OPERATORS WITH SINGULAR CONTINUOUS SPECTRUM: III. ALMOST PERIODIC SCHRÖDINGER OPERATORS

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Abstract. We prove that one-dimensional Schrödinger operators with even almost periodic potential have no point spectrum for a dense G_δ in the hull. This implies purely singular continuous spectrum for the almost Mathieu equation for coupling larger than 2 and a dense G_δ in θ even if the frequency is an irrational with good Diophantine properties.

§1. Introduction

This is a paper that provides yet another place where singular continuous spectrum occurs in the theory of Schrödinger operators and Jacobi matrices (see [5,6,2,10,3]). It is especially interesting because it will provide examples where a non-resonance condition in a KAM argument is not merely needed for technical reasons but necessary.

Our main results, proven in §2, do not deal directly with singular continuous spectrum but only with continuous spectrum.

Theorem 1S. *Let V be an even almost periodic function on $(-\infty,\infty)$ and let Ω be the hull of V and $V_\omega(x)$ the corresponding function for $\omega \in \Omega$. Then there is a dense G_δ, U in Ω (in the natural metric topology), so that if $\omega \in U$, then $H_\omega \equiv \frac{-d^2}{dx^2} + V_\omega(x)$ has no eigenvalues as an operator on $L^2(\mathbb{R})$.*

For the Jacobi case, we let h_0 be the operator on $\ell^2(\mathbb{Z})$ defined by $(h_0 u)(n) = u(n+1) + u(n-1)$.

Theorem 1J. *Let V be an even almost periodic function on $\mathbb{Z}$, Ω its hull, and $V_\omega(n)$ the function associated to $\omega \in \Omega$. Then there is a dense G_δ, U in Ω so that if $\omega \in U$, then $H_\omega = h_0 + V_\omega(n)$ has no eigenvalues as an operator on $\ell^2(\mathbb{Z})$.*

The G_δ set U will be rather explicit—see §2. By combining this with the machinery of [10], we can sometimes get singular continuous spectrum.

Theorem 2. *In the context of Thm. 1, suppose there is a single $\omega \in \Omega$ so that H_ω has no absolutely continuous spectrum. Then for a dense $G_\delta, \tilde{U}$, H_ω has purely singular continuous spectrum.*

Proof. Let $U_1 = \{\omega \in \Omega \mid H_\omega$ has no a.c. spectrum$\}$. By [10], U_1 is a G_δ. By hypothesis, ω_0 and its translates lie in U_1, so U_1 is a dense G_δ. Thus, $\tilde{U} = U_1 \cap U$ is a dense G_δ. □

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Example 1. Consider the Jacobi matrix with

$$V_\theta(n) = \lambda \cos(\pi\beta n + \theta). \tag{1}$$

If $\lambda > 2$, the Lyapunov exponent is positive ([1,7]) so if β is irrational, there is no a.c. spectrum for Lebesgue a.e. θ (see e.g. [1]), so h_θ has purely singular continuous spectrum for a dense G_δ of θ.

Sinai [11] and Fröhlich-Spencer-Wittwer [4] have proven for λ large and β having good Diophantine properties, a.e. θ has pure point spectrum, and Jitomirskaya [8] has proven that for $\lambda \geq 15$. In that case there are intertwined locally uncountable sets of θ with only pure point and with only singular continuous spectrum. For $\lambda = 2$, $\text{spec}(h_\theta)$ has zero measure for many irrational β's [9] and so no a.c. spectrum. We conclude

Theorem 3. *For the example* (1)*, h_θ has purely singular continuous spectrum for a dense G_δ of θ's if β is irrational and $\lambda > 2$ or if the continued fraction expansion of β has unbounded integers and $\lambda = 2$.*

Example 2. Consider the Schrödinger case with $V_\theta(x) = -k[\cos(2\pi x)+\cos(2\pi\beta x+\theta)]$. Then, Fröhlich-Spencer-Wittwer [4] have proven for a.e. θ (k large enough), there is pure point spectrum for low energies. Sorets-Spencer [12] have proven positivity of the Lyapunov exponent for a wider area of low energy. We conclude that for a dense G_δ of θ, there is purely singular continuous spectrum for low energies.

§2. Proof of Theorem 1

We'll consider the Jacobi case in detail and then discuss the changes for the Schrödinger case. Let V_{ω_0} be the even almost periodic function on $\mathbb{Z}$:

$$V_{\omega_0}(-n) = V_{\omega_0}(n).$$

Fix once and for all a number B so

$$B > 4\ln(3 + 2\sup_n |V_{\omega_0}(n)|) \equiv 4\ln\alpha. \tag{2.1}$$

α is chosen so that the matrix $\left(\begin{smallmatrix} E-V(u) & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has norm bounded by α if $|E| \leq 2 + \sup_n |V_{\omega_0}(n)|$.

Let Ω be the hull of V, that is, the closure in $\|\cdot\|_\infty$ of translates of V; it is compact by hypothesis. Define ρ on Ω by

$$\rho(\omega, \omega') \equiv \sup_n(|V_\omega(n) - V_{\omega'}(n)|)$$

and define maps R and T on Ω by

$$V_{R\omega}(n) = V_\omega(-n) \qquad V_{T\omega}(n) = V_\omega(n-1).$$

Lemma 2.1. *Let* $U_n = \underset{|m|>n}{\cup} \{\omega \mid \rho(RT^{2m}\omega,\omega) < e^{-B|m|}\}$ *and let* $U = \overset{\infty}{\underset{n=1}{\cap}} U_n$. *Then* U_n *is a dense open set and* U *is a dense* G_δ *in* Ω.

Proof. Let $\omega_m = T^{-m}\omega_0$. Then $RT^{2m}\omega_m = \omega_m$ since $R\omega_0 = \omega_0$, so $\omega_m \in U_n$ if $|m| > n$. It is easy to see the set of $\{\omega_m \mid |m| > n\}$ is dense in Ω, so U_n is dense. It is clearly open and so $U = \cap U_n$ is a dense G_δ by the Baire category theorem. □

U is the set of ω's for which there exists an infinite sequence m_i with $|m_i| \to \infty$ with $\rho(RT^{2m_i}\omega,\omega) < e^{-B|m_i|}$. For a subsequence, either $m_i \to \infty$ or $m_i \to -\infty$ and by reflection invariance, we can suppose $m_i \to \infty$. Thus, Thm. 1J follows from

Theorem 2.2. *Suppose that* V *is a function obeying*

$$|V(2m_i - n) - V(n)| \le e^{-Bm_i} \tag{2.2}$$

for a sequence $m_i \to \infty$ *where* B *is given by* (2.1). *Then*

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{2.3}$$

has no ℓ^2 *solutions for any* E.

Remark. The intuition behind the proof is that any u obeying (2.3) has to be close to being even or odd about m_i so $u(n) \nrightarrow 0$.

Proof. Suppose not. Then we can find a solution u of (2.3) in ℓ^2 which we normalize, so that

$$\sum_n |u(n)|^2 = 1. \tag{2.4}$$

We let $u_i(n) \equiv u(2m_i - n)$. Let $W(f,g)(n) = f(n+1)g(n) - f(n)g(n+1)$ be the Wronskian as usual, and let

$$\Phi(n) = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}; \qquad \Phi_i(n) = \begin{pmatrix} u_i(n+1) \\ u_i(n) \end{pmatrix}$$

as two component vectors.

Step 1. *Almost constancy of* $W(u,u_i)$

By a standard calculation using (2.3)

$$\begin{aligned} |W(u,u_i)(n) - W(u,u_i)(n-1)| &\le |V(n) - V(2m_i - n)||u(n)u_i(n)| \\ &\le e^{-Bm_i} \end{aligned} \tag{2.5}$$

by (2.2) and (2.4).

Step 2. *Smallness of* $W(u,u_i)$ *for* m_i *large*

Since u and u_i are in ℓ^2 with ℓ^2 norm 1, the Schwarz inequality implies that $\sum_n |W(n)| \le 2$. Thus for some n with $|n| \le e^{Bm_i/2}$, we must have that $|W(n)| \le e^{-Bm_i/2}$. By (2.5) we se that for $|n| \le e^{Bm_i/2}$, we have that

$$|W(n)| \le 3e^{-Bm_i/2} \tag{2.6}$$

and in particular for $n = m_i$.

Now define $u_i^\pm = u \pm u_i$, $\Phi_i^\pm = \Phi \pm \Phi_i$.

Step 3. *Smallness of $\Phi_i^+(m_i)$ or $\Phi_i^-(m_i)$*

Since $W(u_i^-, u_i^+) = 2W(u, u_i)$ and $u_i^-(m_i) = 0$, we see that

$$|u_i^+(m_i)u_i^-(m_i+1)| \le 6e^{-Bm_i/2}$$

so either

$$|u^+(m_i)| \le \sqrt{6}e^{-Bm_i/4} \tag{2.7}$$

or

$$|u_i^-(m_i+1)| \le \sqrt{6}e^{-Bm_i/4}. \tag{2.8}$$

We claim that this means either

$$\|\Phi_i^\pm(m_i)\| \le Ce^{-Bm_i/4} \quad \text{(for one of } + \textit{ or } -). \tag{2.9}$$

If (2.8) holds, (2.9) is immediate since $u_i^-(m_i) = 0$. If (2.7) holds, note that by (2.3)

$$u^+(m_i+1) + \frac{1}{2}(V(m_i) - E)u^+(m_i) = 0$$

so (2.9) holds for Φ_i^+.

Step 4. *Smallness of $\Phi_i^\pm(0)$*

Let $T_i^{(1)}$ be the transfer matrix for (2.3), taking $\Phi(m_i)$ to $\Phi(0)$ and let $T_i^{(2)}$ be the same with $V(2m_i - n)$ so

$$\begin{aligned} T_i^{(1)}\Phi(m_i) &= \Phi(0) \\ T_i^{(2)}\Phi_i(m_i) &= \Phi_i(0). \end{aligned}$$

Writing out T_i as a product and using the definition of α and (2.2), we have that

$$\|T_i^{(1)} - T_i^{(2)}\| \le 2m_i\alpha^{m_i-1}e^{-Bm_i} \le 2m_ie^{-3Bm_i/4}.$$

Writing

$$\begin{aligned} \Phi_i^\pm(0) &= T_i^{(1)}\Phi(m_i) \pm T_i^{(2)}\Phi_i(m_i) \\ &= T_i^{(1)}(\Phi_i^\pm(m_i)) \mp (T_i^{(1)} - T_i^{(2)})\Phi_i(m_i) \end{aligned}$$

we see that

$$\|\Phi_i^\pm(0)\| \le m_ie^{-3Bm_i/4} + C(\alpha e^{-B/4})^{m_i}$$

goes to zero as $m_i \to \infty$.

Step 5. *Completion of the proof*

By the last fact, $\|\Phi(0)\| - \|\Phi(2m_i)\| \to 0$ which is only consistent with $u \in \ell^2$ if $\|\Phi(0)\| = 0$ which implies that $u = 0$. $\square$

For the continuum (Schrödinger case), here are the changes: We can suppose (2.2) holds, but with e^{-Bm_i} replaced by $e^{-m_i^2}$ (any $f(m)$ with $\lim_{i\to\infty} m_i^{-1} \ln f(m)^{-1} = \infty$ will do). We normalize u so that

$$\int \left[u(x)^2 + u'(x)^2\right] dx = 1. \tag{2.10}$$

Step 1. By (2.10) and a Sobolev estimate, u and u' are uniformly bounded so $\left|\frac{dW}{dx}(u, u_i)(x)\right| \leq Ce^{-m_i^2}$ for some C.

Step 2. $\int |W(u, u_i)| dx \leq 2$, so, by the same argument

$$|W(u, u_i)|(x) \leq (2C+1)e^{-m_i^2} \qquad \text{if } |x| \leq e^{m_i^2/2}.$$

Step 3. This is actually easier since $(u_i^+)'(m_i) = 0$ and $u_i^-(m_i) = 0$.

Step 4. This is similar. The transfer matrix is bounded by e^{Cm_i} where C is E-dependent (and goes to infinity as $E \to \infty$) which is always beaten out by $e^{-m_i^2/2}$).

Step 5 is unchanged.

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