OPERATORS WITH SINGULAR CONTINUOUS SPECTRUM: II. RANK ONE OPERATORS

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Abstract. For an operator, A, with cyclic vector φ, we study $A + \lambda P$ where P is the rank one projection onto multiples of φ. If $[\alpha, \beta] \subset \operatorname{spec}(A)$ and A has no a.c. spectrum, we prove that $A + \lambda P$ has purely singular continuous spectrum on (α, β) for a dense G_δ of λ's.

§1. Introduction

The subject of rank one perturbations of self-adjoint operators and the closely related issue of the boundary condition dependence of Sturm-Liouville operators on $[0, \infty)$ has a long history. We're interested here in the connection with Borel-Stieltjes transforms of measures (Im $z > 0$):

$$F(z) = \int \frac{d\rho(x)}{x - z} \tag{1.1}$$

where ρ is a measure with

$$\int (|x| + 1)^{-1} d\rho(x) < \infty. \tag{1.2}$$

In two fundamental papers Aronszajn [1] and Donoghue [5] related F to spectral theory with important later input by Simon-Wolff [13]. In all three works, as in ours, the function (y real)

$$G(y) = \int \frac{d\rho(x)}{(x - y)^2}$$

plays an important role. Note we define G to be $+\infty$ if the integral diverges. Note too if $G(y) < \infty$, then the integral defining F is finite at $z = y$ and so we can and will talk about $F(y)$.

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Donoghue studied the situation

$$A_\lambda = A_0 + \lambda P$$

where $P\psi = (\varphi, \psi)\varphi$ with φ a unit vector cyclic for A. $d\rho$ is then taken to be spectral measure for φ, that is,

$$(\varphi, e^{isA_0}\varphi) = \int e^{isx} d\rho(x).$$

Aronszajn studied the situation

$$H_{\text{formal}} = -\frac{d^2}{dx^2} + V(x)$$

on $[0, \infty)$ where V is such that the operator is limit point at ∞. Then, there is a one-parameter family of operators, H_θ, with boundary condition

$$u(0)\cos\theta + u'(0)\sin\theta = 0.$$

ρ is the conventional Weyl-Titchmarsh-Kodaira spectral measure for a fixed boundary condition, θ_0.

An important result of the Aronszajn-Donoghue theory is

Theorem 1. *E is an eigenvalue of A_λ (resp. H_θ) if and only if*

(i) $G(E) < \infty$

(ii) $F(E) = -\lambda^{-1}$ $\qquad$ (*resp.* $\cot(\theta - \theta_0)$).

Our goal here is to the prove the following pair of theorems:

Theorem 2. *$\{E | G(E) = \infty\}$ is a dense G_δ in $\operatorname{spec}(A_0)$ (resp. H_{θ_0}).*

Theorem 3. *$\{\lambda | A_\lambda$ has no eigenvalues in $\operatorname{spec}(A_0)\}$ (resp. $\{\theta | H_\theta$ has no eigenvalues in $\operatorname{spec}(H_{\theta_0}\}$ is a dense G_δ in $\mathbb{R}$ (resp. $[0, 2\pi]$)).*

While not stated precisely in those terms, Thm. 2 is a generalization of del Rio [4]. Gordon [8,9] has independently obtained these results by different methods.

Thm. 2 is quite easy and appears in §2. Thm. 3 is deeper and depends on some subtle estimates of F found in §3 and applied in §4 to prove Thm. 3.

The interesting applications of Thm. 3 found in §5 concern singular continuous spectrum. For example, suppose A_0 has spectrum $[0, 1]$ but has no a.c. spectrum. By general principles, A_λ has no a.c. spectrum either. Then, Thm. 3 says that for a dense G_δ of λ, A_λ has purely singular continuous spectrum. This is especially interesting because there are examples where the Simon-Wolff theory implies that for Lebesgue a.e. λ, A_λ has only point spectrum. However, this is not always the case. There exist A_0 and P so that $\operatorname{spec}(A_0 + \lambda P)$ is purely singular continuous on $(0, 1)$ for *all* λ. However, Thm. 3 implies that it *cannot* happen that the spectrum is *always* pure point.

§2. Forbidden Energies

In this section we'll essentially prove Thm. 2.

Theorem 2.1. *Let $d\rho$ be a measure obeying* (1.2)*. Let*

$$G(y) = \int \frac{d\rho(x)}{(x-y)^2}.$$

Then, $\{y|G(y)=\infty\}$ is a dense G_δ in $\operatorname{supp}(d\rho)$*, the support of $d\rho$.*

Remarks 1. By the Thm. 1, only E's with $G(E)<\infty$ can be eigenvalues of A_λ or H_θ, so E's with $G(y)=\infty$ are "forbidden energies," that is, energies which cannot be eigenvalues.

2. If $\operatorname{supp}(d\rho)$ $(=\operatorname{spec}(A))$ is perfect (no isolated points), then the theorem says the forbidden energies are locally uncountable in $\operatorname{supp}(d\rho)$.

3. Obviously, $\{y|G(y)=\infty\}\subset\operatorname{supp}(d\rho)$.

4. This says that $\{y\in\operatorname{supp}(d\rho)|G(y)<\infty\}$ has interior empty in $\operatorname{supp}(d\rho)$. Even more so is the interior empty in $\mathbb{R}$. The theorem is a stronger result than interior empty in $\mathbb{R}$ since $\operatorname{supp}(d\rho)$ might itself have interior empty in $\mathbb{R}$.

Proof. The following are fundamental facts about Borel-Stieltjes transforms and their relation to $d\rho$ (see [3]):

(1) $\lim_{\epsilon\downarrow 0} F(E+i\epsilon)\equiv F(E+i0)$ exists and is finite for Lebesgue a.e. E.
(2) $d\rho_{\rm ac}$ is supported on $\{E|\operatorname{Im} F(E+i0)>0\}$.
(3) $d\rho_{\rm sing}$ is supported on $\{E|\lim_{\epsilon\downarrow 0}\operatorname{Im} F(E+i0)=\infty\}$.

If $G(y)<\infty$, it is easy to see that $\lim_{\epsilon\downarrow 0} F(E+i0)$ exists is finite and real. Thus, if $G(y)<\infty$ on an interval $(\alpha,\beta)\subset\mathbb{R}$, then by (2), (3), $d\rho(\alpha,\beta)=0$, that is, $(\alpha,\beta)\cap\operatorname{supp}(d\rho)=\emptyset$. Thus, $\{y|G(y)=\infty\}$ is dense in $\operatorname{supp}(d\rho)$.

That $\{y|G(y)=\infty\}$ is a G_δ follows from the fact that G is lower semicontinuous. To be explicit, let

$$G_m(y) = \int \frac{d\rho(x)}{(x-y)^2+(m^{-1})^2}$$

which is a C^∞ function by (1.2) and $G(y)=\sup_m G_m(y)$. Thus

$$\begin{aligned}\{y|G(y)=\infty\} &= \{y|\forall n, \exists m G_m(y)>n\}\\ &= \bigcap_n\bigcup_m\{y|G_m(y)>n\}\end{aligned}$$

is a G_δ. $\square$

§3. The Main Technical Lemma

In this section we'll prove

Lemma 3.1. *Let $d\rho$ obey* (1.2). *Then*

$$\{F(y) | G(y) < \infty \text{ and } y \in \operatorname{supp}(d\rho)\}$$

is a countable union of nowhere dense subsets of $\mathbb{R}$.

Note that $G(y) < \infty$ implies the integral defining $F(y)$ is absolutely convergent and $F(y)$ is real. The proof will depend critically on the fact that F is the boundary value of an analytic function. That such considerations must enter is seen by

Example. Let $A \subset [0,1]$ be a nowhere dense set of positive measure (e.g., remove the middle open $\frac{1}{4}$ from $[0,1]$, the middle $\frac{1}{9}$ from the remaining two pieces, the middle $\frac{1}{16}$ $\dots$, $\frac{1}{n^2}$ at the $(n-1)$st step). Let

$$\tilde{F}(y) = |A \cap [0,y]|$$

where $|\cdot|$ is Lebesgue measure. Then $\tilde{F}$ is Lipschitz; indeed, if $x < y$, $\left|\tilde{F}(x) - \tilde{F}(y)\right| = |A \cap [x,y]| \leq |x-y|$. But $\tilde{F}[A] = [0, |A|]$ has non-empty interior. Thus for our F, we need more than just Lipschitz properties (our F is certainly not Lipschitz but $F \restriction \{y | G(y) < \alpha\}$ is the restriction of a Lipschitz function to that set).

The idea of the proof will be to break up $\{y | G(y) < \infty,\ y \in \operatorname{supp}(d\rho)\}$ into a countable union of nowhere dense sets, A_n, so that F is a homeomorphism on each of those sets. On each A_n, G will be close to constant. We'll use:

Lemma 3.2. *Let $B \subset \mathbb{R}$ be a nowhere dense set and let $F : B \to \mathbb{R}$ be a function obeying for $x < y$, with $x, y \in B$:*

$$\alpha(y-x) < F(y) - F(x) < \beta(y-x) \tag{3.1}$$

for fixed $\alpha, \beta > 0$. Then $F[B]$ is nowhere dense.

Proof. By (3.1) F has a unique continuum extension to $\bar{B}$ obeying (3.1). $\mathbb{R} \setminus \bar{B}$ is a union of intervals (x_i, y_i) with $x_i, y_i \in \bar{B}$. Extend F to the interval by linear interpolation using slope $\frac{1}{2}(\alpha+\beta)$ on any semi-infinite subintervals of $\mathbb{R} \setminus \bar{B}$. The extended F also obeys (3.1) and so defines a homeomorphism of $\mathbb{R}$ to $\mathbb{R}$. As a homeomorphism, it takes nowhere dense sets to nowhere dense sets. $\square$

Proof of Lemma 3.1. We first break $A = \{y \in \operatorname{supp}(d\rho) | G(y) < \infty\}$ into a countable family of sets A_n so that for each n, there is $a_n > 0$, $\delta_n > 0$ so that

(i) for $y \in A_n$, $\frac{8a_n}{9} < G(y) \leq a_n$;

(ii) for $y \in A_n$, $\int\limits_{|x-y|<\delta_n} \frac{d\rho(x)}{|x-y|^2} \leq \frac{a_n}{21}$;

(iii) $\bigcup\limits_{y \in A_n} [y - \beta\delta_n, y + \beta\delta_n]$ is connected where $\beta = \frac{1}{18}\left(\frac{3}{4}\right)^{3/2}$.

Such a breakup exists for we can first break $\mathbb{R}$ into intervals $\left(\left(\frac{8}{9}\right)^{m+1}, \left(\frac{8}{9}\right)^{m}\right]$ and pigeonhole G by its values. Since $G(y) < \infty$ implies $\lim\limits_{\delta \downarrow 0} \int\limits_{|x-y|<\delta} \frac{d\rho(x)}{(x-y)^2} = 0$, we can break each such set into countably many sets where (ii) holds. Then we can break each such set into countably many sets so that (iii) holds by looking for gaps of size longer than $\delta_n \beta$.

Each A_n is nowhere dense by Thm. 2 and we'll show that on A_n, $y > x$ implies that

$$\frac{1}{3}\, a_n (y-x) < F(y) - F(x) < \frac{5}{3}\, a_n (y-x) \tag{3.2}$$

so that the lemma follows from Lemma 3.2.

Define $\ell_n = \beta \delta_n$ and $\epsilon_n = \sqrt{\frac{3}{4}}\, \ell_n$. For $y \in A_n$, let $\triangle_n(y)$ be the triangle in $\mathbb{C}$ (see Fig. 3.1):

$$\triangle_n(y) = \left\{ z | 0 < \text{Im } z \leq \epsilon_n, \left| \arg(z-y) - \frac{\pi}{2} \right| \leq \frac{\pi}{6} \right\}.$$

This is the equilateral triangle of side ℓ_n with one side parallel to the real axis at distance ϵ_n from that axis and the opposite vertex at y.

For $z \in \triangle_n(y)$, define

$$G(z) = \int \frac{d\rho(x)}{(x-z)^2} = \frac{dF}{dz}.$$

We claim that for $z \in \triangle_n$

$$|G(z) - a_n| \leq \frac{a_n}{3}. \tag{3.3}$$

Accepting (3.3) for the moment, let us prove (3.2). By the fundamental theorem of calculus, (3.3) implies for $z, z' \in \triangle_n(y)$:

$$|F(z) - F(z') - a_n(z-z')| \leq \frac{a_n}{3}\, |z - z'|. \tag{3.4}$$

By hypothesis (iii) on A_n, $\bigcup\limits_{y \in A_n} \triangle_n(y)$ is connected and so, given $y < y' \in A_n$, we can find a finite sequence $y_0 = y < y_1 < \cdots < y_n = y'$ and $z_1, \dots, z_n$ so that (see Fig. 3.2)

$$z_j \in \triangle_n(y_{j-1}) \cap \triangle_n(y_j) \qquad \text{and} \qquad |z_j - y_{j-1}| = |z_j - y_j| = |y_j - y_{j-1}|. \tag{3.5}$$

By (3.4) and (3.5)

$$|F(y) - F(y') - a_n(y-y')| \leq \frac{2a_n}{3}\, (y - y')$$

which is (3.2).

Thus we need only prove (3.3) We write

$$|G(z) - a_n| \leq |b_0| + |b_1| + |b_2| + |b_3|$$

where

$$
\begin{aligned}
b_0 &= G(y) - a_n \\
b_1 &= \int\limits_{|x-y|<\delta_n} \frac{d\rho(x)}{|x-z|^2} \\
b_2 &= \int\limits_{|x-y|<\delta_n} \frac{d\rho(x)}{(x-y)^2} \\
b_3 &= \tilde{G}(z) - \tilde{G}(y)
\end{aligned}
$$

with

$$\tilde{G}(z) = \int\limits_{|x-y|\geq\delta_n} \frac{d\rho(x)}{(x-z)^2}.$$

By hypothesis (i) on A_n, $|b_0| \leq \frac{a_n}{9}$.

By hypothesis (ii) on A_n, $|b_2| \leq \frac{a_n}{21}$. By elementary trigonometry,

$$z \in \triangle(y) \qquad \text{and} \qquad x \in \mathbb{R} \Rightarrow |z-y| \geq \sqrt{\frac{3}{4}}\,|x-y|. \tag{3.6}$$

Thus

$$|b_1| \leq \frac{4}{3}\,|b_2|$$

so

$$|b_1| + |b_2| \leq \frac{7}{3}\frac{a_n}{21} = \frac{a_n}{9}.$$

Finally, using the fundamental theorem of calculus and (3.6)

$$
\begin{aligned}
|b_3| &\leq 2|z-y| \left(\frac{4}{3}\right)^{3/2} \delta_n^{-1} \int\limits_{|x-y|\geq\delta_n} \frac{d\rho(x)}{(x-y)^2} \\
&\leq 2\delta_n^{-1}\ell_n \left(\frac{4}{3}\right)^{3/2} a_n \\
&= 2\beta \left(\frac{4}{3}\right)^{3/2} a_n = \frac{a_n}{9}
\end{aligned}
$$

by definition of the constant β. Thus (3.3) holds. $\square$

§4. Proof of the Main Theorem

Our goal here is to prove Thm. 3 (from §1) and derive some simple abstract consequences of it.

Proof of Thm. 3. The maps $M_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ by $M_1(\lambda) = -\lambda^{-1}$ and $M_2 : [0, \pi) \to \mathbb{R} \cup \{\infty\}$ by $M_2(\theta) = \cot(\theta - \theta_0)$ are homeomorphisms. Thus, by Lemma 3.1,

$$\{\lambda | \exists E \text{ s.t. } G(E) < \infty, E \in \operatorname{spec}(A_0), F(E) = -\lambda^{-1}\}$$

and

$$\{\theta | \exists E \text{ s.t. } G(E) < \infty, E \in \operatorname{spec}(A_0), F(E) = \cot(\theta - \theta_1)\}$$

are countable unions of nowhere dense sets. Its complement is thus a dense set by the Baire category theorem. But by Thm. 1, this is precisely $\{\lambda | A_\lambda$ has no eigenvalues on $\operatorname{spec}(A_0)\}$, which we conclude is dense. By general principles [12], it is also a G_δ. $\square$

Here are some simple corollaries of Thm. 3. We state them in the rank one case but they hold in the $\cot(\theta - \theta_0)$ B.C. case also.

Corollary 4.1. *Suppose that A_0 is an operator with no a.c. spectrum and P is a rank one projection whose range is cyclic for A. Then for a dense G_δ of λ's, $A_\lambda = A + \lambda P$ has only singular continuous spectrum in* $\operatorname{spec}(A_0)^{\text{int}}$.

Proof. A_λ has no a.c. spectrum since the a.c. spectrum is left invariant by finite rank perturbations. $\operatorname{spec}(A_0)$ has no eigenvalues for a dense G_δ of λ. There can be eigenvalues on $\mathbb{R} \setminus \operatorname{spec}(A_0)$ and so point spectrum on $\partial(\operatorname{spec}(A_0))$. But there cannot be point spectrum in $\operatorname{spec}(A_0)^{\text{int}}$.

Corollary 4.2. *Suppose that A_0 is an operator with no a.c. spectrum and an interval $[\alpha, \beta] \subset \operatorname{spec}(A_0)$. Let P be a rank one projection whose range is a cyclic vector for A_0. Then for a dense G_δ of λ's, $A_0 + \lambda P$ has singular continuous spectrum on all of (α, β) and only singular continuous spectrum there.*

Proof. A direct consequence of Cor. 4.1. $\square$

§5. Examples and Consequences

Our first class of examples are half-line Schrödinger operators:

Theorem 5.1. *Let $V(x)$ be a locally L_1 function on $[0, \infty)$ and let $H_\theta = -\frac{d^2}{dx^2} + V(x)$ with θ boundary conditions. Suppose there is some θ_0 and $\alpha < \beta$ so that*

(i) $[\alpha, \beta] \subset \operatorname{spec}(H_{\theta_0})$.

(ii) *for Lebesgue a.e., $E_0 \in [\alpha, \beta]$, there exists a function φ_{E_0} obeying*

$$-\varphi''(x) + V(x)\varphi(x) = E_0 \varphi(x) \tag{5.1}$$

$$\int_0^\infty |\varphi(x)|^2 \, dx < \infty. \tag{5.2}$$

Then:

(a) *For a dense G_δ of E's in $[\alpha, \beta]$, there is no solution of* (5.1) *obeying* (5.2).

(b) *For Lebesgue a.e. θ, H_θ has only point spectrum in (α, β).*

(c) *For a dense G_δ of θ, H_θ has only singular continuous spectrum in (α, β).*

Remarks. 1. (b), (c) say that there are disjoint sets both locally uncountable where H_θ shifts between purely pure point and purely singular continuous spectrum on $[\alpha, \beta]$.

2. It may well happen that there are θ's with spectrum of mixed type.

3. For precursors of (a), see del Rio [4].

Proof. If E is such that (5.1) has a solution obeying (5.2), then φ_E obeys some boundary condition at $x = 0$ and so E is an eigenvalue of some H_θ. Thus (a) follows from Thm. 2.

To prove (b), note that if E_0 has a solution obeying (5.1–2) and E_0 is not an eigenvalue of H_{θ_0}, then $\lim_{\epsilon \downarrow 0} \int_0^\infty |G(0, x; E + i\epsilon)|^2 \, dx < \infty$. Now apply the ideas of Kotani [11] and Simon-Wolff [13].

(c) follows from Thm. 3. $\square$

Example 5.2. Suppose that $[a, b] \subset \operatorname{spec}(-\frac{d^2}{dx^2} + V(x))$ and that for a.e. $E \subset [a, b]$, $\lim_{x\to\infty} \frac{1}{|x|} \ln \|T_E(x)\| = \gamma(E)$ and is positive. Here T is the standard transfer matrix, that is,

$$T_E(x) = \begin{pmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{pmatrix}$$

where φ_i obeys $-u'' + Vu = Eu$ with $\varphi_1(0) = \varphi_2'(0) = 1$ and $\varphi_1'(0) = \varphi_2(0) = 0$. Then (a) implies there must be a dense G_δ of E where either $\lim \frac{1}{|x|} \|T_E(x)\|$ fails to exist or is zero. Thus, a positive limit can *never* exist for all E in an interval. Results of this genre have been found previously by Goldsheid [6] and Carmona [2].

Example 5.3. Consider a one-dimensional random model with localization, for example, the GMP model [7,2]. Then for almost every E in $[\alpha, \infty)$, one knows $\gamma(E)$ exists and is positive. It follows from Thm. 5.1 that for a locally uncountable set of boundary conditions (a Lebesgue typical set), one has pure point spectrum, while for a distinct set of locally uncountable boundary conditions (a Baire typical set), one has singular spectrum. Each spectral type is unstable to change to the other spectral type.

Example 5.4. Let $H = -\frac{d^2}{dx^2} + \cos(\sqrt{x})$ on $L^2(0, \infty)$, a model studied by Stolz [14]. As proven by him for any boundary condition θ:

$$\operatorname{spec}(H_\theta) = [-1, \infty).$$

$\operatorname{spec}(H_\theta)$ is purely absolutely continuous on $(1, \infty)$. Kirsch et al. [10] prove that for a.e. θ, H_θ has pure point spectrum in $[-1, 1]$ only. Our results show that for a dense G_δ of θ, the spectrum is purely singular continuous. Once again you have intertwined purely pure point and purely singular continuous spectrum.

Finally, we consider the case of the Anderson model:

Theorem 5.5. *Let V_ω be a ν-dimensional model with uniform distribution on $[a, b]$ at each set. Suppose that the corresponding Jacobi matrix h_ω has point spectrum in $[\alpha, \beta]$ for a.e. V_ω. Fix $V_\omega(n)$ typical. Then for a.e., choice $V_\omega(0) \in [a, b]$, h_ω has pure point spectrum in $[\alpha, \beta]$; but for a dense G_δ of values of $V_\omega(0)$, h_ω has purely singular spectrum in $[\alpha, \beta]$.*

This follows from Thm. 3.

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References

[1] N. Aronszajn, *On a problem of Weyl in the theory of singular Sturm-Liouville equations*, Am. J. Math. **79** (1957), 597–610.

[2] R. Carmona, *Exponential localization in one-dimensional disordered systems*, Duke Math. J. **49** (1982), 191–213.

[3] R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators*, Birkhäuser, Boston, 1990.

[4] R. del Rio, *A forbidden set for embedded eigenvalues* (to appear Proc. AMS).

[5] W. Donoghue, *On the perturbation of spectra*, Comm. Pure App. Math. **18** (1965), 559–579.

[6] I. Goldsheid, *Asymptotics of the product of random matrices depending on a parameter*, Soviet Math. Dokl. **16** (1975), 1375–1379.

[7] I. Goldsheid, S. Molchanov and L. Pastur, *A pure point spectrum of the stochastic one-dimensional Schrödinger equation*, Func. Anal. Appl. **11** (1977), 1–10.

[8] A. Gordon, *On exceptional value of the boundary phase for the Schrödinger equation of a half-line*, Russian Math Surveys **47** (1992), 260–261.

[9] ———, *Pure point spectrum under 1-parameter perturbations and instability of Anderson localization*, preprint.

[10] W. Kirsch, S. Molchanov and L. Pastur, *One dimensional Schrödinger operators with high potential barriers*, Operator Theory: Advances and Applications, vol. 57, Birkhäuser Verlag, Basel, 1992, pp. 163–170.

[11] S. Kotani, *Lyapunov exponents and spectra for one-dimensional random Schrödinger operators*, Contemp. Math. **50** (1986), 277–286.

[12] B. Simon, *Operators with singular continuous spectrum, I. General operators*, preprint.

[13] B. Simon and T. Wolff, *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Comm. Pure Appl. Math. **39** (1986), 75–90.

[14] G. Stolz, *Bounded solutions and absolute continuity of Sturm-Liouville operators*, J. Math. Anal. Appl. **169** (1992), 210–228.