# SINGULAR CONTINUOUS SPECTRUM IS GENERIC

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Abstract. In a variety of contexts, we prove that singular continuous spectrum is generic in the sense that for certain natural complete metric spaces of operators, those with singular spectrum are a dense $G_\delta$.

In the spectral analysis of various operators of mathematical physics, a key step, often the hardest, is to prove that the operator has no continuous singular spectrum, that is, that the spectral measures for the operators have only pure point and absolutely continuous parts. Examples are the absence of such spectrum for $N$-body Schrödinger operators [3,19] and for the one-dimensional random models [12,7,8,24,18].

Our goal here is to announce results that show that singular continuous spectrum is lying quite close to many operators by proving it is often generic in Baire sense. Detailed proofs and further results will appear in three papers: one for general operators [22], one for rank one perturbations [6], and one for almost periodic Schrödinger operators [23].

Precursors of our results include work on generic ergodic processes [15,21], on special energies for Schr̈odinger operators/Jacobi matrices [11,4,5]. Gordon [13,14] has independently (and presumably, before us) proven Theorem 5. His method of proof is very different from ours.

Recall that the Baire category theorem implies that if $X$ is a complete metric space, a countable intersection of dense $G_\delta$ is still a dense $G_\delta$ and if $X$ is perfect, then any dense $G_\delta$ has uncountable intersection with any open ball.

Our first two results are for one-body Schrödinger operators and for the "generic Anderson model."

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**Theorem 1 ([22]).** *Let $C_\infty(\mathbb{R}^\nu)$ denote the continuous functions on $\mathbb{R}^\nu$ vanishing at infinity in the $\|\cdot\|_\infty$ norm. Then for a dense $G_\delta$ of $V \in C_\infty(\mathbb{R}^\nu)$, $-\Delta + V$ has purely singular continuous spectrum on $(0,\infty)$.*

*Remarks.* 1. If $V(x) = 0(|x|^{-1-\epsilon})$ at infinity, it is known [1,20] that $-\Delta + V$ has absolutely continuous spectrum on $(0,\infty)$ with a possible set of eigenvalues.

2. There is a similar result ([22]) for $\{V \mid (1+x^2)^{\alpha/2} V \in C(\mathbb{R}^\nu)\}$ with norm $|||V||| = \|(1+x^2)^{\alpha/2} V\|_\infty$ so long as $\alpha < \frac{1}{2}$.

3. In one dimension, there is a similar result for Jacobi matrices [22].

**Theorem 2 ([22]).** *Let $a < b$ be fixed and let $\Omega = [a,b]^{\mathbb{Z}^\nu}$, functions $v : \mathbb{Z}^\nu \to [a,b]$ with the (compact metrizable) Tychonoff topology. Given any $v$, let $h(v)$ be the Jacobi matrix on $l^2(\mathbb{Z}^\nu)$ by*

$$(h(v)u)(n) = \sum_{|j|=1} u(n+j) + v(n)u(n). \tag{1}$$

*Then for a dense $G_\delta$ in $\Omega$, $h(u)$ has spectrum $[a-2\nu, b+2\nu]$ and the spectrum is purely singular continuous.*

*Remark.* If $\nu = 1$ or if $\nu$ is arbitrary and $b-a$ is large, it is known that if $\Omega$ is given the product measure $\underset{j\in\mathbb{Z}^\nu}{\mathrm{X}} (b-a)^{-1}\, dx_j$, then for a.e. $v$, $h(v)$ has only pure point spectrum [7,9,8,24,2]. So the generic Baire and generic Lebesgue behavior are very different.

These results are not limited to Schrödinger operators:

**Theorem 3 ([22)].** *Let $X_a$ be the family of all self-adjoint operators, $A$, on a fixed separable Hilbert space, $\mathcal{H}$ with $\|A\| \le a$. Give $X_a$ the metrizable topology of strong convergence. Then $X_a$ has a dense $G_\delta$ of operators with spec$(A) = [-a,a]$ and the spectrum purely singular continuous.*

**Theorem 4 ([22]).** *Let $A$ be a fixed self-adjoint operator. Let $\mathcal{I}_2$ be the Hilbert-Schmidt operators with Hilbert-Schmidt norm. Then for a dense $G_\delta$ of $C$'s in $\mathcal{I}_2$, the set of vectors*

$$\{\psi \mid d\mu^\psi_{(A+C)} \text{ is purely singular continuous}\} \\ \cup \{\psi \mid (A+C)\psi = E\psi; E \in \operatorname{spec}_{\rm disc}(A+C)\}$$

*span $\mathcal{H}$.*

*Remarks.* 1. $d\mu^\psi_D$ is the spectral measure $(\psi, e^{isD}\psi) = \int e^{isE}\, d\mu^\psi_D(E)$.

2. If spec$(A)$ is thin, for example, $A \equiv 0$, the vectors are the discrete eigenvectors. But if spec$(A)$ contains an interval, spec$(A+C)$ will have lots of singular continuous spectrum.

3. This is to be distinguished from the Weyl-von Neumann theorem [26,27,17] that there are $C$'s with $\|C\|_2$ arbitrarily small so that $A+C$ has only point spectrum. Here we see that generically there will be singular continuous spectrum (for $A$ suitable).

4. The result holds if $\mathcal{I}_2$ is replaced by $\mathcal{I}_p$ with $p > 1$. If $A$ has no a.c. spectrum, it even holds for $\mathcal{I}_1$.

These four theorems are rather soft with no hard estimates. More subtle is the case of rank one perturbations. We'll consider two closely related cases:

(a) $A$ is a self-adjoint operator with cyclic vector $\varphi$; let $P$ be the projection onto $\varphi$ and let $A_\lambda = A + \lambda P$.

(b) Let $H$ be the differential operator $-\frac{d^2}{dx^2} + V(x)$ on $[0, \infty)$ assumed to be limit point at infinity. $H_\theta$ is the self-adjoint operator with boundary condition $\cos\theta u(0) + \sin\theta u'(0) = 0$.

**Theorem 5 ([6]).** (a) *Suppose $A$ has an interval $[a, b]$ in its spectrum and the spectrum there has no a.c. component. Then*

(i) *There is a dense $G_\delta$, $C$, in $[a, b]$, so that if $E \in C$, then $E$ is not an eigenvalue of any $A_\lambda$.*

(ii) *For a dense $G_\delta$, $L$, of $\mathbb{R}$, $A_\lambda$ has purely singular spectrum in $[a, b]$ if $\lambda \in L$.*

(b) *Suppose for some $\theta_0$, $H_{\theta_0}$ has an interval $[a, b]$ in its spectrum and the spectrum there has no a.c. component, then*

(i) *There is a dense $G_\delta$, $C$, in $[a, b]$, so that if $E \in C$, then $E$ is not an eigenvalue of any $H_\theta$.*

(ii) *For a dense $G_\delta$, $L$, of $[0, \pi)$, $H_\theta$ has purely singular spectrum in $[a, b]$ if $\theta \in L$.*

*Remarks.* 1. Case (i) implies that under the hypothesis of $E \in C$, either $\lim\limits_{x\to\infty} \frac{1}{x} \ln \|T_E(x)\|$ fails to exist or is 0 when $T(E)$ is the fundamental matrix for the problem. This means that for many cases where one can only prove Lyapunov behavior for a.e. $E$, there really is a set where the Lyapunov behavior fails [11,4,5].

2. There are also results for general $A$ without any hypothesis on spec($A$) or absolute continuous spectrum.

3. These results imply that in the Anderson model in the localized regime, varying $V(0)$ a little can produce singular spectrum. Indeed, there are disjoint, locally uncountable sets with purely pure point spectrum when $V(0)$ is in one set and pure singular continuous spectrum when $V(0)$ is in the other set!

Another subtle class are almost periodic Schrödinger operators. We'll consider functions $V$ on $\mathbb{R}$ or $\mathbb{Z}$ that are even and almost periodic (typical examples are $V(n) = \lambda\cos(\pi\alpha n)$ in the $\mathbb{Z}$ case and $V(x) = \lambda\cos(\pi x) + \mu\cos(\pi\alpha x)$ in the $\mathbb{R}$ case with $\alpha$ irrational) and define

$$H_\omega = -\frac{d^2}{dx^2} + V_\omega(x) \qquad \mathbb{R}\text{ case}$$

$$(H_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n) \qquad \mathbb{Z}\text{ case}$$

where $\omega$ is a point in the hull, $\Omega$, of $V$ and $V_\omega$ the corresponding potential (in the typical cases above, $\Omega = S^1$ and $S^1 \times S^1$ with $V_\theta(n) = \lambda\cos(\pi\alpha n + \theta)$ and $V_{\theta,\psi}(x) = \lambda\cos(\pi x + \theta) + \mu\cos(\pi\alpha x + \psi)$). $\Omega$ is a compact metric space in the Bohr topology.

**Theorem 6 ([23]).** *Let $V$ be an even almost periodic potential on $\mathbb{R}$ or $\mathbb{Z}$. Then*

(a) *For a dense $G_\delta$ in the hull, $H_\omega$ has no point spectrum.*

(b) *If for some point $\omega_0$ in the hull, $H_{\omega_0}$ has no a.c. spectrum, then for a dense $G_\delta$ in the hull, $H_\omega$ has purely singular continuous spectrum.*

**Example ([23]).** In the $\mathbb{Z}$ case, if $V = \lambda\cos(\pi\alpha n+\theta)$ with $\lambda \geq 2$, and $\alpha$ irrational, then it follows that $H_\theta$ has purely singular spectrum for a dense $G_\delta$ of $\theta$. When $\lambda$ is large [25,10,16], it is known that we have pure point spectrum only for a set of $\theta$ of full Lebesgue measure. Once again, we have locally uncountable sets of parameters with point spectrum for one parameter set and singular continuous in the other.

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