Normal forms and nonlinear symmetries

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Abstract

We give some general theorems, and extensions of previous results, concerning the problem of transforming an algebra of vector fields into Poincaré normal form. By means of an unifying algebraic language, we show the possibility of obtaining either a "parallel" or "joint" normal form of the vector fields in a well definite way, which simplifies the construction of normal forms, providing a precise restriction on their structure. The application to the finite dimensional dynamical systems and to their Lie point symmetries is also discussed.

1. Introduction and notations

The problem of transforming a vector field (or an algebra of vector fields) into normal form (in the sense of Poincaré - Dulac - Birkhoff) is an old and important topic [1-9], not only for its algebraic aspects, but also for its applications in the theory of dynamical systems, especially in connection with symmetry properties [1-11].

Quite different approaches and points of view ("algebraic", "analytical", or "dynamical") for this problem can be found in the literature, and it is not rare that the differences in the language make it difficult even the comparison of (apparently unrelated but strongly connected) results.

In this paper we want to propose some general results and extensions of previous statements, in a (essentially) self-contained presentation, using a geometrical approach similar to that in [11], with an abstract and "unifying" algebraic language, but avoiding as far as possible any technicality (Sect. 2). The applications to dynamical systems and their symmetries (Lie point symmetries) are discussed in Sect. 3.

Let us recall some basic definitions and fix some notations. Let $u \in M \subseteq R^n$, where M is a smooth neighbourhood of the origin in R^n, and consider the space $\mathcal{V}$ of analytical vector fields (VF) $\varphi : M \to TM$ in R^n: they are in one-to-one correspondence with the elements of the space V of analytical functions $f : M \to R^n$; in component expansion we shall write (here and in the following, sum over repeated indices, unless otherwise stated)

$$\varphi \equiv f(u)\partial_u \equiv f_i(u)\frac{\partial}{\partial u_i} \qquad (i = 1, \dots, n) \tag{1.1}$$

We assume that $u = 0$ is an isolated fixed point for $f(u)$; $f(u)$ will be also written as a series expansion in the form:

$$f \equiv Au + \widetilde{f} \equiv \sum_{j=1}^{\infty} f_{(j)} \tag{1.2}$$

where $Au \equiv f_{(1)}$ is the linear part of f, $\widetilde{f}$ the nonlinear part, and $f_{(j)} \in V_{(j)}$, the subspace of the homogeneous polynomial functions in V of degree j.

Given two VFs $\varphi = f\partial_u$ and $\psi = g\partial_u$ in $\mathcal{V}$, the notion of Lie commutator $[\varphi, \psi]$ in $\mathcal{V}$ induces a Lie-Poisson bracket $\{f, g\}$ in V:

$$[\varphi, \psi] = \{f, g\}\partial_u \qquad \{f, g\}_k = f_i\partial_i g_k - g_i\partial_i g_k \qquad \left(\partial_i \equiv \frac{\partial}{\partial u_i}\right) \tag{1.3}$$

Also, a notion of scalar product can be introduced in each subspace $V_{(j)}$ [8,11].

Given a $n \times n$ matrix A, we denote by $\mathcal{A} : V \to V$ the homological operator associated to A (which is also the Lie derivative $\mathcal{L}_A$ along the VF $Au\partial_u$):

$$\mathcal{A}(h_k) = (Au)_i \partial_i h_k - (Ah)_k \tag{1.4}$$

where $h = h(u) \in V$. According to the classical Poincaré - Dulac - Birkhoff definition [1], a (nonlinear) term $h(u)$ is said to be *resonant with* A if (see also [8])

$$\mathcal{A}^+(h) \equiv \{A^+ u, h\} = 0 \ , \tag{1.5}$$

and a VF $\varphi = (Au + \widetilde{f})\partial_u$ is said to be in normal form (NF) if all nonlinear terms are resonant with A, i.e. $\widetilde{f} \in \mathrm{Ker}\mathcal{A}^+$. If A is diagonal, with eigenvalues $\alpha_1, \ldots, \alpha_n$, a monomial $h_k(u) = u_1^{m_1} \cdot \ldots \cdot u_n^{m_n}$ of degree j (with m_i integer numbers such that $\sum_i m_i = j, \ m_i \geq 0$) is resonant if $m_i \alpha_i = \alpha_k$, which is the usual "resonance condition" for the eigenvalues [1]. As well known, the relevance of the above definitions is essentially due to the fact that, given a VF φ, all nonresonant terms can be removed by means of a coordinate transformation. As usual in NF theory, these transformations are expressed by means of *formal* series, i.e. no assumption is made on their convergence (cf. [1]).

Notice that for both operators $\mathcal{A}$ and $\mathcal{A}^+$ one has, for each j,

$$\mathcal{A} : V_{(j)} \to V_{(j)} \quad \text{and} \quad \mathcal{A}^+ : V_{(j)} \to V_{(j)} \ . \tag{1.6}$$

If $\mathcal{A} = \mathcal{L}_A$ and $\mathcal{B} = \mathcal{L}_B$ are the homological operators associated to two $n \times n$ matrices A and B, the operator $\mathcal{AB} - \mathcal{BA}$ is just the homological operator $\mathcal{C} = \mathcal{L}_C$ associated to the matrix commutator $C = [A, B]$. In particular, the three statements $\mathcal{AB} = \mathcal{BA}$, $[A, B] = 0$, and $\{Au, Bu\} = 0$ are equivalent.

Finally, if A is any $n \times n$ matrix, we shall denote by

$$A = A_s + A_n \tag{1.7}$$

its (unique) decomposition into commuting semisimple (diagonalizable) and nilpotent part.

2. The algebraic approach.

The main results will be obtained as a consequence of a series of simple lemmas, which can be of some independent interest: even if some of these are not new, it is convenient to give a complete list of all of them, together with a sketch of their proof.

Lemma 1. If $[A,B]=0$ then $[A_s,B]=[A,B_s]=0$.

Proof. This easily follows once the matrix A (or resp. B) is written in its Jordan form.

Lemma 2. Given the matrix A, the set $\mathrm{Ker}\mathcal{A}^+$ of terms $h(u)$ resonant with A, is given by $K(\kappa(u))u$, where K is the most general matrix such that $[K,A^+]=0$ and its entries K_{ij} are functions of the time independent analytical constants of motion $\kappa=\kappa(u)$ of the linear system $\dot{u}=A^+u$ [8,11].

Proof. This follows writing explicitly the equation $\mathcal{A}^+(h)=0$ as a first order PDE, and applying standard procedures [12].

Lemma 3. $\mathrm{Ker}\mathcal{A}\subset\mathrm{Ker}\mathcal{A}_s$; $\mathrm{Ker}\mathcal{A}^+\subset\mathrm{Ker}\mathcal{A}_s$, where $\mathcal{A}_s$ is the homological operator associated to the semisimple part A_s of A.

Proof. According to Lemma 1, if K commutes with A^+, then it also commutes with A_s $(=A_s^+)$. On the other hand, the solutions of the linear systems $\dot{u}=A_su$ and $\dot{u}=A^+u$ are respectively $u(t)=\exp(A_st)u(0)$ and $u(t)=\exp(A^+t)u(0)$; it is easy to be convinced that the (time-independent) constants of motion, which can be expressed in analytic form (polynomial, in this case) of the second system are also constants of motion of the first one (but the converse is not true). The statement of Lemma 3 then follows from Lemma 2.

Lemma 4. If $\varphi=(Au+\widetilde{f})\partial_u$ and $\psi=(Bu+\widetilde{g})\partial_u$ form a 2-dimensional algebra, then it is possible to perform a (formal) coordinate transformation which takes the nonlinear terms $\widetilde{f}$ into normal form with respect to A, and $\widetilde{g}$ with respect to B ("*parallel* normal form").

Proof. Up to a linear transformation, any 2-dimensional algebra satisfies the commutation rule

$$[\varphi,\psi]=c\psi \tag{2.1}$$

where c is any constant (including $c=0$). First of all, we can always put $\widetilde{g}$ into NF with respect to B, so, let us assume (without changing notations)

$$\widetilde{g}\in\mathrm{Ker}\mathcal{B}^+$$

where $\mathcal{B}^+$ is the homological operator associated to B^+. Now, from (2.1), $[A,B]=cB$, or $[A^+,B^+]=-\bar{c}B^+$, which implies, in terms of the homological operators,

$$\mathcal{B}^+\left(\mathcal{A}^+(\widetilde{g})\right) = \mathcal{A}^+\left(\mathcal{B}^+(\widetilde{g})\right) - \bar{c}\,\mathcal{B}^+(\widetilde{g}) = 0$$

i.e. $\mathcal{A}^+(\widetilde{g}) \in \mathrm{Ker}\mathcal{B}^+$, or $\mathcal{A}^+ : \mathrm{Ker}\mathcal{B}^+ \to \mathrm{Ker}\mathcal{B}^+$. This ensures the possibility of performing another transformation, leaving invariant the space $\mathrm{Ker}\mathcal{B}^+$ of the terms resonant with B, in such a way to change the terms $\widetilde{f}$ into NF with respect to A. Then we can choose coordinates in such a way that:

$$\widetilde{f} \in \mathrm{Ker}\mathcal{A}^+ \qquad \text{and} \qquad \widetilde{g} \in \mathrm{Ker}\mathcal{B}^+ \; . \tag{2.2}$$

We can now state the first main result.

Theorem 1. Let $\varphi = f(u)\partial_u = (Au + \widetilde{f})\partial_u,\ \psi = g(u)\partial_u = (Bu + \widetilde{g})\partial_u$ satisfy

$$[\varphi,\ \psi] = 0 \tag{2.3}$$

then, by means of a formal coordinates transformation, $\widetilde{f},\ \widetilde{g}$ can be taken into a "*joint* normal form" (JNF) of this type:

$$\widetilde{f} \in \mathrm{Ker}\mathcal{A}^+ \cap \mathrm{Ker}\mathcal{B}_s \quad \text{and} \quad \widetilde{g} \in \mathrm{Ker}\mathcal{A}_s \cap \mathrm{Ker}\mathcal{B}^+ \; . \tag{2.4}$$

Proof. From Lemmas 3 and 4, we get

$$\widetilde{f} \in \mathrm{Ker}\mathcal{A}^+ \subset \mathrm{Ker}\mathcal{A}_s \; ; \quad \widetilde{g} \in \mathrm{Ker}\mathcal{B}^+ \subset \mathrm{Ker}\mathcal{B}_s \tag{2.5}$$

Let us write now (2.3) step by step, with

$$f(u) = Au + \sum_{j=2}^{\infty} f_{(j)} \quad \text{and} \quad g(u) = Bu + \sum_{j=2}^{\infty} g_{(j)} \tag{2.6}$$

We have first

$$[A, B] = 0 \tag{2.7}$$

and

$$\{Au, g_{(2)}\} - \{Bu, f_{(2)}\} = 0 \quad \text{or} \quad \mathcal{A}(g_{(2)}) = \mathcal{B}(f_{(2)}) \tag{2.8}$$

whereas, for $k > 2$,

$$\{Au, g_{(k)}\} - \{Bu, f_{(k)}\} = \sum_{j=2}^{k-1} \{f_{(j)}, g_{(k-j+1)}\} \tag{2.9}$$

Applying the operator $\mathcal{A}_s$ to (2.8), and using Lemma 1, we obtain, thanks to (2.5)

$$\mathcal{A}_s\left(\mathcal{A}(g_{(2)})\right) = \mathcal{B}\left(\mathcal{A}_s(f_{(2)})\right) = 0$$

and also $\mathcal{A}_s^2(g_{(2)}) = 0$, which implies

$$\mathcal{A}_s(g_{(2)}) = 0 \ ;$$

in fact, being A_s a diagonalizable matrix, we can choose coordinates such that $\mathrm{Ker}\mathcal{A}_s$ is the orthogonal complement to $\mathrm{Ran}\mathcal{A}_s$ in the space $V_{(2)}$. Repeating the argument for the operator $\mathcal{B}_s$ applied to (2.8), we get similarly

$$\mathcal{B}_s(f_{(2)}) = 0 \ .$$

An immediate application of the Jacobi identity shows that if

$$f_{(j)}, \ g_{(i)} \in \mathrm{Ker}\mathcal{A}_s \cap \mathrm{Ker}\mathcal{B}_s \quad \forall \ i,j = 2, \ldots, k-1$$

then the same is true for

$$\{f_{(j)}, g_{(i)}\} \quad \text{and} \quad \sum_{j=2}^{k-1} \{f_{(j)}, g_{(k-j+1)}\} \tag{2.10}$$

This allows us to proceed inductively: applying the operators $\mathcal{A}_s$ and $\mathcal{B}_s$ to (2.10), we can conclude, for all j, that $f_{(j)} \in \mathrm{Ker}\mathcal{B}_s$ and $g_{(j)} \in \mathrm{Ker}\mathcal{A}_s$, which, together with (2.5), gives the result.

The possibility of extending of the above results (namely, Lemma 4 and Theorem 1) to algebras of dimension $d > 2$ is clearly related to the specific commutation properties of the algebra. We consider here some special cases.

Theorem 2. Let us consider a $d-$dimensional algebra $\mathcal{G}$ of VFs spanned by $\varphi_a = f_a \partial_u = (A_a u + \widetilde{f}_a)\partial_u$ $(a = 1, \ldots, d)$. Then:
i) If the algebra $\mathcal{G}$ is *solvable*, then all the nonlinear terms $\widetilde{f}_a$ can be put in parallel NF, namely

$$\widetilde{f}_a \in \mathrm{Ker}\mathcal{A}_a^+ \qquad \text{for each } a = 1, \ldots, d \ . \tag{2.11}$$

ii) If the algebra $\mathcal{G}$ is *nilpotent* (in particular: abelian), then one can put all $\widetilde{f}_a$ into a JNF, precisely (with obvious notations):

$$\widetilde{f}_a \in \Big(\bigcap_{b \neq a} \mathrm{Ker}\mathcal{A}_{b,s} \Big) \cap \mathrm{Ker}\mathcal{A}_a^+ \qquad \text{for each } a = 1, \ldots, d \tag{2.12}$$

iii) In any solvable (resp.: nilpotent, or in particular abelian) *subalgebra* of a generic algebra $\mathcal{G}$, all nonlinear terms can be put parallel NF as in (2.11) (resp.: in JNF as in (2.12)).
Proof. If the algebra is solvable, let us consider the sequence of commutators terminating in 0

$$[\varphi, \varphi] = \varphi^{(1)}, \ [\varphi^{(1)}, \varphi^{(1)}] = \varphi^{(2)}, \ \ldots, \ [\varphi^{(m)}, \varphi^{(m)}] = 0 \tag{2.13}$$

In the ideal $\mathcal{G}^{(m)}$ spanned by $\varphi^{(m)}$ all non linear terms of the VFs can be taken in NF (or even in JNF if dim $\mathcal{G}^{(m)} > 1$: in this case indeed this subalgebra is abelian and Theorem 1 can be directly applied). Using $[\varphi^{(m-1)}, \varphi^{(m-1)}] = \varphi^{(m)}$ and $\mathcal{G}^{(m-1)} \supseteq \mathcal{G}^{(m)}$, we can repeat the argument of Lemma 4 to show that also in $\mathcal{G}^{(m-1)}$ parallel NFs can be obtained, and so on. If now the algebra $\mathcal{G}$ is nilpotent, let us consider the sequence of commutators terminating in 0

$$[\varphi, \varphi] = \varphi^{[1]}, \ [\varphi, \varphi^{[1]}] = \varphi^{[2]}, \ \ldots, \ [\varphi, \varphi^{[m]}] = 0 \tag{2.14}$$

The first part of this Theorem ensures (since nilpotency implies solvability) that all $\widetilde{f}_a$ can be taken in their respective NF: $\widetilde{f}_a \in \mathrm{Ker}\mathcal{A}_a^+$; on the other hand, the last commutator in (2.14) says that all fields in the abelian ideal $\mathcal{G}^{[m]}$ spanned by $\varphi^{[m]}$ commute with all the $\varphi_a \in \mathcal{G}$. Then the procedure followed in the proof of Theorem 1 can be repeated for each $\varphi_a \in \mathcal{G}$, using the last commutator in (2.14) in order to obtain (2.12). Statement iii) is an immediate consequence.

Remark 1. The result in Theorem 1 and its extension in Theorem 2.ii) are generalizations of Theorem 2.2 [1] of Ref. [7], which gives in fact $\widetilde{f}_a \in \bigcap_b \mathrm{Ker}\mathcal{A}_{b,s}$. Notice that actually condition $\widetilde{f} \in \mathrm{Ker}\mathcal{A}^+$ is a rather stronger restriction for $\widetilde{f}$ than $\widetilde{f} \in \mathrm{Ker}\mathcal{A}_s$, the space $\mathrm{Ker}\mathcal{A}^+$ being in general considerably smaller than the space $\mathrm{Ker}\mathcal{A}_s$, as simple examples can easily show. Notice also that, in general, it is not possible to extend the result in Theorem 1 (and in 2.ii) as well) to have also e.g. $\widetilde{f} \in \mathrm{Ker}\mathcal{B}$ or $\widetilde{f} \in \mathrm{Ker}\mathcal{B}^+$. The case of solvable algebras is quite different: e.g., if $[\varphi, \psi] = \psi$, then $[A, B] = B$, but this implies $B_s = 0$, and therefore one gets in this case $\mathcal{B}_s = 0$.

Remark 2. In NFs theory it is usual to give special attention to the case of VFs with *normal* linear part, i.e. $[A, A^+] = 0$ [11]. Here, we will not consider this restriction: the general results given here can of course be specialized to this case (obtaining among others some of the results given in [11]). For instance, eq. (2.12) becomes immediately, with this restriction,

$$\widetilde{f}_a \in \bigcap_b \mathrm{Ker}\mathcal{A}_b \ . \tag{2.12$'$}$$

[1] Any 2-dimensional nilpotent algebra is in fact abelian. Unfortunately, Ref. [7] came to our knowledge only after our paper [11] - which contains results already obtained in [7], although by different methods - was published.

3. Applications to dynamical systems and their symmetry properties

Let us now apply the above algebraic results to the case of (finite dimensional) dynamical systems (DS). With $u = u(t) \in M \subseteq R^n$, let

$$\dot{u} = f(u) = Au + \widetilde{f}(u) \tag{3.1}$$

be a DS, where $\dot{u} = du/dt$ and $f(0) = 0$.

Denoting by $\varphi \equiv f\partial_u$ the VF expressing the dynamical flow of this DS, any VF $\psi = g\partial_u$ such that

$$[\varphi, \psi] = 0 \tag{3.2}$$

is the generator of a Lie-point time-independent (LPTI) symmetry of this DS [13-15] (see also [10-11,16-20] and Ref. therein). Therefore, according to Theorem 1, one can choose coordinates in such a way that both VFs φ and ψ are in JNF (2.4). In concrete cases, once the DS is given, a typical problem is that of finding its LPTI symmetries: then, the set of eq.s (2.7-9) may be used in practice in order to construct recursively step by step the symmetry field, and the JNF condition (2.4) determines the nonlinear terms which may be removed, both in the DS and in the VF describing the symmetry.

Remark 3. Clearly, it is not granted, in general, that all LPTI symmetries of a DS can be written as a series expansion (even if formal; for some examples of "singular" LPTI symmetries, see e.g. [17-18]); however, the method of proceeding step by step may be useful to construct "approximate" symmetries, i.e. "up to the a given (finite) order" [16]). Some sufficient conditions ensuring the existence of (polynomial) LPTI symmetries, and some explicit examples can be found in Ref [10,11,17-20]. A (linear) symmetry which is always present (unless A_s, the semisimple part of A, is $= 0$) is given by the following Proposition.

Proposition 1. Any DS (3.1) which is in NF, i.e. with $\widetilde{f} \in \mathrm{Ker}\mathcal{A}^+$, admits the linear symmetry generated by

$$\sigma = A_s u \partial_u \tag{3.3}$$

If A_s is diagonalized, with real eigenvalues α_i $(i = 1, \ldots, n)$, this symmetry generates the scaling $u_i \to u_i\ \exp(\varepsilon\alpha_i)$ $(\varepsilon \in R)$.

Proof. The symmetry condition (3.2) is certainly satisfied by this σ (3.3), indeed

$$\{Au + \widetilde{f}, A_s u\} = \{\widetilde{f}, A_s u\} = 0$$

as a consequence of the resonance assumption and Lemma 3.

This generalizes Proposition 4 of Ref. [10], where a diagonalizable A was assumed, and another result contained in Ref. [8] which - in the present language - may be stated as follows: if the DS (3.1) is in NF, then $(A^+u)\partial_u$ is a symmetry for the *nonlinear* part of the DS $\dot{u} = \widetilde{f}$ (but not necessarily for the *full* DS $\dot{u} = f$). Another property of LPTI symmetries and NFs, which is a direct consequence of (2.4), is given by the following Proposition.

Proposition 2. If the DS (3.1) admits a LPTI symmetry $\psi = g\partial_u = (Bu + \widetilde{g})\partial_u$, and the fields φ, ψ are in JNF, then ψ is also a symmetry for the linear semisimple part of the DS, i.e. for $\dot{u} = A_s u$ (but the converse is not true: i.e. symmetries of this linear system are not necessarily symmetries for the full DS), and the linear semisimple part $B_s u\partial_u$ provides another symmetry (if $B_s \neq 0$) for the DS:

$$\{A_s u, g\} = 0 \quad \text{and} \quad \{B_s u, f\} = 0 \ . \tag{3.4}$$

Clearly, the set $\{\psi_1, \ldots, \psi_r\}$ of the LPTI symmetries of a DS spans a Lie algebra [2] $\mathcal{G}$: it always contains the VF φ giving the dynamical flow. This algebra may be abelian (as in Proposition 3 below) or not. An interesting example of nonabelian symmetry algebra can be constructed starting from a 4-dimensional problem which has the "quaternionic structure" [21]: the generators of its symmetry span the Lie algebra of the group $SU(2)$ (in real form).

The possibility of taking in parallel or joint NF the VFs $\{\varphi, \psi_1, \ldots, \psi_r\}$ in this algebra depends on the properties of the algebra itself, according to Theorem 2. In any case, however, being $[\varphi, \psi_a] = 0$ for all $a = 1, \ldots, r$, by the definition of symmetry, the JNF is possible for φ and *one* at least of the ψ_a, or better for all the ψ_a which span an abelian subalgebra $\mathcal{H} \subseteq \mathcal{G}$.

Let us consider finally the special case of *linearizable* DSs, i.e. DSs for which all terms are nonresonant and therefore can be removed by a (formal) coordinate transformation.

[2] Actually, multiplying a LPTI symmetry by any constant of motion of the DS gives another symmetry, so one should more correctly speak of a (finite dimensional) *module* (rather than of an infinite dimensional algebra) of symmetries [20]. Here, we are interested in the algebraic structure, and therefore we are considering only "independent" (i.e. pointwise linearly independent) VFs generating symmetries.

Remark 4. According to the definition, a DS (3.1) is linearizable if $\mathrm{Ker}\mathcal{A}^+ = \{0\}$. But a simple consequence of JNF indicates also that a condition ensuring the linearizability of a DS is that it admits a LPTI symmetry $\psi = (Bu + \widetilde{g})\partial_u$ such that $\mathrm{Ker}\mathcal{A}^+ \cap \mathrm{Ker}\mathcal{B}_s = \{0\}$. More in general, the same is true if $\left(\bigcap_h \mathrm{Ker}\mathcal{B}_{h,s}\right) \cap \mathrm{Ker}\mathcal{A}^+ = \{0\}$ where $\psi_h = (B_h u + \widetilde{g}_h)\partial_u \in \mathcal{H}$, and $\mathcal{H}$ is an abelian subalgebra contained in the algebra $\mathcal{G}$ of the admitted symmetries, as said before.

We also have:

Proposition 3. If a DS can be linearized, then it admits n independent *commuting* symmetries, which can be simultaneously taken into *linear* form by a coordinate transformation. If, in particular, the system has a diagonalizable A with real eigenvalues, then - once it is linearized and A is diagonal - the dilations $\zeta_i = u_i\partial_i$ (no sum over i) along each direction i, are n linear commuting symmetries for the system. Conversely, if there is a coordinate system where the DS admits n independent linear commuting symmetries $\sigma_i = B_i u\partial_u$ such that all B_i are semisimple, then the DS can be linearized.

Proof. In the coordinates where the DS is linear, it is easy to construct n linear commuting symmetries $B_i u\partial_u$, simply choosing n independent matrices B_i commuting among themselves and with A (if the matrices $I, A, A^2, \ldots, A^{n-1}$ are linearly independent, they immediately provide the matrices B_i ; even if this is not the case, the existence of n matrices B_i with the required properties is easily verified if A is put in Jordan form). The existence of the n independent scalings ζ_i in the case of diagonal A is obvious. Conversely, given n semisimple commuting matrices B_i, they can be simultaneously diagonalized: $B_i \to$ diag $(\beta_1^{(i)}, \ldots, \beta_n^{(i)})$; now, with respect to the basis spanned by the n (independent) vectors $\beta^{(i)} \equiv (\beta_1^{(i)}, \ldots, \beta_n^{(i)})$, the symmetries $\sigma_i = B_i u\partial_u$ become $\sigma_i \to \zeta_i = u_i\partial_i$ (no sum over i), i.e. the independent dilations along the directions $\beta^{(i)}$. A DS admitting such n symmetries is necessarily linear.

Acknowledgments.
We are grateful to prof. Giuseppe Marmo for useful discussions.

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