supremum over all θ_2–invariant states ϕ on the 2^∞–UHF algebra; and by standard arguments this implies that $h_\omega(\theta_\lambda) \leqslant \log 2 \ \forall \omega \in S^{\theta_\lambda}_{\mathcal{A}_\lambda}$.

But to show that this upper bound is attained by the supremum over $S^{\theta_\lambda}_{\mathcal{A}_\lambda}$, it is necessary to compute $h_\omega(\theta_\lambda)$ for states ω other than the "canonical" trace tr_λ on $\mathcal{A}_\lambda$, and that is not as easy a task as it might seem at first sight. Note from [31, 30] that all the $\mathcal{A}_\lambda$ for $\lambda < \frac{1}{4}$ are isomorphic (as AF algebras), and one could be tempted to use the explicit construction as in [34] for computing $h_\omega(\theta_\lambda)$ with other θ_λ–invariant states $\omega \in S^{\theta_\lambda}_{\mathcal{A}_\lambda}$: There [34], the GNS representation of the 2^∞–UHF algebra with the faithful (infinite tensor product) Powers states with eigenvalues $\{t, 1-t\}$ of the individual $M_2(\mathbb{C})$–state factors, giving rise to the respective Powers factor (cf. [3]) of type $\mathrm{III}_{t/(1-t)}$ for $t \in (0, \frac{1}{2})$, was shown to have as the centralizer of that respective Powers state exactly the hyperfinite II_1–factor generated by $\mathcal{A}_\lambda$ in this particular representation, for the respective $\lambda = t(1-t)$.

But although θ_λ is always given by the respective restriction of the 2–shift θ_2 to $\mathcal{A}_\lambda$ in the respective representation of [34], and although $\hbar_{\mathcal{A}_\infty}(\theta_\lambda) \equiv \log 2$ for $\lambda \in (0, \frac{1}{4})$ (which might suggest that the sufficient condition (2.13,i) is actually fulfilled), the θ_λ are "unfortunately" *not* necessarily C*–algebraically conjugate via those $*$–isomorphisms of the $\mathcal{A}_\lambda$ for pairs of different $\lambda < \frac{1}{4}$. If they *were* all pairwise conjugate, we could simply use that for any fixed θ_λ, and for all other possible $\lambda' \neq \lambda$, we would have the conjugacy–invariance from [8]: $h_{\omega_{\lambda'}}(\theta_\lambda) = h_{\mathrm{tr}_{\lambda'}}(\theta_{\lambda'})$, where $\omega_{\lambda'}$ denotes the "pullback" to $\mathcal{A}_\lambda$ of the respective Powers state for λ' (restricted to $\mathcal{A}_{\lambda'}$ in the respective construction of [34]). By the results for the Connes–Størmer entropy cited before we know that $\lim_{\lambda' \to 1/4-} h_{\mathrm{tr}_{\lambda'}}(\theta_{\lambda'}) = \log 2$, which would then imply that also $\lim_{\lambda' \to 1/4-} h_{\omega_{\lambda'}}(\theta_\lambda) = \log 2$ holds.

So this "cheap trick" cannot be applied here; but on the other hand, at least on the face of it, the different θ_λ are also *not* necessarily pairwise *non*–conjugate via the $*$–isomorphisms of the different $\mathcal{A}_\lambda$, although the analogous non–conjugacy is in fact realized for their respective natural extensions $\bar{\theta}_\lambda$ to the respective hyperfinite II_1–factor $\mathcal{R}$ generated by the GNS representation of $\mathcal{A}_\lambda$ with the respective canonical trace state tr_λ: The important point of the construction in [34] is just the aforementioned fact that $\bar{\theta}_\lambda$ on $\mathcal{R}$ is the Connes-Krieger-Størmer "non–commutative Bernoulli shift" (cf. [9]) with weights $\{t, 1-t\}$ for $\lambda = t(1-t)$, and the natural extension of tr_λ is always the *unique* trace tr on $\mathcal{R}$. To repeat it now, the Connes–Størmer entropy (CNT entropy with the trace) $h_{\mathrm{tr}}(\bar{\theta}_\lambda) = -t\log t - (1-t)\log(1-t)$ shows that all the automorphisms $\bar{\theta}_\lambda$ of $\mathcal{R}$ are pairwise non–conjugate, but this does *not* yet imply the analogous non–conjugacy for the respective restrictions θ_λ on $\mathcal{A}_\lambda$: Not any of the $*$–isomorphisms between $\mathcal{A}_\lambda \subset \mathcal{R}$ and $\mathcal{A}_{\lambda'} \subset \mathcal{R}$ for $\lambda \neq \lambda'$ can be extended to an isomorphism of $\mathcal{R}$ onto itself, because none of those C*–isomorphisms preserves the unique trace tr on $\mathcal{R}$ (restricted to the $\mathcal{A}_\lambda$ respectively $\mathcal{A}_{\lambda'}$, where it gives the C*–traces tr_λ respectively $\mathrm{tr}_{\lambda'}$ with *different* trace vectors on the Bratteli diagrams). By the way, if θ_λ and $\theta_{\lambda'}$ for $\lambda \neq \lambda'$ are non–conjugate via $\mathcal{A}_\lambda \cong \mathcal{A}_{\lambda'}$, this shows that the "$\mathcal{A}_\infty$–topological" entropy $\hbar_{\mathcal{A}_\infty}(\theta_\lambda)$ is a "weaker" conjugacy invariant than the Connes–Størmer entropy $h_{\mathrm{tr}}(\bar{\theta}_\lambda)$.

But still, the opening question remains: Can one compute $h_\omega(\theta_\lambda) \to \log 2$ for a family of states $\omega \in S^{\theta_\lambda}_{\mathcal{A}_\lambda}$ (with $\omega \neq \mathrm{tr}_\lambda$, throughout, $\forall \lambda < \frac{1}{4}$)? Put differently, the problem is that for general θ_2–invariant faithful states ϕ_λ (the subscript λ here indicates the possible adaptation of ϕ_λ for the fixed $\lambda = t(1-t)$, not any concrete construction of ϕ_λ) on the 2^∞–UHF algebra, *different* from the respective Powers state with weights $\{t, 1-t\}$, the relation between the generated von Neumann algebra in the GNS construction with ϕ_λ on the one hand, and the enveloping von Neumann algebra of $\mathcal{A}_\lambda$ (the latter constructed as in

[34] inside the 2^∞–UHF algebra) within this GNS representation for ϕ_λ on the other hand, seems generally not to be obtainable by exactly the same method as in [34], even if ϕ_λ is tracial on $\mathcal{A}_\lambda$ and one takes again the ϕ_λ–centralizer of the full GNS–bicommutant of the 2^∞–UHF algebra (so for example with ϕ_λ given by the unique C*–trace on the latter simple algebra, for which the CNT entropy of the full shift θ_2 is $\log 2$, as desired). Some modified method seems necessary to relate the entropy to be calculated, $h_{\phi_\lambda \restriction \mathcal{A}_\lambda}(\theta_\lambda)$, to the entropy $h_{\phi_\lambda}(\theta_2)$ of the full shift as in (4.8), whereupon one could use the results of [8, 7]. The other, direct way would be to find sufficiently close *lower* bounds (or even exact values) for $h_{\phi_\lambda \restriction \mathcal{A}_\lambda}(\theta_\lambda) > 0$ by explicit calculation methods for the CNT entropy [8], and then to show that these bounds still tend to $\log 2$ for suitably chosen $\phi_\lambda \restriction \mathcal{A}_\lambda$.

As stated already in the corresponding concluding remark in [13], it will presumably be still much harder in the case of any computable *non*–AF example $(\mathcal{A}, \theta)$ to find a corresponding AA–sequence τ as in (2.11) such that again $\hbar_\tau(\theta) = \sup_{\omega \in S^\theta_{\mathcal{A}}} h_\omega(\theta)$ holds. In the meantime, since [13] had been submitted for publication, Heide Narnhofer [37] has succeeded in "estimating" the Connes–Størmer entropy (i.e. CNT entropy with the trace state) for the $\mathrm{SL}(2, \mathbb{Z})$–action ("CAT map") by single automorphisms on the irrational rotation C*–algebra $\mathcal{A}_\theta$ (θ now the rotation parameter!):

For "almost all" values of the irrational rotation parameter θ, [37] finds entropy *zero* for *any* CAT map (and for the "exceptional" values $\theta \in \mathbb{R} \setminus \mathbb{Q}$, still the *upper* bound given by the corresponding classical CAT–entropy, i.e. the logarithm of the larger $\mathrm{SL}(2, \mathbb{Z})$–eigenvalue).

Despite some serious efforts, we have not been able to estimate the "τ–topological" entropy (2.11) of this non–commutative CAT map for any *non*–trivial sequence $\tau \equiv (\tau_n)_{n \in \mathbb{N}}$, $\tau_n : \mathcal{B}_\backslash \to \mathcal{A}_\theta$ (even with *Abelian* finite–dimensional $\mathcal{B}_\backslash$, or $\tau_n \in \mathcal{P}_\infty(\mathcal{A}_\theta)$ with our notation from section II). The problem comes from the fact that invertibility in $\mathcal{A}_\theta$ is very hard to control, see for example [38], and cf. Arveson's recent C*–algebraic numerical work [23]. –

We recall that Watatani [39] had used the "Pontriagin dual" of the *classical* topological entropy on compact Abelian *groups* (such as $\mathbb{T}^{\nvDash}$) to compute the "K_1–entropy" of the $\mathcal{A}_\theta$–induced $\mathrm{SL}(2,\mathbb{Z})$–action by group automorphisms on the discrete Abelian K_1–group $\mathrm{K}_1(\mathcal{A}_\theta)$ (cf. [33]), where using the explicit construction for $\mathrm{K}_1(\mathcal{A}_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}$ due to Pimsner and Voiculescu [40, (2.5)] it then follows rather trivially that the result is again the classical topological (group) entropy of the $\mathbb{T}^{\nvDash}$–CAT map (for *all* values of $\theta \in \mathbb{R} \setminus \mathbb{Q}$!).

Clearly, this "K_1–entropy" of [39] is not suited at all for AF–algebras $\mathcal{A}$ where $K_1(\mathcal{A}) \equiv /$ (see [33]), for which in turn the analogous "K_0–entropy" [39] might give some non–trivial results. –

On the other hand, Thomsen's approach [19] as discussed and cited before, defining the entropy by the supremum over $\mathcal{P}(\mathcal{B})$ (see (3.11, 3 & 4)) for an invariant *local* C^**–algebra* $\mathcal{B} \subset \mathcal{A}$ (cf. also [33]) might not be well suited for $\mathcal{A} = \mathcal{A}_\theta$ instead of an AF algebra $\mathcal{A}$ as in [19], as $\mathrm{SL}(2,\mathbb{Z})$–action invariant local C*–subalgebras of $\mathcal{A}_\theta$ (for example, subalgebras $\mathcal{B}$ of "smooth" elements) might be equally hard to control as all of $\mathcal{A}$ concerning questions of invertibility. Further work should show if the more pragmatic "AA entropy" proposed here is better suited for that purpose.

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Dedication . This work is dedicated to the memory of Alfred Wehrl, whose untimely death struck me while being here in Berkeley. Fredl had been my first guiding teacher in mathematical physics, then my co–operative colleague and helpful friend.

In his most frequently cited, excellent review [27], he wrote in section IV (Related concepts) after part A on dynamical entropies (not topological, but only measure–theoretic; also mentioning Emch, Lindblad and Connes–Størmer in the "quantum case") at the end of the short section B:

> *Some concepts measuring the amount of information have been described. The list is not exhaustive and it is left to everyone to invent new such quantities. However, it will be very hard to establish their physical meaning.*

I hope that the present work can cope with this latter sentence in the affirmative.

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